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Fixed Point Theorem for Cyclic (μ, ψ, ϕ)-Weakly Contractions via a New Function

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Abstract. In this paper, we introduce a generalization of cyclic (μ, ψ, ϕ)-weakly contraction via a new function and derive the existence of fixed point for such mappings in the setup of complete metric spaces. Our results extend and improve some fixed point theorems in the literature.

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1 Introduction

It is well known that the fixed point theorem of Banach, for contraction mappings, is one of the pivotal result in analysis. It has been used in different fields of mathematics. Fixed point problems involving different types of contractive inequalities have been studied by many authors (see [1]-[18] and references cited therein).

Ya. I. Alber and S. Guerre-Delabriere [1] introduced the concept of weakly contractive mappings and proved the existence of fixed points for single-valued weakly contractive mappings in Hilbert spaces. Thereafter, in 2001, B. E. Rhoades [18] proved the fixed point theorem which is one of the generalizations of Banach's Contraction Mapping Principle, because the weakly contractions contains contractions as a special case and he also showed that some results of [1] are true for any Banach space. In fact, weakly contractive mappings are closely related to the mappings of D. W. Boyd and T. S. W. Wong [2] and of S. Reich types [17].

In [15], W. A. Kirk et al. introduced the following notion of cyclic representation and characterized the Banach Contraction Principle in the context of cyclic mapping.

Definition 1.1. [15] *Let X be a non-empty set and $T\colon X \to X$ an operator. By definition, $X = \cup_{i=1}^{m} X_i$ is a cyclic representation of X with respect to T if:*

1. *X_i with $i = 1, \dots, m$ are non-empty sets,*

2. *$T(X_1) \subset X_2, \dots, T(X_{m-1}) \subset X_m$, $T(X_m) \subset X_1$.*

M. Pacurar and I.A. Rus [16] proved the following important result o in fixed point theory. We state an analogue of this result as follows.

Theorem 1.1. [16] *Let (X, d) be a complete metric space, $m \in \mathbb{N}$, $A_1, A_2, \dots, A_m$ be non-empty closed subsets of X with , $A_{m+1} = A_1$, $Y = \cup_{i=1}^{m} A_i$, $\phi\colon [0, \infty) \to [0, \infty)$ be monotone increasing continuous functions with*

$$\begin{cases} \phi(t) > 0, & \text{if } t > 0, \\ \phi(0) = 0, \end{cases}$$

and $T\colon Y \to Y$ be an operator. Assume that:

1. *$\cup_{i=1}^{m} A_i$ is a cyclic representation of Y with respect to T,*

2. *$d(Tx, Ty) \leq d(x, y) - \phi(d(x, y))$,*

for any $x \in A_i$, $y \in A_{i+1}$ with $i = 1, 2, \dots, m$. Then T has a unique fixed point $z \in \cap_{i=1}^{n} A_i$.

In 2013, S. Chandok and V. Popa [7] introduced the notion of cyclic (μ, ψ, ϕ)-weakly contraction mappings and they also derived a fixed point theorem for such cyclic contractions, in the framework of complete metric spaces.

In this paper, we introduce a generalization of cyclic (μ, ψ, ϕ)-weakly contraction and derive the existence of a fixed point for such mappings in the setup of complete metric spaces. Our results extend and improve some fixed point theorems in the literature.

2 Background

Throughout this paper, $\mathbb{N}$ stands for the set of all positive integers and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. We introduce our notion of cyclic (μ, ψ, ϕ)-weakly contraction mappings in metric space.

Let θ denote the set of all monotone increasing continuous functions $\mu\colon [0,\infty) \to [0,\infty)$, with

$$\begin{cases} \mu(t) > 0, & \text{if } t > 0, \\ \mu(0) = 0, & \\ \mu(t_1 + t_2) \leq \mu(t_1) + \mu(t_2), & \text{for all } t_1, t_2 \in [0,\infty). \end{cases}$$

Let Φ denote the set of all continuous functions $\phi\colon [0,\infty) \to [0,\infty)$ such that

$$\begin{cases} \phi(t) > 0, & \text{if } t > 0, \\ \phi(0) \geq 0. & \end{cases}$$

Let $C - class$ denote the set of all functions $f : [0,\infty)^2 \to [0,\infty)$ such that

1. f is continuous which is increasing in first variable,
2. for all $t, s \in [0,\infty)$, $f(s,t,) \leq s$,
3. $f(s,t) = s \Longrightarrow s = 0$ or $t = 0$.

Let Ψ denote the set of all functions $\psi : [0,\infty)^5 \to [0,\infty)$ such that

1. ψ is continuous,
2. ψ is strictly increasing in all the variables,
3. For all $t \in (0,\infty)$, we have $\psi(t,t,t,0,2t) \leq t$, $\psi(t,t,t,2t,0) \leq t$, $\psi(0,0,t,t,0) \leq t$, $\psi(0,t,0,0,t) \leq t$ and $\psi(t,0,0,t,t) \leq t$.

Example 2.1. Let $s, t \in [0,\infty)$, then the following functions are of $C-class$:

1. $f(s,t) = s - t$, $\quad f(s,t) = s \quad \Rightarrow \quad t = 0$,
2. $f(s,t) = \frac{s-t}{1+t}$, $\quad f(s,t) = s \quad \Rightarrow \quad t = 0$,
3. $f(s,t) = \frac{s}{1+t}$, $\quad f(s,t) = s \quad \Rightarrow \quad s = 0$ or $t = 0$,
4. $f(s,t) = s - \frac{t}{1+t}$, $\quad f(s,t) = s \quad \Rightarrow \quad t = 0$,
5. $f(s,t) = s \log_{a+t} a$, $a \in (1,\infty)$, $\quad f(s,t) = s \quad \Rightarrow \quad s = 0$ or $t = 0$,

6. $f(s,t) = s - \frac{s}{1+t}$, $\quad f(s,t) = s \quad \Rightarrow \quad s = 0.$

Definition 2.1. *Let (X, d) be a metric space, m be a natural number, $A_1, A_2 \dots, A_m$ be non-empty subsets of X and $Y = \cup_{i=1}^{m} A_i$. An operator $T\colon Y \to Y$ is called a cyclic (μ, ψ, ϕ)-f-weakly contraction if*

1. *$\cup_{i=1}^{m} A_i$ is a cyclic representation of Y with respect to T,*
2. $$\mu\big(d(Tx, Ty)\big) \le f\left(\psi\Big(\mu\big(d(x,y)\big), \mu\big(d(x,Tx)\big), \mu\big(d(y,Ty)\big), \mu\big(d(x,Ty)\big), \mu\big(d(y,Tx)\big)\Big), \phi\big(M(x,y)\big)\right),$$

   for all $x \in A_i$, $y \in A_{i+1}$, $i = 1, 2, \dots, m$ where $A_{m+1} = A_1$, $\mu \in \theta$, $\phi \in \Phi$, $\psi \in \Psi$ and $M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty)\}$.

Definition 2.2. *Let (X, d) be a metric space, m be a natural number, $A_1, A_2 \dots, A_m$ be non-empty subsets of X and $Y = \cup_{i=1}^{m} A_i$. An operator $T\colon Y \to Y$ is called a cyclic (μ, ψ, ϕ)-raitional-weakly contraction if*

1. *$\cup_{i=1}^{m} A_i$ is a cyclic representation of Y with respect to T,*
2. $$\mu\big(d(Tx, Ty)\big) \le \frac{\psi\Big(\mu\big(d(x,y)\big), \mu\big(d(x,Tx)\big), \mu\big(d(y,Ty)\big), \mu\big(d(x,Ty)\big), \mu\big(d(y,Tx)\big)\Big)}{1 + \phi\big(M(x,y)\big)}$$

for all $x \in A_i$, $y \in A_{i+1}$, $i = 1, 2, \dots, m$ where $A_{m+1} = A_1$, $\mu \in \theta$, $\phi \in \Phi$, $\psi \in \Psi$ and $M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty)\}$.

3 Main results

To state and prove our main results, we need the following lemma in the sequel.

Lemma 3.1. *For every positive real number ϵ, there exists a natural number n such that if $r, q \ge n$ with $r - q \equiv 1 \pmod{m}$, then $d(x_r, x_q) < \epsilon$.*

Proof. Assume the contrary. Thus there exists $\epsilon > 0$ such that for any $n \in \mathbb{N}$, we can find $r_n > q_n \geq n$ with $r_n - q_n \equiv 1 \pmod m$ satisfying $d(x_{r_n}, x_{q_n}) \geq \epsilon$.

Now, we take $n > 2m$. Then, corresponding to $q_n \geq n$, we can choose r_n in such that it is a smallest integer with $r_n > q_n$ satisfying $r_n - q_n \equiv 1 \pmod m$ and $d(x_{r_n}, x_{q_n}) \geq \epsilon$. Therefore, $d(x_{r_{n-m}}, x_{q_n}) < \epsilon$. By using the triangular inequality, we have

$$\begin{aligned} \epsilon &\leq d(x_{q_n}, x_{r_n}) \\ &\leq d(x_{q_n}, x_{r_{n-m}}) + \sum_{i=1}^{m} d(x_{r_{n-i}}, x_{r_{n-i+1}}) \\ &< \epsilon + \sum_{i=1}^{m} d(x_{r_{n-i}}, x_{r_{n-i+1}}). \end{aligned}$$

Letting $n \to \infty$ and using $d(x_{n+1}, x_n) \to 0$, we obtain

$$\lim d(x_{q_n}, x_{r_n}) = \epsilon. \tag{3.1}$$

Again, by the triangular inequality,

$$\begin{aligned} \epsilon &\leq d(x_{q_n}, x_{r_n}) \\ &\leq d(x_{q_n}, x_{q_{n+1}}) + d(x_{q_{n+1}}, x_{r_{n+1}}) + d(x_{r_{n+1}}, x_{r_n}) \\ &\leq d(x_{q_n}, x_{q_{n+1}}) + d(x_{q_{n+1}}, x_{q_n}) + d(x_{q_n}, x_{r_n}) + d(x_{r_n}, x_{r_{n+1}}) + d(x_{r_{n+1}}, x_{r_n}). \end{aligned}$$

Letting $n \to \infty$ and using $d(x_{n+1}, x_n) \to 0$, we get

$$\lim d(x_{q_{n+1}}, x_{r_{n+1}}) = \epsilon. \tag{3.2}$$

Consider

$$\begin{aligned} d(x_{q_n}, Tx_{r_n}) &= d(x_{q_n}, x_{r_{n+1}}) \\ &\leq d(x_{q_n}, x_{r_n}) + d(x_{r_n}, x_{r_{n+1}}), \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} d(x_{r_n}, Tx_{q_n}) &= d(x_{r_n}, x_{q_{n+1}}) \\ &\leq d(x_{r_n}, x_{q_n}) + d(x_{q_n}, x_{q_{n+1}}). \end{aligned} \tag{3.4}$$

Taking $n \to \infty$ in the inequalities (3.3) and (3.4), we have

$$\lim_{n\to\infty} d(x_{q_n}, Tx_{r_n}) = \epsilon, \tag{3.5}$$

and

$$\lim_{n\to\infty} d(x_{r_n}, Tx_{q_n}) = \epsilon. \tag{3.6}$$

As x_{q_n} and x_{r_n} lie in different adjacently labeled sets A_i and A_{i+1} for certain $1 \leq i \leq m$, using the fact T is a cyclic (μ, ψ, ϕ)-weakly contraction, we obtain

$$\begin{aligned}
\mu(\epsilon) &\leq \mu\big(d(x_{q_{n+1}}, x_{r_{n+1}})\big) \\
&= \mu\big(d(Tx_{q_n}, Tx_{r_n})\big) \\
&\leq f\Big(\psi\Big(\mu\big(d(x_{q_n}, x_{r_n})\big), \mu\big(d(x_{q_n}, Tx_{q_n})\big), \mu\big(d(x_{r_n}, Tx_{r_n})\big) \\
&\quad , \mu\big(d(x_{q_n}, Tx_{r_n})\big), \mu\big(d(x_{r_n}, Tx_{q_n})\big)\Big), \phi\big(M(x_{q_n}, x_{r_n})\big)\Big) \\
&= f\Big(\psi\Big(\mu\big(d(x_{q_n}, x_{r_n})\big), \mu\big(d(x_{q_n}, x_{q_{n+1}})\big), \mu\big(d(x_{r_n}, x_{r_{n+1}})\big) \\
&\quad , \mu\big(d(x_{q_n}, x_{r_{n+1}})\big), \mu\big(d(x_{r_n}, x_{q_{n+1}})\big)\Big), \phi\big(M(x_{q_n}, x_{r_n})\big)\Big),
\end{aligned}$$

where $M(x_{q_n}, x_{r_n}) = \max\big\{d(x_{q_n}, x_{r_n}), d(x_{q_n}, Tx_{q_n}), d(x_{r_n}, Tx_{r_n})\big\}$

Letting $n \to \infty$ in the last inequality, by using (3.5), (3.6), the continuity of μ and ϕ and the property of ψ, we get that

$$\begin{aligned}
\mu(\varepsilon) &\leq f\Big(\psi\big(\mu(\varepsilon), \mu(0), \mu(0), \mu(\varepsilon), \mu(\varepsilon)\big), \phi(\varepsilon)\Big) \\
&\leq f\big(\mu(\varepsilon), \phi(\varepsilon)\big).
\end{aligned}$$

Consequently, we obtain that $\mu(\varepsilon) = 0$ and $\phi(\varepsilon) = 0$ which is a contradiction with $\varepsilon > 0$. Hence, the result is proved. □

Theorem 3.2. *Let (X, d) be a complete metric space, $f : [0, \infty)^2 \longrightarrow \mathbb{R}$ be a function of C-class, $m \in \mathbb{N}$, $A_1, A_2, \ldots, A_m$ be non-empty closed subsets of X and $Y = \cup_{i=1}^m A_i$. Suppose that T is a cyclic (μ, ψ, ϕ)-f-weakly contraction. Then T has a fixed point $z \in \cap_{i=1}^n A_i$.*

Proof. We can construct a sequence $x_{n+1} = Tx_n$ with $n \in \mathbb{N}_0$. If there exists $n_0 \in \mathbb{N}$ such that $x_{n_0+1} = x_{n_0}$, hence the result. Indeed, we can see that $Tx_{n_0} = x_{n_0+1} = x_{n_0}$. Now, we assume that $x_{n+1} \neq x_n$ for any $n \in \mathbb{N}_0$. As $X = \cup_{i=1}^m A_i$, for any $n > 0$, there exists $i_n \in \{1, 2, \ldots, m\}$ such that $x_{n-1} \in A_{i_n}$ and $x_n \in A_{i_{n+1}}$. Since T is a cyclic (μ, ψ, ϕ)-weakly contraction, we have

$$\begin{aligned}
\mu\big(d(x_{n+1}, x_n)\big) &= \mu\big(d(Tx_n, Tx_{n-1})\big) \\
&\leq f\Big(\psi\Big(\mu\big(d(x_n, x_{n-1})\big), \mu\big(d(x_n, Tx_n)\big), \mu\big(d(x_{n-1}, Tx_{n-1})\big) \\
&\quad , \mu\big(d(x_n, Tx_{n-1})\big), \mu\big(d(x_{n-1}, Tx_n)\big)\Big), \phi\Big(M(x_n, x_{n-1})\Big)\Big) \\
&= f\Big(\psi\Big(\mu\big(d(x_n, x_{n-1})\big), \mu\big(d(x_n, x_{n+1})\big), \mu\big(d(x_{n-1}, x_n)\big) \\
&\quad , \mu\big(d(x_n, x_n)\big), \mu\big(d(x_{n-1}, x_{n+1})\big)\Big), \phi\Big(M(x_n, x_{n-1})\Big)\Big)
\end{aligned}$$

where $M(x_n, x_{n-1}) = \max\big\{d(x_n, x_{n-1}), d(x_n, x_{n+1}), d(x_{n-1}, x_n)\big\}$.

If $M(x_n, x_{n-1}) = d(x_n, x_{n+1})$, then

$$\begin{aligned}\mu\big(d(x_{n+1}, x_n)\big) &\leq f\Big(\psi\Big(\mu\big(d(x_n, x_{n+1})\big), \mu\big(d(x_n, x_{n+1})\big), \mu\big(d(x_n, x_{n+1})\big), 0,\\ &\qquad 2\mu\big(d(x_n, x_{n+1})\big)\Big), \phi\Big(d(x_n, x_{n+1})\Big)\Big)\\ &\leq f\Big(\mu\big(d(x_n, x_{n+1})\big), \phi\big(d(x_n, x_{n+1})\big)\Big)\\ &\leq \mu\big(d(x_n, x_{n+1})\big),\end{aligned}$$

which is a contradiction. Hence,

$$\mu\big(d(x_{n+1}, x_n)\big) \leq f\Big(\mu\big(d(x_n, x_{n-1})\big), \phi\big(d(x_n, x_{n-1})\big)\Big) \tag{3.7}$$

and

$$d(x_{n+1}, x_n) \leq d(x_n, x_{n-1}).$$

Thus, $\{d(x_{n+1}, x_n)\}$ is a monotone decreasing sequence of non-negative real numbers and hence is convergent. Therefore, there exists $r \geq 0$ such that $d(x_{n+1}, x_n) \to r$.

Letting $n \to \infty$ in (3.7) and using the continuity of μ and ϕ, we obtain that

$$\mu(r) \leq f\big(\mu(r), \phi(r)\big).$$

This implies that $\phi(r) = 0$ and $r = 0$. Thus, we have

$$d(x_{n+1}, x_n) \to 0. \tag{3.8}$$

To prove that $\{x_n\}$ is a Cauchy sequence, we use Lemma 3.1. Indeed, we show that $\{x_n\}$ is a Cauchy sequence in Y. Fix $\epsilon > 0$. By Lemma 3.1, we can find $n_0 \in \mathbb{N}$ such that $r, q \geq n_0$ with $r - q \equiv 1 \pmod{m}$

$$d(x_r, x_q) \leq \frac{\epsilon}{2}. \tag{3.9}$$

Since $\lim d(x_n, x_{n+1}) = 0$, we can also find $n_1 \in \mathbb{N}$ such that

$$d(x_n, x_{n+1}) \leq \frac{\epsilon}{2m}, \tag{3.10}$$

for any $n \geq n_1$.

Assume that $r, s \geq \max\{n_0, n_1\}$ and $s > r$. Then there exists $k \in \{1, 2, \ldots, m\}$ such that $s - r \equiv k \pmod{m}$. Hence, $s - r + t = 1 \pmod{m}$, for $t = m - k + 1$. So, we have

$$d(x_r, x_s) \leq d(x_r, x_{s+j}) + d(x_{s+j}, x_{s+j-1}) + \ldots + d(x_{s+1}, x_s). \tag{3.11}$$

Using (3.9), (3.10) and (3.11), we obtain

$$d(x_r, x_s) \leq \frac{\epsilon}{2} + j \times \frac{\epsilon}{2m} \leq \frac{\epsilon}{2} + m \times \frac{\epsilon}{2m} = \epsilon. \tag{3.12}$$

Therefore, $\{x_n\}$ is a Cauchy sequence in Y. Since Y is closed in X, then Y is also complete space and there exists $x \in Y$ such that $\lim x_n = x$.

Next, we prove that x is a fixed point of T. For this purpose, we have $Y = \cup_{i=1}^{m} A_i$ is a cyclic representation of Y with respect to T. So, the sequence $\{x_n\}$ has infinite terms in each A_i for $i = \{1, 2, \ldots, m\}$. Suppose that $x \in A_i$, $Tx \in A_{i+1}$ and take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ with $x_{n_k} \in A_i$. By using the contractive condition, we obtain that

$$\begin{aligned}
\mu\big(d(x_{n_k+1}, Tx)\big) &= \mu\big(d(Tx_{n_k}, Tx)\big) \\
&\leq f\Big(\psi\Big(\mu\big(d(x_{n_k}, x)\big), \mu\big(d(x_{n_k}, Tx_{n_k})\big), \mu\big(d(x, Tx)\big) \\
&\quad , \mu\big(d(x_{n_k}, Tx)\big), \mu\big(d(x, Tx_{n_k})\big)\Big), \phi\big(M(x_{n_k}, x)\big)\Big) \\
&= f\Big(\psi\Big(\mu\big(d(x_{n_k}, x)\big), \mu\big(d(x_{n_k}, x_{n_k+1})\big), \mu\big(d(x, Tx)\big) \\
&\quad , \mu\big(d(x_{n_k}, Tx)\big), \mu\big(d(x, x_{n_k+1})\big)\Big), \phi\big(M(x_{n_k}, x)\big)\Big),
\end{aligned}$$

where $M(x_{n_k}, x) = \max\big\{d(x_{n_k}, x), d(x_{n_k}, x_{n_k+1}), d(x, Tx)\big\}$. Letting $n \to \infty$ and using continuity of μ and ϕ, we have

$$\begin{aligned}
\mu\big(d(x, Tx)\big) &\leq f\Big(\psi\Big(\mu(0), \mu(0), \mu\big(d(x, Tx)\big), \mu\big(d(x, Tx)\big), \mu(0)\Big), \phi\big(d(x, Tx)\big)\Big) \\
&\leq f\Big(\mu\big(d(x, Tx)\big), \phi\big(d(x, Tx)\big)\Big),
\end{aligned}$$

which is a contradiction unless $d(x, Tx) = 0$. Thus, x is a fixed point of T.

For the uniqueness of the fixed point, we suppose that x_1 and x_2 $(x_1 \neq x_2)$ are two fixed points of T. Using the contractive condition and the continuity of μ and ψ, we get

$$\begin{aligned}
\mu\big(d(x_1, x_2)\big) &= \mu\big(d(Tx_1, Tx_2)\big) \\
&\leq f\Big(\psi\Big(\mu\big(d(x_1, x_2)\big), \mu\big(d(x_1, Tx_1)\big), \mu\big(d(x_2, Tx_2)\big) \\
&\quad , \mu\big(d(x_1, Tx_2)\big), \mu\big(d(x_2, Tx_1)\big)\Big), \phi\big(M(x_1, x_2)\big)\Big) \\
&= f\Big(\psi\Big(\mu\big(d(x_1, x_2)\big), \mu\big(d(x_1, x_1)\big), \mu\big(d(x_2, x_2)\big), \mu\big(d(x_1, x_2)\big) \\
&\quad , \mu\big(d(x_2, x_1)\big)\Big), \phi\big(M(x_1, x_2)\big)\Big) \\
&= f\Big(\psi\Big(\mu\big(d(x_1, x_2)\big), \mu(0), \mu(0), \mu\big(d(x_1, x_2)\big), \mu\big(d(x_2, x_1)\big)\Big) \\
&\quad , \phi\big(M(x_1, x_2)\big)\Big),
\end{aligned}$$

where $M(x_1, x_2) = \max\{d(x_1, x_2), d(x_1, x_1), d(x_2, x_2)\}$.

Therefore, we deduce that

$$\begin{aligned} \mu\big(d(x_1, x_2)\big) &\leq f\Big(\psi\Big(\mu\big(d(x_1, x_2)\big), 0, 0, \mu\big(d(x_1, x_2)\big), \mu\big(d(x_2, x_1)\big)\Big), \phi\big(d(x_1, x_2)\big)\Big) \\ &\leq f\Big(\mu\big(d(x_1, x_2)\big), \phi\big(d(x_1, x_2)\big)\Big) \end{aligned}$$

which is a contradiction unless $x_1 = x_2$. □

4 Applications

Form Theorem 3.2, we can obtain the following corollaries as natural results.

Corollary 4.1. *Let (X, d) be a complete metric space, $m \in \mathbb{N}$, $A_1, A_2, \ldots,$ A_m be non-empty closed subsets of X and $Y = \cup_{i=1}^{m} A_i$. Suppose that T is a cyclic (μ, ψ, ϕ)-rational-weakly contraction. Then T has a fixed point $z \in \cap_{i=1}^{n} A_i$.*

Proof. Taking

$$f(s, t) = \frac{s}{1 + t}$$

for all $s, t \in [0, \infty)$ in Theorem 3.2, we get the desired result. □

Corollary 4.2. *[*[7] *Theorem 2.2] Let (X, d) be a complete metric space, $m \in \mathbb{N}$, $A_1, A_2, \ldots, A_m$ be non-empty closed subsets of X and $Y = \cup_{i=1}^{m} A_i$. Suppose that T is a cyclic (μ, ψ, ϕ)-weakly contraction. Then T has a fixed point $z \in \cap_{i=1}^{n} A_i$.*

Proof. The proof follows from Theorem 3.2 by taking

$$f(s, t) = s - t$$

for all $s, t \in [0, \infty)$ and we get the desired result. □

By similar method, we can prove the following consequences.

Corollary 4.3. *Let (X, d) be a complete metric space, $m \in \mathbb{N}$, $A_1, A_2, \ldots,$ A_m be non-empty closed subsets of X and $Y = \cup_{i=1}^{m} A_i$. Suppose that $T\colon Y \to Y$ is an operator such that*

1. *$\cup_{i=1}^{m} A_i$ is a cyclic representation of Y with respect to T,*

2.

$$\mu\big(d(Tx,Ty)\big) \leq \psi\Big(\mu\big(d(x,y)\big),\mu\big(d(x,Tx)\big),\mu\big(d(y,Ty)\big),\ \mu\big(d(x,Ty)\big),\mu\big(d(y,Tx)\big)\Big)\times\log_{a+\phi\big(M(x,y)\big)} a,$$

for any $x\in A_i$*,* $y\in A_{i+1}$*,* $i=1,2,\ldots,m$ *where* $A_{m+1}=A_1$*,* $\mu\in\theta$*,* $\phi\in\Phi$*,* $\psi\in\Psi$ *and*

$$M(x,y)=\max\Big\{d(x,y),d(x,Tx),d(y,Ty)\Big\}.$$

Then T *has a fixed point* $z\in\cap_{i=1}^{n}A_i$*.*

Corollary 4.4. *Let* (X,d) *be a complete metric space,* $m\in\mathbb{N}$*,* $A_1,A_2,\ldots,A_m$ *be non-empty closed subsets of* X *and* $Y=\cup_{i=1}^{m}A_i$*. Suppose that* $T\colon Y\to Y$ *is an operator such that*

1. $\cup_{i=1}^{m}A_i$ *is a cyclic representation of* Y *with respect to* T*,*
2. $\mu\big(d(Tx,Ty)\big)\leq$

$$\psi\Big(\mu\big(d(x,y)\big),\mu\big(d(x,Tx)\big),\mu\big(d(y,Ty)\big),\mu\big(d(x,Ty)\big),\mu\big(d(y,Tx)\big)\Big)$$

$$-\frac{\psi\Big(\mu\big(d(x,y)\big),\mu\big(d(x,Tx)\big),\mu\big(d(y,Ty)\big),\mu\big(d(x,Ty)\big),\mu\big(d(y,Tx)\big)\Big)}{1+\phi\big(M(x,y)\big)},$$

for any $x\in A_i$*,* $y\in A_{i+1}$*,* $i=1,2,\ldots,m$ *where* $A_{m+1}=A_1$*,* $\mu\in\theta$*,* $\phi\in\Phi$*,* $\psi\in\Psi$ *and*

$$M(x,y)=\max\Big\{d(x,y),d(x,Tx),d(y,Ty)\Big\}.$$

Then T *has a fixed point* $z\in\cap_{i=1}^{n}A_i$*.*

If $\mu(a)=a$ in Corollary 4.2, then we have the following corollary.

Corollary 4.5. *Let* (X,d) *be a complete metric space,* $m\in\mathbb{N}$*,* $A_1,A_2,\ldots,A_m$ *be non-empty closed subsets of* X *and* $Y=\cup_{i=1}^{m}A_i$*. Suppose that* $T\colon Y\to Y$ *is an operator such that*

1. $\cup_{i=1}^{m}A_i$ *is a cyclic representation of* Y *with respect to* T*,*
2. $d(Tx,Ty)\leq\psi\Big(d(x,y),d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx)\Big)-\phi\Big(M(x,y)\Big),$

for any $x \in A_i$, $y \in A_{i+1}$, $i = 1, 2, \ldots, m$ *where* $A_{m+1} = A_1$, $\phi \in \Phi$, $\psi \in \Psi$ *and*

$$M(x, y) = \max\Big\{d(x, y), d(x, Tx), d(y, Ty)\Big\}.$$

Then T *has a fixed point* $z \in \cap_{i=1}^{n} A_i$.

If $\mu(a) = a$ in Theorem 3.2, then we have the following corollary.

Corollary 4.6. *Let* (X, d) *be a complete metric space,* $m \in \mathbb{N}$, $A_1, A_2, \ldots, A_m$ *be non-empty closed subsets of* X *and* $Y = \cup_{i=1}^{m} A_i$. *Suppose that* $T\colon Y \to Y$ *is an operator such that*

1. $\cup_{i=1}^{m} A_i$ *is a cyclic representation of* Y *with respect to* T,
2. $d(Tx, Ty) \leq f\left(\psi\Big(d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\Big), \phi\Big(M(x, y)\Big)\right)$

for any $x \in A_i$, $y \in A_{i+1}$, $i = 1, 2, \ldots, m$ *where* $A_{m+1} = A_1$, $\phi \in \Phi$, $\psi \in \Psi$ *and*

$$M(x, y) = \max\Big\{d(x, y), d(x, Tx), d(y, Ty)\Big\}.$$

Then T *has a fixed point* $z \in \cap_{i=1}^{n} A_i$.

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