FIXED POINT THEOREM OF CONE EXPANSION AND COMPRESSION OF FUNCTIONAL TYPE

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Abstract. The fixed point theorem of cone expansion and compression of norm type is generalized by replacing the norms with two functionals satisfying certain conditions to produce a fixed point theorem of cone expansion and compression of functional type. We conclude with an application verifying the existence of a positive solution to a discrete second-order conjugate boundary value problem.

Dedicated to Allan Peterson on the occasion of his 60th birthday.

1. Preliminaries

There are many fixed point theorems. See [7] for an introduction to the study and applications of fixed point theorems. In this paper we will generalize the fixed point theorem of cone expansion and compression of norm type. The generalization allows the user to choose two functionals that satisfy certain conditions which are used in place of the norm. In applications to boundary value problems the functionals will typically be the minimum or maximum of the function over a specific interval. Hence in boundary value problem applications the functionals usually do not satisfy the triangle inequality property of a norm. The flexibility of using functionals instead of norms allows the theorem to be used in a wider variety of situations. In particular, in applications to boundary value problems it allows for improved sufficiency conditions for the existence of a positive solution.
In this section we will state the definitions that are used in the remainder of the paper.

Definition 1. Let E be a real Banach space. A nonempty closed convex set $P \subset E$ is called a *cone* if it satisfies the following two conditions:

(i) $x \in P, \lambda \geq 0$ implies $\lambda x \in P$;
(ii) $x \in P, -x \in P$ implies $x = 0$.

Every cone $P \subset E$ induces an ordering in E given by

$$x \leq y \text{ if and only if } y - x \in P.$$

Definition 2. An operator is called *completely continuous* if it is continuous and maps bounded sets into precompact sets.

Definition 3. A map α is said to be a *nonnegative continuous functional* on a cone P of a real Banach space E if

$$\alpha : P \to [0, \infty)$$

[1]Research was supported by a Concordia College Summer Study Grant
1991 *Mathematics Subject Classification.* 47H10, 39A10.
Key words and phrases. fixed point theorems, boundary value problem, difference equations.

is continuous.

Let α and γ be nonnegative continuous functionals on P; then, for positive real numbers r and R, we define the following sets:

$$P(\gamma, R) = \{x \in P : \gamma(x) < R\} \tag{1}$$

and

$$P(\gamma, \alpha, r, R) = \{x \in P : r < \alpha(x) \text{ and } \gamma(x) < R\}. \tag{2}$$

Definition 4. Let D be a subset of a real Banach space E. If $r : E \to D$ is continuous with $r(x) = x$ for all $x \in D$, then D is a *retract* of E, and the map r is a *retraction*. The *convex hull* of a subset D of a real Banach space X is given by

$$conv(D) = \left\{ \sum_{i=1}^{n} \lambda_i x_i : x_i \in D,\ \lambda_i \in [0,1],\ \sum_{i=1}^{n} \lambda_i = 1, \text{ and } n \in \mathbb{N} \right\}.$$

The following theorem is due to Dugundji and a proof can be found in [4, p44].

Theorem 1. *For Banach spaces X and Y, let $D \subset X$ be closed and let*

$$F : D \to Y$$

be continuous. Then F has a continuous extension

$$\tilde{F} : X \to Y$$

such that

$$\tilde{F}(X) \subset \overline{conv(F(D))}.$$

Corollary 2. *Every closed convex set of a Banach space is a retract of the Banach space.*

Note that it follows from Corollary 1 that a cone P of a real Banach space E is a retract of E.

2. Fixed Point Index

The following theorem, which establishes the existence and uniqueness of the fixed point index, is from [5, p82-86]; an elementary proof can be found in [4, p58&238]. The proof of the generalization of the fixed point theorem of norm type in the next section will invoke the properties of the fixed point index.

Theorem 3. *Let X be a retract of a real Banach space E. Then, for every bounded relatively open subset U of X and every completely continuous operator $A : \overline{U} \to X$ which has no fixed points on ∂U (relative to X), there exists an integer $i(A, U, X)$ satisfying the following conditions:*

- (G1) *Normality: $i(A, U, X) = 1$ if $Ax \equiv y_0 \in U$ for any $x \in \overline{U}$;*
- (G2) *Additivity: $i(A, U, X) = i(A, U_1, X) + i(A, U_2, X)$ whenever U_1 and U_2 are disjoint open subsets of U such that A has no fixed points on $\overline{U} - (U_1 \cup U_2)$;*
- (G3) *Homotopy Invariance: $i(H(t, \cdot), U, X)$ is independent of $t \in [0, 1]$ whenever $H : [0,1] \times \overline{U} \to X$ is completely continuous and $H(t, x) \neq x$ for any $(t, x) \in [0,1] \times \partial U$;*
- (G4) *Permanence: $i(A, U, X) = i(A, U \cap Y, Y)$ if Y is a retract of X and $A(\overline{U}) \subset Y$;*

(G5) *Excision:* $i(A, U, X) = i(A, U_0, X)$ *whenever* U_0 *is an open subset of* U *such that* A *has no fixed points in* $\overline{U} - U_0$*;*

(G6) *Solution: If* $i(A, U, X) \neq 0$*, then* A *has at least one fixed point in* U*.*

Moreover, $i(A, U, X)$ *is uniquely defined.*

3. Fixed Point Theorems

The proof of the following fixed point results can be found in [5, p88-89].

Lemma 4. *Let* P *be a cone in a real Banach space* E*,* Ω *a bounded open subset of* E *with* $0 \in \Omega$*, and* $A : P \cap \overline{\Omega} \to P$ *a completely continuous operator. If*

$$Ax \neq \mu x$$

for all $x \in P \cap \partial\Omega$ *and* $\mu \geq 1$*, then*

$$i(A, P \cap \Omega, P) = 1.$$

Lemma 5. *Let* P *be a cone in a real Banach space* E*,* Ω *a bounded open subset of* E*, and* $A : P \cap \overline{\Omega} \to P$ *a completely continuous operator. If*

(*i*) $\inf_{P \cap \partial\Omega} \|Ax\| > 0$

and

(*ii*) $Ax \neq \mu x$ *for all* $x \in P \cap \partial\Omega$ *and* $\mu \in (0, 1]$*,*

then

$$i(A, P \cap \Omega, P) = 0.$$

The following theorem is a generalization of the fixed point theorem of cone expansion and compression of norm type.

Theorem 6. *Let* P *be a cone in a real Banach space* E*, and let* α *and* γ *be nonnegative continuous functionals on* P*. Assume* $P(\gamma, \alpha, r, R)$ *as in* (2) *is a nonempty bounded subset of* P*,*

$$A : \overline{P(\gamma, \alpha, r, R)} \to P$$

is a completely continuous operator with

$$\inf_{x \in \partial P(\gamma, \alpha, r, R)} \|Ax\| > 0,$$

and

$$\overline{P(\alpha, r)} \subseteq P(\gamma, R)$$

for these sets as in (1)*. If one of the two conditions*

(*H*1) $\alpha(Ax) \leq r$ *for all* $x \in \partial P(\alpha, r)$*,* $\gamma(Ax) \geq R$ *for all* $x \in \partial P(\gamma, R)$*, and for all* $y \in \partial P(\alpha, r)$*,* $z \in \partial P(\gamma, R)$*,* $\lambda \geq 1$*, and* $\mu \in (0, 1]$ *the functionals satisfy the properties*

$$\alpha(\lambda y) \geq \lambda\alpha(y), \ \ \gamma(\mu z) \leq \mu\gamma(z), \ \text{and} \ \ \alpha(0) = 0$$

or

(*H*2) $\alpha(Ax) \geq r$ *for all* $x \in \partial P(\alpha, r)$*,* $\gamma(Ax) \leq R$ *for all* $x \in \partial P(\gamma, R)$*, and for all* $y \in \partial P(\alpha, r)$*,* $z \in \partial P(\gamma, R)$*,* $\lambda \in (0, 1]$ *and* $\mu \geq 1$ *the functionals satisfy the properties*

$$\alpha(\lambda y) \leq \lambda\alpha(y), \ \ \gamma(\mu z) \geq \mu\gamma(z), \ \text{and} \ \ \gamma(0) = 0$$

is satisfied, then A has at least one positive fixed point x^* such that

$$r \leq \alpha(x^*) \ \text{ and } \ \gamma(x^*) \leq R.$$

Proof. If there exists an $x \in \partial P(\gamma, \alpha, r, R)$ such that $Ax = x$, then there is nothing to prove; thus suppose that $Ax \neq x$ for all $x \in \partial P(\gamma, \alpha, r, R)$. By Dugundji's Theorem (Theorem 1), A has a completely continuous extension

$$A : \overline{P(\gamma, R)} \to P.$$

Suppose condition $(H1)$ is satisfied; the proof when $(H2)$ is satisfied is nearly identical and will be omitted.

Claim 1: $Ay \neq \lambda y$ for all $y \in \partial P(\alpha, r)$ and $\lambda \geq 1$.
To the contrary, suppose there exists a $y_0 \in \partial P(\alpha, r)$ and $\lambda_0 > 1$ (since A has no fixed points on the boundary) such that

$$Ay_0 = \lambda_0 y_0.$$

Then

$$\alpha(Ay_0) = \alpha(\lambda_0 y_0) \geq \lambda_0 \alpha(y_0) > \alpha(y_0) = r,$$

which is a contradiction of the first part of $(H1)$.

Note that $0 \in P(\alpha, r)$ by assumption, hence by Lemma 3

$$i(A, P(\alpha, r), P) = 1.$$

Claim 2: $Az \neq \mu z$ for all $z \in \partial P(\gamma, R)$ and $\mu \in (0, 1]$.
Suppose to the contrary that there exists a $z_0 \in \partial P(\gamma, R)$ and $\mu_0 \in (0, 1]$ such that

$$Az_0 = \mu_0 z_0.$$

Then

$$\gamma(Az_0) = \gamma(\mu_0 z_0) \leq \mu_0 \gamma(z_0) < \gamma(z_0) = R,$$

which is a contradiction of the second part of $(H1)$.

Hence by Lemma 4

$$i(A, P(\gamma, R), P) = 0$$

since we have assumed that

$$\inf_{x \in \partial P(\gamma, R)} \|Ax\| > 0.$$

Thus by the additivity $(G2)$ of the fixed point index

$$i(A, P(\gamma, \alpha, r, R), P) = i(A, P(\gamma, R), P) - i(A, P(\alpha, r), P) = -1 \neq 0,$$

and by the solution property $(G6)$ of the fixed point index the operator A has a fixed point $x^* \in P(\gamma, \alpha, r, R)$. Thus in any event the operator A has a fixed point x^* such that

$$r \leq \alpha(x^*) \ \text{ and } \ \gamma(x^*) \leq R.$$

$\square$

4. Application

Throughout the remainder of this paper we apply the above result to a second-order difference equation; see [6] for an introduction to the general subject, and [1, 2, 3] for representative examples of applying fixed point theorems to difference equations.
Consider the discrete second-order nonlinear conjugate boundary value problem,

$$\Delta^2 x(t-1) + f(x(t)) = 0 \text{ for all } t \in [a+1, b+1] \tag{3}$$

$$x(a) = 0 = x(b+2) \tag{4}$$

where $f : \mathbb{R} \to \mathbb{R}$ is continuous and f is nonnegative for $x \geq 0$; $\mathbb{R}$ denotes the real numbers. Here we assume a and b are integers with $b > a+2$, and we define the discrete interval $[a, b+2]$ by

$$[a, b+2] := \{a, a+1, ..., b+1, b+2\}.$$

(Other intervals may be discrete or continuous; the type of interval will be obvious by context.) Define the Banach space (with sup norm)

$$E = \{x \mid x : [a, b+2] \to \mathbb{R}\}$$

and the corresponding cone

$$P = \left\{ x \in E \;\middle|\; \begin{array}{l} x \text{ is concave, symmetric,} \\ x(t) \geq 0 \text{ for all } t \in [a, b+2] \text{ and} \\ x(t) \geq \frac{t-a}{\tau-a}\|x\| \text{ for all } t \in [a, \tau] \end{array} \right\}$$

where

$$\tau := \left\lfloor \frac{b+2+a}{2} \right\rfloor.$$

The solutions of (3), (4) are the fixed points of the operator A defined on E by

$$Ax(t) = \sum_{s=a+1}^{b+1} G(t,s) f(x(s))$$

for $t \in [a, b+2]$, where

$$G(t,s) = \begin{cases} \dfrac{(t-a)(b+2-s)}{b+2-a} & \text{if } t \leq s \\[2ex] \dfrac{(s-a)(b+2-t)}{b+2-a} & \text{if } s \leq t \end{cases}$$

is the Greens function on $[a, b+2] \times [a+1, b+1]$ for the operator L defined by

$$Lx(t) = -\Delta^2 x(t-1)$$

with boundary conditions

$$x(a) = 0 = x(b+2).$$

A key property of this Greens function is that if $a \leq t \leq \tau$ and $s \in [a+1, b+1]$, then

$$\frac{G(t,s)}{G(\tau,s)} \geq \frac{t-a}{\tau-a}.$$

Let
$$\tau_1 := a + \left\lceil \frac{\tau - a}{2} \right\rceil \quad \text{and} \quad \tau_2 := b + 2 - \left\lceil \frac{\tau - a}{2} \right\rceil;$$
thus $\tau_1 < \tau < \tau_2$, and
$$\frac{\tau_1 - a}{\tau - a} \geq \frac{1}{2}.$$
Define the constants
$$M := \sum_{s=a+1}^{b+1} G(\tau, s),$$
$$m_l := \sum_{s=\tau_1}^{\tau_2} G(\tau_1, s),$$
and
$$m_u := \sum_{s=\tau_1}^{\tau_2} G(\tau, s);$$
also define the nonnegative continuous functionals
$$\alpha(x) := \max_{t \in [a, b+2]} x(t),$$
$$\psi(x) := \min_{t \in [\tau_1, \tau_2]} x(t),$$
and
$$\gamma(x) := e^{\psi(x)} - 1.$$
We now prove our main existence theorem for the discrete conjugate boundary value problem.

Theorem 7. *Suppose there exist positive real numbers r and R, where $e^r - 1 < R$, and continuous $f : \mathbb{R} \to \mathbb{R}$, where $f(x) \geq 0$ for $x \geq 0$, such that the following conditions are met:*

(i) $f(r) \neq 0$,
(ii) $f(w) \leq \frac{r}{M}$ *for* $w \in [0, r]$,
(iii) $f(w) \geq \frac{1}{m_l} \ln(Jw)$ *for* $w \in [\ln(R+1), 2\ln(R+1)]$

where $J := \frac{R+1}{ln(R+1)}$. *Then, the discrete second-order conjugate boundary value problem* (3), (4) *has at least one positive solution* x^* *such that*
$$r \leq \alpha(x^*) \ \text{ and } \ \gamma(x^*) \leq R.$$

Proof. We will invoke the fixed point theorem of cone compression and expansion of functional type (Theorem 5) once we have shown that the hypotheses $(H1)$ have been satisfied. We have that P is a cone in the Banach space E with the sup norm and $A : P \to P$ since $G(t,s)$ is nonnegative on its domain, $f(x) \geq 0$ on $[0, \infty)$, $-\Delta^2 Ax(t-1) = f(x(t))$, $Ax(t) = Ax(b+2+a-t)$ for all $t \in [a, b+2]$, and $Ax(t) \geq \frac{t-a}{\tau-a} Ax(\tau)$ for $t \in [a, \tau]$.

Using elementary calculus techniques one can show that for $\mu \in (0,1]$ and $w = \ln(R+1)$
$$e^{\mu w} - 1 \leq \mu(e^w - 1);$$

thus if $z \in \partial P(\gamma, R)$, then $\psi(z) = \ln(R+1)$ and

$$\gamma(\mu z) = e^{\psi(\mu z)} - 1 = e^{\mu\psi(z)} - 1 \leq \mu(e^{\psi(z)} - 1) = \mu\gamma(z)$$

for $\mu \in (0, 1]$. Also for $y \in \partial P(\alpha, r)$ and $\lambda \geq 1$,

$$\alpha(\lambda y) = \lambda\alpha(y)$$

with $\alpha(0) = 0$.

If $x \in \overline{P(\alpha, r)}$, then

$$\gamma(x) = e^{\psi(x)} - 1 \leq e^r - 1 < R,$$

so that $x \in P(\gamma, R)$ and $\overline{P(\alpha, r)} \subseteq P(\gamma, R)$.

If $x \in \partial P(\gamma, \alpha, r, R)$ then

$$\begin{aligned} \|Ax\| &= \max_{t \in [a, b+2]} \sum_{s=a+1}^{b+1} G(t, s) f(x(s)) \\ &= \sum_{s=a+1}^{b+1} G(\tau, s) f(x(s)) \\ &\geq G(\tau, \tau) f(x(\tau)) \\ &\geq G(\tau, \tau) \min\{f(r), f(\ln(R+1))\} \end{aligned}$$

since $\ln(R+1) \leq x(\tau) \leq r$; hence

$$\inf_{x \in \partial P(\gamma, \alpha, r, R)} \|Ax\| > 0.$$

If $x \in \partial P(\alpha, r)$ then $\alpha(Ax) \leq r$, since

$$\begin{aligned} \alpha(Ax) &= \max_{t \in [a, b+2]} \sum_{s=a+1}^{b+1} G(t, s) f(x(s)) \\ &= \sum_{s=a+1}^{b+1} G(\tau, s) f(x(s)) \\ &\leq \frac{r}{M} \sum_{s=a+1}^{b+1} G(\tau, s) \\ &= r \end{aligned}$$

using (ii). Let $x \in \partial P(\gamma, R)$. By (1) and the symmetry of x,

$$x(\tau_1) = \ln(R+1) \leq x(t)$$

for all $t \in [\tau_1, \tau_2]$. Since $x \in P$,

$$x(t) \leq ||x|| \leq \frac{\tau - a}{t - a} x(t);$$

in particular, for $t = \tau_1$, so that

$$\ln(R+1) \leq x(t) \leq 2\ln(R+1)$$

for all $t \in [\tau_1, \tau_2]$. It follows that

$$\begin{aligned}
\gamma(Ax) &= \exp\left(\min_{t\in[\tau_1,\tau_2]} \sum_{s=a+1}^{b+1} G(t,s) f(x(s)) \right) - 1 \\
&= \exp\left(\sum_{s=a+1}^{b+1} G(\tau_1,s) f(x(s)) \right) - 1 \\
&\geq \exp\left(\sum_{s=\tau_1}^{\tau_2} G(\tau_1,s) f(x(s)) \right) - 1 \\
&= \prod_{s=\tau_1}^{\tau_2} (\exp f(x(s)))^{G(\tau_1,s)} - 1 \\
&\geq \prod_{s=\tau_1}^{\tau_2} (Jx(s))^{\frac{G(\tau_1,s)}{m_l}} - 1 \\
&\geq \prod_{s=\tau_1}^{\tau_2} (Jx(\tau_1))^{\frac{G(\tau_1,s)}{m_l}} - 1 \\
&= (Jx(\tau_1))^{\sum_{s=\tau_1}^{\tau_2} \frac{G(\tau_1,s)}{m_l}} - 1 \\
&= Jx(\tau_1) - 1 \\
&= J\ln(R+1) - 1 \\
&= R.
\end{aligned}$$

Therefore $\gamma(Ax) \geq R$, and the hypotheses of the fixed point theorem of cone compression and expansion of functional type are satisfied. Thus the discrete second-order conjugate problem has a positive solution x^* such that

$$r \leq \alpha(x^*) \text{ and } \gamma(x^*) \leq R.$$

Moreover, by the choice of our functionals, we have

$$r \leq \max_{t\in[a,b+2]} x^*(t) \text{ and } \min_{t\in[\tau_1,\tau_2]} x^*(t) \leq \ln(R+1).$$

□

Note, this is an improvement over the standard application of the fixed point theorem of cone compression and expansion of norm type to this boundary value problem whenever R

satisfies the inequality

$$\ln(R+1) \geq \frac{m_u \ \ln(2)}{2m_l - m_u} > 0.$$

References

[1] D. Anderson, R. Avery, and A. Peterson, Three Positive Solutions to a Discrete Focal Boundary Value Problem, *J. Comput. Appl. Math* **88** (1998), No 1, 103-118.

[2] Richard I. Avery, *Multiple Positive Solutions to Boundary Value Problems*, Dissertation, University of Nebraska-Lincoln, 1997.

[3] R.I. Avery and A.C. Peterson, Multiple positive solutions of a discrete second order conjugate problem, *PanAmerican Mathematical Journal* **8** (1998), 1-12.

[4] Klaus Deimling, *Nonlinear Functional Analysis*, Springer-Verlag, New York, 1985.

[5] Dajun Guo and V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, Academic Press, San Diego, 1988.

[6] Walter G. Kelley and Allan C. Peterson, *Difference Equations: An Introduction with Applications*, 2e, Harcourt/Academic Press, San Diego, 2001.

[7] Eberhard Zeidler, *Nonlinear Functional Analysis and its Applications I : Fixed-Point Theorems*, Springer-Verlag, New York, 1986.

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