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FIXED POINT THEOREMS AND CHARACTERIZATIONS OF METRIC COMPLETENESS

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1. Introduction

Let X be a metric space with metric d. A mapping T from X into itself is called *contractive* if there exists a real number $r \in [0,1)$ such that $d(Tx,Ty) \leq rd(x,y)$ for every $x,y \in X$. It is well know that if X is a complete metric space, then every contractive mapping from X into itself has a unique fixed point in X. However, we exhibit a metric space X such that X is not complete and every contractive mapping from X into itself has a fixed point in X; see Section 4. On the other hand, in [1], Caristi proved the following theorem: Let X be a complete metric space and let $\phi : X \to (-\infty,\infty)$ be a lower semicontinuous function, bounded from below. Let $T : X \to X$ be a mapping satisfying

$$d(x,Tx) \leq \phi(x) - \phi(Tx)$$

for every $x \in X$. Then T has a fixed point in X. Later, characterizations of metric completeness have been discussed by Weston [8], Takahashi [7], Park and Kang [6] and others. For example, Park and Kang [6] proved the following: Let X be a metric space. Then X is complete if and only if for every selfmap T of X with a uniformly continuous function $\phi : X \to [0,\infty)$ such that

$$d(x,Tx) \leq \phi(x) - \phi(Tx)$$

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for every $x \in X$, T has a fixed point in X. Recently, Kada, Suzuki and Takahashi [4] introduced the concept of w-distance on a metric space X (see Section 2) and improved Caristi's fixed point theorem [1], Ekeland's variational principle [3], and the nonconvex minimization theorem according to Takahashi [7].

In this paper, using the concept of w-distance, we first establish fixed point theorems for set-valued mappings on complete metric spaces which are connected with Nadler's fixed point theorem [5] and Edelstein's fixed point theorem [2]. Next, we give characterizations of metric completeness. One of them is as follows: A convex subset D of a normed linear space is complete if and only if every contractive mapping from D into itself has a fixed point in D.

2. Preliminaries

Throughout this paper, we denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{R}$ the set of real numbers. Let X be a metric space with metric d. Then a function $p : X \times X \to [0, \infty)$ is called a *w-distance* on X if the following are satisfied:

(1) $p(x, z) \le p(x, y) + p(y, z)$ for any $x, y, z \in X$;
(2) for any $x \in X$, $p(x, \cdot) : X \to [0, \infty)$ is lower semicontinuous;
(3) for any $\varepsilon > 0$, there exists $\delta > 0$ such that $p(z, x) \le \delta$ and $p(z, y) \le \delta$ imply $d(x, y) \le \varepsilon$.

The metric d is a w-distance on X. Some other examples of w-distances are given in [4]. We have the following lemmas regarding w-distance.

Lemma 1. *Let X be a metric space with metric d, let p be a w-distance on X, and let q be a function from $X \times X$ into $[0, \infty)$ satisfying* (1), (2) *in the definition of w-distance. Suppose that $q(x, y) \ge p(x, y)$ for every $x, y \in X$. Then q is also a w-distance on X. In particular, if q satisfies* (1), (2) *in the definition of w-distance and $q(x, y) \ge d(x, y)$ for every $x, y \in X$, then q is a w-distance on X.*

Proof. We show that q satisfies (3). Let $\varepsilon > 0$. Since p is a w-distance, there exists a positive number δ such that $p(z, x) \le \delta$ and $p(z, y) \le \delta$ imply $d(x, y) \le \varepsilon$. Then $q(z, x) \le \delta$ and $q(z, y) \le \delta$ imply $d(x, y) \le \varepsilon$. $\square$

Lemma 2. *Let F be a bounded and closed subset of a metric space X. Assume that F contains at least two points and c is a constant with $c \ge \delta(F)$, where $\delta(F)$ is the diameter of F. Then the function $p : X \times X \to [0, \infty)$ defined by*

$$p(x, y) = \begin{cases} d(x, y) & \text{if } x, y \in F, \\ c & \text{if } x \notin F \text{ or } y \notin F, \end{cases}$$

is a w-distance on X.

Proof. If $x, y, z \in F$, we have

$$p(x,z) = d(x,z) \le d(x,y) + d(y,z) = p(x,y) + p(y,z).$$

In the other case, we have

$$p(x,z) \le c \le p(x,y) + p(y,z).$$

Let $x \in X$. If $\alpha \ge c$, we have $\{y \in X : p(x,y) \le \alpha\} = X$. Let $\alpha < c$. If $x \in F$, then $p(x,y) \le \alpha$ implies $y \in F$. So, we have

$$\{y \in X : p(x,y) \le \alpha\} = \{y \in X : d(x,y) \le \alpha\} \cap F.$$

If $x \notin F$, we have $\{y \in X : p(x,y) \le \alpha\} = \emptyset$. In each case, the set $\{y \in X : p(x,y) \le \alpha\}$ is closed. Therefore $p(x, \cdot\,) : X \to [0,\infty)$ is lower semicontinuous. Let $\varepsilon > 0$. Then there exists $n_0 \in \mathbb{N}$ such that $0 < \varepsilon/n_0 < c$. Let $\delta = \varepsilon/(2n_0)$. Then $p(z,x) \le \delta$ and $p(z,y) \le \delta$ imply $x, y, z \in F$. So, we have

$$d(x,y) \le d(x,z) + d(y,z) = p(z,x) + p(z,y) \le \frac{\varepsilon}{2n_0} + \frac{\varepsilon}{2n_0} = \frac{\varepsilon}{n_0} \le \varepsilon. \qquad \square$$

Let $\varepsilon \in (0,\infty]$. A metric space X with metric d is called *ε-chainable* [2] if for every $x, y \in X$ there exists a finite sequence $\{u_0, u_1, \dots, u_k\}$ in X such that $u_0 = x$, $u_k = y$ and $d(u_i, u_{i+1}) < \varepsilon$ for $i = 0, 1, \dots, k-1$. Such a sequence is called an *ε-chain* in X linking x and y.

Lemma 3. *Let $\varepsilon \in (0,\infty]$ and let X be an ε-chainable metric space with metric d. Then the function $p : X \times X \to [0,\infty)$ defined by*

$$p(x,y) = \inf\left\{\sum_{i=0}^{k-1} d(u_i, u_{i+1}) : \{u_0, u_1, \dots, u_k\} \text{ is an } \varepsilon\text{-chain linking } x \text{ and } y\right\}$$

is a w-distance on X.

Proof. Note that p is well-defined because X is ε-chainable. Let $x, y, z \in X$ and let $\eta > 0$ be arbitrary. Then there exist ε-chains $\{u_0, u_1, \dots, u_k\}$ linking x and y and $\{v_0, v_1, \dots, v_l\}$ linking y and z such that

$$\sum_{i=0}^{k-1} d(u_i, u_{i+1}) \le p(x,y) + \eta \quad \text{and} \quad \sum_{i=0}^{l-1} d(v_i, v_{i+1}) \le p(y,z) + \eta.$$

Since $\{u_0, u_1, \dots, u_k, v_1, v_2, \dots, v_l\}$ is an ε-chain linking x and z, we have

$$p(x,z) \le \sum_{i=0}^{k-1} d(u_i, u_{i+1}) + \sum_{i=0}^{l-1} d(v_i, v_{i+1}) \le p(x,y) + p(y,z) + 2\eta.$$

Since $\eta > 0$ is arbitrary, we have $p(x,z) \le p(x,y) + p(y,z)$.

Let us prove (2). Let $x, y \in X$ and let $\{y_n\}$ be a sequence in X with $y_n \to y$. Choose $n_0 \in \mathbb{N}$ such that $d(y, y_n) < \varepsilon$ for every $n \ge n_0$. Let $\eta > 0$ be arbitrary and let $n \geq n_0$. Then there exists an ε-chain $\{u_0, u_1, \ldots, u_k\}$ linking x and y_n such that

$$\sum_{i=0}^{k-1} d(u_i, u_{i+1}) \leq p(x, y_n) + \eta.$$

Since $d(y, y_n) < \varepsilon$, $\{u_0, u_1, \ldots, u_k, y\}$ is an ε-chain linking x and y. So, we have

$$p(x,y) \leq \sum_{i=0}^{k-1} d(u_i, u_{i+1}) + d(y_n, y) \leq p(x, y_n) + \eta + d(y_n, y)$$

and hence

$$p(x,y) \leq \liminf_{n\to\infty} p(x, y_n) + \eta.$$

Since $\eta > 0$ is arbitrary, we have

$$p(x,y) \leq \liminf_{n\to\infty} p(x, y_n).$$

This implies that $p(x, \cdot)$ is lower semicontinuous. Since $p(x,y) \geq d(x,y)$ for every $x, y \in X$, by Lemma 1, p is a w-distance. □

The following lemma was proved in [4].

Lemma 4 ([4]). *Let X be a metric space with metric d and let p be a w-distance on X. Let $\{x_n\}$ and $\{y_n\}$ be sequences in X, let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[0,\infty)$ converging to 0, and let $x, y, z \in X$. Then the following hold:*

1. *if $p(x_n, y) \leq \alpha_n$ and $p(x_n, z) \leq \beta_n$ for any $n \in \mathbb{N}$, then $y = z$; in particular, if $p(x,y) = 0$ and $p(x,z) = 0$, then $y = z$;*
2. *if $p(x_n, y_n) \leq \alpha_n$ and $p(x_n, z) \leq \beta_n$ for any $n \in \mathbb{N}$, then $\{y_n\}$ converges to z;*
3. *if $p(x_n, x_m) \leq \alpha_n$ for any $n, m \in \mathbb{N}$ with $m > n$, then $\{x_n\}$ is a Cauchy sequence;*
4. *if $p(y, x_n) \leq \alpha_n$ for any $n \in \mathbb{N}$, then $\{x_n\}$ is a Cauchy sequence.*

3. Fixed point theorems

Let X be a metric space with metric d. A set-valued mapping T from X into itself is called *weakly contractive* or *p-contractive* if there exist a w-distance p on X and $r \in [0,1)$ such that for any $x_1, x_2 \in X$ and $y_1 \in Tx_1$ there is $y_2 \in Tx_2$ with $p(y_1, y_2) \leq r p(x_1, x_2)$.

Theorem 1. *Let X be a complete metric space and let T be a set-valued p-contractive mapping from X into itself such that for any $x \in X$, Tx is a nonempty closed subset of X. Then there exists $x_0 \in X$ such that $x_0 \in Tx_0$ and $p(x_0, x_0) = 0$.*

Proof. Let p be a w-distance on X and let $r \in [0,1)$ be such that for any $x_1, x_2 \in X$ and $y_1 \in Tx_1$, there exists $y_2 \in Tx_2$ with $p(y_1, y_2) \le rp(x_1, x_2)$. Fix $u_0 \in X$ and $u_1 \in Tu_0$. Then there exists $u_2 \in Tu_1$ such that $p(u_1, u_2) \le rp(u_0, u_1)$. Thus, we have a sequence $\{u_n\}$ in X such that $u_{n+1} \in Tu_n$ and $p(u_n, u_{n+1}) \le rp(u_{n-1}, u_n)$ for every $n \in \mathbb{N}$. For any $n \in \mathbb{N}$, we have

$$p(u_n, u_{n+1}) \le rp(u_{n-1}, u_n) \le r^2 p(u_{n-2}, u_{n-1}) \le \ldots \le r^n p(u_0, u_1)$$

and hence, for any $n, m \in \mathbb{N}$ with $m > n$,

$$\begin{aligned} p(u_n, u_m) &\le p(u_n, u_{n+1}) + p(u_{n+1}, u_{n+2}) + \cdots + p(u_{m-1}, u_m) \\ &\le r^n p(u_0, u_1) + r^{n+1} p(u_0, u_1) + \cdots + r^{m-1} p(u_0, u_1) \\ &\le \frac{r^n}{1-r} p(u_0, u_1). \end{aligned}$$

By Lemma 4, $\{u_n\}$ is a Cauchy sequence. Hence $\{u_n\}$ converges to a point $v_0 \in X$. Fix $n \in \mathbb{N}$. Since $\{u_m\}$ converges to v_0 and $p(u_n, \cdot\,)$ is lower semicontinuous, we have

$$(*) \qquad p(u_n, v_0) \le \liminf_{m \to \infty} p(u_n, u_m) \le \frac{r^n}{1-r} p(u_0, u_1).$$

By hypothesis, we also have $w_n \in Tv_0$ such that $p(u_n, w_n) \le rp(u_{n-1}, v_0)$. So, for any $n \in \mathbb{N}$,

$$p(u_n, w_n) \le rp(u_{n-1}, v_0) \le \frac{r^n}{1-r} p(u_0, u_1).$$

By Lemma 4, $\{w_n\}$ converges to v_0. Since Tv_0 is closed, we have $v_0 \in Tv_0$. For such v_0, there exists $v_1 \in Tv_0$ such that $p(v_0, v_1) \le rp(v_0, v_0)$. Thus, we also have a sequence $\{v_n\}$ in X such that $v_{n+1} \in Tv_n$ and $p(v_0, v_{n+1}) \le rp(v_0, v_n)$ for every $n \in \mathbb{N}$. So, we have

$$p(v_0, v_n) \le rp(v_0, v_{n-1}) \le \ldots \le r^n p(v_0, v_0).$$

By Lemma 4, $\{v_n\}$ is a Cauchy sequence. Hence $\{v_n\}$ converges to a point $x_0 \in X$. Since $p(v_0, \cdot\,)$ is lower semicontinuous, $p(v_0, x_0) \le \liminf_{n \to \infty} p(v_0, v_n) \le 0$ and hence $p(v_0, x_0) = 0$. Then, for any $n \in \mathbb{N}$,

$$p(u_n, x_0) \le p(u_n, v_0) + p(v_0, x_0) \le \frac{r^n}{1-r} p(u_0, u_1).$$

So, using $(*)$ and Lemma 4, we obtain $v_0 = x_0$ and hence $p(v_0, v_0) = 0$. □

Let X be a metric space with metric d and let T be a mapping from X into itself. Then T is called *weakly contractive* or *p-contractive* if there exist a w-distance p on X and $r \in [0,1)$ such that $p(Tx, Ty) \le rp(x, y)$ for every $x, y \in X$. In the case of $p = d$, T is called *contractive.*

Theorem 2. *Let X be a complete metric space. If a mapping T from X into itself is p-contractive, then T has a unique fixed point $x_0 \in X$. Further the x_0 satisfies $p(x_0, x_0) = 0$.*

Proof. Let p be a w-distance and let $r \in [0,1)$ be such that $p(Tx, Ty) \leq rp(x,y)$ for every $x, y \in X$. Then from Theorem 1, there exists $x_0 \in X$ with $Tx_0 = x_0$ and $p(x_0, x_0) = 0$. If $y_0 = Ty_0$, then

$$p(x_0, y_0) = p(Tx_0, Ty_0) \leq rp(x_0, y_0)$$

and hence $p(x_0, y_0) = 0$. So, by $p(x_0, x_0) = 0$ and Lemma 4, we have $x_0 = y_0$. □

Using Theorem 1, we will prove a fixed point theorem which generalizes Nadler's fixed point theorem for set-valued mappings and Edelstein's fixed point theorem on an ε-chainable metric space. Before proving it, we give some definitions and notations. Let X be a metric space with metric d. For $x \in X$ and $A \subset X$, set $d(x, A) = \inf\{d(x,y) : y \in A\}$. Denote by $\mathrm{CB}(X)$ the class of all nonempty bounded closed subsets of X. Let H be the Hausdorff metric with respect to d, i.e.,

$$H(A, B) = \max\{\sup_{u \in A} d(u, B), \sup_{v \in B} d(v, A)\}$$

for every $A, B \in \mathrm{CB}(X)$. Let $\varepsilon \in (0, \infty]$. A mapping T from X into $\mathrm{CB}(X)$ is said to be (ε, σ)-*uniformly locally contractive* [2] if there exists $\sigma \in [0,1)$ such that $H(Tx, Ty) \leq \sigma d(x,y)$ for every $x, y \in X$ with $d(x,y) < \varepsilon$. In particular, T is said to be *contractive* when $\varepsilon = \infty$.

Theorem 3. *Let $\varepsilon \in (0, \infty]$ and let X be a complete and ε-chainable metric space with metric d. Suppose that a mapping T from X into $\mathrm{CB}(X)$ is (ε, σ)-uniformly locally contractive. Then there exists $x_0 \in X$ with $x_0 \in Tx_0$.*

Proof. Define a function p from $X \times X$ into $[0, \infty)$ as follows:

$$p(x,y) = \inf\left\{\sum_{i=0}^{k-1} d(u_i, u_{i+1}) : \{u_0, u_1, \ldots, u_k\} \text{ is an } \varepsilon\text{-chain linking } x \text{ and } y\right\}.$$

From Lemma 3, p is a w-distance on X. We prove that T is p-contractive. Choose a real number r such that $\sigma < r < 1$. Let $x_1, x_2 \in X$, $y_1 \in Tx_1$ and $\eta > 0$. Then there exists an ε-chain $\{u_0, u_1, \ldots, u_k\}$ linking x_1 and x_2 such that

$$\sum_{i=0}^{k-1} d(u_i, u_{i+1}) \leq p(x_1, x_2) + \eta.$$

Put $v_0 = y_1$. Since T is (ε, σ)-uniformly locally contractive, there exists $v_1 \in Tu_1$ such that

$$d(v_0, v_1) \leq rd(u_0, u_1) < r\varepsilon \leq \varepsilon.$$

In a similar way, we define an ε-chain $\{v_0, v_1, \dots, v_k\}$ linking y_1 and v_k such that $v_i \in Tu_i$ for every $i = 0, 1, \dots, k$ and

$$d(v_i, v_{i+1}) \le rd(u_i, u_{i+1}) < \varepsilon$$

for every $i = 0, 1, \dots, k-1$. Putting $y_2 = v_k$, since $y_2 \in Tx_2$ and $\{v_0, v_1, \dots, v_k\}$ is an ε-chain linking y_1 and y_2, we have

$$p(y_1, y_2) \le \sum_{i=0}^{k-1} d(v_i, v_{i+1}) \le \sum_{i=0}^{k-1} rd(u_i, u_{i+1}) \le rp(x_1, x_2) + r\eta < rp(x_1, x_2) + \eta.$$

Since $\eta > 0$ is arbitrary, we have $p(y_1, y_2) \le rp(x_1, x_2)$. So, T is a p-contractive set-valued mapping from X into itself. Theorem 1 now gives the desired result.□

As direct consequences of Theorem 3, we obtain the following.

COROLLARY 1 (Nadler [5]). *Let X be a complete metric space and let T be a contractive set-valued mapping from X into* $\mathrm{CB}(X)$. *Then there exists $x_0 \in X$ with $x_0 \in Tx_0$.*

PROOF. We may assume that there exists $\sigma \in [0, 1)$ such that $H(Tx, Ty) \le \sigma d(x, y)$ for every $x, y \in X$. Since T is (∞, σ)-uniformly locally contractive and X is ∞-chainable, using Theorem 3, we obtain the desired result. □

COROLLARY 2 (Edelstein [2]). *Let $\varepsilon \in (0, \infty]$ and let X be a complete and ε-chainable metric space with metric d. Suppose that a mapping T from X into itself is (ε, σ)-uniformly locally contractive. Then T has a unique fixed point.*

4. Characterizations of metric completeness

In this section, we discuss characterizations of metric completeness. We first give the following example.

EXAMPLE. Define subsets of $\mathbb{R}^2$ as follows:

$$A_n = \{(t, t/n) : t \in (0, 1]\} \quad \text{for every } n \in \mathbb{N}, \quad S = \bigcup_{n \in \mathbb{N}} A_n \cup \{0\}.$$

Then S is not complete and every continuous mapping on S has a fixed point in S.

PROOF. It is clear that S is not complete. Let T be a continuous mapping from S into itself. If $T0 = 0$, then 0 is a fixed point of T. Assume that $T0 \in A_j$ for some $j \in \mathbb{N}$ and define a mapping U on $A_j \cup \{0\}$ as follows:

$$Ux = \begin{cases} Tx & \text{if } Tx \in A_j, \\ 0 & \text{if } Tx \notin A_j. \end{cases}$$

Then U is continuous. In fact, let $\{x_n\}$ be a sequence in $A_j \cup \{0\}$ which converges to x_0. Then $\{Tx_n\}$ converges to Tx_0. If $Tx_0 \in A_j$, then $\{Ux_n\}$ also converges to $Tx_0 = Ux_0$. Otherwise $\{Ux_n\}$ converges to 0 and $Ux_0 = 0$. Hence U is continuous. On the other hand, $A_j \cup \{0\}$ is compact and convex. So, U has a fixed point z_0 in $A_j \cup \{0\}$. It is clear that $z_0 \neq 0$ and z_0 is a fixed point of T. □

Motivated by this example, we obtain the following.

THEOREM 4. *Let X be a metric space. Then X is complete if and only if every weakly contractive mapping from X into itself has a fixed point in X.*

PROOF. Since the "only if" part is proved in Theorem 2, we need only prove the "if" part. Assume that X is not complete. Then there exists a sequence $\{x_n\}$ in X which is Cauchy and does not converge. So, we have $\lim_{m\to\infty} d(x_n, x_m) > 0$ for any $n \in \mathbb{N}$ and also $\lim_{n\to\infty} \lim_{m\to\infty} d(x_n, x_m) = 0$. Then, for any $c > 0$, we can choose a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that, for any $i \in \mathbb{N}$,

$$\lim_{m\to\infty} d(x_{n_i}, x_m) > c \lim_{m\to\infty} d(x_{n_{i+1}}, x_m)$$

and hence

$$\lim_{j\to\infty} d(x_{n_i}, x_{n_j}) > c \lim_{j\to\infty} d(x_{n_{i+1}}, x_{n_j}).$$

So, we may assume that there exists a sequence $\{x_n\}$ in X satisfying the following conditions:

(1) $\{x_n\}$ is Cauchy;
(2) $\{x_n\}$ does not converge;
(3) $\lim_{n\to\infty} d(x_i, x_n) > 3 \lim_{n\to\infty} d(x_{i+1}, x_n)$ for any $i \in \mathbb{N}$.

Put $F = \{x_n : n \in \mathbb{N}\}$. Then F is bounded and closed. So, the function $p : X \times X \to [0,\infty)$ defined by

$$p(x,y) = \begin{cases} d(x,y) & \text{if } x, y \in F, \\ 2\delta(F) & \text{if } x \notin F \text{ or } y \notin F, \end{cases}$$

is a w-distance on X by Lemma 2. Further, $p(x,y) = p(y,x)$ for any $x, y \in X$. Define a mapping T from X into itself as follows:

$$Tx = \begin{cases} x_1 & \text{if } x \notin F, \\ x_{i+1} & \text{if } x = x_i. \end{cases}$$

Then it is clear that T has no fixed point in X. To complete the proof, it is sufficient to show that T is p-contractive. If $x \notin F$ or $y \notin F$, then

$$p(Tx, Ty) \leq \delta(F) = \frac{1}{2} \cdot 2\delta(F) = \frac{1}{2} p(x,y) \leq \frac{2}{3} p(x,y).$$

Let $x, y \in F$. Then, without loss of generality, we may assume that $x = x_i, y = x_j$ and $i < j$. We have

$$\begin{aligned} d(x_i, x_j) &\geq \lim_{n\to\infty} d(x_i, x_n) - \lim_{n\to\infty} d(x_j, x_n) \\ &\geq \lim_{n\to\infty} d(x_i, x_n) - \lim_{n\to\infty} d(x_{i+1}, x_n) \\ &\geq 2 \lim_{n\to\infty} d(x_{i+1}, x_n). \end{aligned}$$

On the other hand,

$$\begin{aligned} d(x_{i+1}, x_{j+1}) &\leq \lim_{n\to\infty} d(x_{i+1}, x_n) + \lim_{n\to\infty} d(x_{j+1}, x_n) \\ &\leq \lim_{n\to\infty} d(x_{i+1}, x_n) + \lim_{n\to\infty} d(x_{i+2}, x_n) \\ &\leq \frac{4}{3} \lim_{n\to\infty} d(x_{i+1}, x_n). \end{aligned}$$

Therefore we have

$$\begin{aligned} p(Tx, Ty) &= p(Tx_i, Tx_j) = d(x_{i+1}, x_{j+1}) \leq \frac{4}{3} \lim_{n\to\infty} d(x_{i+1}, x_n) \\ &\leq \frac{4}{3} \cdot \frac{1}{2} d(x_i, x_j) = \frac{2}{3} d(x_i, x_j) = \frac{2}{3} p(x_i, x_j) = \frac{2}{3} p(x, y). \qquad \square \end{aligned}$$

Theorem 5. *Let X be a normed linear space and let D be a convex subset of X. Then D is complete if and only if every contractive mapping from D into itself has a fixed point in D.*

Before proving Theorem 5, we need two lemmas.

Lemma 5. *Let X be a normed linear space and let D be a convex subset of X with $0 \in \overline{D}$, where $\overline{D}$ is the closure of D. Then for any $x \in D \setminus \{0\}$, there exists $y \in D$ such that $2\|y\| = \|x\|$ and $\|x - y\| \leq 2\|x\| - 2\|y\|$.*

Proof. Let $x \in D \setminus \{0\}$. Then, since $0 \in \overline{D}$, we obtain an element $z \in D$ with $\|z\| \leq \|x\|/3$. So, there exist $y \in D$ and $t \in [0, 1]$ such that $y = tz + (1-t)x$ and $\|y\| = \|x\|/2$. From

$$\frac{\|x\|}{2} = \|y\| \leq t\|z\| + (1-t)\|x\| \leq t\frac{\|x\|}{3} + (1-t)\|x\|,$$

we have $1/2 \leq t/3 + (1-t)$ and hence $t \leq 3/4$. Then we obtain

$$\begin{aligned}\|x-y\| &= t\|x-z\| \le \frac{3}{4}\|x-z\| \le \frac{3}{4}\|x\| + \frac{3}{4}\|z\| \\ &\le \frac{3}{4}\|x\| + \frac{1}{4}\|x\| = \|x\| = \|x\| + (\|x\| - 2\|y\|) = 2\|x\| - 2\|y\|. \qquad \square\end{aligned}$$

Lemma 6. *Let X be a normed linear space and let D be a convex subset of X with $0 \in \overline{D} \setminus D$. Then there exist a sequence $\{v_n\}$ in D and a mapping w from $(0,\infty)$ into D satisfying the following conditions:*

(1) $\|v_n\| = \|v_1\|/2^{n-1}$ *for every* $n \in \mathbb{N}$*;*
(2) $w(\|v_n\|) = v_n$ *for every* $n \in \mathbb{N}$*;*
(3) $\|w(s) - w(t)\| \le 2|s-t|$ *for every* $s,t \in (0,\infty)$*;*
(4) $\|w(t)\| \le t$ *for every* $t \in (0,\infty)$.

Proof. Let $v_1 \in D$. Then from $v_1 \ne 0$ and Lemma 5 there exists $v_2 \in D$ such that $2\|v_2\| = \|v_1\|$ and $\|v_1 - v_2\| \le 2\|v_1\| - 2\|v_2\|$. Thus, we can find a sequence $\{v_n\}$ in D such that

$$\|v_n\| = \frac{1}{2^{n-1}}\|v_1\| \quad \text{and} \quad \|v_{n-1} - v_n\| \le 2\|v_{n-1}\| - 2\|v_n\|.$$

Note that $\|v_n\| \to 0$ and $\|v_{n+1}\| < \|v_n\|$ for every $n \in \mathbb{N}$. Define a mapping w from $(0,\infty)$ into D as follows:

$$w(t) = \begin{cases} v_1 & \text{if } \|v_1\| < t, \\ \dfrac{t - \|v_{n+1}\|}{\|v_n\| - \|v_{n+1}\|} v_n + \dfrac{\|v_n\| - t}{\|v_n\| - \|v_{n+1}\|} v_{n+1} & \text{if } \|v_{n+1}\| < t \le \|v_n\| \\ & \text{for some } n \in \mathbb{N}. \end{cases}$$

Then it is clear that $w(\|v_n\|) = v_n$ for every $n \in \mathbb{N}$. We shall show (3). In fact, if $\|v_1\| \le s \le t$, it is obvious that $\|w(t) - w(s)\| \le 2(t-s)$ and if $\|v_{n+1}\| \le s \le t \le \|v_n\|$ for some $n \in \mathbb{N}$, we have

$$\|w(s) - w(t)\| = \frac{t-s}{\|v_n\| - \|v_{n+1}\|}\|v_n - v_{n+1}\| \le 2(t-s).$$

Further, if $\|v_{m+1}\| < s \le \|v_m\| \le \|v_n\| \le t < \|v_{n-1}\|$ for some $m,n \in \mathbb{N}$ with $m \ge n \ge 1$, where $\|v_0\| = \infty$, we have

$$\begin{aligned}\|w(s) - w(t)\| &\le \|w(s) - w(\|v_m\|)\| \\ &\quad + \sum_{i=n}^{m-1} \|w(\|v_{i+1}\|) - w(\|v_i\|)\| + \|w(\|v_n\|) - w(t)\| \\ &\le 2(\|v_m\| - s) + \sum_{i=n}^{m-1} 2(\|v_i\| - \|v_{i+1}\|) + 2(t - \|v_n\|) = 2(t-s).\end{aligned}$$

We shall show (4). In fact, if $\|v_1\| < t$, it is obvious that $\|w(t)\| = \|v_1\| \le t$. And if $\|v_{n+1}\| < t \le \|v_n\|$ for some $n \in \mathbb{N}$, we have

$$\|w(t)\| \le \frac{t - \|v_{n+1}\|}{\|v_n\| - \|v_{n+1}\|}\|v_n\| + \frac{\|v_n\| - t}{\|v_n\| - \|v_{n+1}\|}\|v_{n+1}\| = t. \qquad \square$$

Proof of Theorem 5. Since the "only if" part is well known, we need only prove the "if" part. Suppose that D is not complete. We denote the completion of X by $\widehat{X}$ and the closure of D in $\widehat{X}$ by $\widehat{D}$. Since D is not complete, we obtain $z_0 \in \widehat{D} \setminus D$. Since $D - z_0$ is convex in $\widehat{X}$ and the closure of $D - z_0$ in $\widehat{X}$ includes 0, there exists a mapping w from $(0, \infty)$ into $D - z_0$ satisfying (3) and (4) of Lemma 6. Now, define a mapping T from D into itself as follows:

$$T(x) = w\left(\frac{\|x - z_0\|}{4}\right) + z_0 \quad \text{for every } x \in D.$$

Then we have, for any $x, y \in D$,

$$\begin{aligned} \|Tx - Ty\| &= \left\| w\left(\frac{\|x - z_0\|}{4}\right) - w\left(\frac{\|y - z_0\|}{4}\right)\right\| \\ &\le 2\left|\frac{\|x - z_0\|}{4} - \frac{\|y - z_0\|}{4}\right| \le \frac{1}{2}\|x - y\|. \end{aligned}$$

Further, we have, for every $x \in D$,

$$\|Tx - z_0\| = \left\| w\left(\frac{\|x - z_0\|}{4}\right)\right\| \le \frac{\|x - z_0\|}{4} < \|x - z_0\|.$$

So, T has no fixed point in D. $\square$

As a direct consequence of Theorem 5, we obtain the following.

Corollary 3. *Let X be a normed linear space. Then X is a Banach space if and only if every contractive mapping from X into itself has a fixed point in X.*

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