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FIXED POINT THEOREMS BY ALTERING DISTANCES BETWEEN THE POINTS

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In this paper we have established some fixed point theorems in complete and compact metric spaces.

1. Introduction

Let R^+ be the set of nonnegative real numbers and N the set of positive integers.

Delbosco [1] and Skof [8] have established fixed point theorems for selfmaps of complete metric spaces by altering the distances between the points with the use of a function $\varphi : R^+ \to R^+$ satisfying the following properties:

1. φ is continuous and strictly increasing in R^+ ;
2. $\varphi(t) = 0$ if and only if $t = 0$;
3. $\varphi(t) \geq M.t^\mu$ for every $t > 0$, where $M > 0$, $\mu > 0$ are constant.

We denote the set of above functions φ with Φ .

Precisely in [8, corol. 2] the following theorem was proved:

THEOREM 1. *Let T be a selfmap of a complete metric space (X,d) and $\varphi \in \Phi$ such that for every x,y in X,*

$$(A) \qquad \varphi(d(Tx,Ty)) \leq a.\varphi(d(x,y)) + b.\varphi(d(x,Tx)) + c.\varphi(d(y,Ty))$$

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where $0 \leqslant a + b + c < 1$. *Then* T *has a unique fixed point.*

In [1], the author has considered functions $\varphi \in \Phi$ such that $\varphi(t) = t^n$, $n \in N$, for every $t \geqslant 0$.

REMARK 1. Note that φ is not necessarily a metric: for example, $\varphi(t) = t^2$.

REMARK 2. By symmetry of metric d, we may assume $b = c$ in (A). The purpose of this paper is to study a stronger condition than (A) and to remove the hypothesis (3) which seems superfluous. Furthermore, our main theorem is an improvement upon some fixed point theorems of Rakotch [5], Reich [6], and a result of Fisher [3] in compact metric spaces. Other related results can be found in Sessa [7].

2. Main theorem

We shall prove a fixed point theorem offering a condition closely related to that used by Massa [4] in Banach spaces. Strictly speaking, the following theorem holds:

THEOREM 2. *Let* (X,d) *be a complete metric space,* T *a selfmap of* X, *and* $\varphi : R^+ \to R^+$ *an increasing, continuous function satisfying property (2). Furthermore, let* a,b,c *be three decreasing functions from* $R^+ \setminus \{0\}$ *into* $[0,1[$ *such that* $a(t) + 2b(t) + c(t) < 1$ *for every* $t > 0$. *Suppose that* T *satisfies the following condition:*

$$\text{(B)}\quad \begin{aligned}\varphi(d(Tx,Ty)) \leqslant\ & a(d(x,y)).\varphi(d(x,y)) + b(d(x,y)).\{\varphi(d(x,Tx)) + \\ & \varphi(d(y,Ty)\} + c(d(x,y)).\min\{\varphi(d(x,Ty)),\ \varphi(d(y,Tx))\},\end{aligned}$$

where $x,y \in X$ *and* $x \neq y$. *Then* T *has a unique fixed point.*

Proof. Let x_0 be a point of X. We define

$$(*)\qquad x_{n+1} = Tx_n\,, \qquad \tau_n = d(x_n, x_{n+1}),\ \text{for all}\ n \in N \cup \{0\}\,.$$

We first prove that T has a fixed point. We may assume $\tau_n > 0$ for each n. From (B), we obtain:

$$\begin{aligned}\varphi(\tau_{n+1}) \leqslant\ & a(\tau_n).\varphi(\tau_n) + b(\tau_n).\{\varphi(\tau_n) + \varphi(\tau_{n+1})\} + \\ & c(\tau_n).\min\{\varphi(d(x_n,x_{n+2})),\ \varphi(d(x_{n+1},x_{n+1}))\}.\end{aligned}$$

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Hence we obtain:

$$(2.1)\qquad \varphi(\tau_{n+1}) \leqslant \frac{a(\tau_n)+b(\tau_n)}{1 - b(\tau_n)} \,.\, \varphi(\tau_n) < \varphi(\tau_n).$$

Since φ is increasing, $\{\tau_n\}$ is a decreasing sequence.

We put $\lim\limits_{n\to\infty} \tau_n = \tau$ and suppose that $\tau > 0$. By (2.1), then $\tau_n \geqslant \tau$ implies that

$$\varphi(\tau_{n+1}) \leqslant \frac{a(\tau) + b(\tau)}{1 - b(\tau)} \,.\, \varphi(\tau_n) \,.$$

By letting $n \to \infty$, since φ is continuous, we have:

$$\varphi(\tau) \leqslant \frac{a(\tau) + b(\tau)}{1 - b(\tau)} \,.\, \varphi(\tau) < \varphi(\tau) \,,$$

which is inadmissible. So $\tau = 0$. Now we prove that $\{x_n\}$ is a Cauchy sequence. Suppose it is not. Then there exist $\varepsilon > 0$ and two sequences $\{p(n)\}$, $\{q(n)\}$ such that for every $n \in N \cup \{0\}$, we find that $p(n) > q(n) \geqslant n$, $d(x_{p(n)},x_{q(n)}) \geqslant \varepsilon$ and $d(x_{p(n)-1},x_{q(n)}) < \varepsilon$.

For each $n \geqslant 0$, we put $s_n = d(x_{p(n)},x_{q(n)})$. Then we have

$$\varepsilon \leqslant s_n \leqslant d(x_{p(n)-1},x_{p(n)}) + d(x_{p(n)-1},x_{q(n)}) < \tau_{p(n)-1} + \varepsilon.$$

Since $\{\tau_n\}$ converges to 0, $\{s_n\}$ converges to ε.

Furthermore, the triangular inequality implies, for each $n \geqslant 0$, $-\tau_{p(n)} - \tau_{q(n)} + s_n \leqslant d(x_{p(n)+1}, x_{q(n)+1}) \leqslant \tau_{p(n)} + \tau_{q(n)} + s_n$, and therefore also the sequence $\{d(x_{p(n)+1},x_{q(n)+1})\}$ converges to ε.

From (B), we also deduce:

$$\varphi(d(x_{p(n)+1},x_{q(n)+1})) \leqslant a(s_n).\varphi(s_n) + b(s_n).\{\varphi(\tau_{p(n)})+ \varphi(\tau_{q(n)})\} + c(s_n).\min\{\varphi(d(x_{p(n)},x_{q(n)+1})),\varphi(d(x_{q(n)},x_{p(n)+1}))\} \leqslant a(\varepsilon).\varphi(s_n) + b(\varepsilon).\{\varphi(\tau_{p(n)}) + \varphi(\tau_{q(n)})\} + c(\varepsilon) \,.\, \varphi(s_n + \tau_{q(n)} + \tau_{p(n)}) \,.$$

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For $n \to \infty$ we are left with

$$\varphi(\varepsilon) \leqslant \{a(\varepsilon) + c(\varepsilon)\}.\varphi(\varepsilon) < \varphi(\varepsilon) ,$$

which is absurd. Therefore $\{x_n\}$ is a Cauchy sequence. By completeness of X, $\{x_n\}$ converges to some point z. Now we show that z is a fixed point of T. Since each $\tau_n > 0$, there is a subsequence $\{x_{h(n)}\}$ of $\{x_n\}$ such that $x_{h(n)} \neq z$ for each $n \geqslant 0$ and we put $\rho_n = d(z,x_n)$.

Since $b < 1/2$, we obtain from (B):

$$\varphi(d(x_{h(n)+1},Tz)) \leqslant a(\rho_{h(n)}).\varphi(\rho_{h(n)}) + b(\rho_{h(n)}).\{\varphi(\tau_{h(n)}) + \varphi(d(z,Tz))\}$$
$$+ c(\rho_{h(n)}).min\{\varphi(\rho_{h(n)+1}),\varphi(d(x_{h(n)},Tz))\}$$
$$< \varphi(\rho_{h(n)}) + 1/2\{\varphi(\tau_{h(n)}) + \varphi(d(z,Tz))\} + \varphi(\rho_{h(n)} + \tau_{h(n)}).$$

Since $\{\rho_n\}$ converges to 0, for $n \to \infty$ the last inequality yields

$$\limsup_{n\to\infty} \varphi(d(x_{h(n)+1},Tz)) \leqslant 1/2\ \varphi(d(z,Tz)). \tag{2.2}$$

On the other hand, the triangular inequality implies that

$$d(z,Tz) \leqslant \rho_{h(n)} + \tau_{h(n)} + d(x_{h(n)+1},Tz) ,$$

which in turn implies that

$$\varphi(d(z,Tz)) \leqslant \limsup_{n\to\infty} \varphi(d(x_{h(n)+1},Tz)). \tag{2.3}$$

From (2.2) and (2.3), then we deduce

$$\varphi(d(z,Tz)) \leqslant 1/2\ \varphi(d(z,Tz)) ;$$

that is, $\varphi(d(z,Tz)) = 0$ and therefore $d(z,Tz) = 0$.

If T has two distinct fixed points x,y in X, then $\varphi(d(x,y)) = \varphi(d(Tx,Ty)) \leqslant \{a(d(x,y)) + c(d(x,y))\}.\varphi(d(x,y)) < \varphi(d(x,y))$, a contradiction. This completes the proof.

REMARK 3. Note that we have not supposed the continuity of T.

3. Some consequences and examples

If we assume $c = 0$ in Theorem 2 and take a, b as constants, we obtain Theorem 1. The following examples show that condition (B) is more general than condition (A) :

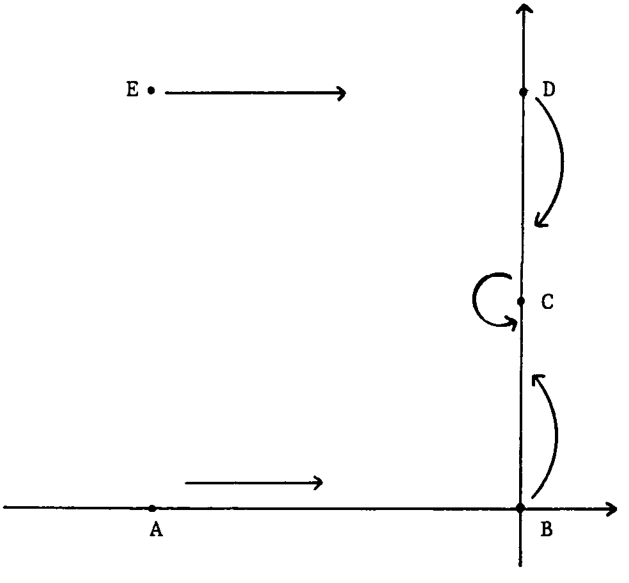
EXAMPLE 1. Let X be the subset of R^2 defined by

$$X = \{A, B, C, D, E\},$$

where $A \equiv (-1,0)$, $B \equiv (0,0)$, $C \equiv (0,1/2)$, $D \equiv (0,1)$, $E \equiv (-1,1)$.

Let $T : X \to X$ be given by

$$T(A) = B, \; T(B) = T(C) = T(D) = C, \; T(E) = D .$$



Then T satisfies condition (B) by letting:

$$a(t) = 3/4, \; b(t) = 0, \; c(t) = 1/5 \text{ and } \varphi(t) = t^2 \text{ for any } t \in R^+ .$$

However, T does not satisfy condition (A). For otherwise, choosing $x = A$ and $y = E$, we would have

$$\varphi(d(TA,TE)) = \varphi(1) \leqslant a.\varphi(1) + b.\varphi(1) + c.\varphi(1) < \varphi(1) ,$$

which is a contradiction.

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In Theorem 2 if we assume $c = 0$ and $\varphi(t) = t$ for every $t \geq 0$, we obtain the following condition indebted to Reich [6],

$$\text{(C)} \quad d(Tx,Ty) \leq a(d(x,y)).d(x,y) + b(d(x,y)).\{d(x,Tx) + d(y,Ty)\}.$$

The example given below proves that condition (B) is more general than condition (C) :

EXAMPLE 2. Consider the set $X = \{1,2,3,4\}$ equipped with the metric d which is defined by

$$d(1,2) = 2/5, \quad d(1,3) = 1/5, \quad d(1,4) = 3/5,$$
$$d(2,3) = 2/5, \quad d(2,4) = 1, \quad d(3,4) = \sqrt{2}/2.$$

Let T be a selfmap of X such that

$$T(1) = T(3) = T(4) = 3, \qquad T(2) = 4.$$

Here all the assumptions of Theorem 2 are satisfied with

$$a(t) = 1/16, \ b(t) = 1/3, \ c(t) = 1/16 \ \text{ and } \ \varphi(t) = t^4 \ \text{ for any } \ t \in R^+.$$

But the condition (C) is not fulfilled, otherwise for $x = 1$ and $y = 2$, and all functions a,b from $R^+ \setminus \{0\}$ into $[0,1[$ with $a + 2b < 1$, we would have

$$d(T1,T2) = \sqrt{2}/2 \leq a(2/5).2/5 + b(2/5).6/5 \leq a(2/5).3/5 + 2b(2/5).3/5 < 3/5,$$

which is a contradiction as $\sqrt{2}/2 > 3/5$.

If we assume $b = c = 0$ in Theorem 2, we get the following:

THEOREM 3. *Let (X,d) be a complete metric space, T a selfmap of X and $\varphi : R^+ \to R^+$ be an increasing, continuous function for which property (2) holds. Let a be a decreasing function from $R^+ \setminus \{0\}$ into $[0,1[$ such that*

$$\text{(D)} \qquad \varphi(d(Tx,Ty)) \leq a(d(x,y)).\varphi(d(x,y)),$$

where $x,y \in X$ and $x \neq y$. Then T has a unique fixed point.

REMARK 4. For $\varphi(t)=t$ Theorem 3 yields Rakotch's fixed point theorem [5].

4. A result in compact metric spaces

In a paper of Fisher [3], the following theorem has been given:

THEOREM 4. *Let* T *be a continuous selfmap of a compact metric space* (X,d) *such that*

$$\text{(E)} \qquad d(Tx,Ty) < 1/2\{d(x,Tx) + d(y,Ty)\}$$

for all distinct x,y *in* X. *Then* T *has a unique fixed point.*

Following the fundamental idea of our work presented in section 2 we now generalize Theorem 4 as follows:

THEOREM 5. *Let* T *be a continuous selfmap of a metric space* (X,d) *such that for some* $x_o \in X$ *the sequence* $\{T^n x_o\}$ *has a cluster point* $z \in X$. *Let there exist a continuous function* $\varphi : R^+ \to R^+$ *satisfying property (2). Furthermore, for all distinct* x,y *in* X *the inequality*

$$\text{(F)} \qquad \varphi(d(Tx,Ty)) < c.\varphi(d(x,y)) + \left(\frac{1-c}{2}\right)\{\varphi(d(x,Tx)) + \varphi(d(y,Ty))\}$$

holds, where $0 \leqslant c \leqslant 1$. *Then* z *is the unique fixed point of* T.

Proof. If $T^n x_o = T^{n+1} x_o$ for some $n \in N$, then $z = T^k x_o$ for all $k \geqslant n$ and therefore the thesis. So we may assume that $T^n x_o \neq T^{n+1} x_o$ for every $n \in N$. Let $\{k(n)\}$ be a sequence of positive integers such that $\{T^{k(n)} x_o\}$ converges to z. By maintaining the notations (*) of Theorem 2, and using the continuity of T, we have

$$\lim_{n\to+\infty} x_{k(n)+1} = T(z) \quad \text{and} \quad \lim_{n\to+\infty} x_{k(n)+2} = T^2(z) .$$

As φ is continuous, it also follows that

$$\text{(4.1)} \qquad \varphi(d(z,Tz)) = \lim_{n\to+\infty} \varphi(\tau_{k(n)}) = \lim_{n\to+\infty} \varphi(\tau_{k(n)+1}) = \varphi(d(Tz,T^2z)).$$

Now we claim that $z = Tz$, otherwise, by condition (F) when $x = z$ and $y = Tz$, we have

$$\varphi(d(Tz,T^2z)) < c.\varphi(d(z,Tz)) + \frac{1-c}{2}\{\varphi(d(z,Tz)) + \varphi(d(Tz,T^2z))\}.$$

This last inequality implies that

$$\varphi(d(Tz,T^2z)) < \varphi(d(z,Tz)) ,$$

which contradicts (4.1).

Property (2) assures the uniqueness of the fixed point.

REMARK 5. If $\varphi(t) = t$ for any $t \geq 0$ and $c = 1$, Theorem 5 becomes a well-known result of Edelstein [2].

REMARK 6. If $\varphi(t) = t$ for any $t \geq 0$ and $c = 0$, Theorem 5 reduces to Theorem 4 as every sequence in a compact metric space necessarily has a cluster point.

Using the following example, we show that condition (F) is more general than condition (E):

EXAMPLE 3. Consider the set $X = \{1,2,3,4\}$ with the metric d defined as

$$d(1,2) = 9\sqrt{3}, \qquad d(1,3) = 3\sqrt{3}, \qquad d(1,4) = 12\sqrt{3},$$

$$d(2,3) = 9\sqrt{3}, \qquad d(2,4) = 21\sqrt{3}, \qquad d(3,4) = 21.$$

Let $T : X \to X$ be defined by

$$T(1) = T(3) = T(4) = 3 , \quad T(2) = 4.$$

Then condition (F) is clearly verified for $\varphi(t) = t^2$ and $c = 1/3$. But condition (E) does not hold because for $x = 1$ and $y = 2$, we have:

$$d(T1,T2) = 21 > 12\sqrt{3} = \frac{1}{2}(3\sqrt{3} + 21\sqrt{3}) = \frac{1}{2}\{d(1,T1) + d(2,T2)\}.$$

The idea of this example appears in [1].

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References

[1] D. Delbosco, "Un'estensione di un teorema sul punto fisso di S. Reich", *Rend. Sem. Mat. Univers. Politecn. Torino*, 35 (1976-77), 233-238.

[2] M. Edelstein, "On fixed and periodic points under contractive mappings", *J. London Math. Soc.*, 37 (1962), 74-79.

[3] B. Fisher, "A fixed point mapping", *Bull. Calcutta Math. Soc.* 68 (1976), 265-266.

[4] S. Massa, "Un'osservazione su un teorema di Browder-Roux-Soardi", *Boll. U.M.I.*, *(4)* 7 (1973), 151-155.

[5] E. Rakotch, "A note on contractive mappings", *Proc. Amer. Math. Soc.*, 13 (1962), 459-465.

[6] S. Reich, "Kannan's fixed point theorem", *Boll. U.M.I. (4)* 4 (1971), 1-11.

[7] S. Sessa, "Un teorema orbitale di punto fisso ed una sua conseguenza", *Le Matematiche*, Catania (Italy), to appear.

[8] F. Skof, "Teorema di punti fisso per applicazioni negli spazi metrici", *Atti. Accad. Sci. Torino*, 111 (1977), 323-329.

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