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## FIXED POINT THEOREMS FOR EXPANSION MAPPINGS SATISFYING IMPLICIT RELATIONS

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**ABSTRACT.** In this note a fixed point theorem for expansion mappings is established in a complete metric space under certain conditions. Further a common fixed point theorem for pair of weakly compatible expansion mappings is established. In this theorem the completeness of space is replaced with a set of four alternative conditions for functions satisfying implicit relations. These theorems extend and improve results of S. M. Kang [4], M. A. Khan et al. [5], B. E. Rhoades [11] and T. Taniguchi [12].

### 1. Introduction

Wang et al. [13] proved some fixed point theorems on expansion mappings, which correspond some contractive mappings. Further, by using functions, Khan et al. [5] generalized the result of [13]. Also Rhoades [11] and Taniguchi [12] generalized the result of [13] for pair of mappings. Kang [4] generalized the result of Khan et al. [5], Rhoades [11] and Taniguchi [12] for expansion mappings.

Popa [10] improved results of Jha et al. [1], Pant et al. [6], [7] for Meir and Keeler type mappings by taking weak compatibility property and replacing the completeness of the space with a set of four alternative conditions for functions satisfying implicit relations.

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The objective of this paper is to prove common fixed point theorem for surjective mappings satisfying some expansion conditions, which extend corresponding result of Kang [4] and Khan et al. [5]. In the sequel, we introduce some implicit relations in section 4 which are found to be viable, productive and powerful tool in finding the existence of common fixed point for non-surjective mappings satisfying certain expansion type conditions.

## 2. Preliminaries

Throughout this paper,  $\mathbb{R}$  and  $\mathbb{N}$  denote the set of real numbers and the set of natural numbers, respectively. We use the following definitions in the proof of our main theorems.

**Definition 2.1.** Let  $X$  be a topological space and  $f : X \rightarrow \mathbb{R}$  a real valued mapping on  $X$ . Then  $f$  is called *upper semi-continuous* on  $X$  iff  $f^{-1}(-\infty, t)$  is open in  $X$  for every  $t \in \mathbb{R}$ . A mapping  $f$  is called *lower semi-continuous* if  $-f$  is upper semi-continuous.

**Definition 2.2.**[11] The self maps  $S$  and  $T$  of a metric space  $(X, d)$  are said to be weak compatible if  $Sx = Tx$  implies  $STx = TSx$ .

Kang [4] proved the following theorem :

Let  $\mathbb{R}_+$  be the set of all non-negative real numbers and let  $\Phi$  denote the family of all real functions  $\phi : \mathbb{R}_+^3 \rightarrow \mathbb{R}_+$  satisfying the following conditions  $(C_1)$  and  $(C_2)$  according to Khan et al. [5] :

$(C_1)$   $\phi$  is lower semi-continuous in each coordinate variable,

$(C_2)$  Let  $v, w \in \mathbb{R}_+$  be such that either  $v \geq \phi(v, w, w)$  or  $v \geq \phi(w, v, w)$ .

Then  $v \geq hw$ , where  $\phi(1, 1, 1) = h > 1$ .

**Theorem 2.3.** Let  $A$  and  $B$  be surjective mappings from a complete metric space  $(X, d)$  into itself satisfying

$$d(Ax, By) \geq \phi(d(Ax, x), d(By, y), d(x, y))$$

for all  $x, y \in X$  with  $x \neq y$ , where  $\phi \in \Phi$ . Then  $A$  and  $B$  have a common fixed point in  $X$ .

## 3. Main Results

Let  $\mathbb{R}_+$  be the set of all non-negative real numbers and let  $\Phi$  denote the family of all real valued functions  $\phi : \mathbb{R}_+^5 \rightarrow \mathbb{R}_+$  satisfying the following conditions :

$(C_1)$   $\phi$  is *lower semi-continuous* in each coordinate variable,

$(C_2)$   $\phi$  is non-increasing in second and third coordinate variables,

(C<sub>3</sub>) Let  $v, w \in \mathbb{R}_+$  be such that either  $v \geq \phi(v, v + w, 0, w, w)$  or  $v \geq \phi(w, 0, v + w, v, w)$ . Then  $v \geq hw$ , where  $\phi(1, 1, 1, 1, 1) = h > 1$ .

Now we prove our theorem as follows :

**Theorem 3.1.** Let A and B be surjective mappings from a complete metric space  $(X, d)$  into itself satisfying

$$(3.1) \quad d(Ax, By) \geq \phi(d(Ax, x), d(Ax, y), d(By, x), d(By, y), d(x, y))$$

for all  $x, y \in X$  with  $x \neq y$ , where  $\phi \in \Phi$ . Then A and B have a common fixed point in X.

**Proof.** Let  $x_0$  be an arbitrary point in X. Since A and B are surjective, we choose a point  $x_1$  in X such that  $Ax_1 = x_0$  and for this point  $x_1$ , there exists a point  $x_2$  in X such that  $Bx_2 = x_1$ . By this way, we can define a sequence  $\{x_n\}$  in X such that

$$(3.2) \quad Ax_{2n+1} = x_{2n} \text{ and } Bx_{2n+2} = x_{2n+1}.$$

Suppose that  $x_{2n} = x_{2n+1}$  for  $n \geq 0$ . Then  $x_{2n}$  is a fixed point of A. If  $x_{2n+1} \neq x_{2n+2}$  then, from (3.1), we have

$$\begin{aligned} d(x_{2n}, x_{2n+1}) &= d(Ax_{2n+1}, Bx_{2n+2}) \\ &\geq \phi(d(Ax_{2n+1}, x_{2n+1}), d(Ax_{2n+1}, x_{2n+2}), \\ &\quad d(Bx_{2n+2}, x_{2n+1}), d(Bx_{2n+2}, x_{2n+2}), d(x_{2n+1}, x_{2n+2})) \\ &\geq \phi(d(x_{2n}, x_{2n+1}), d(x_{2n}, x_{2n+2}), \\ &\quad d(x_{2n+1}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}), d(x_{2n+1}, x_{2n+2})) \\ &\geq \phi(d(x_{2n}, x_{2n+1}), d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2}), 0 \\ &\quad d(x_{2n+1}, x_{2n+2}), d(x_{2n+1}, x_{2n+2})), \end{aligned}$$

which implies, by (C<sub>3</sub>)

$$d(x_{2n}, x_{2n+1}) \geq hd(x_{2n+1}, x_{2n+2}).$$

This yields a contradiction, and so  $x_{2n+1} = x_{2n+2}$ . Thus  $x_{2n}$  is a common fixed point of A and B. If  $x_{2n+1} = x_{2n+2}$  for some  $n \geq 0$ , it is similarly verified that  $x_{2n+1}$  is a common fixed point of A and B. Without loss of generality, we can suppose  $x_n \neq x_{n+1}$ , for each  $n \geq 0$ . From (3.1), we have

$$\begin{aligned} d(x_{2n}, x_{2n+1}) &= d(Ax_{2n+1}, Bx_{2n+2}) \\ &\geq \phi(d(Ax_{2n+1}, x_{2n+1}), d(Ax_{2n+1}, x_{2n+2}), \\ &\quad d(Bx_{2n+2}, x_{2n+1}), d(Bx_{2n+2}, x_{2n+2}), d(x_{2n+1}, x_{2n+2})), \end{aligned}$$

which implies, from (C<sub>3</sub>)

$$d(x_{2n+1}, x_{2n+2}) \leq \frac{1}{h}d(x_{2n}, x_{2n+1}).$$

Similarly

$$\begin{aligned}
d(x_{2n+1}, x_{2n+2}) &= d(Bx_{2n+2}, Ax_{2n+3}) \\
&\geq \phi(d(Ax_{2n+3}, x_{2n+3}), d(Ax_{2n+3}, x_{2n+2}), \\
&\quad d(Bx_{2n+2}, x_{2n+3}), d(Bx_{2n+2}, x_{2n+2}), d(x_{2n+3}, x_{2n+2})) \\
&\geq \phi(d(x_{2n+2}, x_{2n+3}), d(x_{2n+2}, x_{2n+2}), \\
&\quad d(x_{2n+1}, x_{2n+3}), d(x_{2n+1}, x_{2n+2}), d(x_{2n+3}, x_{2n+2})),
\end{aligned}$$

which implies, by  $(C_3)$

$$d(x_{2n+1}, x_{2n+2}) \geq h d(x_{2n+2}, x_{2n+3}),$$

or

$$d(x_{2n+2}, x_{2n+3}) \leq \frac{1}{h} d(x_{2n+1}, x_{2n+2}).$$

Therefore, we obtain

$$d(x_{n+1}, x_{n+2}) \leq \frac{1}{h} d(x_n, x_{n+1})$$

Since  $h > 1$ , by Lemma of Jungck [2],  $\{x_n\}$  is a Cauchy sequence and hence it converges to some point  $z$  in  $X$ . Consequentially, the sub-sequences  $\{x_{2n}\}$ ,  $\{x_{2n+1}\}$  and  $\{x_{2n+2}\}$  also converge to  $z$ .

Since  $A$  and  $B$  are surjective, there exist two points  $v$  and  $w$  in  $X$  such that  $z = Av$  and  $z = Bw$ . Thus, using (3.1), we have

$$\begin{aligned}
d(x_{2n}, z) &= d(Ax_{2n+1}, Bw) \\
&\geq \phi(d(Ax_{2n+1}, x_{2n+1}), d(Ax_{2n+1}, w), \\
&\quad d(Bw, x_{2n+1}), d(Bw, w), d(x_{2n+1}, w)) \\
&\geq \phi(d(x_{2n}, x_{2n+1}), d(x_{2n}, w), \\
&\quad d(Bw, x_{2n+1}), d(Bw, w), d(x_{2n+1}, w)).
\end{aligned}$$

Letting  $n \rightarrow \infty$

$$\begin{aligned}
0 = d(z, z) &\geq \phi(d(z, z), d(z, w), d(z, z), d(z, w), d(z, w)) \\
&\geq \phi(0, d(z, w), 0, d(z, w), d(z, w)) \\
&\geq \phi(0, 0 + d(z, w), 0, d(z, w), d(z, w)),
\end{aligned}$$

which implies, by  $(C_3)$

$$0 \geq h d(z, w),$$

so that  $z = w$ . Similarly, we have  $z = v$ . Therefore,  $A$  and  $B$  have a common fixed point in  $X$ .

**Remark 3.2.** Our Theorem 3.1 extends corresponding results of Kang [4] and for  $A = B$  it extends result of Khan et al. [5].

**Corollary 3.3.** Let  $A$  and  $B$  be surjective mappings from a complete metric space  $(X, d)$  into itself satisfying

$$d(Ax, By) \geq a d(Ax, x) + b d(By, y) + c d(x, y) + e d(Ax, y) + f d(By, x)$$

for all  $x, y \in X$  with  $x \neq y$ , where  $a, b, c, e, f$  are non-negative real numbers with  $a + e < 1$ ,  $f + b < 1$ ,  $a + b + c + e + f > 1$ . Then  $A$  and  $B$  have a common fixed point in  $X$ .

**Proof.** Let  $h = a + b + c + e + f$  and  $\phi(t_1, t_2, t_3, t_4, t_5) = at_1 + bt_2 + ct_3 + et_4 + ft_5$  for every  $t_1, t_2, t_3, t_4, t_5 \in \mathbb{R}_+$ .

If  $v \geq av + ev + fw + bw + cw$  for some  $v, w \in \mathbb{R}_+$ , then

$$\begin{aligned} v(1 - a - e) &\geq (f + b + c)w \\ v &\geq \frac{(f + b + c)}{(1 - a - e)}w \\ &\geq (a + b + c + e + f)w \\ &\geq hw. \end{aligned}$$

If  $v \geq aw + ew + fw + bw + cw$  for some  $v, w \in \mathbb{R}_+$ , then similarly we have

$$v \geq hw.$$

Therefore  $\phi \in \Phi$ . And so the proof of the Corollary is complete by Theorem 3.1.

**Remark 3.4.** (I) If we take  $e = f = 0$  in Corollary 3.3 we get Corollary 2.3 of Kang [4].

(II) The surjective condition of mappings  $A$  and  $B$  can not be dropped in Theorem 3.1 and Corollary 3.3 as shown below :

**Example 3.5.** Let  $X = [0, \infty)$  with the Euclidean metric  $d$ . Define  $A$  and  $B : X \rightarrow X$  by  $Ax = h(x + 1)$  and

$$Bx = \begin{cases} hx & \text{if } x < \frac{1}{2} \\ h(x + 1) & \text{if } x \geq \frac{1}{2} \end{cases}$$

for each  $x$  in  $X$ , where  $h > 1$ . Consider  $\phi(t_1, t_2, t_3, t_4, t_5) = ht_5$  for every  $t_1, t_2, t_3, t_4, t_5 \in \mathbb{R}_+$ , where  $h > 1$ . Then  $\phi \in \Phi$  and we have

$$\begin{aligned} d(Ax, By) &= \begin{cases} h|x - y + 1| & \text{if } y < \frac{1}{2} \\ h|x - y| & \text{if } y \geq \frac{1}{2} \end{cases} \\ &\geq h|x - y| \\ &\geq hd(x, y) \\ &= \phi(d(Ax, x), d(Ax, y), d(By, x), d(By, y), d(x, y)) \end{aligned}$$

for all  $x$  and  $y$  in  $X$ . In this example all the hypothesis of Theorem 3.1 are satisfied except that  $A$  and  $B$  are surjective, but  $A$  and  $B$  have no common fixed point in  $X$ . Therefore, the surjectivity of mappings  $A$  and  $B$  is a necessary condition in Theorem 3.1 .

Next, let  $\Phi^*$  denotes the family of all real functions  $\psi : \mathbb{R}_+^5 \rightarrow \mathbb{R}_+$  satisfying  $(C_1), (C_2)$  and  $(C_4)$  Let  $v, w \in \mathbb{R}_+ - \{0\}$  be such that either  $v \geq \psi(v, v + w, 0, w, w)$  or  $v \geq \psi(w, 0, v + w, v, w)$ . Then  $v \geq hw$ , where  $\psi(1, 1, 1, 1, 1) = h > 1$ .

**Theorem 3.6.** Let  $A$  and  $B$  be continuous surjective mappings from a complete metric space  $(X, d)$  into itself satisfying

$$d(Ax, By) \geq \psi(d(Ax, x), d(Ax, y), d(By, x), d(By, y), d(x, y))$$

for all  $x, y$  in  $X$  with  $x \neq y$ , where  $\psi \in \Phi^*$ . Then  $A$  or  $B$  has a fixed point or  $A$  and  $B$  have a common fixed point in  $X$ .

**Proof.** Let  $\{x_n\}$  be a sequence in  $X$  defined by

$$Ax_{2n+1} = x_{2n} \text{ and } Bx_{2n+2} = x_{2n+1}.$$

If  $x_n = x_{n+1}$  for some  $n \geq 0$ , then  $A$  or  $B$  has a fixed point in  $X$ . Now, we suppose that  $x_n \neq x_{n+1}$  for each  $n \geq 0$ . As in the proof of Theorem 3.1, it can be shown that  $\{x_n\}$  is a Cauchy sequence and hence it converges to some point  $z$  in  $X$ . Consequently, the sub-sequences  $\{x_{2n}\}$ ,  $\{x_{2n+1}\}$  and  $\{x_{2n+2}\}$  also converges to  $z$ . Since  $A$  and  $B$  are continuous, we get

$$Ax_{2n+1} = x_{2n} \rightarrow Az \text{ and } Bx_{2n+2} = x_{2n+1} \rightarrow Bz \text{ as } n \rightarrow \infty.$$

Thus,  $A$  and  $B$  have a common fixed point in  $X$ .

From Theorem 3.6, we obtain the following corollary :

**Corollary 3.7.** Let  $A$  and  $B$  be continuous surjective mappings from a complete metric space  $(X, d)$  into itself satisfying

$$d(Ax, By) \geq h \min\{d(Ax, x), d(By, y), d(x, y), \max\{d(Ax, y), d(By, x)\}\}$$

for all  $x$  and  $y$  in  $X$  with  $x \neq y$ , where  $h > 1$ . Then the same conclusion of Theorem 3.6 holds.

**Remark 3.8.** If  $\max\{d(Ax, y), d(By, x)\}$  is greater than or equal to any one of  $d(Ax, x), d(By, y), d(x, y)$  for all  $x, y \in X$ , then we get the Corollary 2.7 of Kang [4]. For  $A = B$ , our Theorem 3.6 extends corresponding results of Rhoades [11] and Khan et al. [5].

Finally, let  $\Psi$  denotes the family of all real valued functions  $\psi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ , satisfying the following conditions :

$(C_5)$   $\psi$  is upper semi – continuous and non decreasing,

$(C_6)$   $\psi(t) < t$  for each  $t > 0$ .

**Theorem 3.9.** Let  $A$  and  $B$  be continuous surjective mappings from a complete metric space  $(X, d)$  into itself satisfying

$$\psi(d(Ax, By)) \geq \min\{d(Ax, x), d(By, y), d(x, y), \max\{d(Ax, y), d(By, x)\}\}$$

for all  $x, y$  in  $X$  with  $x \neq y$ , where  $\psi \in \Psi$  and  $\sum \psi^n(t) < \infty$  for each  $t > 0$ . Then  $A$  or  $B$  has a fixed point or  $A$  and  $B$  have a common fixed point in  $X$ .

**Proof.** Let  $\{x_n\}$  be a sequence in  $X$  defined by

$$Ax_{2n+1} = x_{2n} \text{ and } Bx_{2n+2} = x_{2n+1}.$$

If  $x_n = x_{n+1}$  for some  $n \geq 0$ , then  $A$  or  $B$  has a fixed point in  $X$ . Now, we suppose that  $x_n \neq x_{n+1}$  for each  $n \geq 0$ . Then

$$\begin{aligned} \psi(d(x_{2n}, x_{2n+1})) &= \psi(d(Ax_{2n+1}, Bx_{2n+2})) \\ &\geq \min\{d(Ax_{2n+1}, x_{2n+1}), d(Bx_{2n+2}, x_{2n+2}), d(x_{2n+1}, x_{2n+2}), \\ &\quad \max\{d(Ax_{2n+1}, x_{2n+2}), d(Bx_{2n+2}, x_{2n+1})\}\} \\ &\geq \min\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}), d(x_{2n+1}, x_{2n+2}), \\ &\quad \max\{d(x_{2n}, x_{2n+2}), d(x_{2n+1}, x_{2n+2})\}\} \\ &\geq \min\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}), d(x_{2n+1}, x_{2n+2}), \\ &\quad \max\{d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2}), d(x_{2n+1}, x_{2n+2})\}\} \\ &\geq \min\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}), d(x_{2n}, x_{2n+1}) \\ &\quad + d(x_{2n+1}, x_{2n+2})\} \\ &= d(x_{2n+1}, x_{2n+2}). \end{aligned}$$

Similarly we have

$$\begin{aligned} \psi(d(x_{2n+1}, x_{2n+2})) &= \psi(d(Ax_{2n+3}, Bx_{2n+2})) \\ &\geq d(x_{2n+2}, x_{2n+3}). \end{aligned}$$

Therefore,  $d(x_{n+1}, x_{n+2}) \leq \psi(d(x_n, x_{n+1}))$ .

For any  $n > m \geq 0$  we have

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x_{m+1}) + d(x_{m+1}, x_n) \\ &\leq d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+2}) \\ &\quad + d(x_{m+2}, x_{m+3}) + \dots + d(x_{n-1}, x_n) \\ &\leq \psi^m(d(x_0, x_1)) + \dots + \psi^{n-1}(d(x_0, x_1)). \end{aligned}$$

From  $\sum \psi^n < \infty$  for each  $t > 0$ , it follows that  $\{x_n\}$  is a Cauchy sequence in  $X$ . And as the proof of Theorem 3.6, it is obvious that  $A$  and  $B$  have a common fixed point in  $X$ .

#### 4. Implicit Relations

In this section, we use implicit relations satisfied by a quadruple of mappings. The essence of implicit relations as a tool in finding common fixed point of mappings lies on the fact that these relations help us to ensure coincidence point of pair of mappings that ultimately leads to the existence of common fixed points of a quadruple of mappings satisfying weak compatibility criterion.

Let  $F_6$  be the set of all continuous functions  $F(t_1, \dots, t_6) : \mathbb{R}_+^6 \rightarrow \mathbb{R}$  satisfying the following conditions :

$$(F_1) : F(u, 0, u, 0, 0, u) \geq 0 \text{ implies } u = 0;$$

$$(F_2) : F(u, 0, 0, u, u, 0) \geq 0 \text{ implies } u = 0.$$

The function  $F(t_1, \dots, t_6) : \mathbb{R}_+^6 \rightarrow \mathbb{R}$  satisfies the condition  $(F_u)$  if

$$(F_u) : F(u, u, 0, 0, u, u) \leq 0, \text{ for all } u > 0.$$

**Example 4.1.** Let  $F(t_1, \dots, t_6) = pt_2 + q(t_3 + t_4) + r(t_5 + t_6) - t_1$ , where  $p, q, r \geq 0$ ,  $0 \leq q + r < 1$  and  $0 \leq p + 2r \leq 1$ .

$$(F_1) : F(u, 0, u, 0, 0, u) = u(q + r - 1) \geq 0 \text{ implies } u = 0;$$

$$(F_2) : F(u, 0, 0, u, u, 0) = u(q + r - 1) \geq 0 \text{ implies } u = 0 :$$

and

$$(F_u) : F(u, u, 0, 0, u, u) = u(p + 2r - 1) \leq 0; \text{ for each } u > 0.$$

**Example 4.2.** Let  $F(t_1, \dots, t_6) = \min\{(t_2 + t_3)/2, k(t_4 + t_5)/2, t_6\} - t_1$ ; where  $0 \leq k < 1$ .

$$(F_1) : F(u, 0, u, 0, 0, u) = \min\{u/2, 0, u\} - u \geq 0 \text{ implies } u = 0;$$

$$(F_2) : F(u, 0, 0, u, u, 0) = \min\{0, ku, 0\} - u \geq 0 \text{ implies } u = 0;$$

$$(F_u) : F(u, u, 0, 0, u, u) = \min\{u/2, ku/2, u\} - u \leq 0; \text{ for each } u > 0.$$

**Example 4.3.** Let  $F(t_1, \dots, t_6) = \max\{k_1 t_2, k_2(t_3 + t_5)/2, (t_4 + t_6)/2\} - t_1$ ; where  $0 \leq k_1 < 1$ ,  $1 \leq k_2 < 2$ .

$$(F_1) : F(u, 0, u, 0, 0, u) = \max\{0, k_2 u/2, u/2\} - u \geq 0 \text{ implies } u = 0;$$

$$(F_2) : F(u, 0, 0, u, u, 0) = \max\{0, k_2 u/2, u/2\} - u \geq 0 \text{ implies } u = 0;$$

$$(F_u) : F(u, u, 0, 0, u, u) = \max\{k_1 u, k_2 u/2, u/2\} - u \leq 0; \text{ for each } u > 0.$$

**Example 4.4.** Let  $F(t_1, \dots, t_6) = h \cdot \max\{t_2, t_3, t_4, t_5, t_6\} - t_1$ ; where  $0 \leq h < 1$ .

$$(F_1) : F(u, 0, u, 0, 0, u) = h \cdot \max\{0, u, 0, 0, u\} - u \geq 0 \text{ implies } u = 0;$$

$$(F_2) : F(u, 0, 0, u, u, 0) = h \cdot \max\{0, 0, u, u, 0\} - u \geq 0 \text{ implies } u = 0;$$

$$(F_u) : F(u, u, 0, 0, u, u) = h \cdot \max\{u, 0, 0, u, u\} - u \leq 0; \text{ for each } u > 0.$$



**Example 4.5.** Let  $F(t_1, \dots, t_6) = pt_2^2 + t_3t_4 + qt_5^2 + rt_6^2 - t_1^2$ ; where  $p, q, r \geq 0$ ,  $0 \leq p + q + r < 1$ .

$$(F_1) : F(u, 0, u, 0, 0, u) = ru^2 - u^2 \geq 0 \text{ implies } u = 0;$$

$$(F_2) : F(u, 0, 0, u, u, 0) = qu^2 - u^2 \geq 0 \text{ implies } u = 0;$$

$$(F_u) : F(u, u, 0, 0, u, u) = (p + q + r)u^2 - u^2 \leq 0; \text{ for each } u > 0.$$

**Example 4.6.** Let  $F(t_1, \dots, t_6) = k(t_2^3 + t_3^3 + t_4^3 + t_5^3 + t_6^3) - t_1^3$ ; where  $0 \leq k \leq 1/3$ .

$$(F_1) : F(u, 0, u, 0, 0, u) = (2k - 1)u^3 \geq 0 \text{ implies } u = 0;$$

$$(F_2) : F(u, 0, 0, u, u, 0) = (2k - 1)u^3 \geq 0 \text{ implies } u = 0;$$

$$(F_u) : F(u, u, 0, 0, u, u) = (3k - 1)u^3 \leq 0; \text{ for each } u > 0.$$

## 5. Application of Implicit Relations in Fixed Point Theory

In this section we prove a common fixed point theorem for a quadruple of expansion mappings satisfying implicit relations.

**Theorem 5.1.** Let  $A, B, S$  and  $T$  be the self mappings of a metric space  $(X, d)$ , such that

$$(I) \ A(X) \subset S(X) \text{ and } B(X) \subset T(X),$$

$$(II) \ d(Tx, Sy) \geq h \min\{d(Ax, By), d(Tx, Ax), d(Sy, By),$$

$$\frac{1}{2} \max\{d(Tx, By), d(Sy, Ax)\}\},$$

where  $h > 1$ .

(III) there exists  $F \in F_6$  such that

$F(d(Ax, By), d(Tx, Sy), d(Tx, Ax), d(Sy, By), d(Tx, By), d(Sy, Ax)) > 0$   
for all  $x, y$  in  $X$ ,  $x \neq y$ .

If one of  $A(X)$ ,  $B(X)$ ,  $S(X)$  or  $T(X)$  is complete subspace of  $X$ , then

(IV)  $A$  and  $T$  have a coincidence point,

(V)  $B$  and  $S$  have a coincidence point.

Moreover, if the pairs  $(A, T)$  and  $(B, S)$  are weakly compatible, then  $A, B, S$  and  $T$  have a unique common fixed point.

**Proof.** Since (I) holds, we can define a sequence by choosing an arbitrary point  $x_0$  in  $X$ , such that

$$y_{2n} = Ax_{2n} = Sx_{2n+1} \text{ and } y_{2n+1} = Bx_{2n+1} = Tx_{2n+2}$$

for  $n = 0, 1, 2, \dots$

Now we first prove that  $\{y_n\}$  is a Cauchy sequence in  $X$ . For this, put  $x = x_{2n}$  and  $y = x_{2n+1}$  in (II), we get

$$d(Tx_{2n}, Sx_{2n+1}) \geq h \min\{d(Ax_{2n}, Bx_{2n+1}), d(Tx_{2n}, Ax_{2n}), \\ d(Sx_{2n+1}, By), \frac{1}{2} \max\{d(Tx_{2n}, Bx_{2n+1}), d(Sx_{2n+1}, Ax_{2n})\}\},$$

or

$$d(y_{2n-1}, y_{2n}) \geq h \min\{d(y_{2n}, y_{2n+1}), d(y_{2n-1}, y_{2n}), \\ d(y_{2n}, y_{2n+1}), \frac{1}{2} \max\{d(y_{2n-1}, y_{2n+1}), d(y_{2n}, y_{2n})\}\} \\ \geq h \min\{d(y_{2n}, y_{2n+1}), d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1}), \\ \frac{1}{2} \max\{d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1}), 0\}\}.$$

If

$$d(y_{2n}, y_{2n+1}) > d(y_{2n-1}, y_{2n}),$$

then we have

$$d(y_{2n-1}, y_{2n}) \geq h \min\{d(y_{2n}, y_{2n+1}), d(y_{2n-1}, y_{2n})\} \\ \geq h d(y_{2n-1}, y_{2n}),$$

a contradiction. Therefore

$$d(y_{2n}, y_{2n+1}) \leq d(y_{2n-1}, y_{2n}).$$

This gives

$$d(y_{2n}, y_{2n+1}) \leq \frac{1}{h} d(y_{2n-1}, y_{2n}).$$

Similarly, for  $x = x_{2n+2}$  and  $y = x_{2n+1}$  with (II), we get

$$d(y_{2n+1}, y_{2n+2}) \leq \frac{1}{h} d(y_{2n}, y_{2n+1}).$$

Thus we have

$$d(y_n, y_{n+1}) \leq \frac{1}{h} d(y_{n-1}, y_n) \\ \leq \frac{1}{h^2} d(y_{n-2}, y_{n-1}) \\ \dots$$

...

$$< \frac{1}{h^n} d(y_0, y_1).$$

Moreover, for every integer  $p > 0$ , we have

$$\begin{aligned} d(y_n, y_{n+p}) &\leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + \cdots + d(y_{n+p-1}, y_{n+p}) \\ &\leq \frac{1}{h^n} \left[ 1 + \frac{1}{h} + \cdots + \frac{1}{h^{p-1}} \right] d(y_n, y_{n+1}) \\ &\leq \frac{1}{h^n} \left[ \frac{1}{1 - \frac{1}{h}} \right] d(y_0, y_1). \end{aligned}$$

This means that  $d(y_n, y_{n+p}) \rightarrow 0$  as  $n \rightarrow \infty$ . Therefore  $\{y_n\}$  is a Cauchy sequence in  $X$ .

Now suppose that  $S(X)$  is a complete subspace of  $X$ , then the sequence  $y_{2n} = Sx_{2n+1}$ , is a Cauchy sequence in  $S(X)$  and hence has a limit  $u$  (say). Let  $v \in S^{-1}u$ , then  $Sv = u$ . Since  $\{y_{2n}\}$  converges to  $u$ , it follows that  $\{y_{2n+1}\}$  also converges to  $u$ . Setting  $x = x_{2n}$  and  $y = v$  in (III), we have

$$\begin{aligned} F(d(Ax_{2n}, Bv), d(Tx_{2n}, Sv), d(Tx_{2n}, Ax_{2n}), d(Sv, Bv), \\ d(Tx_{2n}, Bv), d(Sv, Ax_{2n})) > 0, \end{aligned}$$

which implies

$$\begin{aligned} F(d(y_{2n}, Bv), d(y_{2n-1}, Sv), d(y_{2n-1}, y_{2n}), d(Sv, Bv), \\ d(y_{2n-1}, Bv), d(Sv, y_{2n})) > 0, \end{aligned}$$

letting  $n \rightarrow \infty$ , we obtain

$$F(d(u, Bv), d(u, Sv), d(u, u), d(Sv, Bv), d(u, Bv), d(Sv, u)) \geq 0,$$

or

$$F(d(u, Bv), d(u, u), d(u, u), d(u, Bv), d(u, Bv), d(u, u)) \geq 0,$$

or

$$F(d(u, Bv), 0, 0, d(u, Bv), d(u, Bv), 0) \geq 0.$$

By  $(F_2)$ , we have  $u = Bv$ . Thus  $B$  and  $S$  have a coincidence point. Since  $B(X) \subset T(X)$ ,  $u = Bv$  implies  $u \in T(X)$ .

Let  $w \in T^{-1}u$ , then  $Tw = u$ .

Now by setting  $x = w$  and  $y = x_{2n+1}$  in (III), we obtain

$$\begin{aligned} F(d(Aw, Bx_{2n+1}), d(Tw, Sx_{2n+1}), d(Tw, Aw), d(Sx_{2n+1}, Bx_{2n+1}), \\ d(Tw, Bx_{2n+1}), d(Sx_{2n+1}, Aw)) > 0, \end{aligned}$$

which implies

$$\begin{aligned} F(d(Aw, y_{2n+1}), d(Tw, y_{2n}), d(Tw, Aw), d(y_{2n}, y_{2n+1}), \\ d(Tw, y_{2n+1}), d(y_{2n}, Aw)) > 0, \end{aligned}$$

letting  $n \rightarrow \infty$ , we obtain

$$F(d(Aw, u), d(u, u), d(u, Aw), d(u, u), d(u, u), d(u, Aw)) \geq 0,$$

or

$$F(d(Aw, u), 0, d(u, Aw), 0, 0, d(u, Aw)) \geq 0,$$

we obtain  $Aw = u$ , by  $(F_1)$ . It means  $A$  and  $T$  have a coincidence point.

Similarly if we assume that  $T(X)$  is complete, then we can easily establish

that both  $B, S$  and  $A, T$  have a coincidence point.

The remaining two cases are the same as the previous cases.

Thus results (IV) and (V) are completely established. Now by  $u = Sv = Bv$  and weak compatibility of  $(B, S)$ , we have

$$Bu = BSv = SBv = Su.$$

Similarly by  $u = Tw = Aw$  and weak compatibility of  $(A, T)$ , we have

$$Au = ATw = TAw = Tu.$$

By (III), we have

$$F(d(Aw, Bu), d(Tw, Su), d(Tw, Aw), d(Su, Bu), d(Tw, Bu), d(Su, Aw)) > 0,$$

or

$$F(d(u, Bu), d(u, Su), d(u, u), d(Su, Bu), d(u, Bu), d(Su, u)) > 0,$$

or

$$F(d(u, Bu), d(u, Bu), 0, 0, d(u, Bu), d(Bu, u)) > 0,$$

which contradicts  $(F_u)$ . Thus  $u = Bu$ . Similarly we can show that  $Au = u$ . Therefore  $u = Au = Bu = Su = Tu$  and thus  $u$  is a common fixed point of  $A, B, S$  and  $T$ .

For uniqueness of common fixed point, let  $z$  and  $v$  be two common fixed points of  $A, B, S$  and  $T$ . Then from (III), we have

$$F(d(Az, Bv), d(Tz, Sv), d(Tz, Az), d(Sv, Bv), d(Tz, Bv), d(Sv, Az)) > 0,$$

or

$$F(d(z, v), d(z, v), 0, 0, d(z, v), d(v, z)) > 0,$$

which contradicts  $(F_u)$ . Thus  $z = v$ .

**Remark 5.2.** Theorem 5.1 improves Theorem 2 and Theorem 3 of Taniguchi [12] and Corollary 3.7 above.

**Corollary 5.3.** Let  $A, B, S$  and  $T$  be the self mappings of a complete metric space satisfying conditions (I), (II) and (III) of Theorem 5.1. Then (IV) and (V) hold. Moreover, if the pairs  $(A, T)$  and  $(B, S)$  are compatible (compatible of type (A), compatible of type (B), compatible of type (P)) then  $A, B, S$  and  $T$  have a common fixed point.

**Proof.** It follows by Theorem 5.1 and the fact that every compatible (compatible of type (A), compatible of type (B), compatible of type (P)) pair of mappings is weakly compatible (see [3], [8], [9]).

**Corollary 5.4.** Let the pairs  $(A, T)$  and  $(B, S)$  be compatible mappings of a complete metric space into itself, such that

$$(I) A(X) \subset S(X) \text{ and } B(X) \subset T(X),$$

$$(II) d(Tx, Sy) \geq a d(Ax, By) + b d(Tx, Ax) + c d(Sy, By)$$

$$-e [d(Tx, By) + d(Sy, Ax)],$$

for all  $x, y$  in  $X$ ,  $x \neq y$ , where  $a, b, c, e$  are non-negative real numbers with  $0 < \max\{e - b, e - c\} < 1$  and  $a + b + c - 2e > 1$ . If one of the mappings  $A, B, S$  and  $T$  is continuous then  $A, B, S$  and  $T$  have a unique common fixed point.

**Proof.** It follows by Corollary 5.2 and Example 4.1 above.

**Remark 5.5.** Corollary 5.4 improves corresponding Theorem 1 of Taniguchi [12] and Corollary 3.3 above.

**Example 5.6.** Let  $X = \{0, 1, \frac{1}{3}, \frac{1}{3^2}, \frac{1}{3^3}, \dots\}$  be a metric space with the usual metric  $d(x, y) = |x - y|$  for all  $x, y$  in  $X$ . Define mappings  $A, T : X \rightarrow X$  by

$$A(0) = \frac{1}{3^2}, \quad A\left(\frac{1}{3^n}\right) = \frac{1}{3^{n+2}} \quad \text{and} \quad T(0) = \frac{1}{3}, \quad T\left(\frac{1}{3^n}\right) = \frac{1}{3^{n+1}},$$

for  $n = 0, 1, 2, \dots$ , respectively. Also let  $A = B$  and  $S = T$ . Then clearly

$$A(X) = \left\{\frac{1}{3^2}, \frac{1}{3^3}, \frac{1}{3^4}, \dots\right\} \subset \left\{\frac{1}{3}, \frac{1}{3^2}, \frac{1}{3^3}, \dots\right\} = T(X)$$

Define a continuous function  $F = \mathbb{R}_+^6 \rightarrow \mathbb{R}$  by

$$F(t_1, \dots, t_6) = pt_2^2 + \frac{qt_5t_6}{t_2^2 + t_4^2 + r} - t_1^2;$$

where  $0 \leq p \leq \frac{1}{2}$ ,  $0 \leq q \leq \frac{1}{2}$ ,  $r \in \mathbb{N}$ .

Then

$$(F_1) : F(u, 0, u, 0, 0, u) = p0 + \frac{q0}{u^2+0+r} - u^2 \geq 0 \text{ implies } u = 0;$$

$$(F_2) : F(u, 0, 0, u, u, 0) = p0 + \frac{q0}{0+u^2+r} - u^2 \geq 0 \text{ implies } u = 0;$$

$$(F_u) : F(u, u, 0, 0, u, u) = pu^2 + \frac{qu^2}{0+0+r} - u^2$$

$$= \left(p + \frac{q}{r} - 1\right)u^2 \leq 0, \text{ for all } u > 0.$$

Thus  $F$  satisfies  $F_1, F_2$  and  $F_u$ .

Furthermore, for  $x = 0, y = 1$  we have

$$\begin{aligned} & F\left(d(Ax, By), d(Tx, Sy), d(Tx, Ax), d(Sy, By), d(Tx, By), d(Sy, Ax)\right) \\ &= F\left(d(Ax, By), d(Tx, Sy), d(Tx, Ax), d(Sy, By), d(Tx, By), d(Sy, Ax)\right) \\ &= F\left(d\left(\frac{1}{3^2}, \frac{1}{3^2}\right), d\left(\frac{1}{3}, \frac{1}{3}\right), d\left(\frac{1}{3}, \frac{1}{3^2}\right), d\left(\frac{1}{3}, \frac{1}{3^2}\right), d\left(\frac{1}{3}, \frac{1}{3^2}\right), d\left(\frac{1}{3}, \frac{1}{3^2}\right)\right) \\ &= F\left(0, 0, \frac{2}{9}, \frac{2}{9}, \frac{2}{9}, \frac{2}{9}\right) \\ &= 0 + \frac{q\frac{2}{9}\frac{2}{9}}{\left(\frac{2}{9}\right)^2 + \left(\frac{2}{9}\right)^2 + r} - 0 \\ &= \frac{4q}{8+81r} \\ &> 0. \end{aligned}$$

Similarly, for  $x = 0, y = \frac{1}{3^m}$ , we obtain

$$\begin{aligned}
& F(d(Ax, By), d(Tx, Sy), d(Tx, Ax), d(Sy, By), d(Tx, By), d(Sy, Ax)) \\
&= F\left(d(A0, B\frac{1}{3^m}), d(T0, S\frac{1}{3^m}), d(T0, A0), d(S\frac{1}{3^m}, B\frac{1}{3^m}), d(T0, B\frac{1}{3^m}), \right. \\
&\quad \left. d(S\frac{1}{3^m}, A0)\right) \\
&= F\left(d\left(\frac{1}{3^2}, \frac{1}{3^{m+2}}\right), d\left(\frac{1}{3}, \frac{1}{3^{m+1}}\right), d\left(\frac{1}{3}, \frac{1}{3^2}\right), d\left(\frac{1}{3^m}, \frac{1}{3^{m+2}}\right), d\left(\frac{1}{3}, \frac{1}{3^{m+2}}\right), \right. \\
&\quad \left. d\left(\frac{1}{3^{m+1}}, \frac{1}{3^2}\right)\right) \\
&= F\left(\frac{1}{3^2}\left|1 - \frac{1}{3^m}\right|, \frac{1}{3}\left|1 - \frac{1}{3^m}\right|, \frac{1}{3}\left|1 - \frac{1}{3}\right|, \frac{1}{3^m}\left|1 - \frac{1}{3^{m+1}}\right|, \frac{1}{3}\left|1 - \frac{1}{3^{m+1}}\right|, \right. \\
&\quad \left. \frac{1}{3}\left|\frac{1}{3^m} - \frac{1}{3}\right|\right) \\
&= p\frac{1}{9}\left|1 - \frac{1}{3^m}\right|^2 + \frac{q\frac{1}{9}\left|1 - \frac{1}{3^{m+1}}\right|\left|\frac{1}{3^m} - \frac{1}{3}\right|}{\frac{1}{9}\frac{4}{9} + \frac{1}{3^m}\frac{64}{81} + r} - \frac{1}{9^2}\left|1 - \frac{1}{3^m}\right|^2 \\
&= \left(p - \frac{1}{9}\right)\frac{1}{9}\left|1 - \frac{1}{3^m}\right|^2 + \frac{q\frac{1}{9}\left|1 - \frac{1}{3^{m+1}}\right|\left|\frac{1}{3^m} - \frac{1}{3}\right|}{\frac{4}{9^2} + \frac{64}{3^{2m+4}} + r} \\
&> 0.
\end{aligned}$$

Also for  $x = \frac{1}{3^n}$  and  $y = \frac{1}{3^m}$  ( $n, m = 0, 1, 2, \dots, n \neq m$ ), we have

$$\begin{aligned}
& F(d(Ax, By), d(Tx, Sy), d(Tx, Ax), d(Sy, By), d(Tx, By), d(Sy, Ax)) \\
&= F\left(d\left(\frac{1}{3^{n+2}}, \frac{1}{3^{m+2}}\right), d\left(\frac{1}{3^n}, \frac{1}{3^m}\right), d\left(\frac{1}{3^{n+1}}, \frac{1}{3^{m+2}}\right), \right. \\
&\quad \left. d\left(\frac{1}{3^{m+1}}, \frac{1}{3^{m+2}}\right), d\left(\frac{1}{3^{n+1}}, \frac{1}{3^{m+2}}\right), d\left(\frac{1}{3^{m+1}}, \frac{1}{3^{n+2}}\right)\right) \\
&= F\left(\frac{1}{3^2}\left|\frac{1}{3^m} - \frac{1}{3^n}\right|, \frac{1}{3^{n+1}}\left|1 - \frac{1}{3}\right|, \frac{1}{3^{m+1}}\left|1 - \frac{1}{3}\right|, \right. \\
&\quad \left. \frac{1}{3^{m+1}}\left|1 - \frac{1}{3}\right|, \frac{1}{3}\left|\frac{1}{3^n} - \frac{1}{3^{m+1}}\right|, \frac{1}{3}\left|\frac{1}{3^m} - \frac{1}{3^{n+1}}\right|\right) \\
&= p\frac{1}{3^2}\left|\frac{1}{3^n} - \frac{1}{3^m}\right|^2 + \frac{q\frac{1}{3}\frac{1}{3}\left|\frac{1}{3^n} - \frac{1}{3^{m+1}}\right|\left|\frac{1}{3^m} - \frac{1}{3^{n+1}}\right|}{\frac{2^2}{3^{2(n+2)}} + \frac{2^2}{3^{2(m+2)}} + r} \\
&\quad - \frac{1}{3^2}\left|\frac{1}{3^n} - \frac{1}{3^m}\right|^2 \\
&= p\frac{1}{3^2}\left(p - \frac{1}{3^2}\right)\left|\frac{1}{3^n} - \frac{1}{3^m}\right|^2 + \frac{q\frac{1}{3}\frac{1}{3}\left|\frac{1}{3^n} - \frac{1}{3^{m+1}}\right|\left|\frac{1}{3^m} - \frac{1}{3^{n+1}}\right|}{\frac{2^2}{3^{2(n+2)}} + \frac{2^2}{3^{2(m+2)}} + r} \\
&> 0
\end{aligned}$$

for all  $\frac{1}{9} \leq p$ . Thus all the conditions of theorem 5.1 are satisfied except the completeness of the subspace  $A(X)$  and  $B(X)$ . Note that  $A$  and  $T$  have

no coincidence point. Here it is interesting to note that in Theorem 5.1 the completeness of the space can not ensure the existence of coincidence point as the space  $X$  is complete in the given example. Also note that  $A$  and  $T$  are not continuous at the origin.

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