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Fixed point theorems for generalized $(\alpha_* - \psi)$-Ćirić-type contractive multivalued operators in b-metric spaces

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Abstract

In this paper we introduce the notion of $(\alpha_* - \psi)$- Ćirić-type contractive multivalued operator and investigate the existence and uniqueness of fixed point for such a mapping in b-metric spaces. The well-posedness of the fixed point problem and the Ulam-Hyres stability is also studied.

Keywords: α_*-ψ-contractive multivalued operator, fixed point, $b-$metric space, well-posedness, Ulam-Hyers stability.
2010 MSC: 46T99, 47H10, 54H25.

1. Introduction and Preliminaries

Recently, in [27], Samet *et al.* proved some fixed point results for $(\alpha - \psi)$ −contractive and $\alpha-$ admissible mapping. Asl *et al.* in [4], generalize these notions by introducing the notions of $(\alpha_* - \psi)-$ contractive and α_*-admissible mapping and proved some fixed point results in complete metric spaces. Ali and Kamran, in [1], generalized the notion of $(\alpha_* - \psi)-$ contractive mappings.

For more details about the $(\alpha - \psi)-$ contractions, $\alpha-$admissible mappings, $(\alpha_* - \psi)$ −contractions and α_*-admissible mappings, see e.g. [1, 2, 3, 8, 15, 17, 18, 20, 26, 28].

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The purpose of this paper is to introduce the notion of generalized $(\alpha_* - \psi)$ −contractive multivalued mapping and to prove some fixed point results in b-metric spaces.

Let us recall now some essential definitions and fundamental results. We begin with the definition of a b-metric space.

Definition 1.1 ([12])**.** Let X be a set and let $s \geq 1$ be a given real number. A functional $d : X \times X \to [0, \infty)$ is said to be a b-metric if the following conditions are satisfied:

1. $d(x, y) = 0$ if and only if $x = y$;
2. $d(x, y) = d(y, x)$;
3. $d(x, z) \leq s[d(x, y) + d(y, z)]$

for all $x, y, z \in X$. In this case the pair (X, d) is called a b-metric space.

Remark 1.2*.* The class of b-metric spaces is larger than the class of metric spaces since a b-metric space is a metric space when s=1. For more details and examples on b-metric spaces, see e.g. [5, 10, 11, 12, 13, 16].

For the sake of completeness we state the following examples.

Example 1.3 ([5])**.** Let X be a set with the cardinal $card(X) \geq 3$. Suppose that $X = X_1 \cup X_2$ is a partition of X such that $card(X_1) \geq 2$. Let $s > 1$ be arbitrary. Then, the functional $d : X \times X \to [0, \infty)$ defined by:

$$d(x,y) := \begin{cases} 0, & x = y \\ 2s, & x, y \in X_1 \\ 1, & otherwise \end{cases}$$

is a b-metric on X with coefficient $s > 1$.

Example 1.4. Let X=$\{0, 1, 2\}$ and $d : X \times X \to \mathbb{R}_+$ such that $d(0,1) = d(1,0) = d(0,2) = d(2,0) = 1, d(1,2) = d(2,1) = \alpha \geq 2, d(0,0) = d(1,1) = d(2,2) = 0$. Then

$$d(x,y) \leq \frac{\alpha}{2}\left[d(x,z) + d(z,y)\right] \text{ for } x, y, z \in X.$$

Then (X, d) is a b-metric space. If $\alpha > 2$ the ordinary triangle inequality does not hold and (X, d) is not a metric space.

Definition 1.5. Let (X, d) be a $b-metric$ space with constant s. Then the sequence $(x_n)_{n\in\mathbb{N}} \subset X$ is called:

1. convergent if and only if there exists $x \in X$ such that $d(x_n, x) \to 0,$ as $n \to \infty$;
2. Cauchy if and only if $d(x_n, x_m) \to 0,$ as $n, m \to \infty$.

Definition 1.6. The $b - metric$ space (X, d) is complete if every Cauchy sequence in X converges.

Let us consider the following families of subsets of a b-metric space (X, d):

$$\mathcal{P}(X) = \{Y \mid Y \subset X\}, P(X) := \{Y \in \mathcal{P}(X) \mid Y \neq \emptyset\}, P_b(X) := \{Y \in P(X) \mid Y \text{ is bounded}\},$$

$$P_{cl}(X) := \{Y \in P(X) \mid Y \text{ is closed}\}, P_{cp}(X) := \{Y \in P(X) \mid Y \text{ is compact}\}.$$

Let us define *the gap functional* $D : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\}$, as:

$$D(A, B) = \inf\{d(a, b) \mid a \in A,\ b \in B\}.$$

In particular, if $x_0 \in X$, then $D(x_0, B) := D(\{x_0\}, B)$.

The excess generalized functional $\rho : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\}$, as:

$$\rho(A, B) = \sup\{D(a, B) \mid a \in A\}.$$

The Pompeiu-Hausdorff generalized functional: $H : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\}$, as:

$$H(A, B) = \max\{\rho(A, B), \rho(B, A)\}.$$

The generalized diameter functional: $\delta : P(X) \times P(X) \to \mathbb{R}_+ \cup \{\infty\}$, as:

$$\delta(A, B) = \sup\{d(a, b) \mid a \in A,\ b \in B\}.$$

In particular $\delta(A) := \delta(A, A)$ is the diameter of the set A.

It is known (see Czerwik [12]) that $(P_{b,cl}(X), H)$ is a complete b-metric space implies that (X, d) is a complete b-metric space. In the sequel, the following results are useful for some of the proofs in the paper.

Lemma 1.7 ([12])**.** *Let (X, d) be a b-metric space with constant $s > 1$ and let $A, B \in P(X)$. We suppose that there exists $\eta > 0$ such that:*

(i) for each $a \in A$ there is $b \in B$ such that $d(a, b) \leq \eta$;

(ii) for each $b \in B$ there is $a \in A$ such that $d(a, b) \leq \eta$.

Then, $H(A, B) \leq \eta$.

Lemma 1.8 ([12])**.** *Let (X, d) be a b-metric space with constant $s > 1$, $A \in P(X)$ and $x \in X$. Then $D(x, A) = 0$ if and only if $x \in \overline{A}$.*

Lemma 1.9 ([12])**.** *Let (X, d) be a b-metric space with constant s and let $\{x_k\}_{k=0}^{n} \subset X$. Then*

1. *$D(x, A) \leq s[d(x, y) + D(y, A)]$ for all $x, y \in X$ and $A \subset X$.*
2. *$d(x_n, x_0) \leq sd(x_0, x_1) + ... + s^{n-1}d(x_{n-2}, x_{n-1}) + s^n d(x_{n-1}, x_n)$.*
3. *$H(A, C) \leq s[H(A, B) + H(B, C)]$ for all $A, B, C \in P(X)$.*

Lemma 1.10. *Let (X, d) be a b-metric space with constant $s > 1$ and $B \in P_{cl}(X)$. Assume that there exists $x \in X$ such that $D(x, B) > 0$. Then there exists $y \in B$ such that*

$$d(x, y) < qD(x, B),$$

where $q > 1$.

Proof. Because $D(x, B) = \inf\{d(x, y) \mid y \in B\}$ we have that for $\varepsilon > 0$, there exists $y \in B$ such that

$$d(x, y) < D(x, B) + \varepsilon.$$

If we choose $\varepsilon = (q - 1)D(x, B) > 0$ then we reach the conclusion. □

A mapping $\varphi : [0, \infty) \to [0, \infty)$ is called a *comparison function* if it is increasing and $\varphi^n(t) \to 0$, $n \to \infty$, for any $t \in [0, \infty)$. We denote by Φ, the class of the comparison functions $\varphi : [0, \infty) \to [0, \infty)$. For more details and examples see e.g. [7, 23].

We recall the following essential result.

Lemma 1.11 ([7, 23])**.** *If $\varphi : [0, \infty) \to [0, \infty)$ is a comparison function, then:*

(1) each iterate φ^k of φ, $k \geq 1$, is also a comparison function;

(2) φ is continuous at 0;

(3) $\varphi(t) < t$, for any $t > 0$.

Later, Berinde [7] introduced the concept of (c)-*comparison function* in the following way.

Definition 1.12 ([7]). A function $\varphi : [0, \infty) \to [0, \infty)$ is said to be a (c)-comparison function if

(1) φ is increasing;
(2) there exists $k_0 \in \mathbb{N}$, $a \in (0, 1)$ and a convergent series of nonnegative terms $\sum_{k=1}^{\infty} v_k$ such that $\varphi^{k+1}(t) \le a\varphi^k(t) + v_k$ for $k \ge k_0$ and any $t \in [0, \infty)$.

The notion of a (c)-comparison function was improved as a (b)-comparison function by Berinde [6], in order to extend some fixed point results to the class of b-metric spaces.

Definition 1.13 ([6]). Let $s \ge 1$ be a real number. A mapping $\varphi : [0, \infty) \to [0, \infty)$ is called a (b)-comparison function if the following conditions are fulfilled:

(1) φ is monotone increasing;
(2) there exist $k_0 \in \mathbb{N}$, $a \in (0, 1)$ and a convergent series of nonnegative terms $\sum_{k=1}^{\infty} v_k$ such that $s^{k+1}\varphi^{k+1}(t) \le as^k\varphi^k(t) + v_k$, for $k \ge k_0$ and any $t \in [0, \infty)$.

We denote by Ψ_b the class of (b)-comparison functions. It is obvious that the concept of (b)-comparison function reduces to that of (c)-comparison function when $s = 1$.

The following lemma has a crucial role in the proof of our main result.

Lemma 1.14 ([5]). *If $\varphi : [0, \infty) \to [0, \infty)$ is a (b)-comparison function, then we have the followings:*

(1) the series $\sum_{k=0}^{\infty} s^k\varphi^k(t)$ converges for any $t \in \mathbb{R}_+$;
(2) the function $s_b : [0, \infty) \to [0, \infty)$ defined by $s_b(t) = \sum_{k=0}^{\infty} s^k\varphi^k(t)$, $t \in [0, \infty)$, is increasing and continuous at 0.

We note that any (b)-comparison function is a comparison function due to the above Lemma.

We will need the following Generalized Cauchy lemma proved by Păcurar in [21].

Lemma 1.15. *Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a $b-$comparison function with constant $s \ge 1$ and $a_n \in \mathbb{R}_+$, $n \in \mathbb{N}$ such that $a_n \to 0$, as $n \to \infty$ then*

$$\sum_{k=0}^{\infty} s^{n-k}\varphi^{n-k}(a_k) \to 0, \ as\ n \to \infty.$$

Let us denote by Ψ the family of nondecreasing functions $\psi : [0, \infty) \to [0, \infty)$ such that $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for each $t > 0$, where ψ^n is the n-th iterate of ψ. It is clear that if $\Psi \subset \Phi$ (see e.g. [14]) and hence, by Lemma 1.11, (3), for $\psi \in \Psi$ we have $\psi(t) < t$, for any $t > 0$.

Let (X, d) be a b-metric space with constant $s > 1$ and let $T : X \to P(X)$ a multivalued operator. $x \in X$ is called fixed point for T if and only if $x \in Tx$. The set $Fix\,(T) = \{x \in X : x \in Tx\}$ is called the fixed point set of T.

Definition 1.16 ([4]). Let $T : X \to P(X)$ and $\alpha : X \times X \to [0, \infty)$. We say that T is α_*-admissible if

$$x, y \in X, \quad \alpha(x, y) \ge 1 \Longrightarrow \alpha_*(T(x), T(y)) \ge 1,$$

where $\alpha_*(A, B) = inf\{\alpha(a, b),\ a \in A,\ b \in B\}$.

Definition 1.17 ([4]). Let (X, d) be a metric space and $T : X \to P(X)$ be a multivalued operator. We say that T is an $(\alpha_* - \psi)$-contractive multivalued operator if there exist two functions $\alpha : X \times X \to [0, \infty)$ and $\psi \in \Psi$, such that

$$\alpha_*(T(x), T(y))H(T(x), T(y)) \le \psi(d(x, y)) \text{ for all } x, y \in X, \tag{1.1}$$

where $\alpha_*(A, B) = inf\{\alpha(a, b),\ a \in A,\ b \in B\}$.

Inspired from Definition 1.17 we introduce the following contraction types.

Definition 1.18. Let (X, d) be a b-metric space and $T : X \to P_{cl}(X)$ be a multivalued operator. We say that T is an generalized $(\alpha_* - \psi)$-contractive multivalued operator of type (b) if there exist two functions $\alpha : X \times X \to [0, \infty)$ and $\psi \in \Psi_b$, such that

$$\alpha_*(T(x), T(y))H(T(x), T(y)) \leq \psi(M(x, y)) \text{ for all } x, y \in X, \tag{1.2}$$

where

$$M(x, y) = \max\left\{d(x, y), D(x, Tx), D(y, Ty), \frac{D(x, Ty) + D(y, Tx)}{2s}\right\},$$

and $\alpha_*(A, B) = inf\{\alpha(a, b),\ a \in A,\ b \in B\}$.

Definition 1.19. Let (X, d) be a b-metric space and $T : X \to P_{cl}(X)$ be a multivalued operator. We say that T is an $(\alpha_* - \psi)$-contractive multivalued operator of type (b) if there exist two functions $\alpha : X \times X \to [0, \infty)$ and $\psi \in \Psi_b$, such that

$$\alpha_*(T(x), T(y))H(T(x), T(y)) \leq \psi(d(x, y)), \text{ for all } x, y \in X, \tag{1.3}$$

where $\alpha_*(A, B) = inf\{\alpha(a, b),\ a \in A,\ b \in B\}$.

2. Fixed point results

Theorem 2.1. *Let (X, d) be a complete b-metric space with constant $s > 1$ and $d : X \times X \to \mathbb{R}_+$ a continuous b-metric. Let $T : X \to P_{cl}(X)$ be a generalized $(\alpha_* - \psi)$-contractive multivalued operator of type-(b) with $\psi(t) < \frac{t}{s}, \forall t > 0$, satisfying the following conditions:*

(i) T is α_*-admissible;

(ii) there exist $x_0 \in X$ and $x_1 \in T(x_0)$ such that $\alpha(x_0, x_1) \geq 1$;

(iii) if $(x_n)_{n\in\mathbb{N}}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ and $x_n \to x$ then $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$.

Then T has a fixed point.

Proof. From (ii) we have that there exist $x_0 \in X$ and $x_1 \in T(x_0)$ such that $\alpha(x_0, x_1) \geq 1$. Then by the generalized $(\alpha_* - \psi)$-contraction condition we have

$$\alpha_*(T(x_0), T(x_1))H(T(x_0), T(x_1)) \leq \psi(M(x_0, x_1)), \tag{2.1}$$

where

$$M(x_0, x_1) = \max\left\{d(x_0, x_1), D(x_0, Tx_0), D(x_1, Tx_1), \frac{D(x_0, Tx_1) + D(x_1, Tx_0)}{2s}\right\}.$$

Because $x_1 \in T(x_0)$, we have that $D(x_1, Tx_0) = 0$. On the other hand $D(x_0, Tx_0) \leq d(x_0, x_1)$ hence,

$$M(x_0, x_1) = \max\left\{d(x_0, x_1), D(x_1, Tx_1), \frac{D(x_0, Tx_1)}{2s}\right\}.$$

We have

$$\frac{D(x_0, Tx_1)}{2s} \leq \frac{1}{2}\left(d(x_0, x_1) + D(x_1, Tx_1)\right) \leq \max\left\{d(x_0, x_1), D(x_1, Tx_1)\right\}.$$

Thus, we obtain that

$$M(x_0, x_1) = \max\left\{d(x_0, x_1), D(x_1, Tx_1)\right\}.$$

Suppose that $M(x_0, x_1) = D(x_1, Tx_1)$.

$$\begin{aligned} 0 < D(x_1, Tx_1) &\leq H(T(x_0), T(x_1)) \\ &\leq \alpha_*(T(x_0), T(x_1))H(T(x_0), T(x_1)) \\ &\leq \psi(M(x_0, x_1)) = \psi(D(x_1, Tx_1)), \end{aligned}$$

which is a contradiction. Hence, we have that $M(x_0, x_1) = d(x_0, x_1)$, and (2.1) becomes

$$\alpha_*(T(x_0), T(x_1))H(T(x_0), T(x_1)) \leq \psi(d(x_0, x_1)). \tag{2.2}$$

Using Lemma 1.10, for $q > 1$, there exists $x_2 \in T(x_1)$ such that

$$d(x_1, x_2) < qD(x_1, T(x_1)),$$

and hence

$$\begin{aligned} d(x_1, x_2) &< qH(T(x_0), T(x_1)) \\ &\leq q\alpha_*(T(x_0), T(x_1))H(T(x_0), T(x_1)). \end{aligned} \tag{2.3}$$

From (2.2) and (2.3) we obtain that

$$d(x_1, x_2) < q\psi(d(x_0, x_1)). \tag{2.4}$$

Because ψ is increasing, from (2.4) we have

$$\psi(d(x_1, x_2)) < \psi\left(q\psi(d(x_0, x_1))\right). \tag{2.5}$$

Let us consider $q_1 = \frac{\psi(q\psi(d(x_0,x_1)))}{\psi d(x_1,x_2))} > 1$.

Since T is α_*-admissible we have $\alpha_*(T(x_0), T(x_1)) \geq 1$. Using the definition of α_* and the fact that $x_1 \in T(x_0)$ and $x_2 \in T(x_1)$, we shall obtain that

$$\alpha\left(x_1, x_2\right) \geq 1,$$

and because T is α_*-admissible we shall have

$$\alpha_*(T(x_1), T(x_2)) \geq 1.$$

By the generalized $(\alpha_* - \psi)$-contraction condition we have

$$\alpha_*(T(x_1), T(x_2))H(T(x_1), T(x_2)) \leq \psi(M(x_1, x_2)), \tag{2.6}$$

where

$$M(x_1, x_2) = \max\left\{d(x_1, x_2), D(x_1, Tx_1), D(x_2, Tx_2), \frac{D(x_1, Tx_2) + D(x_2, Tx_1)}{2s}\right\}.$$

It easy to see that $M(x_1, x_2) = d(x_1, x_2)$, and (2.6) becomes

$$\alpha_*(T(x_1), T(x_2))H(T(x_1), T(x_2))) \leq \psi(d(x_1, x_2))).$$

For $q_1 > 1$, there exists $x_3 \in T(x_2)$ such that

$$\begin{aligned} d(x_2, x_3) < q_1 D(x_2, T(x_2)) &\leq q_1 H(T(x_1), T(x_2)) \\ &\leq q_1\alpha_*(T(x_1), T(x_2))H(T(x_1), T(x_2)) \\ &\leq q_1\psi(d(x_1, x_2)). \end{aligned}$$

From the definition of q_1 we shall obtain that

$$d(x_2, x_3) < \psi\left(q\psi(d(x_0, x_1))\right).$$

Hence, using the monotonicity of ψ we shall have

$$\psi\left(d(x_2,x_3)\right) < \psi^2\left(q\psi(d(x_0,x_1))\right).$$

Let us, now, consider $q_2 = \frac{\psi^2(q\psi(d(x_0,x_1)))}{\psi(d(x_2,x_3))} > 1$.

Since T is α_*-admissible we have $\alpha_*(T(x_1),T(x_2)) \geq 1$. Using the definition of α_* and the fact that $x_2 \in T(x_1)$ and $x_3 \in T(x_2)$, we shall obtain that

$$\alpha\left(x_2,x_3\right) \geq 1,$$

and because T is α_*-admissible we shall have

$$\alpha_*(T(x_2),T(x_3)) \geq 1.$$

By the generalized $(\alpha_* - \psi)$-contraction condition we have

$$\alpha_*(T(x_2),T(x_3))H(T(x_2),T(x_3)) \leq \psi(d(x_2,x_3)). \tag{2.7}$$

For $q_2 > 1$, there exists $x_4 \in T(x_3)$ such that

$$\begin{aligned} d(x_3,x_4) &< q_2 D(x_3,T(x_3)) \leq q_2 H(T(x_2),T(x_3)) \\ &\leq q_2 \alpha_*(T(x_2),T(x_3))H(T(x_2),T(x_3)) \\ &\leq q_2 \psi(M(x_2,x_3)). \end{aligned}$$

Again, as above, it is easy to see that $M(x_2,x_3) = d(x_2,x_3)$. Hence

$$d(x_3,x_4) < q_2\psi(d(x_2,x_3)) \leq \psi^2\left(q\psi(d(x_0,x_1))\right).$$

Using the monotonicity of ψ we shall have

$$\psi\left(d(x_2,x_3)\right) \leq \psi^3\left(q\psi(d(x_0,x_1))\right).$$

By an inductive procedure we have that there exists $x_{n+1} \in T(x_n)$ such that $\alpha(x_n,x_{n+1}) \geq 1$ and

$$d(x_n,x_{n+1}) < \psi^{n-1}(q\psi(d(x_0,x_1))) \quad \text{for each} \quad n \in \mathbb{N}.$$

We shall prove that $(x_n)_{n\in\mathbb{N}^*}$ is a Cauchy sequence.

$$\begin{aligned} d(x_n,x_{n+p}) &\leq s d(x_n,x_{n+1}) + s^2 d(x_{n+1},x_{n+2}) + \ldots + s^p \cdot d(x_{n+p-1},x_{n+p}) \\ &< s \cdot \psi^{n-1}(q\psi(d(x_0,x_1))) + s^2 \cdot \psi^n(q\psi(d(x_0,x_1))) + \ldots + s^p \cdot \psi^{n+p-2}(q\psi(d(x_0,x_1))) \\ &= \frac{1}{s^{n-2}}\left\{s^{n-1}\psi^{n-1}(q\psi(d(x_0,x_1))) + \ldots + s^{n+p-2} \cdot \psi^{n+p-2}(q\psi(d(x_0,x_1)))\right\} \\ &= \frac{1}{s^{n-2}} \cdot \sum_{k=n-1}^{n+p-2} s^k \cdot \psi^k(q\psi(d(x_0,x_1))). \end{aligned}$$

Denoting $S_n = \sum_{k=0}^{n} s^k \psi^k(q\psi(d(x_0,x_1)))$, $n \geq 1$ we obtain:

$$d(x_n,x_{n+p}) \leq \frac{1}{s^{n-2}}[S_{n+p-2} - S_{n-2}], \ n \geq 2, \ p \geq 2. \tag{2.8}$$

Using Lemma 1.14 we conclude that the series $\sum_{k=0}^{\infty} s^k \psi^k(q\psi(d(x_0,x_1)))$ is convergent.

Thus, there exists $S = \lim\limits_{n\to\infty} S_n$ and this will imply $d(x_n,x_{n+p}) \to 0$, as $n \to \infty$.

In this way we obtain that $(x_n)_{n\in\mathbb{N}^*}$ is a Cauchy sequence in the b-metric space (X,d). Since (X,d) is complete, there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$.

$$\begin{aligned} D(x^*,T(x^*)) &\leq sd(x^*,x_{n+1}) + sD(x_{n+1},T(x^*)) \\ &\leq sd(x^*,x_{n+1}) + sH(T(x_n),T(x^*)) \\ &\leq sd(x^*,x_{n+1}) + s\psi(M(x_n,x^*), \end{aligned}$$

where

$$M(x_n,x^*) = \max\left\{d(x_n,x^*), D(x_n,Tx_n), D(x^*,Tx^*), \frac{D(x_n,Tx^*)+D(x^*,Tx_n)}{2s}\right\}.$$

Letting $n \to \infty$ we obtain that $d(x_n,x^*) \to 0$, $D(x_n,Tx_n) < d(x_n,x_{n+1}) \to 0$, $D(x^*,Tx_n) \to 0$. Hence $M(x_n,x^*) \to D(x^*,Tx^*)$, as $n \to \infty$.

From the properties of ψ we have that

$$D(x^*,T(x^*)) \leq s\psi(D(x^*,Tx^*)) < s\frac{D(x^*,Tx^*)}{s} = D(x^*,Tx^*),$$

which is a contradiction, so $D(x^*,T(x^*)) = 0$ and since $T(x^*)$ is closed we obtain $x^* \in T(x^*)$. □

Theorem 2.2. *Adding to the hypotheses of* Theorem 2.1*, the condition:* $\alpha(x^*,y^*) \geq 1$ *for all* $x^*,y^* \in Fix(T)$*, we obtain that* $x^* = y^*$.

Proof. From the conditions of Theorem 2.1. we have that T has a fixed point.

Suppose now that there exist $x^*,y^* \in Fix(T)$, $x^* \neq y^*$. Hence $d(x^*,y^*) \neq 0$. Moreover $D(x^*,T(y^*)) > 0$.

Using Lemma 1.10, with $q = s$, we obtain

$$d(x^*,y^*) < sD(x^*,T(y^*)) \tag{2.9}$$

We have that $\alpha(x^*,y^*) \geq 1$, and because T is α_*-admissible we shall obtain that $\alpha_*(T(x^*),T(y^*)) \geq 1$. Now from 2.9 we have

$$\begin{aligned} d(x^*,y^*) &< sD(x^*,T(y^*)) \leq sH(T(x^*),T(y^*)) \\ &\leq s\alpha_*(T(x^*),T(y^*))H(T(x^*),T(y^*)) \\ &\leq s\psi(M(x^*,y^*)), \end{aligned}$$

where

$$\begin{aligned} M(x^*,y^*) &= \max\left\{d(x^*,y^*), D(x^*,T(x^*)), D(y^*,Ty^*), \frac{D(x^*,Ty^*)+D(y^*,Tx^*)}{2s}\right\} \\ &= \max\left\{d(x^*,y^*), \frac{d(x^*,y^*)}{s}\right\} = d(x^*,y^*). \end{aligned}$$

Hence, we obtain

$$d(x^*,y^*) \leq s\psi(d(x^*,y^*)) < d(x^*,y^*),$$

which is a contradiction. Hence $x^* = y^*$ and, thus, T has a unique fixed point. □

We now state the following consequences of our results.

Corollary 2.3. *Let* (X,d) *be a complete b-metric space with constant* $s > 1$*, and* $d : X \times X \to \mathbb{R}_+$ *a continuous b-metric. Let* $T : X \to P_{cl}(X)$ *be a generalized* $(\alpha_* - \psi)$*-contractive multivalued operator of type-*(b) *with* $\psi(t) < \frac{t}{s}, \forall t > 0$*, satisfying the following conditions:*

(i) T *is* α_**-admissible;*

(ii) There exist $x_0 \in X$ and $x_1 \in T(x_0)$ such that $\alpha(x_0, x_1) \geq 1$;

(iii) If $(x_n)_{n\in\mathbb{N}}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ and $x_n \to x$ then $\alpha(x_n, x) \geq 1$, for all $n \in \mathbb{N}$.

Then T has a fixed point.

The proof is verbatim of Theorem 2.1, and hence we omitted.

Theorem 2.4. *Adding to the hypotheses of* Corollary 2.3, *the condition: $\alpha(x^*, y^*) \geq 1$ for all $x^*, y^* \in Fix(T)$, we obtain that $x^* = y^*$.*

In what follows, we state the consequence of in the context of metric space. For this purpose, we state the following notion that is inspired from Definition 1.18 we introduce the following contraction types.

Definition 2.5. Let (X, d) be a b-metric space and $T : X \to P_{cl}(X)$ be a multivalued operator. We say that T is an generalized $(\alpha_* - \psi)$-contractive multivalued operator if there exist two functions $\alpha : X \times X \to [0, \infty)$ and $\psi \in \Psi$, such that

$$\alpha_*(T(x), T(y))H(T(x), T(y)) \leq \psi(M(x, y)), \text{ for all } x, y \in X, \tag{2.10}$$

where

$$M(x, y) = \max\left\{d(x, y), D(x, Tx), D(y, Ty), \frac{D(x, Ty) + D(y, Tx)}{2}\right\},$$

and $\alpha_*(A, B) = \inf\{\alpha(a, b),\ a \in A,\ b \in B\}$.

If $s = 1$ in Theorem 2.1, then we get the following result in the context of metric space.

Corollary 2.6. *Let (X, d) be a complete metric space. Let $T : X \to P_{cl}(X)$ be a generalized $(\alpha_* - \psi)$-contractive multivalued operator. Suppose also that it satisfying the following conditions:*

(i) T is α_*-admissible;

(ii) there exist $x_0 \in X$ and $x_1 \in T(x_0)$ such that $\alpha(x_0, x_1) \geq 1$;

(iii) if $(x_n)_{n\in\mathbb{N}}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ and $x_n \to x$, then $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$.

Then T has a fixed point.

The following results (which is a main result of [4]) follows immediately.

Corollary 2.7. *Let (X, d) be a complete metric space. Let $T : X \to P_{cl}(X)$ be a $(\alpha_* - \psi)$-contractive multivalued operator. Suppose also that it satisfying the following conditions:*

(i) T is α_*-admissible;

(ii) there exist $x_0 \in X$ and $x_1 \in T(x_0)$ such that $\alpha(x_0, x_1) \geq 1$;

(iii) if $(x_n)_{n\in\mathbb{N}}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ and $x_n \to x$, then $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$.

Then T has an fixed point.

Example 2.8. Let $X = \{-1, -2, -3\} \cup [0, \infty)$ and $d : X \times X \to \mathbb{R}_+$ such that $d(x, y) = |x - y|$ for all $x, y \in [0, \infty)$ and $d(x, -1) = d(x, -2) = 0$ for all $x \in [0, \infty)$ and $d(-3, -1) = d(-1, -3) = d(-3, -2) = d(-2, -3) = 1, d(-1, -2) = d(-2, -1) = A \geq 2, d(-3, -3) = d(-1, -1) = d(-2, -2) = 0$. Then

$$d(x, y) \leq \frac{A}{2}\left[d(x, z) + d(z, y)\right] \text{ for } x, y, z \in X.$$

Then (X,d) is a b-metric space. If $A>2$ the ordinary triangle inequality does not hold and (X,d) is not a metric space.

Let $\psi(t)=\frac{t}{4}$ and define now $Tx=\begin{cases}[0,\frac{x}{8}] & if\ 0\leq x\leq 1,\\ (\frac{x}{2},\frac{x}{4}) & otherwise,\end{cases}$ and $\alpha(x,y)=\begin{cases}1 & if\ x,y\in[0,1]\\ 0 & otherwise.\end{cases}$

It is enough to examine two cases:

Case (I). Suppose $x,y\in[0,1]$. Then,

$$\alpha_*(T(x),T(y))H(T(x),T(y))=|\frac{x}{8}-\frac{y}{8}|\leq\frac{|x-y|}{4}=\psi(d(x,y))\leq\psi(M(x,y)),\ \text{for all } x,y\in X, \tag{2.11}$$

Case (II). Suppose $x,y\in X\subset[0,1]$. Then,

$$\alpha_*(T(x),T(y))H(T(x),T(y))=0\leq\psi(M(x,y)),\ \text{for all } x,y\in X, \tag{2.12}$$

T is an $(\alpha_*-\psi)$-contractive multivalued operator of type (b). On the other hand, for $\alpha(x,y)\geq 1$, we have $x,y\in[0,1)$ and hence $\alpha_*(x,y)\geq 1$. That is, T is α_*-admissible. For any sequence $\{x_n\}$ with $x_n\to x$ and $\alpha(x_n,x_{n+1})\geq 1$, we have $x_n,x\in[0,1]$ and hence $\alpha(x_n,x)\geq 1$. So all hypothesis of Theorem 2.1 are satisfied and T has a fixed point.

3. Well-posedness of the fixed point problem

In this section we present a well-posedness result for the fixed point problem.

Definition 3.1. Let (X,d) be a b-metric space with constant $s>1$ and $T:X\to P_{cl}(X)$ be a multivalued operator. The fixed point problem for T with respect to D is well-posed if

(a) $FixT=\{x^*\}$;

(b) If $(x_n)_{n\in\mathbb{N}}$ is a sequence such that $D(x_n,T(x_n))\to 0$, as $n\to\infty$, then $x_n\to x^*$, as $n\to\infty$.

Theorem 3.2. *Let (X,d) be a complete b-metric space with constant $s>1$. Suppose that all the hypotheses of* Theorem 2.2 *hold. Additionally we suppose that*

(i) the function ψ is continuous;

(ii) for any sequence $(x_n)_{n\in\mathbb{N}}$, with $D(x_n,T(x_n))\to 0$, as $n\to\infty$, we have $\alpha(x_n,x^*)\geq 1$ for all $n\in\mathbb{N}$, where $x^*\in FixT$.

In these conditions the fixed point problem for T with respect to D is well-posed.

Proof. Let $(x_n)_{n\in\mathbb{N}}$ is a sequence such that $D(x_n,T(x_n))\to 0$, as $n\to\infty$. We are in conditions of Theorem 2.2. Thus $FixT=\{x^*\}$.

From (ii) we have that $\alpha(x_n,x^*)\geq 1$ for all $n\in\mathbb{N}$, and because T is α_*-admissible we shall obtain that $\alpha_*(T(x_n),T(x^*))\geq 1$.

We shall prove that $x_n\to x^*$, as $n\to\infty$.

We have

$$\begin{aligned} d(x_n,x^*) &\leq sH(T(x_n),T(x^*))+sD(x_n,T(x_n))\\ &\leq s\alpha_*(T(x_n),T(x^*))H(T(x_n),T(x^*))+sD(x_n,T(x_n))\\ &\leq s\psi(M(x_n,x^*))+sD(x_n,T(x_n)). \end{aligned}$$

Hence we have

$$d(x_n,x^*)\leq s\psi(M(x_n,x^*))+sD(x_n,T(x_n)), \tag{3.1}$$

where

$$M(x_n, x^*) = \max\left\{d(x_n, x^*), D(x_n, Tx_n), D(x^*, Tx^*), \frac{D(x_n, Tx^*) + D(x^*, Tx_n)}{2s}\right\}.$$

Because $\frac{D(x_n,Tx^*)+D(x^*,Tx_n)}{2s} \leq d(x_n, x^*) + \frac{1}{2}D(x_n, Tx_n)$, we shall obtain that

$$M(x_n, x^*) \leq \max\left\{d(x_n, x^*), D(x_n, Tx_n), d(x_n, x^*) + \frac{1}{2}D(x_n, Tx_n)\right\}.$$

Let us suppose that there exists $\delta > 0$ such that $d(x_n, x^*) \to \delta$, as $n \to \infty$. Hence $\lim\limits_{n\to\infty} M(x_n, x^*) \leq \delta$.

If in (3.1), $n \to \infty$, then using the continuity of the function ψ, we have

$$\delta \leq s\psi(\delta) < \delta,$$

which is a contradiction.

Thus $\delta = 0$ which implies that $x_n \to x^*$, as $n \to \infty$. □

4. Ulam-Hyers stability

Definition 4.1. Let (X, d) be a b-metric space and $T : X \to P(X)$ be a multivalued operator. The fixed point inclusion

$$x \in T(x),\ x \in X \tag{4.1}$$

is called generalized Ulam-Hyers stable if and only if there exists $\varsigma : \mathbb{R}_+ \to \mathbb{R}_+$ which is increasing and continuous in 0 and $\varsigma(0) = 0$, such that for each $\varepsilon > 0$ and for each solution $y^* \in X$ of the inequality

$$D(y, T(y)) \leq \varepsilon, \tag{4.2}$$

there exists a solution x^* of the fixed point inclusion (4.1) such that

$$d(y^*, x^*) \leq \varsigma(\varepsilon).$$

If there exists $c > 0$ such that $\varsigma(t) := c \cdot t$, for each $t \in \mathbb{R}_+$, then the fixed point inclusion (4.1) is said to be Ulam-Hyers stable.

Remark 4.2. The definition of generalized Ulam-Hyers stability uses a function ψ instead of ς. We work with ς because ψ is used to denote $(\alpha_* - \psi)$-contraction.

For other results regarding the Ulam-Hyers stability see also [9], [19], [22], [24], [25].

Theorem 4.3. *Let (X, d) be a complete b-metric space with constant $s > 1$. Suppose that all the hypotheses of* Theorem 2.1 *hold. Additionally we suppose that*

(i) $\psi(t) < \frac{t}{2s}$ and the function $\beta : [0, \infty) \to [0, \infty)$, $\beta(r) := r - 2s\psi(r)$ is increasing and onto;
(ii) for any solution $y^* \in X$ of (18) *we have $\alpha(x^*, y^*) \geq 1$, where $x^* \in Fix(T)$.*

In this conditions the fixed point inclusion (4.1) *is generalized Ulam-Hyers stable.*

Proof. We are in the conditions of Theorem 2.1, hence there exists $x^* \in Fix(T)$. Let $\varepsilon > 0$ and y^* be a solution of (4.2).

From (ii) we have that $\alpha(x^*, y^*) \geq 1$, and because T is α_*-admissible we shall obtain that $\alpha_*(T(x^*), T(y^*)) \geq 1$.

We have

$$\begin{aligned} d(x^*, y^*) &\leq sH(T(x^*), T(y^*)) + sD(y^*, T(y^*)) \\ &\leq s\alpha_*(T(x^*), T(y^*))H(T(x^*), T(y^*)) + sD(y^*, T(y^*)) \\ &\leq s\psi(M(x^*, y^*)) + s\varepsilon, \end{aligned}$$

where

$$M(x^*, y^*) = \max \left\{ d(x^*, y^*), D(x^*, T(x^*)), D(y^*, Ty^*), \frac{D(x^*, Ty^*) + D(y^*, Tx^*)}{2s} \right\}.$$

We have

$$\begin{aligned} \frac{D(x^*, Ty^*) + D(y^*, Tx^*)}{2s} &\leq \frac{1}{2s} \left(sd(x^*, y^*) + sD(y^*, Ty^*) + d(x^*, y^*) \right) \\ &= \frac{s+1}{2s} d(x^*, y^*) + \frac{1}{2} D(y^*, Ty^*) \\ &\leq d(x^*, y^*) + \frac{1}{2}\varepsilon \\ &\leq \max \left\{ 2d(x^*, y^*), \varepsilon \right\}. \end{aligned}$$

From here we have

$$M(x^*, y^*) \leq \max \left\{ 2d(x^*, y^*), \varepsilon \right\}.$$

It is obvious that if $2d(x^*, y^*) \leq \varepsilon$ the result is proved. Suppose that $\max \left\{ 2d(x^*, y^*), \varepsilon \right\} = 2d(x^*, y^*)$. Hence $M(x^*, y^*) \leq 2d(x^*, y^*)$ and

$$\begin{aligned} 2d(x^*, y^*) &\leq 2s\psi(2d(x^*, y^*))) + 2s\varepsilon \\ 2d(x^*, y^*) - 2s\psi(2d(x^*, y^*)) &\leq 2s\varepsilon. \end{aligned} \tag{4.3}$$

From (4.3) we get that

$$\beta \left(2d(x^*, y^*) \right) \leq 2s\varepsilon,$$

and hence

$$d(x^*, y^*) \leq \frac{1}{2} \beta^{-1} \left(2s\varepsilon \right).$$

Hence (4.1) is generalized Ulam-Hyers stable. □

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