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Fixed point theorems for twisted (α, β)-ψ-contractive type mappings and applications

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Abstract. The purpose of this paper is to discuss the existence and uniqueness of fixed points for new classes of mappings defined on a complete metric space. The obtained results generalize some recent theorems in the literature. Several applications and interesting consequences of our theorems are also given.

1. Introduction and Preliminaries

We know by the Banach contraction principle [2], which is a classical and powerful tool in nonlinear analysis, that a self-mapping f on a complete metric space (X, d) such that $d(fx, fy) \leq c\, d(x, y)$ for all $x, y \in X$, where $c \in (0, 1)$, has a unique fixed point. Since then, the Banach contraction principle has been generalized in several directions, see the papers [1, 6, 11, 13, 14, 18, 20, 21] and references cited therein. Therefore, in the recent fixed point theory for self-mappings, there are some new concepts which can be mutually related so that the properties of each one might be combined for such kind of self-mappings [9, 10, 16, 19]. On the other hand, an important direction in generalizing the Banach contraction principle has been initiated by Ran and Reurings [17] who considered a partial ordering $\preceq$ on the metric space (X, d) and required that the contractive condition is satisfied only for comparable elements, that is, we have $d(fx, fy) \leq c\, d(x, y)$ for all $x, y \in X$ with $x \preceq y$. Later on, many authors followed this new approach and obtained results with significant applications, for example, to the existence of solutions for matrix equations or ordinary differential equations [8, 12, 15].

In view of the above considerations, the principal motivation of this paper is to relate some results in the literature by discussing the existence and uniqueness of fixed points for new classes of mappings defined on a complete metric space. In particular, we use our results to obtain fixed points for some new classes of cyclic mappings and cyclic ordered mappings. We conclude the paper by giving an application to functional equations.

Following the direction in [18], we give some notions and notations useful in the sequel.
Denote with Ψ the family of nondecreasing functions $\psi : [0, +\infty) \to [0, +\infty)$ such that $\sum_{n=1}^{\infty} \psi^n(t) < +\infty$ for all $t > 0$, where ψ^n is the n-th iterate of ψ. The next lemma is obvious.

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Lemma 1.1. *If $\psi \in \Psi$, then $\psi(0) = 0$ and $\psi(t) < t$ for all $t > 0$.*

The following definitions play a key role in our paper.

Definition 1.2 ([18]). *Let $f : X \to X$ and $\alpha : X \times X \to [0, +\infty)$. We say that f is an α-admissible mapping if*

$$x, y \in X, \quad \alpha(x, y) \geq 1 \quad \Longrightarrow \quad \alpha(fx, fy) \geq 1.$$

Definition 1.3. *Let $f : X \to X$ and $\alpha, \beta : X \times X \to [0, +\infty)$. We say that f is a twisted (α, β)-admissible mapping if*

$$x, y \in X, \quad \begin{cases} \alpha(x, y) \geq 1, \\ \beta(x, y) \geq 1, \end{cases} \quad \Longrightarrow \quad \begin{cases} \alpha(fy, fx) \geq 1, \\ \beta(fy, fx) \geq 1. \end{cases}$$

Definition 1.4. *Let (X, d) be a metric space and $f : X \to X$ be a twisted (α, β)-admissible mapping. Then f is said to be a*

(a) twisted (α, β)-ψ-contractive mapping of type (I), if

$$\alpha(x, y)\beta(x, y)d(fx, fy) \leq \psi(d(x, y)) \tag{1}$$

holds for all $x, y \in X$, where $\psi \in \Psi$.

(b) twisted (α, β)-ψ-contractive mapping of type (II), if there is $0 < \ell \leq 1$ such that

$$\Big(\alpha(x, y)\beta(x, y) + \ell\Big)^{d(fx, fy)} \leq (1 + \ell)^{\psi(d(x, y))} \tag{2}$$

holds for all $x, y \in X$, where $\psi \in \Psi$.

(c) twisted (α, β)-ψ-contractive mapping of type (III), if there is $\ell \geq 1$ such that

$$\Big(d(fx, fy) + \ell\Big)^{\alpha(x, y)\beta(x, y)} \leq \psi(d(x, y)) + \ell \tag{3}$$

holds for all $x, y \in X$, where $\psi \in \Psi$.

2. Main Results

In this section we give some theorems linking the above concepts. Our first main result is the following fixed point theorem useful to generalize the Banach contraction principle and many other related results.

Theorem 2.1. *Let (X, d) be a complete metric space and let $f : X \to X$ be a continuous twisted (α, β)-ψ-contractive mapping of type (I) or (II) or (III). If there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$ and $\beta(x_0, fx_0) \geq 1$, then f has a fixed point in X.*

Proof. Let $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$ and $\beta(x_0, fx_0) \geq 1$. Define a sequence $\{x_n\}$ by $x_n = f^n x_0 = fx_{n-1}$ for all $n \in \mathbb{N}$. Since f is a twisted (α, β)-admissible mapping and $\beta(x_0, x_1) = \beta(x_0, fx_0) \geq 1$, then $\beta(x_2, x_1) = \beta(fx_1, fx_0) \geq 1$ which implies $\beta(x_2, x_3) = \beta(fx_1, fx_2) \geq 1$. By continuing this process, we get $\beta(x_{2n}, x_{2n+1}) \geq 1$ and $\beta(x_{2n}, x_{2n-1}) \geq 1$ for all $n \in \mathbb{N}$. Similarly, we have $\alpha(x_{2n}, x_{2n+1}) \geq 1$ and $\alpha(x_{2n}, x_{2n-1}) \geq 1$ for all $n \in \mathbb{N}$. Now, we distinguish the following cases:

(a) Let f be a twisted (α, β)-ψ-contractive mapping of type (I). Then by (1) with $x = x_{2n}$ and $y = x_{2n+1}$ we have

$$\begin{aligned} d(x_{2n+1}, x_{2n+2}) &\leq \alpha(x_{2n}, x_{2n+1})\beta(x_{2n}, x_{2n+1})d(x_{2n+1}, x_{2n+2}) \\ &\leq \psi(d(x_{2n}, x_{2n+1})). \end{aligned}$$

Then $d(x_{2n+1}, x_{2n+2}) \leq \psi(d(x_{2n}, x_{2n+1}))$. Similarly, by (1) with $x = x_{2n}$ and $y = x_{2n-1}$ we have $d(x_{2n+1}, x_{2n}) \leq \psi(d(x_{2n}, x_{2n-1}))$. In view of these inequalities, for all $n \in \mathbb{N}$ we can deduce that

$$d(x_n, x_{n+1}) \leq \psi^n(d(x_0, x_1)).$$

(b) Let f be a twisted (α,β)-ψ-contractive mapping of type (II). Then by (2) with $x = x_{2n}$ and $y = x_{2n+1}$ we have

$$\begin{aligned}(1+\ell)^{d(x_{2n+1},x_{2n+2})} &\le (\alpha(x_{2n},x_{2n+1})\beta(x_{2n},x_{2n+1})+\ell)^{d(x_{2n+1},x_{2n+2})}\\ &\le (1+\ell)^{\psi(d(x_{2n},x_{2n+1}))}.\end{aligned}$$

Then $d(x_{2n+1},x_{2n+2}) \le \psi(d(x_{2n},x_{2n+1}))$. Similarly, by (2) with $x = x_{2n}$ and $y = x_{2n-1}$ we have $d(x_{2n+1},x_{2n}) \le \psi(d(x_{2n},x_{2n-1}))$. Again, for all $n \in \mathbb{N}$ we can deduce that

$$d(x_n,x_{n+1}) \le \psi^n(d(x_0,x_1)).$$

(c) Let f be a twisted (α,β)-ψ-contractive mapping of type (III). Then by (3) with $x = x_{2n}$ and $y = x_{2n+1}$ we have

$$\begin{aligned}d(x_{2n+1},x_{2n+2})+\ell &\le (d(x_{2n+1},x_{2n+2})+\ell)^{\alpha(x_{2n},x_{2n+1})\beta(x_{2n},x_{2n+1})}\\ &\le \psi(d(x_{2n},x_{2n+1}))+\ell.\end{aligned}$$

Then $d(x_{2n+1},x_{2n+2}) \le \psi(d(x_{2n},x_{2n+1}))$. Similarly, by (3) with $x = x_{2n}$ and $y = x_{2n-1}$ we have $d(x_{2n+1},x_{2n}) \le \psi(d(x_{2n},x_{2n-1}))$.

Thus, in all cases, for all $n \in \mathbb{N}$ we can deduce easily that

$$d(x_n,x_{n+1}) \le \psi^n(d(x_0,x_1)).$$

Fix $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\sum_{n\ge N} \psi^n(d(x_0,x_1)) < \epsilon.$$

Let $m,n \in \mathbb{N}$ with $m > n \ge N$. Then by using the triangular inequality we get

$$d(x_n,x_m) \le \sum_{k=n}^{m-1} d(x_k,x_{k+1}) \le \sum_{n\ge N} \psi^n(d(x_0,x_1)) < \epsilon$$

and consequently $\lim_{m,n\to+\infty} d(x_n,x_m) = 0$. Hence $\{x_n\}$ is a Cauchy sequence. Since X is complete, then there is $z \in X$ such that $x_n \to z$ as $n \to +\infty$. Finally, since f is continuous then we have

$$fz = \lim_{n\to+\infty} fx_n = \lim_{n\to+\infty} x_{n+1} = z$$

and so z is a fixed point of f. □

Similarly, one can obtain the same conclusion under an alternative assumption. Precisely, in the following theorem we omit the continuity condition on f but use an adjunctive condition on X.

Theorem 2.2. *Let (X,d) be a complete metric space and let $f : X \to X$ be a twisted (α,β)-ψ-contractive mapping of type (I) or (II) or (III). Also suppose that the following conditions hold:*

(i) there exists $x_0 \in X$ such that $\alpha(x_0,fx_0) \ge 1$ and $\beta(x_0,fx_0) \ge 1$,

(ii) if $\{x_n\}$ is a sequence in X such that $\alpha(x_{2n},x_{2n+1}) \ge 1$ and $\beta(x_{2n},x_{2n+1}) \ge 1$ for all $n \in \mathbb{N}\cup\{0\}$ and $x_n \to x$ as $n \to +\infty$, then $\alpha(x_{2n},x) \ge 1$ and $\beta(x_{2n},x) \ge 1$ for all $n \in \mathbb{N}\cup\{0\}$.

Then f has a fixed point.

Proof. Let $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$ and $\beta(x_0, fx_0) \geq 1$. Proceeding as in the proof of Theorem 2.1, we know that there is $z \in X$ such that $x_n \to z$ as $n \to +\infty$, $\alpha(x_{2n}, x_{2n+1}) \geq 1$ and $\beta(x_{2n}, x_{2n+1}) \geq 1$. We shall prove that $z = fz$. Assume to the contrary that $z \neq fz$. From (ii) we have $\alpha(x_{2n}, z) \geq 1$ and $\beta(x_{2n}, z) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$. Now, we distinguish the following cases:

(a) Let f be a twisted (α, β)-ψ-contractive mapping of type (I). Then by (2) with $x = x_{2n}$ and $y = z$ we have

$$\begin{aligned} d(fx_{2n}, fz) &\leq \alpha(x_{2n}, z)\beta(x_{2n}, z)d(fx_{2n}, fz) \\ &\leq \psi(d(x_{2n}, z)). \end{aligned}$$

(b) Let f be a twisted (α, β)-ψ-contractive mapping of type (II). Then by (2) with $x = x_{2n}$ and $y = z$ we have

$$\begin{aligned} (1+\ell)^{d(fx_{2n}, fz)} &\leq (\alpha(x_{2n}, z)\beta(x_{2n}, z) + \ell)^{d(fx_{2n}, fz)} \\ &\leq (1+\ell)^{\psi(d(x_{2n}, z))}, \end{aligned}$$

that implies $d(fx_{2n}, fz) \leq \psi(d(x_{2n}, z))$.

(c) Let f be a twisted (α, β)-ψ-contractive mapping of type (III). Then by (2) with $x = x_{2n}$ and $y = z$ we have

$$\begin{aligned} d(fx_{2n}, fz) + \ell &\leq (d(fx_{2n}, fz) + \ell)^{\alpha(x_{2n}, z)\beta(x_{2n}, z)} \\ &\leq \psi(d(x_{2n}, z)) + \ell, \end{aligned}$$

that implies $d(fx_{2n}, fz) \leq \psi(d(x_{2n}, z))$.

Therefore, in all cases, by using the triangular inequality we can write

$$\begin{aligned} d(z, fz) &\leq d(z, fx_{2n}) + d(fx_{2n}, fz) \\ &= d(z, x_{2n+1}) + d(fx_{2n}, fz) \\ &\leq d(z, x_{2n+1}) + \psi(d(x_{2n}, z)). \end{aligned}$$

By taking the limit as $n \to +\infty$ in the above inequality, since ψ is continuous in $t = 0$, we have $d(z, fz) = 0$, that is, $z = fz$. □

Example 2.3. *Let $X = \mathbb{R}$ be endowed with the usual metric $d(x, y) = |x - y|$ for all $x, y \in X$ and $f : X \to X$ be defined by*

$$fx = \begin{cases} -\dfrac{1}{4}x & \text{if } x \in [-1, 1], \\ \sqrt[3]{\dfrac{x+1}{x^2+1}} & \text{if } x \in \mathbb{R}\setminus[-1, 1]. \end{cases}$$

Define also $\alpha, \beta : X \times X \to [0, +\infty)$ by

$$\alpha(x, y) = \beta(x, y) = \begin{cases} 1 & \text{if } x \in [0, 1] \text{ and } y \in [-1, 0], \\ 0 & \text{otherwise}, \end{cases}$$

and $\psi : [0, +\infty) \to [0, +\infty)$ by $\psi(t) = \frac{1}{2}t$ for all $t \geq 0$.
We prove that Theorem 2.2 can be applied to f.

Proof. Let $\alpha(x,y) \geq 1$ for $x, y \in X$. Then $x \in [0,1]$ and $y \in [-1,0]$, and so $fy \in [0,1]$ and $fx \in [-1,0]$, that is, $\alpha(fy, fx) \geq 1$. Also, assume $\beta(x,y) \geq 1$ for $x, y \in X$. Therefore $x \in [0,1]$ and $y \in [-1,0]$, and hence $fy \in [0,1]$ and $fx \in [-1,0]$, that is, $\beta(fy, fx) \geq 1$. Clearly, $\alpha(0, f0) \geq 1$ and $\beta(0, f0) \geq 1$. Now, let $\{x_n\}$ be a sequence in X such that $\alpha(x_{2n}, x_{2n+1}) \geq 1$ and $\beta(x_{2n}, x_{2n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$ as $n \to +\infty$. This implies that $\{x_{2n+1}\} \subset [-1,0]$ and $\{x_{2n}\} \subset [0,1]$. Thus, $x = 0$ and so $\alpha(x_{2n}, x) \geq 1$ and $\beta(x_{2n}, x) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$. Moreover, for $x \in [0,1]$ and $y \in [-1,0]$ we have

$$\begin{aligned} \alpha(x,y)\beta(x,y)d(fx,fy) &= |fx - fy| \\ &= \frac{1}{4}|x-y| \\ &\leq \frac{1}{2}|x-y| = \psi(d(x,y)). \end{aligned}$$

Otherwise, $\alpha(x,y)\beta(x,y) = 0$ and (1) trivially holds. Then f is a twisted (α,β)-ψ-contractive mapping of type (I) and, by Theorem 2.2, f has a fixed point. □

Example 2.4. *Let X, d, α and β be as in Example 2.3 and $f : X \to X$ be defined by*

$$fx = \begin{cases} -\dfrac{1}{4\pi}(x + x^2) & \text{if } x \in [-1,1], \\[2ex] \dfrac{x^2 - \cos(x^5)}{2 + \sin(x)} & \text{if } x \in \mathbb{R}\setminus[-1,1]. \end{cases}$$

Define also $\psi : [0,+\infty) \to [0,+\infty)$ by $\psi(t) = \frac{1}{4}t$ for all $t \geq 0$.
We prove that Theorem 2.2 *can be applied to f.*

Proof. Proceeding as in the proof of Example 2.3, we deduce that f is a twisted (α,β)-admissible mapping and that the conditions (i) and (ii) of Theorem 2.2 hold. Moreover, if $x \in [0,1]$, $y \in [-1,0]$ and $0 < \ell \leq 1$ we have

$$\begin{aligned} \Big(\alpha(x,y)\beta(x,y) + \ell\Big)^{d(fx,fy)} &= (1+\ell)^{d(fx,fy)} \\ &= (1+\ell)^{\frac{1}{4\pi}|x-y||x+y+1|} \\ &\leq (1+\ell)^{\frac{3}{4\pi}|x-y|} \\ &\leq (1+\ell)^{\frac{1}{4}|x-y|} = (1+\ell)^{\psi(d(x,y))}. \end{aligned}$$

Otherwise, $\alpha(x,y)\beta(x,y) = 0$ and (2) trivially holds. Hence, f is a twisted (α,β)-ψ-contractive mapping of type (II) and by Theorem 2.2, f has a fixed point. □

Example 2.5. *Let $X = [0,+\infty)$ be endowed with the usual metric $d(x,y) = |x-y|$ for all $x, y \in X$ and $f : X \to X$ be defined by*

$$fx = \begin{cases} \dfrac{1}{8}x^4 & \text{if } x \in [0,1], \\[2ex] \dfrac{1}{x} - \dfrac{1}{1+x} & \text{if } x \in (1,+\infty). \end{cases}$$

Define also $\alpha, \beta : X \times X \to [0,+\infty)$ by

$$\alpha(x,y) = \beta(x,y) = \begin{cases} 1 & \text{if } x, y \in [0,1], \\ 0 & \text{otherwise,} \end{cases}$$

and $\psi : [0, +\infty) \to [0, +\infty)$ *by* $\psi(t) = \frac{1}{2}t$ *for all* $t \geq 0$.
We prove that Theorem 2.2 *can be applied to* f.

Proof. Proceeding as in the proof of Example 2.3, we deduce that f is a twisted (α, β)-admissible mapping and that the conditions (i) and (ii) of Theorem 2.2 hold. Moreover, if $x, y \in [0, 1]$ and $\ell \geq 1$, we have

$$\begin{aligned}\left(d(fx, fy) + \ell\right)^{\alpha(x,y)\beta(x,y)} &= \frac{1}{8}|x^4 - y^4| + \ell \\ &= \frac{1}{8}|x - y||x + y||x^2 + y^2| + \ell \\ &\leq \frac{1}{2}|x - y| + \ell = \psi(d(x, y)) + \ell.\end{aligned}$$

Otherwise, $\alpha(x, y)\beta(x, y) = 0$ and (3) trivially holds. Hence, f is a twisted (α, β)-ψ-contractive mapping of type (III) and, by Theorem 2.2, f has a fixed point. □

In the next result, we consider an hypothesis useful to obtain the uniqueness of the fixed point.

Theorem 2.6. *Assume that all the hypotheses of Theorem 2.1 (respectively Theorem 2.2) hold. Adding the following condition:*

(iii) for all $x, y \in X$ *with* $x \neq y$*, there exists* $v \in X$ *such that* $\alpha(x, v) \geq 1$, $\alpha(y, v) \geq 1$ *and* $\beta(x, v) \geq 1$, $\beta(y, v) \geq 1$,

we obtain the uniqueness of the fixed point of f.

Proof. Suppose that z and z^* are two fixed points of f such that $z \neq z^*$. By condition (iii), there exists v such that $\alpha(z, v) \geq 1$ and $\alpha(z^*, v) \geq 1$. Therefore, since f is a twisted (α, β)-admissible mapping, we deduce that $\alpha(f^{2n}z, f^{2n}v) \geq 1$, $\alpha(f^{2n-1}v, f^{2n-1}z) \geq 1$ and $\alpha(f^{2n}z^*, f^{2n}v) \geq 1$, $\alpha(f^{2n-1}v, f^{2n-1}z^*) \geq 1$. Similarly, we get $\beta(f^{2n}z, f^{2n}v) \geq 1$, $\beta(f^{2n-1}v, f^{2n-1}z) \geq 1$ and $\beta(f^{2n}z^*, f^{2n}v) \geq 1$, $\beta(f^{2n-1}v, f^{2n-1}z^*) \geq 1$.

Now, if f is a twisted (α, β)-ψ-contractive mapping of type (I), then by (1) with $x = f^{2n}z$ and $y = f^{2n}v$ we have

$$\begin{aligned}d(ff^{2n}z, ff^{2n}v) &\leq \alpha(f^{2n}z, f^{2n}v)\beta(f^{2n}z, f^{2n}v)d(ff^{2n}z, ff^{2n}v) \\ &\leq \psi(d(f^{2n}z, f^{2n}v)).\end{aligned}$$

Similarly by (1) with $x = f^{2n-1}v$ and $y = f^{2n-1}z$ we get

$$d(ff^{2n-1}z, ff^{2n-1}v) \leq \psi(d(f^{2n-1}z, f^{2n-1}v)).$$

Hence for all $n \in \mathbb{N}$ we have

$$d(ff^n z, ff^n v) \leq \psi(d(f^n z, f^n v)),$$

or equivalently,

$$d(z, f^{n+1}v) \leq \psi^n(d(z, v)).$$

Of course, we get the same conclusion if we suppose that f is a twisted (α, β)-ψ-contractive mapping of type (II) or (III) and so we omit the details. By taking the limit as $n \to +\infty$ in the above inequality we obtain

$$\lim_{n\to+\infty} d(z, f^{n+1}v) = 0.$$

Using a similar argument we also get

$$\lim_{n\to+\infty} d(z^*, f^{n+1}v) = 0.$$

From the last two limits and the triangular inequality we have

$$d(z, z^*) \leq \lim_{n\to+\infty}[d(z, f^{n+1}v) + d(z^*, f^{n+1}v)] = 0,$$

that is, $z = z^*$. □

The following result of Boyd and Wong [7] is a consequence of Theorem 2.1.

Corollary 2.7. *Let (X, d) be a complete metric space and let $f : X \to X$ be a continuous mapping. If there exists $\psi \in \Psi$ such that*

$$d(fx, fy) \leq \psi(d(x, y)) \tag{4}$$

holds for all $x, y \in X$, then f has a unique fixed point in X.

Proof. By taking $\alpha(x, y) = \beta(x, y) = 1$ for all $x, y \in X$ in Theorem 2.1, we deduce that f has a fixed point in X. The uniqueness of the fixed point follows easily from (4) and so we omit the details. □

Clearly, if $\psi(t) = ct$ for all $t > 0$ where $c \in (0, 1)$, then Corollary 2.7 reduces to the Banach contraction principle [2]; recall that contractions are always continuous.

3. Cyclic Results

In this section we show how is possible to apply the results in the main section for proving, in a natural way, some analogous fixed point results involving a cyclic mapping. First, for our further use, we adapt Definition 1.4 as follows:

Definition 3.1. *Let (X, d) be a metric space and A, B be two nonempty and closed subsets of X. Let $\alpha : X \times X \to [0, +\infty)$ and $f : A \cup B \to A \cup B$, with $fA \subseteq B$ and $fB \subseteq A$, such that $\alpha(fy, fx) \geq 1$ if $\alpha(x, y) \geq 1$, where $x \in A$ and $y \in B$. Thus f is said to be a*

(a) cyclic α-ψ-contractive mapping of type (I), if

$$\alpha(x, y)d(fx, fy) \leq \psi(d(x, y)) \tag{5}$$

holds for all $x \in A$ and $y \in B$, where $\psi \in \Psi$.

(b) cyclic α-ψ-contractive mapping of type (II), if there is $0 < \ell \leq 1$ such that

$$\left(\alpha(x, y) + \ell\right)^{d(fx, fy)} \leq (1 + \ell)^{\psi(d(x, y))} \tag{6}$$

holds for all $x \in A$ and $y \in B$, where $\psi \in \Psi$.

(c) cyclic α-ψ-contractive mapping of type (III), if there is $\ell \geq 1$ such that

$$\left(d(fx, fy) + \ell\right)^{\alpha(x, y)} \leq \psi(d(x, y)) + \ell \tag{7}$$

holds for all $x \in A$ and $y \in B$, where $\psi \in \Psi$.

Now, we prove the following result for a continuous cyclic mapping.

Theorem 3.2. *Let (X, d) be a complete metric space and A, B be two nonempty and closed subsets of X such that $f : A \cup B \to A \cup B$ is a continuous cyclic α-ψ-contractive mapping of type (I) or (II) or (III). If there exists $x_0 \in A$ such that $\alpha(x_0, fx_0) \geq 1$, then f has a fixed point in $A \cap B$.*

Proof. Let $Y = A \cup B$ and $\beta : Y \times Y \to [0, +\infty)$ be the function defined by

$$\beta(x, y) = \begin{cases} 1 & \text{if } x \in A \text{ and } y \in B, \\ 0 & \text{otherwise.} \end{cases}$$

Then (Y, d) is a complete metric space and f is a twisted (α, β)-admissible mapping. Now, if $x_0 \in A$ is such that $\alpha(x_0, fx_0) \geq 1$, then also $\beta(x_0, fx_0) \geq 1$ and hence all the hypotheses of Theorem 2.1 hold with $X = Y$. Consequently, f has a fixed point in $A \cup B$, say z. Since $z \in A$ implies $z = fz \in B$ and $z \in B$ implies $z = fz \in A$, then $z \in A \cap B$. □

Also for cyclic α-ψ-contractive mappings, we can omit the continuity condition as is shown by the following theorem.

Theorem 3.3. *Let (X, d) be a complete metric space and A, B be two nonempty and closed subsets of X such that $f : A \cup B \to A \cup B$ is a cyclic α-ψ-contractive mapping of type (I) or (II) or (III). Also suppose that the following conditions hold:*

(i) there exists $x_0 \in A$ such that $\alpha(x_0, fx_0) \geq 1$;

(ii) if $\{x_n\}$ is a sequence in X such that $\alpha(x_{2n}, x_{2n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$ as $n \to +\infty$, then $\alpha(x_{2n}, x) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$.

Then f has a fixed point in $A \cap B$.

Proof. Let $Y = A \cup B$ and define the function $\beta : Y \times Y \to [0, +\infty)$ as in the proof of Theorem 3.2. Let $\{x_n\}$ be a sequence in Y such that $\alpha(x_{2n}, x_{2n+1}) \geq 1$ and $\beta(x_{2n}, x_{2n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$ as $n \to +\infty$, then $x_{2n} \in A$ and $x_{2n+1} \in B$. Now, since B is closed, then $x \in B$ and hence $\alpha(x_{2n}, x) \geq 1$ and $\beta(x_{2n}, x) \geq 1$. We deduce that all the hypotheses of Theorem 2.2 are satisfied with $X = Y$ and hence f has a fixed point. □

As an immediate consequence of Theorem 3.2 we present the following result of Boyd-Wong type [7, 14].

Corollary 3.4. *Let (X, d) be a complete metric space and A, B be two nonempty and closed subsets of X such that $f : A \cup B \to A \cup B$ is continuous, $fA \subseteq B$ and $fB \subseteq A$. If there exists $\psi \in \Psi$ such that*

$$d(fx, fy) \leq \psi(d(x, y)) \tag{8}$$

holds for all $x \in A$ and $y \in B$, then f has a unique fixed point in $A \cap B$.

Proof. By taking $\alpha(x, y) = 1$ for all $x \in A$ and $y \in B$ in Theorem 3.2, we deduce that f has a fixed point in $A \cap B$. The uniqueness of the fixed point follows easily from (8) and so we omit the details. □

Clearly, if $\psi(t) = ct$ for all $t > 0$ where $c \in (0, 1)$, then Corollary 3.4 reduces to Theorem 1.1 of [14].

4. Cyclic ordered results

By using the similar arguments to those presented in the previous section, we are able to obtain results in the setting of ordered complete metric spaces.

Definition 4.1. *Let $(X, d, \preceq)$ be an ordered metric space and A, B be two nonempty and closed subsets of X. Let $\alpha : X \times X \to [0, +\infty)$ and $f : A \cup B \to A \cup B$, with $fA \subseteq B$ and $fB \subseteq A$, such that $\alpha(fy, fx) \geq 1$ if $\alpha(x, y) \geq 1$, where $x \in A$ and $y \in B$. Then f is said to be a*

(a) cyclic ordered α-ψ-contractive mapping of type (I), if

$$\alpha(x, y)d(fx, fy) \leq \psi(d(x, y)) \tag{9}$$

holds for all $x \in A$ and $y \in B$ with $x \preceq y$, where $\psi \in \Psi$.

(b) cyclic ordered α-ψ-contractive mapping of type (II), if there is $0 < \ell \leq 1$ such that

$$\Big(\alpha(x, y) + \ell\Big)^{d(fx, fy)} \leq (1 + \ell)^{\psi(d(x, y))} \tag{10}$$

holds for all $x \in A$ and $y \in B$ with $x \preceq y$, where $\psi \in \Psi$.

(c) cyclic ordered α-ψ-contractive mapping of type (III), if there is $\ell \geq 1$ such that

$$\Big(d(fx, fy) + \ell\Big)^{\alpha(x, y)} \leq \psi(d(x, y)) + \ell \tag{11}$$

holds for all $x \in A$ and $y \in B$ with $x \preceq y$, where $\psi \in \Psi$.

Theorem 4.2. *Let $(X, d, \preceq)$ be an ordered complete metric space and A, B be two nonempty and closed subsets of X such that $f : A \cup B \to A \cup B$ is a decreasing continuous cyclic ordered α-ψ-contractive mapping of type (I) or (II) or (III). If there exists $x_0 \in A$ such that $\alpha(x_0, fx_0) \geq 1$ and $x_0 \preceq fx_0$, then f has a fixed point in $A \cap B$.*

Proof. Consider the complete metric space (Y, d) where $Y = A \cup B$ and define the function $\beta : Y \times Y \to [0, +\infty)$ by

$$\beta(x, y) = \begin{cases} 1 & \text{if } x \in A \text{ and } y \in B \text{ with } x \preceq y, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, (1) (respectively, (2) or (3)) holds for all $x, y \in Y$. Let $\beta(x, y) \geq 1$ for $x, y \in X$, then $x \in A$ and $y \in B$ with $x \preceq y$. It follows that $fx \in B$ and $fy \in A$ with $fy \preceq fx$, since f is decreasing. Therefore $\beta(fy, fx) \geq 1$, that is, f is a twisted (α, β)-admissible mapping. Now, let $\alpha(x_0, fx_0) \geq 1$ with $x_0 \in A$ and $x_0 \preceq fx_0$. From $x_0 \in A$ we have $fx_0 \in B$ with $x_0 \preceq fx_0$, that is, $\beta(x_0, fx_0) \geq 1$. Then all the hypotheses of Theorem 2.1 hold with $X = Y$ and f has a fixed point in $A \cup B$, say z. Since $z \in A$ implies $z = fz \in B$ and $z \in B$ implies $z = fz \in A$, then $z \in A \cap B$. □

Theorem 4.3. *Let $(X, d, \preceq)$ be an ordered complete metric space and A, B be two nonempty and closed subsets of X such that $f : A \cup B \to A \cup B$ is a cyclic ordered α-ψ-contractive mapping of type (I) or (II) or (III). Also suppose that the following conditions hold:*

(i) there exists $x_0 \in A$ such that $\alpha(x_0, fx_0) \geq 1$ and $x_0 \preceq fx_0$;

(ii) if $\{x_n\}$ is a sequence in X such that $\alpha(x_{2n}, x_{2n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$ as $n \to +\infty$, then $\alpha(x_{2n}, x) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$;

(iii) if $\{x_n\}$ is a sequence in X such that $x_{2n} \preceq x_{2n+1}$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$ as $n \to +\infty$, then $x_{2n} \preceq x$ for all $n \in \mathbb{N} \cup \{0\}$.

Then f has a fixed point in $A \cap B$.

Proof. Consider the complete metric space (Y, d) where $Y = A \cup B$ and define the function $\beta : Y \times Y \to [0, +\infty)$ as in the proof of Theorem 4.2. Let $\{x_n\}$ be a sequence in X such that $\alpha(x_{2n}, x_{2n+1}) \geq 1$ and $\beta(x_{2n}, x_{2n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$ as $n \to +\infty$, then $x_{2n} \in A$ and $x_{2n+1} \in B$ with $x_{2n} \preceq x_{2n+1}$. Since B is closed and by (iii), we deduce that $x \in B$ and $x_{2n} \preceq x$, that is, $\beta(x_{2n}, x) \geq 1$. Since $\alpha(x_{2n}, x) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$, then all the hypotheses of Theorem 2.2 are satisfied and hence f has a fixed point. □

5. Application to functional equations

In this section we denote by $B(W)$ the space of all bounded real-valued functions defined on the set W. Now, $B(W)$ endowed with the sup metric $d(h, k) = \sup_{x \in W} |hx - kx|$ for all $h, k \in B(W)$ is a complete metric space.
In this setting, we discuss the problem of dynamic programming related to multistage process [3–5]. Indeed, this problem reduces to the problem of solving the functional equation

$$Q(x) = \sup_{y \in D} \{f(x, y) + K(x, y, Q(\tau(x, y)))\}, \quad x \in W, \tag{12}$$

where $\tau : W \times D \to W$, $f : W \times D \to \mathbb{R}$, $K : W \times D \times \mathbb{R} \to \mathbb{R}$.
Specifically, we will prove the following theorem.

Theorem 5.1. *Let $K : W \times D \times \mathbb{R} \to \mathbb{R}$ and $f : W \times D \to \mathbb{R}$ be two bounded functions and let $A : B(W) \to B(W)$ be defined by*

$$A(h)(x) = \sup_{y \in D} \{f(x, y) + K(x, y, h(\tau(x, y)))\}, \tag{13}$$

for all $h \in B(W)$ and $x \in W$. Assume that there exists $\theta : B(W) \times B(W) \to \mathbb{R}$ such that

(i) *$\theta(h,k)\geq 0 \Rightarrow \theta(A(k),A(h))\geq 0$, where $h,k\in B(W)$,*

(ii) *$\left|K(x,y,h(x))-K(x,y,k(x))\right|\leq \psi(|h(x)-k(x)|)$, where $\psi\in\Psi$, $h,k\in B(W)$, $\theta(h,k)\geq 0$, $x\in W$ and $y\in D$.*

Also suppose that

(iii) *if $\{h_n\}$ is a sequence in $B(W)$ such that $\theta(h_{2n},h_{2n+1})\geq 0$ for all $n\in\mathbb{N}\cup\{0\}$ and $h_n\to h^*$ as $n\to+\infty$, then $\theta(h_{2n},h^*)\geq 0$ for all $n\in\mathbb{N}\cup\{0\}$,*

(iv) *there exists $h_0\in B(W)$ such that $\theta(h_0,A(h_0))\geq 0$.*

Then the functional equation (12) *has a bounded solution.*

Proof. Note that $(B(W),d)$ is a complete metric space. Let ε be an arbitrary positive number and $h_1,h_2\in B(W)$ such that $\theta(h_1,h_2)\geq 0$, then there exist $y_1,y_2\in D$ such that

$$\begin{aligned}
A(h_1)(x) &< f(x,y_1)+K(x,y_1,h_1(\tau(x,y_1)))+\varepsilon, &(14)\\
A(h_2)(x) &< f(x,y_2)+K(x,y_2,h_2(\tau(x,y_2)))+\varepsilon, &(15)\\
A(h_1)(x) &\geq f(x,y_2)+K(x,y_2,h_1(\tau(x,y_2))), &(16)\\
A(h_2)(x) &\geq f(x,y_1)+K(x,y_1,h_2(\tau(x,y_1))). &(17)
\end{aligned}$$

Now, from (14) and (17), it follows easily that

$$\begin{aligned}
A(h_1)(x)-A(h_2)(x) &< K(x,y_1,h_1(\tau(x,y_1)))-K(x,y_1,h_2(\tau(x,y_1)))+\varepsilon\\
&\leq \left|K(x,y_1,h_1(\tau(x,y_1)))-K(x,y_1,h_2(\tau(x,y_1)))\right|+\varepsilon\\
&\leq \psi(|h_1(x)-h_2(x)|)+\varepsilon.
\end{aligned}$$

Hence we get

$$A(h_1)(x)-A(h_2)(x)<\psi(|h_1(x)-h_2(x)|)+\varepsilon. \tag{18}$$

Similarly, from (15) and (16) we obtain

$$A(h_2)(x)-A(h_1)(x)<\psi(|h_1(x)-h_2(x)|)+\varepsilon. \tag{19}$$

Therefore, from (18) and (19) we have

$$|A(h_1)(x)-A(h_2)(x)|<\psi(|h_1(x)-h_2(x)|)+\varepsilon, \tag{20}$$

that implies,

$$d(A(h_1),A(h_2))\leq\psi(d(h_1,h_2))+\varepsilon, \tag{21}$$

Since $\varepsilon>0$ is arbitrary, then

$$d(A(h_1),A(h_2))\leq\psi(d(h_1,h_2)).$$

Define

$$\alpha(h,k)=\beta(h,k)=\begin{cases}1 & \text{if } \theta(h,k)\geq 0, \text{ where } h,k\in B(W),\\ 0 & \text{otherwise.}\end{cases}$$

Consequently, we have

$$\alpha(h_1,h_2)\beta(h_1,h_2)d(A(h_1),A(h_2))\leq\psi(d(h_1,h_2)),$$

that is, A is a twisted (α,β)-ψ-contractive mapping of type (I) with $\alpha(h,k)=1$ for all $h,k\in B(W)$. Thus, by Theorem 2.2, A has a fixed point, that is, the functional equation (12) has a bounded solution. □

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