Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH Open Access

# Fixed point theorems for weakly contractive mappings in partially ordered metric-like spaces

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**Abstract**
In this article, we establish some fixed point theorems for weakly contractive mappings defined in ordered metric-like spaces. We provide an example and some applications in order to support the useability of our results. These results generalize some well-known results in the literature. We also derive some new fixed point results in ordered partial metric spaces.
**MSC:** Primary 54H25; secondary 47H10

**Keywords:** fixed point; weak contraction; partially ordered set; partial metric space; metric-like space

## 1 Introduction and preliminaries

During the last decades many authors have worked on domain theory in order to equip semantics domain with a notion of distance. In 1994, Matthews [1] introduced the notion of a partial metric space as a part of the study of denotational semantics of dataflow networks and showed that the Banach contraction principle can be generalized to the partial metric context for applications in program verification. Later on, many researchers studied fixed point theorems in partial metric spaces as well as ordered partial metric spaces.

A partial metric on a nonempty set $X$ is a function $p : X \times X \to \mathbb{R}^+$ such that for all $x, y, z \in X$:

(p1) $x = y \Leftrightarrow p(x,x) = p(x,y) = p(y,y)$;
(p2) $p(x,x) \le p(x,y)$;
(p3) $p(x,y) = p(y,x)$;
(p4) $p(x,y) \le p(x,z) + p(z,y) - p(z,z)$.

The pair $(X,p)$ is then called a partial metric space. A sequence $\{x_n\}$ in a partial metric space $(X,p)$ converges to a point $x \in X$ if $\lim_{n\to\infty} p(x_n,x) = p(x,x)$. A sequence $\{x_n\}$ of elements of $X$ is called p-Cauchy if the limit $\lim_{m,n\to\infty} p(x_m,x_n)$ exists and is finite. The partial metric space $(X,p)$ is called complete if for each p-Cauchy sequence $\{x_n\}_{n=0}^{\infty}$, there is some $x \in X$ such that

$$p(x,x) = \lim_{n\to\infty} p(x_n,x) = \lim_{n,m\to\infty} p(x_n,x_m).$$

A basic example of a partial metric space is the pair $(\mathbb{R}^+, p)$, where $p(x,y) = \max\{x,y\}$ for all $x, y \in \mathbb{R}^+$. For some other examples of partial metric spaces, see [2–16].

Another important development is reported in fixed point theory via ordered metric spaces. The existence of a fixed point in partially ordered sets has been considered recently in [17–32]. Tarski's theorem is used in [25] to show the existence of solutions for fuzzy equations and in [27] to prove existence theorems for fuzzy differential equations. In [26, 29] some applications to ordinary differential equations and to matrix equations are presented, respectively. In [19–21, 30] some fixed point theorems were proved for a mixed monotone mapping in a metric space endowed with partial order and the authors applied their results to problems of existence and uniqueness of solutions for some boundary value problems.

Recently, Amini-Harandi [33] introduced the notion of a metric-like space which is a new generalization of a partial metric space. The purpose of this paper is to present some fixed point theorems involving weakly contractive mappings in the context of ordered metric-like spaces. The presented theorems extend some recent results in the literature.

Weakly contractive mappings and mappings satisfying other weak contractive inequalities have been discussed in several works, some of which are noted in [34–40]. Alber and Guerre-Delabriere in [34] suggested a generalization of the Banach contraction mapping principle by introducing the concept of a weak contraction in Hilbert spaces. Rhoades [35] showed that the result which Alber *et al.* had proved in Hilbert spaces [34] was also valid in complete metric spaces.

**Definition 1** A mapping $\sigma : X \times X \to \mathbb{R}^+$, where $X$ is a nonempty set, is said to be metric-like on $X$ if for any $x, y, z \in X$, the following three conditions hold true:

($\sigma$1) $\sigma(x,y) = 0 \Rightarrow x = y$;
($\sigma$2) $\sigma(x,y) = \sigma(y,x)$;
($\sigma$3) $\sigma(x,z) \leq \sigma(x,y) + \sigma(y,z)$.

The pair $(X,\sigma)$ is then called a metric-like space. Then a metric-like on $X$ satisfies all of the conditions of a metric except that $\sigma(x,x)$ may be positive for $x \in X$. Each metric-like $\sigma$ on $X$ generates a topology $\tau_\sigma$ on $X$ whose base is the family of open $\sigma$-balls

$$B_\sigma(x,\varepsilon) = \{y \in X : |\sigma(x,y) - \sigma(x,x)| < \varepsilon\} \quad \text{for all } x \in X \text{ and } \varepsilon > 0.$$

Then a sequence $\{x_n\}$ in the metric-like space $(X,\sigma)$ converges to a point $x \in X$ if and only if $\lim_{n\to\infty} \sigma(x_n,x) = \sigma(x,x)$.

Let $(X,\sigma)$ and $(Y,\tau)$ be metric-like spaces, and let $F : X \to Y$ be a continuous mapping. Then

$$\lim_{n\to\infty} x_n = x \quad \Rightarrow \quad \lim_{n\to\infty} Fx_n = Fx.$$

A sequence $\{x_n\}_{n=0}^{\infty}$ of elements of $X$ is called $\sigma$-Cauchy if the limit $\lim_{m,n\to\infty} \sigma(x_m,x_n)$ exists and is finite. The metric-like space $(X,\sigma)$ is called complete if for each $\sigma$-Cauchy sequence $\{x_n\}_{n=0}^{\infty}$, there is some $x \in X$ such that

$$\lim_{n\to\infty} \sigma(x_n,x) = \sigma(x,x) = \lim_{m,n\to\infty} \sigma(x_m,x_n).$$

Every partial metric space is a metric-like space. Below we give another example of a metric-like space.

**Example 1** Let $X = \{0,1\}$ and

$$\sigma(x,y) = \begin{cases} 2 & \text{if } x = y = 0; \\ 1 & \text{otherwise.} \end{cases}$$

Then $(X,\sigma)$ is a metric-like space, but since $\sigma(0,0) \nleq \sigma(0,1)$, then $(X,\sigma)$ is not a partial metric space.

**Remark 1** Let $X = \{0,1\}$, and $\sigma(x,y) = 1$ for each $x,y \in X$, and $x_n = 1$ for each $n \in \mathbb{N}$. Then it is easy to see that $x_n \to 0$ and $x_n \to 1$, and so in metric-like spaces the limit of a convergent sequence is not necessarily unique.

## 2 Main results

Throughout the rest of this paper, we denote by $(X,\preceq,\sigma)$ a complete partially ordered metric-like space, *i.e.*, $\preceq$ is a partial order on the set $X$ and $\sigma$ is a complete metric-like on $X$.

A mapping $F : X \to X$ is said to be nondecreasing if $x,y \in X$, $x \preceq y \Rightarrow Fx \preceq Fy$.

**Theorem 1** *Let $(X,\preceq,\sigma)$ be a complete partially ordered metric-like space. Let $F : X \to X$ be a continuous and nondecreasing mapping such that for all comparable $x,y \in X$*

$$\psi\bigl(\sigma(Fx,Fy)\bigr) \le \psi\bigl(M(x,y)\bigr) - \phi\bigl(M(x,y)\bigr), \tag{2.1}$$

*where $M$ is given by*

$$M(x,y) = \max\bigl\{\sigma(x,y),\sigma(x,Fx),\sigma(y,Fy),\sigma(x,x),\sigma(y,y),\bigl[\sigma(x,Fy)+\sigma(Fx,y)\bigr]/2\bigr\},$$

*and*

(a) $\psi : [0,\infty) \to [0,\infty)$ *is a continuous monotone nondecreasing function with* $\psi(t) = 0$ *if and only if* $t = 0$;

(b) $\phi : [0,\infty) \to [0,\infty)$ *is a lower semi-continuous function with* $\phi(t) = 0$ *if and only if* $t = 0$.

*If there exists $x_0 \in X$ with $x_0 \preceq Fx_0$, then $F$ has a fixed point.*

*Proof* Since $F$ is a nondecreasing function, we obtain by induction that

$$x_0 \preceq Fx_0 \preceq F^2x_0 \preceq \cdots \preceq F^nx_0 \preceq F^{n+1}x_0 \preceq \cdots.$$

Put $x_{n+1} = Fx_n$. Then, for each integer $n = 0,1,2,\ldots$, as the elements $x_{n+1}$ and $x_n$ are comparable, from (2.1) we get

$$\begin{aligned}\psi\bigl(\sigma(x_{n+1},x_n)\bigr) &= \psi\bigl(\sigma(Fx_n,Fx_{n-1})\bigr) \\ &\le \psi\bigl(M(x_n,x_{n-1})\bigr) - \phi\bigl(M(x_n,x_{n-1})\bigr),\end{aligned} \tag{2.2}$$

which implies $\psi(\sigma(x_{n+1},x_n)) \le \psi(M(x_n,x_{n-1}))$. Using the monotone property of the $\psi$-function, we get

$$\sigma(x_{n+1},x_n) \le M(x_n,x_{n-1}). \tag{2.3}$$

Now, from the triangle inequality, for $\sigma$ we have

$$\begin{aligned} M(x_n, x_{n-1}) &= \max\bigl\{\sigma(x_n, x_{n-1}), \sigma(x_n, Fx_n), \sigma(x_{n-1}, Fx_{n-1}), \sigma(x_n, x_n), \\ &\quad \sigma(x_{n-1}, x_{n-1}), \bigl[\sigma(x_n, Fx_{n-1}) + \sigma(Fx_n, x_{n-1})\bigr]/2\bigr\} \\ &= \max\bigl\{\sigma(x_n, x_{n-1}), \sigma(x_n, x_{n+1}), \sigma(x_{n-1}, x_n), \sigma(x_n, x_n), \\ &\quad \sigma(x_{n-1}, x_{n-1}), \bigl[\sigma(x_n, x_n) + \sigma(x_{n+1}, x_{n-1})\bigr]/2\bigr\} \\ &\leq \max\bigl\{\sigma(x_n, x_{n-1}), \sigma(x_{n+1}, x_n), \bigl[\sigma(x_n, x_{n+1}) + \sigma(x_n, x_{n-1})\bigr]/2\bigr\} \\ &= \max\bigl\{\sigma(x_n, x_{n-1}), \sigma(x_{n+1}, x_n)\bigr\}. \end{aligned}$$

If $\sigma(x_{n+1}, x_n) > \sigma(x_n, x_{n-1})$, then $M(x_n, x_{n-1}) = \sigma(x_{n+1}, x_n) > 0$. By (2.2) it furthermore implies that

$$\psi\bigl(\sigma(x_{n+1}, x_n)\bigr) \leq \psi\bigl(\sigma(x_{n+1}, x_n)\bigr) - \phi\bigl(\sigma(x_{n+1}, x_n)\bigr),$$

which is a contradiction. So, we have

$$\sigma(x_{n+1}, x_n) \leq M(x_n, x_{n-1}) \leq \sigma(x_n, x_{n-1}). \tag{2.4}$$

Therefore, the sequence $\{\sigma(x_{n+1}, x_n)\}$ is monotone nonincreasing and bounded. Thus, there exists $r \geq 0$ such that

$$\lim_{n\to\infty} \sigma(x_{n+1}, x_n) = \lim_{n\to\infty} M(x_n, x_{n-1}) = r. \tag{2.5}$$

We suppose that $r > 0$. Then, letting $n \to \infty$ in the inequality (2.2), we get

$$\psi(r) \leq \psi(r) - \phi(r),$$

which is a contradiction unless $r = 0$. Hence,

$$\lim_{n\to\infty} \sigma(x_{n+1}, x_n) = 0. \tag{2.6}$$

Next we prove that $\{x_n\}$ is a $\sigma$-Cauchy sequence. Suppose that $\{x_n\}$ is not a $\sigma$-Cauchy sequence. Then, there exists $\varepsilon > 0$ for which we can find subsequences $\{x_{m_k}\}$ and $\{x_{n_k}\}$ of $\{x_n\}$ with $n_k > m_k > k$ such that

$$\sigma(x_{n_k}, x_{m_k}) \geq \varepsilon. \tag{2.7}$$

Further, corresponding to $m_k$, we can choose $n_k$ in such a way that it is the smallest integer with $n_k > m_k$ satisfying (2.7). Then

$$\sigma(x_{n_k-1}, x_{m_k}) < \varepsilon. \tag{2.8}$$

Using (2.7), (2.8) and the triangle inequality, we have

$$\varepsilon \leq \sigma(x_{n_k}, x_{m_k}) \leq \sigma(x_{n_k}, x_{n_k-1}) + \sigma(x_{n_k-1}, x_{m_k}) < \sigma(x_{n_k}, x_{n_k-1}) + \varepsilon.$$

Letting $k \to \infty$ and using (2.6), we obtain

$$\lim_{k\to\infty} \sigma(x_{n_k}, x_{m_k}) = \varepsilon. \tag{2.9}$$

Again, the triangle inequality gives us

$$\sigma(x_{n_k-1}, x_{m_k}) \le \sigma(x_{n_k-1}, x_{n_k}) + \sigma(x_{n_k}, x_{m_k}),$$
$$\sigma(x_{n_k}, x_{m_k}) \le \sigma(x_{n_k}, x_{n_k-1}) + \sigma(x_{n_k-1}, x_{m_k}).$$

Then we have

$$\bigl|\sigma(x_{n_k-1}, x_{m_k}) - \sigma(x_{n_k}, x_{m_k})\bigr| \le \sigma(x_{n_k}, x_{n_k-1}).$$

Letting $k \to \infty$ in the above inequality and using (2.6) and (2.9), we get

$$\lim_{k\to\infty} \sigma(x_{n_k-1}, x_{m_k}) = \varepsilon. \tag{2.10}$$

Similarly, we can show that

$$\begin{aligned}\lim_{k\to\infty} \sigma(x_{n_k}, x_{m_k-1}) &= \lim_{k\to\infty} \sigma(x_{n_k-1}, x_{m_k-1}) \\ &= \lim_{k\to\infty} \sigma(x_{n_k}, x_{m_k+1}) \\ &= \lim_{k\to\infty} \sigma(x_{n_k+1}, x_{m_k}) = \varepsilon. \end{aligned} \tag{2.11}$$

As

$$\begin{aligned} M(x_{n_k-1}, x_{m_k-1}) = \max\bigl\{&\sigma(x_{n_k-1}, x_{m_k-1}), \sigma(x_{n_k-1}, x_{n_k}), \\ &\sigma(x_{m_k-1}, x_{m_k}), \sigma(x_{n_k-1}, x_{n_k-1}), \sigma(x_{m_k-1}, x_{m_k-1}), \\ &\bigl[\sigma(x_{n_k-1}, x_{m_k}) + \sigma(x_{n_k}, x_{m_k-1})\bigr]/2\bigr\} \end{aligned}$$

using (2.6) and (2.9)-(2.11), we have

$$\lim_{k\to\infty} M(x_{n_k-1}, x_{m_k-1}) = \max\{\varepsilon, 0, 0, 0, 0, \varepsilon\} = \varepsilon. \tag{2.12}$$

As $n_k > m_k$ and $x_{n_k-1}$ and $x_{m_k-1}$ are comparable, setting $x = x_{n_k-1}$ and $y = x_{m_k-1}$ in (2.1), we obtain

$$\begin{aligned}\psi\bigl(\sigma(x_{n_k}, x_{m_k})\bigr) &= \psi\bigl(\sigma(Fx_{n_k-1}, Fx_{m_k-1})\bigr) \\ &\le \psi\bigl(M(x_{n_k-1}, x_{m_k-1})\bigr) - \phi\bigl(M(x_{n_k-1}, x_{m_k-1})\bigr).\end{aligned}$$

Letting $k \to \infty$ in the above inequality and using (2.9) and (2.12), we get

$$\psi(\varepsilon) \le \psi(\varepsilon) - \phi(\varepsilon),$$

which is a contradiction as $\varepsilon > 0$. Hence $\{x_n\}$ is a $\sigma$-Cauchy sequence. By the completeness of $X$, there exists $z \in X$ such that $\lim_{n\to\infty} x_n = z$, that is,

$$\lim_{n\to\infty} \sigma(x_n, z) = \sigma(z,z) = \lim_{m,n\to\infty} \sigma(x_m, x_n) = 0. \tag{2.13}$$

Moreover, the continuity of $F$ implies that

$$\lim_{n\to\infty} \sigma(x_{n+1}, z) = \lim_{n\to\infty} \sigma(Fx_n, z) = \sigma(Fz, z) = 0$$

and this proves that $z$ is a fixed point. □

Notice that the continuity of $F$ in Theorem 1 is not necessary and can be dropped.

**Theorem 2** *Under the same hypotheses of Theorem* 1 *and without assuming the continuity of $F$, assume that whenever $\{x_n\}$ is a nondecreasing sequence in $X$ such that $x_n \to x \in X$ implies $x_n \preceq x$ for all $n \in \mathbb{N}$, then $F$ has a fixed point in $X$.*

*Proof* Following similar arguments to those given in Theorem 1, we construct a nondecreasing sequence $\{x_n\}$ in $X$ such that $x_n \to z$ for some $z \in X$. Using the assumption of $X$, we have $x_n \preceq z$ for every $n \in \mathbb{N}$. Now, we show that $Fz = z$. By (2.1), we have

$$\begin{aligned}\psi\big(\sigma(Fz, x_{n+1})\big) &= \psi\big(\sigma(Fz, Fx_n)\big) \\ &\le \psi\big(M(z, x_n)\big) - \phi\big(M(z, x_n)\big),\end{aligned} \tag{2.14}$$

where

$$\begin{aligned}\sigma(Fz,z) \le M(z,x_n) &= \max\big\{\sigma(z,x_n), \sigma(Fz,z), \sigma(x_n,x_{n+1}), \\ &\qquad \sigma(z,z), \sigma(x_n,x_n), \big[\sigma(z,x_{n+1}) + \sigma(Fz,x_n)\big]/2\big\} \\ &\le \max\big\{\sigma(z,x_n), \sigma(Fz,z), \sigma(x_n,x_{n+1}), \\ &\qquad \sigma(z,z), \sigma(x_n,x_n), \big[\sigma(z,x_{n+1}) + \sigma(Fz,z) + \sigma(z,x_n)\big]/2\big\}.\end{aligned}$$

Taking limit as $n \to \infty$, by (2.13), we obtain

$$\lim_{n\to\infty} M(z, x_n) = \sigma(Fz, z).$$

Therefore, letting $n \to \infty$ in (2.14), we get

$$\psi\big(\sigma(Fz,z)\big) \le \psi\big(\sigma(Fz,z)\big) - \phi\big(\sigma(Fz,z)\big),$$

which is a contradiction unless $\sigma(Fz, z) = 0$. Thus $Fz = z$. □

Next theorem gives a sufficient condition for the uniqueness of the fixed point.

**Theorem 3** *Let all the conditions of Theorem* 1 (*resp. Theorem* 2) *be fulfilled and let the following condition be satisfied: For arbitrary two points $x, y \in X$, there exists $z \in X$ which is comparable with both $x$ and $y$. Then the fixed point of $F$ is unique.*

*Proof* Suppose that there exist $z, x \in X$ which are fixed points. We distinguish two cases.

Case 1. If $x$ is comparable to $z$, then $F^n x = x$ is comparable to $F^n z = z$ for $n = 0, 1, 2, \ldots$ and

$$\begin{aligned}\psi\bigl(\sigma(z,x)\bigr) &= \psi\bigl(\sigma\bigl(F^n z, F^n x\bigr)\bigr)\\ &\le \psi\bigl(M\bigl(F^{n-1}z, F^{n-1}x\bigr)\bigr) - \phi\bigl(M\bigl(F^{n-1}z, F^{n-1}x\bigr)\bigr)\\ &\le \psi\bigl(M(z,x)\bigr) - \phi\bigl(M(z,x)\bigr),\end{aligned} \tag{2.15}$$

where

$$\begin{aligned}M(z,x) &= \max\bigl\{\sigma(z,x), \sigma(z,Fz), \sigma(x,Fx), \sigma(z,z), \sigma(x,x), \bigl[\sigma(z,Fx) + \sigma(Fz,x)\bigr]/2\bigr\}\\ &= \max\bigl\{\sigma(z,x), \sigma(z,z), \sigma(x,x), \bigl[\sigma(z,x) + \sigma(z,x)\bigr]/2\bigr\} = \sigma(z,x).\end{aligned} \tag{2.16}$$

Using (2.15) and (2.16), we have

$$\psi\bigl(\sigma(z,x)\bigr) \le \psi\bigl(\sigma(z,x)\bigr) - \phi\bigl(\sigma(z,x)\bigr),$$

which is a contradiction unless $\sigma(z,x) = 0$. This implies that $z = x$.

Case 2. If $x$ is not comparable to $z$, then there exists $y \in X$ comparable to $x$ and $z$. The monotonicity of $F$ implies that $F^n y$ is comparable to $F^n x = x$ and $F^n z = z$, for $n = 0, 1, 2, \ldots$. Moreover,

$$\begin{aligned}\psi\bigl(\sigma\bigl(z, F^n y\bigr)\bigr) &= \psi\bigl(\sigma\bigl(F^n z, F^n y\bigr)\bigr)\\ &\le \psi\bigl(M\bigl(F^{n-1}z, F^{n-1}y\bigr)\bigr) - \phi\bigl(M\bigl(F^{n-1}z, F^{n-1}y\bigr)\bigr),\end{aligned} \tag{2.17}$$

where

$$\begin{aligned}M\bigl(F^{n-1}z, F^{n-1}y\bigr) &= \max\bigl\{\sigma\bigl(F^{n-1}z, F^{n-1}y\bigr), \sigma\bigl(F^{n-1}z, F^n z\bigr), \sigma\bigl(F^{n-1}y, F^n y\bigr),\\ &\qquad \sigma\bigl(F^{n-1}z, F^{n-1}z\bigr), \sigma\bigl(F^{n-1}y, F^{n-1}y\bigr),\\ &\qquad \bigl[\sigma\bigl(F^{n-1}z, F^n y\bigr) + \sigma\bigl(F^n z, F^{n-1}y\bigr)\bigr]/2\bigr\}\\ &= \max\bigl\{\sigma\bigl(z, F^{n-1}y\bigr), \sigma(z,z), \sigma\bigl(F^{n-1}y, F^n y\bigr),\\ &\qquad \sigma\bigl(F^{n-1}y, F^{n-1}y\bigr), \bigl[\sigma\bigl(z, F^n y\bigr) + \sigma\bigl(z, F^{n-1}y\bigr)\bigr]/2\bigr\}\\ &\le \max\bigl\{\sigma\bigl(z, F^{n-1}y\bigr), \sigma\bigl(z, F^n y\bigr)\bigr\}\end{aligned} \tag{2.18}$$

for $n$ sufficiently large, because $\sigma(F^{n-1}y, F^{n-1}y) \to 0$ and $\sigma(F^{n-1}y, F^n y) \to 0$ when $n \to \infty$. Similarly as in the proof of Theorem 1, it can be shown that $\sigma(z, F^n y) \le M(z, F^{n-1}y) \le \sigma(z, F^{n-1}y)$. It follows that the sequence $\{\sigma(z, F^n y)\}$ is nonnegative decreasing and, consequently, there exists $\alpha \ge 0$ such that

$$\lim_{n\to\infty} \sigma\bigl(z, F^n y\bigr) = \lim_{n\to\infty} M\bigl(z, F^{n-1}y\bigr) = \alpha.$$

We suppose that $\alpha > 0$. Then letting $n \to \infty$ in (2.17), we have

$$\psi(\alpha) \le \psi(\alpha) - \phi(\alpha)$$

which is a contradiction. Hence $\alpha = 0$. Similarly, it can be proved that

$$\lim_{n\to\infty} \sigma\left(x, F^n y\right) = 0.$$

Now, passing to the limit in $\sigma(x,z) \le \sigma(x,F^n y) + \sigma(F^n y, z)$, it follows that $\sigma(x,z) = 0$, so $x = z$, and the uniqueness of the fixed point is proved. □

Now, we present an example to support the useability of our results.

**Example 2** Let $X = \{0,1,2\}$ and a partial order be defined as $x \preceq y$ whenever $y \le x$, and define $\sigma : X \times X \to \mathbb{R}^+$ as follows:

$$\sigma(0,0) = 1, \qquad \sigma(1,1) = 3, \qquad \sigma(2,2) = 0, \qquad \sigma(1,0) = \sigma(0,1) = 9,$$
$$\sigma(2,0) = \sigma(0,2) = 4, \qquad \sigma(2,1) = \sigma(1,2) = 5.$$

Then $(X, \preceq, \sigma)$ is a complete partially ordered metric-like space.

Let $F : X \to X$ be defined by

$$F0 = 1, \qquad F1 = 2 \quad \text{and} \quad F2 = 2.$$

Define $\psi, \phi : [0,\infty) \to [0,\infty)$ by $\psi(t) = t$ and $\phi(t) = \frac{t}{2}$. We next verify that the function $F$ satisfies the inequality (2.1). For that, given $x, y \in X$ with $x \preceq y$, so $y \le x$. Then we have the following cases.

Case 1. If $x = 1$, $y = 0$, then

$$\sigma(F1, F0) = \sigma(2,1) = 5$$

and

$$\begin{aligned} M(1,0) &= \max\bigl\{\sigma(1,0), \sigma(1,F1), \sigma(0,F0), \\ &\qquad \sigma(1,1), \sigma(0,0), \bigl[\sigma(1,F0) + \sigma(F1,0)\bigr]/2\bigr\} \\ &= \max\left\{9, 5, 3, 1, \frac{3+4}{2}\right\} = 9. \end{aligned}$$

As $\psi(\sigma(F1,F0)) = 5 < 9 - \frac{9}{2} = \psi(M(1,0)) - \phi(M(1,0))$, the inequality (2.1) is satisfied in this case.

Case 2. If $x = 2$, $y = 0$, then

$$\sigma(F2, F0) = \sigma(2,1) = 5$$

and

$$\begin{aligned} M(2,0) &= \max\bigl\{\sigma(2,0), \sigma(2,F2), \sigma(0,F0), \\ &\qquad \sigma(2,2), \sigma(0,0), \bigl[\sigma(2,F0) + \sigma(F2,0)\bigr]/2\bigr\} \\ &= \max\left\{4, 0, 9, 1, \frac{5+4}{2}\right\} = 9. \end{aligned}$$

As $\psi(\sigma(F2,F0)) = 5 < 9 - \frac{9}{2} = \psi(M(2,0)) - \phi(M(2,0))$, the inequality (2.1) is satisfied in this case.

Case 3. If $x = 2$, $y = 1$, then as $\sigma(F2,F1) = 0$ and $M(2,1) = 5$, the inequality (2.1) is satisfied in this case.

Case 4. If $x = 0$, $y = 0$, then as $\sigma(F0,F0) = 3$ and $M(0,0) = 9$, the inequality (2.1) is satisfied in this case.

Case 5. If $x = 1$, $y = 1$, then as $\sigma(F1,F1) = 0$ and $M(1,1) = 5$, the inequality (2.1) is satisfied in this case.

Case 6. If $x = 2$, $y = 2$, then as $\sigma(F2,F2) = 0$ and $M(2,2) = 0$, the inequality (2.1) is satisfied in this case.

So, $F$, $\psi$ and $\phi$ satisfy all the hypotheses of Theorem 1. Therefore $F$ has a unique fixed point. Here 2 is the unique fixed point of $F$.

If we take $\psi(t) = t$ in Theorem 1, we have the following corollary.

**Corollary 1** *Let $(X,\preceq,\sigma)$ be a complete partially ordered metric-like space. Let $F : X \to X$ be a nondecreasing mapping such that for all comparable $x, y \in X$ with*

$$\sigma(Fx,Fy) \le M(x,y) - \phi\bigl(M(x,y)\bigr),$$

*where $M$ is given by*

$$M(x,y) = \max\bigl\{\sigma(x,y),\sigma(x,Fx),\sigma(y,Fy),\sigma(x,x),\sigma(y,y),\bigl[\sigma(x,Fy)+\sigma(Fx,y)\bigr]/2\bigr\},$$

*$\phi : [0,\infty) \to [0,\infty)$ is lower semi-continuous, and $\phi(t) = 0$ if and only if $t = 0$. If there exists $x_0 \in X$ with $x_0 \preceq Fx_0$ and one of the following two conditions is satisfied:*

(a) *$F$ is continuous in $(X,\sigma)$;*

(b) *$\{x_n\}$ is a nondecreasing sequence in $X$ such that $x_n \to x \in X$ implies $x_n \preceq x$ for all $n \in \mathbb{N}$.*

*Then $F$ has a fixed point. Moreover, if the following condition is satisfied: For arbitrary two points $x, y \in X$, there exists $z \in X$ which is comparable with both $x$ and $y$, then the fixed point of $F$ is unique.*

If we take $\phi(t) = (1-k)t$ for $k \in [0,1)$ in Corollary 1, we have the following corollary.

**Corollary 2** *Let $(X,\preceq,\sigma)$ be a complete partially ordered metric-like space. Let $F : X \to X$ be a nondecreasing mapping such that for all comparable $x, y \in X$*

$$\sigma(Fx,Fy) \le kM(x,y),$$

*where $M$ is given by*

$$M(x,y) = \max\bigl\{\sigma(x,y),\sigma(x,Fx),\sigma(y,Fy),\sigma(x,x),\sigma(y,y),\bigl[\sigma(x,Fy)+\sigma(Fx,y)\bigr]/2\bigr\},$$

*and $k \in [0,1)$. If there exists $x_0 \in X$ with $x_0 \preceq Fx_0$ and one of the following two conditions is satisfied:*

(a) *$F$ is continuous in $(X,\sigma)$;*

(b) $\{x_n\}$ *is a nondecreasing sequence in* $X$ *such that* $x_n \to x \in X$ *implies* $x_n \preceq x$ *for all* $n \in \mathbb{N}$.

*Then* $F$ *has a fixed point. Moreover, if the following condition is satisfied: For arbitrary two points* $x, y \in X$, *there exists* $z \in X$ *which is comparable with both* $x$ *and* $y$, *then the fixed point of* $F$ *is unique.*

The following corollary improves Theorem 2.7 in [33].

**Corollary 3** *Let* $(X, \preceq, \sigma)$ *be a complete partially ordered metric-like space. Let* $F : X \to X$ *be a nondecreasing mapping such that for all comparable* $x, y \in X$

$$\sigma(Fx, Fy) \leq \sigma(x, y) - \phi\big(\sigma(x, y)\big),$$

*where* $\phi : [0,\infty) \to [0,\infty)$ *is a lower semi-continuous, and* $\phi(t) = 0$ *if and only if* $t = 0$. *If there exists* $x_0 \in X$ *with* $x_0 \preceq Fx_0$ *and one of the following two conditions is satisfied:*

(a) $F$ *is continuous in* $(X, \sigma)$;

(b) $\{x_n\}$ *is a nondecreasing sequence in* $X$ *such that* $x_n \to x \in X$ *implies* $x_n \preceq x$ *for all* $n \in \mathbb{N}$.

*Then* $F$ *has a fixed point. Moreover, if the following condition is satisfied: For arbitrary two points* $x, y \in X$, *there exists* $z \in X$ *which is comparable with both* $x$ *and* $y$, *then the fixed point of* $F$ *is unique.*

The following corollary improves Theorem 2.1 in [10].

**Corollary 4** *Let* $(X, \preceq, p)$ *be a complete partially ordered partial metric space. Let* $F : X \to X$ *be a nondecreasing mapping such that for all comparable* $x, y \in X$

$$\psi\big(p(Fx, Fy)\big) \leq \psi\big(M(x, y)\big) - \phi\big(M(x, y)\big),$$

*where* $M$ *is given by*

$$M(x, y) = \max\big\{p(x, y), p(x, Fx), p(y, Fy), p(x, x), p(y, y), \big[p(x, Fy) + p(Fx, y)\big]/2\big\},$$

$\psi, \phi : [0,\infty) \to [0,\infty)$, $\psi$ *is continuous monotone nondecreasing,* $\phi$ *is lower semi-continuous, and* $\psi(t) = \phi(t) = 0$ *if and only if* $t = 0$. *If there exists* $x_0 \in X$ *with* $x_0 \preceq Fx_0$ *and one of the following two conditions is satisfied:*

(a) $F$ *is continuous in* $(X, p)$;

(b) $\{x_n\}$ *is a nondecreasing sequence in* $X$ *such that* $x_n \to x \in X$ *implies* $x_n \preceq x$ *for all* $n \in \mathbb{N}$.

*Then* $F$ *has a fixed point. Moreover, the set of fixed points of* $F$ *is well ordered if and only if* $F$ *has one and only one fixed point.*

## 3 Applications

Denote by $\Lambda$ the set of functions $\alpha : [0, +\infty) \to [0, +\infty)$ satisfying the following hypotheses:

(h1) $\alpha$ is a Lebesgue-integrable mapping on each compact subset of $[0, +\infty)$;

(h2) For every $\varepsilon > 0$, we have

$$\int_0^{\varepsilon} \alpha(s)\,ds > 0.$$

We have the following results.

**Corollary 5** *Let $(X, \preceq, \sigma)$ be a complete partially ordered metric-like space. Let $F : X \to X$ be a continuous and nondecreasing mapping such that for all comparable $x, y \in X$*

$$\int_0^{\sigma(Fx,Fy)} \alpha_1(s)\,ds \leq \int_0^{M(x,y)} \alpha_1(s)\,ds - \int_0^{M(x,y)} \alpha_2(s)\,ds,$$

*where $\alpha_1, \alpha_2 \in \Lambda$. If there exists $x_0 \in X$ with $x_0 \preceq Fx_0$, then $F$ has a fixed point.*

*Proof* Follows from Theorem 1 by taking $\psi(t) = \int_0^t \alpha_1(s)\,ds$ and $\phi(t) = \int_0^t \alpha_2(s)\,ds$. □

**Corollary 6** *Under the same hypotheses of Corollary 5 and without assuming the continuity of $F$, assume that whenever $\{x_n\}$ is a nondecreasing sequence in $X$ such that $x_n \to x \in X$ implies $x_n \preceq x$ for all $n \in \mathbb{N}$, then $F$ has a fixed point in $X$.*

*Proof* Follows from Theorem 2 by taking $\psi(t) = \int_0^t \alpha_1(s)\,ds$ and $\phi(t) = \int_0^t \alpha_2(s)\,ds$. □

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
Both authors contributed equally and significantly in writing this paper. Both authors read and approved the final manuscript.

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**Acknowledgements**
Dedicated to Professor Hari M Srivastava.

Received: 25 November 2012 Accepted: 21 February 2013 Published: 12 March 2013

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doi:10.1186/1687-1812-2013-51
**Cite this article as:** Işık and Türkoğlu: **Fixed point theorems for weakly contractive mappings in partially ordered metric-like spaces.** *Fixed Point Theory and Applications* 2013 **2013**:51.