Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 245872, 15 pages
doi:10.1155/2012/245872

Research Article

Fixed Point Theorems in Ordered Banach Spaces and Applications to Nonlinear Integral Equations

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Received 5 March 2012; Accepted 20 June 2012

Academic Editor: Simeon Reich

We present some new common fixed point theorems for a pair of nonlinear mappings defined on an ordered Banach space. Our results extend several earlier works. An application is given to show the usefulness and the applicability of the obtained results.

1. Introduction

The problem of existence of common fixed points to a pair of nonlinear mappings is now a classical theme. The applications to differential and integral equations made it more interesting. A considerable importance has been attached to common fixed point theorems in ordered sets [1, 2]. In a recent paper, Dhage [3] proved some common fixed point theorems for pairs of condensing mappings in an ordered Banach space. More recently, Hussain et al. [4] extended the results of Dhage to 1-set contractive mappings. In the present paper we pursue the investigations started in the aforementioned papers and prove some new common fixed point theorems under weaker assumptions. Also we present some common fixed point results using the weak topology of a Banach space. The use of the concepts of ws-compactness and ww-compactness increases the usefulness of our results in many practical situations especially when we work in nonreflexive Banach spaces. We will illustrate this fact by proving the existence of nonnegative integrable solutions for an implicit integral equation.

For the remainder of this section we gather some notations and preliminary facts. Let X be a real Banach space, ordered by a cone K. A cone K is a closed convex subset of X with $\lambda K \subseteq K$ $(\lambda \geq 0)$, and $K \cap (-K) = \{0\}$. As usual $x \leq y \Leftrightarrow y - x \in K$.

Definition 1.1. Let X be an ordered Banach space with order $\leq$. A mapping $T : D(T) \subseteq X \to X$ is said to be isotone increasing if for all $x, y \in D(T)$, we have that $x \leq y$ implies $Tx \leq Ty$.

Definition 1.2. Let M be a nonempty subset of an ordered Banach space X with order $\leq$. Two mappings $S, T : M \to M$ are said to be weakly isotone increasing if $Sx \leq TSx$ and $Tx \leq STx$ hold for all $x \in M$. Similarly, we say that S and T are weakly isotone decreasing if $Tx \geq STx$ and $Sx \geq TSx$ hold for all $x \in M$. The mappings S and T are said to be weakly isotone if they are either weakly isotone increasing or weakly isotone decreasing.

In our considerations the following definition will play an important role. Let $\mathcal{B}(X)$ denote the collection of all nonempty bounded subsets of X and $\mathcal{W}(X)$ the subset of $\mathcal{B}(X)$ consisting of all weakly compact subsets of X. Also, let B_r denote the closed ball centered at 0 with radius r.

Definition 1.3 (see [5]). A function $\psi : \mathcal{B}(X) \to \mathbb{R}_+$ is said to be a measure of weak noncompactness if it satisfies the following conditions.

(1) The family $\ker(\psi) = \{M \in \mathcal{B}(X) : \psi(M) = 0\}$ is nonempty and $\ker(\psi)$ is contained in the set of relatively weakly compact sets of X.

(2) $M_1 \subseteq M_2 \Rightarrow \psi(M_1) \leq \psi(M_2)$.

(3) $\psi(\overline{\text{co}}(M)) = \psi(M)$, where $\overline{\text{co}}(M)$ is the closed convex hull of M.

(4) $\psi(\lambda M_1 + (1-\lambda)M_2) \leq \lambda\psi(M_1) + (1-\lambda)\psi(M_2)$ for $\lambda \in [0,1]$.

(5) If $(M_n)_{n\geq 1}$ is a sequence of nonempty weakly closed subsets of X with M_1 bounded and $M_1 \supseteq M_2 \supseteq \cdots \supseteq M_n \supseteq \cdots$ such that $\lim_{n\to\infty}\psi(M_n) = 0$, then $M_\infty := \bigcap_{n=1}^{\infty} M_n$ is nonempty.

The family $\ker\psi$ described in (1) is said to be the kernel of the measure of weak noncompactness ψ. Note that the intersection set M_∞ from (5) belongs to $\ker\psi$ since $\psi(M_\infty) \leq \psi(M_n)$ for every n and $\lim_{n\to\infty}\psi(M_n) = 0$. Also, it can be easily verified that the measure ψ satisfies

$$\psi\left(\overline{M^w}\right) = \psi(M), \tag{1.1}$$

where $\overline{M^w}$ is the weak closure of M.

A measure of weak noncompactness ψ is said to be *regular* if

$$\psi(M) = 0 \quad \text{iff } M \text{ is relatively weakly compact}, \tag{1.2}$$

subadditive if

$$\psi(M_1 + M_2) \leq \psi(M_1) + \psi(M_2), \tag{1.3}$$

homogeneous if

$$\psi(\lambda M) = |\lambda|\psi(M), \quad \lambda \in \mathbb{R}, \tag{1.4}$$

and *set additive* (or *have the maximum property*) if

$$\psi(M_1 \cup M_2) = \max\big(\psi(M_1), \psi(M_2)\big). \tag{1.5}$$

The first important example of a measure of weak noncompactness has been defined by De Blasi [6] as follows:

$$w(M) = \inf\{r > 0 : \text{there exists } W \in \mathcal{W}(X) \text{ with } M \subseteq W + B_r\}, \tag{1.6}$$

for each $M \in \mathcal{B}(X)$.

Notice that $w(\cdot)$ is regular, homogeneous, subadditive, and set additive (see [6]).

By a measure of noncompactness on a Banach space X we mean a map $\psi : \mathcal{B}(X) \to \mathbb{R}_+$ which satisfies conditions (1)–(5) in Definition 1.3 relative to the strong topology instead of the weak topology. The concept of a measure of noncompactness was initiated by the fundamental papers of Kuratowski [7] and Darbo [8]. Measures of noncompactness are very useful tools in nonlinear analysis [9] especially the so-called Kuratowski measure of noncompactness [7] and Hausdorff (or ball) measure of noncompactness [10].

Definition 1.4. Let X be a Banach space and ψ a measure of (weak) noncompactness on X. Let $A : D(A) \subseteq X \to X$ be a mapping. If $A(D(A))$ is bounded and for every nonempty bounded subset M of $D(A)$ with $\psi(M) > 0$, we have, $\psi(A(M))) < \psi(M)$, then A is called ψ-condensing. If there exists k, $0 \le k \le 1$, such that $A(D(A))$ is bounded and for each nonempty bounded subset M of $D(A)$, we have $\psi(A(M)) \le k\psi(M)$, then A is called k-ψ-contractive.

Remark 1.5. Clearly, every k-ψ-contractive map with $k < 1$ is ψ-condensing and every ψ-condensing map is 1-ψ-contractive.

Definition 1.6 (see [11]). A map $A : D(A) \to X$ is said to be ws-compact if it is continuous and for any weakly convergent sequence $(x_n)_{n\in\mathbb{N}}$ in $D(A)$ the sequence $(Ax_n)_{n\in\mathbb{N}}$ has a strongly convergent subsequence in X.

Remark 1.7. The concept of ws-compact mappings arises naturally in the study of both integral and partial differential equations (see [11–19]).

Definition 1.8. A map $A : D(A) \to X$ is said to be ww-compact if it is continuous and for any weakly convergent sequence $(x_n)_{n\in\mathbb{N}}$ in $D(A)$ the sequence $(Ax_n)_{n\in\mathbb{N}}$ has a weakly convergent subsequence in X.

Definition 1.9. Let X be a Banach space. A mapping $T : D(T) \subseteq X \to X$ is called a nonlinear contraction if there exists a continuous and nondecreasing function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\|Tx - Ty\| \le \varphi(\|x - y\|) \tag{1.7}$$

for all $x, y \in D(T)$, where $\varphi(r) < r$ for $r > 0$.

Remark 1.10. It is easy to prove that a nonlinear contraction mapping is a nonlinear set-contraction with respect to the Kuratowskii measure of noncompactness. We will prove that the same property holds for the De Blasi measure of weak noncompactness provided that the nonlinear contraction mapping is ww-compact.

Lemma 1.11. *Let T be a ww-compact nonlinear contraction mapping on a Banach X. Then for each bounded subset M of X one has*

$$w(TM) \le \phi(w(M)). \tag{1.8}$$

Here, w is the De Blasi measure of weak noncompactness.

Proof. Let M be a bounded subset of X and $r > w(M)$. There exist $0 \le r_0 < r$ and a weakly compact subset W of X such that $M \subseteq W + B_{r_0}$. Since T is a ϕ-nonlinear contraction, then

$$TM \subseteq TW + B_{\phi(r_0)} \subseteq \overline{TW}^{w} + B_{\phi(r_0)}. \tag{1.9}$$

Moreover, since T is ww-compact, then $\overline{TW}^{w}$ is weakly compact. Accordingly,

$$w(TM) \le \phi(r_0) \le \phi(r). \tag{1.10}$$

Letting $r \to w(M)$ and using the continuity of ϕ we deduce that

$$w(TM) \le \phi(w(M)). \tag{1.11}$$

□

The following theorem is a sharpening of [3, Theorem 2.1] and [4, Theorem 3.1].

Theorem 1.12. *Let X be an ordered Banach space and ψ a set additive measure of noncompactness on X. Let M be a nonempty closed convex subset of X and $S, T : M \to M$ be two continuous mappings satisfying the following:*

(i) *T is 1-ψ-contractive,*

(ii) *S is ψ-condensing,*

(iii) *S and T are weakly isotone.*

Then T and S have a common fixed point.

Proof. Let $x \in M$ be fixed. Consider the sequence $\{x_n\}$ defined by

$$x_0 = x, \quad x_{2n+1} = Sx_{2n}, \quad x_{2n+2} = Tx_{2n}, \quad n = 0, 1, 2, \dots. \tag{1.12}$$

Suppose first that S and T are weakly isotone increasing on M. Then from (1.12) it follows that

$$x_1 \le x_2 \le \cdots \le x_n \le \cdots. \tag{1.13}$$

Set $A = \{x_1, x_2, \dots, x_n, \dots\}$, $A_1 = \{x_2, x_4, \dots, x_{2n}, \dots\}$, and $A_2 = \{x_1, x_3, \dots, x_{2n+1}, \dots\}$. Clearly

$$A = A_1 \cup A_2, \quad A_2 = S(A_1) \cup \{x_1\}, \quad A_1 = T(A_2). \tag{1.14}$$

We show that A_1 is relatively compact. Suppose the contrary, then $\psi(A_1) > 0$. Since T is 1-ψ-contractive and S is ψ-condensing, it follows from (1.14) that

$$\begin{gathered} \psi(A_2) = \psi(S(A_1) \cup \{x_1\}) = \psi(S(A_1)) < \psi(A_1), \\ \psi(A_1) = \psi(T(A_2)) \leq \psi(A_2). \end{gathered} \tag{1.15}$$

This is impossible. Thus A_1 is relatively compact. The continuity of S implies that $A_2 = S(A_1) \cup \{x_1\}$ is relatively compact. Consequently A is relatively compact. Since $\{x_n\}$ is monotone increasing in A, then it is convergent. Let x^* be its limit. Using the continuity of S and T we get

$$\begin{gathered} x^* = \lim_{n\to\infty} x_{2n+1} = \lim_{n\to\infty} S(x_{2n}) = S\left(\lim_{n\to\infty} x_{2n}\right) = S(x^*), \\ x^* = \lim_{n\to\infty} x_{2n+2} = \lim_{n\to\infty} T(x_{2n+1}) = T\left(\lim_{n\to\infty} x_{2n+1}\right) = T(x^*). \end{gathered} \tag{1.16}$$

To complete the proof we consider the case where S and T are weakly isotone decreasing on M. In this case, the sequence $\{x_n\}$ is monotone decreasing and then converges to a common fixed point of S and T. □

Remark 1.13. In [3, Theorem 2.1] T is assumed to be set condensing, while S is assumed to be affine, X is assumed to be reflexive and $I - T$ is demiclosed in [4, Theorem 3.1].

Remark 1.14. As an application of Theorem 1.12, the modified versions of Theorems 3.6–3.12 in [4] can be proved similarly.

As easy consequences of Theorem 1.12 we obtain the following results.

Corollary 1.15. *Let M be a nonempty closed convex subset of a Banach space X and let $S, T : M \to M$ be two completely continuous mappings. Also assume T and S are weakly isotone mappings. Then T and S have a common fixed point.*

Corollary 1.16. *Let M be a nonempty closed convex subset of a Banach space X and $S, T : X \to X$ be two continuous mappings. Assume T is nonexpansive, that is, $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in X$, and S is a nonlinear contraction. Also assume that S and T are weakly isotone mappings, $T(M) \subseteq M$ and $S(M) \subseteq M$. Then T and S have a common fixed point in M.*

Corollary 1.17. *Let M be a nonempty closed convex subset of a Banach space X and ψ a set additive measure of noncompactness on X. Let $S, T : X \to X$ be two continuous mappings satisfying the following:*

(i) *$T(M) \subseteq M$ and $S(M) \subseteq M$,*

(ii) *T is a nonexpansive mapping,*

(iii) *S is ψ-condensing,*

(iv) *T and S are weakly isotone.*

Then S and T have a common fixed point in M.

Note that in Corollary 1.17 we do not have uniqueness. However, if we assume T is shrinking, we obtain uniqueness. Recall that $T : D(T) \subseteq X \to X$ is shrinking if

$$\|Tx - Ty\| < \|x - y\| \quad \text{whenever } x, y \in D(T) \text{ with } x \neq y. \tag{1.17}$$

Corollary 1.18. *Let M be a nonempty closed convex subset of a Banach space X and ψ a set additive measure of noncompactness on X. Let $S, T : X \to X$ be two continuous mappings satisfying the following:*

(i) *$T(M) \subseteq M$ and $S(M) \subseteq M$,*

(ii) *T is a shrinking mapping,*

(iii) *S is ψ-condensing,*

(iv) *T and S are weakly isotone.*

Then S and T have a unique common fixed point in M.

Proof. From Corollary 1.17 it follows that there exists $x \in M$ such that $Sx = Tx = x$. Now from (1.17) we infer that T cannot have two different fixed points. This implies that S and T have a unique common fixed point. □

Corollary 1.19. *Let M be a nonempty closed convex subset of a Banach space X and $S, T : X \to X$ be two mappings satisfying the following:*

(i) *$T(M) \subseteq M$ and $S(M) \subseteq M$,*

(ii) *T is a shrinking mapping,*

(iii) *S is a nonlinear contraction,*

(iv) *T and S are weakly isotone.*

Then S and T have a unique common fixed point in M which is the unique fixed point of S.

Theorem 1.20. *Let X be an ordered Banach space and ψ a set additive measure of weak noncompactness on X. Let M be a nonempty closed convex subset of X and $S, T : M \to M$ be two sequentially weakly continuous mappings satisfying the following:*

(i) *T is 1-ψ-contractive,*

(ii) *S is ψ-condensing,*

(iii) *S and T are weakly isotone.*

Then T and S have a common fixed point in M.

Proof. Let $x \in M$ be fixed. Consider the sequence $\{x_n\}$ defined by

$$x_0 = x, \quad x_{2n+1} = Sx_{2n}, \quad x_{2n+2} = Tx_{2n}, \quad n = 0, 1, 2, \ldots. \tag{1.18}$$

Suppose first that S and T are weakly isotone increasing on M. Then from (1.18) it follows that

$$x_1 \le x_2 \le \cdots \le x_n \le \cdots . \tag{1.19}$$

As in the proof of Theorem 1.12 we set $A = \{x_1, x_2, \ldots, x_n, \ldots\}$, $A_1 = \{x_2, x_4, \ldots, x_{2n}, \ldots\}$, and $A_2 = \{x_1, x_3, \ldots, x_{2n+1}, \ldots\}$. Clearly

$$A = A_1 \cup A_2, \quad A_2 = S(A_1) \cup \{x_1\}, \quad A_1 = T(A_2). \tag{1.20}$$

We show that A_1 is relatively weakly compact. Suppose the contrary, then $\psi(A_1) > 0$. Since T is 1-ψ-contractive and S is ψ-condensing, it follows from (1.20) that

$$\begin{gathered} \psi(A_2) = \psi(S(A_1) \cup \{x_1\}) = \psi(S(A_1)) < \psi(A_1), \\ \psi(A_1) = \psi(T(A_2)) \le \psi(A_2). \end{gathered} \tag{1.21}$$

This is impossible. Thus A_1 is relatively weakly compact. The weak sequential continuity of S implies that $A_2 = S(A_1) \cup \{x_1\}$ is relatively weakly compact. The same reasoning as in the proof of Theorem 1.12 gives the desired result. □

As easy consequences of Theorem 1.20 we obtain the following results.

Corollary 1.21. *Let M be a nonempty closed convex subset of a Banach space X and let $S, T : M \to M$ be two weakly completely continuous mappings. Also assume T and S are weakly isotone mappings. Then T and S have a common fixed point in M.*

Corollary 1.22. *Let M be a nonempty closed convex subset of a Banach space X and let $S, T : X \to X$ be two sequentially weakly continuous mappings. Assume T is nonexpansive and S is a nonlinear contraction. Also assume that S and T are weakly isotone mappings, $T(M) \subseteq M$ and $S(M) \subseteq M$. Then T and S have a common fixed point in M.*

Corollary 1.23. *Let M be a nonempty closed convex subset of a Banach space X and ψ a set additive measure of weak noncompactness on X. Let $S, T : X \to X$ be two sequentially weakly continuous mappings satisfying the following:*

(i) *T is a nonexpansive mapping,*

(ii) *S is ψ-condensing,*

(iii) *T and S are weakly isotone,*

(iv) *$T(M) \subseteq M$ and $S(M) \subseteq M$.*

Then S and T have a common fixed point.

Proof. This follows from Theorem 1.20 on the basis of Lemma 1.11. □

Corollary 1.24. *Let M be a nonempty closed convex subset of a Banach space X and ψ a set additive measure of weak noncompactness on X. Let $S, T : X \to X$ be two sequentially weakly continuous mappings satisfying the following:*

(i) *T is a shrinking mapping,*

(ii) *S is ψ-condensing,*

(iii) *T and S are weakly isotone,*

(iv) *$T(M) \subseteq M$ and $S(M) \subseteq M$.*

Then S and T have a unique common fixed point in M.

Corollary 1.25. *Let M be a nonempty closed convex subset of a Banach space X and let $S, T : X \to X$ be two sequentially weakly continuous mappings satisfying the following:*

(i) *T is a shrinking mapping,*

(ii) *S is a nonlinear contraction,*

(iii) *T and S are weakly isotone,*

(iv) *$T(M) \subseteq M$ and $S(M) \subseteq M$.*

Then S and T have a unique common fixed point in M which is the unique fixed point of S.

Remark 1.26. It is worth noting that, in some applications, the weak sequential continuity is not easy to be verified. The ws-compactness seems to be a good alternative (see [16] and the references therein). In the following we provide a version of Theorem 1.20 where the weak sequential continuity is replaced with ws-compactness.

Theorem 1.27. *Let X be an ordered Banach space and ψ a set additive measure of weak noncompactness on X. Let M be a nonempty closed convex subset of X and $S, T : M \to M$ two continuous mappings satisfying the following:*

(i) *T is 1-ψ-contractive,*

(ii) *S is ws-compact and ψ-condensing,*

(iii) *S and T are weakly isotone.*

Then T and S have a common fixed point.

Proof. Let $x \in M$ be fixed. Consider the sequence $\{x_n\}$ defined by

$$x_0 = x, \quad x_{2n+1} = Sx_{2n}, \quad x_{2n+2} = Tx_{2n}, \quad n = 0, 1, 2, \ldots. \tag{1.22}$$

Suppose first that S and T are weakly isotone increasing on M. Then from (1.22) it follows that

$$x_1 \leq x_2 \leq \cdots \leq x_n \leq \cdots. \tag{1.23}$$

Set $A = \{x_1, x_2, \ldots, x_n, \ldots\}$, $A_1 = \{x_2, x_4, \ldots, x_{2n}, \ldots\}$, and $A_2 = \{x_1, x_3, \ldots, x_{2n+1}, \ldots\}$. Clearly

$$A = A_1 \cup A_2, \quad A_2 = S(A_1) \cup \{x_1\}, \quad A_1 = T(A_2). \tag{1.24}$$

Similar reasoning as in Theorem 1.20 gives that A_1 is relatively weakly compact. From the ws-compactness of S it follows that $A_2 = S(A_1) \cup \{x_1\}$ is relatively compact. Now the continuity of T yields that $A_1 = T(A_2)$ is relatively compact. Consequently, $A = A_1 \cup A_2$ is relatively compact. The rest of the proof runs as in the proof of Theorem 1.12. □

As easy consequences of Theorem 1.27 we obtain the following results.

Corollary 1.28. *Let $S, T : X \to X$ be two continuous mappings. Assume T is nonexpansive and ww-compact and S is a ws-compact nonlinear contraction. Also assume that S and T are weakly isotone mappings. Then T and S have a common fixed point.*

Proof. This follows from Theorem 1.27 on the basis of Lemma 1.11. □

Corollary 1.29. *Let M be a nonempty closed convex subset of a Banach space X and ψ a set additive measure of weak noncompactness on X. Let $S, T : X \to X$ be two continuous mappings satisfying the following the following:*

(i) *T is a ww-compact nonexpansive mapping,*

(ii) *S is ws-compact and ψ-condensing,*

(iii) *T and S are weakly isotone,*

(iv) *$T(M) \subseteq M$ and $S(M) \subseteq M$.*

Then S and T have a common fixed point in M.

Proof. This follows From Theorem 1.27 on the basis of Lemma 1.11. □

Corollary 1.30. *Let M be a nonempty closed convex subset of a Banach space X and ψ a set additive measure of weak noncompactness on X. Let $S, T : X \to X$ be two continuous mappings satisfying the following:*

(i) *T is a ww-compact shrinking mapping,*

(ii) *S is ws-compact and ψ-condensing,*

(iii) *T and S are weakly isotone,*

(iv) *$T(M) \subseteq M$ and $S(M) \subseteq M$.*

Then S and T have a unique common fixed point in M.

Corollary 1.31. *Let M be a nonempty closed convex subset of a Banach space X and let $S, T : X \to X$ two continuous mappings satisfying the following:*

(i) *T is a ww-compact and shrinking mapping,*

(ii) *S is a ws-compact nonlinear contraction,*

(iii) *T and S are weakly isotone,*

(iv) *$T(M) \subseteq M$ and $S(M) \subseteq M$.*

Then S and T have a unique common fixed point in M which is the unique fixed point of S.

Note that if X is a Banach Lattice, then the ws-compactness in Theorem 1.27 can be removed, as it is shown in the following result.

Theorem 1.32. *Let X be a Banach lattice and ψ a set additive measure of weak noncompactness on X. Let M be a nonempty closed convex subset of X and $S, T : M \to M$ two continuous mappings satisfying the following:*

(i) *T is 1-ψ-contractive,*

(ii) *S is ψ-condensing,*

(iii) *S and T are weakly isotone.*

Then T and S have a common fixed point in M.

Proof. Let $x \in M$ be fixed. Consider the sequence $\{x_n\}$ defined by

$$x_0 = x, \quad x_{2n+1} = Sx_{2n}, \quad x_{2n+2} = Tx_{2n}, \quad n = 0, 1, 2, \ldots. \tag{1.25}$$

Suppose first that S and T are weakly isotone increasing on M. Then from (1.25) it follows that

$$x_1 \leq x_2 \leq \cdots \leq x_n \leq \cdots. \tag{1.26}$$

Set $A = \{x_1, x_2, \ldots, x_n, \ldots\}$, $A_1 = \{x_2, x_4, \ldots, x_{2n}, \ldots\}$, and $A_2 = \{x_1, x_3, \ldots, x_{2n+1}, \ldots\}$. Clearly

$$A = A_1 \cup A_2, \quad A_2 = S(A_1) \cup \{x_1\}, \quad A_1 = T(A_2). \tag{1.27}$$

Similar reasoning as in Theorem 1.20 gives that A_1 is relatively weakly compact. Thus $\{x_n\}$ has a weakly convergent subsequence, say $\{x_{n_k}\}$. Since in a Banach lattice every weakly convergent increasing sequence is norm-convergent we infer that $\{x_{n_k}\}$ is norm-convergent. Consequently A_1 is relatively compact. Similar reasoning as in the proof of Theorem 1.12 gives the desired result. □

2. Application to Implicit Integral Equations

The purpose of this section is to study the existence of integrable nonnegative solutions of the integral equation given by

$$p(t, x(t)) = \int_0^1 \zeta(t, s) f(s, x(s)) ds, \quad t \in [0, 1]. \tag{2.1}$$

Integral equations like (2.1) were studied in [20] in $L^2[0,1]$ and in [4] in $L^p[0,1]$ with $1 < p < +\infty$. In this section, we look for a nonnegative solution to (2.1) in $L^1[0,1]$. For the remainder we gather some definitions and results from the literature which will be used in the sequel. Recall that a function $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is said to be a Carathéodory function if

(i) for any fixed $u \in \mathbb{R}$, the function $t \to f(t, u)$ is measurable from $[0,1]$ to $\mathbb{R}$,

(ii) for almost any $x \in [0,1]$, the function $f(t, \cdot) : \mathbb{R} \to \mathbb{R}$ is continuous.

Let $\mathbf{m}[0,1]$ be the set of all measurable functions $x : [0,1] \to \mathbb{R}$. If f is a Carathéodory function, then f defines a mapping $\mathbf{N}_f : \mathbf{m}[0,1] \to \mathbf{m}[0,1]$ by $\mathbf{N}_f(x)(t) := f(t, x(t))$. This mapping is called the superposition (or Nemytskii) operator generated by f. The next two lemmas are of foremost importance for our subsequent analysis.

Lemma 2.1 (see [21, 22]). *Let $f : [0,1]\times\mathbb{R} \to \mathbb{R}$ be a Carathéodory function. Then the superposition operator $\mathbf{N}_f$ maps continuously $L^1[0,1]$ into itself if and only if there exist a constant $b \geq 0$ and a function $a(\cdot) \in L^1_+[0,1]$ such that*

$$\left|f(t,u)\right| \leq a(t) + b|u|, \tag{2.2}$$

where $L^1_+[0,1]$ denotes the positive cone of the space $L^1[0,1]$.

Lemma 2.2 (see [14]). *If $f : [0,1]\times\mathbb{R} \to \mathbb{R}$ is a Carathéodory function and $\mathbf{N}_f$ maps continuously $L^1[0,1]$ into itself, then $\mathbf{N}_f$ is ww-compact.*

Remark 2.3. Although the Nemytskii operator $\mathbf{N}_f$ is ww-compact, generally it is not weakly continuous. In fact, only linear functions generate weakly continuous Nemytskii operators in L^1 spaces (see, for instance, [23, Theorem 2.6]).

The problem of existence of nonnegative integrable solutions to (2.1) will be discussed under the following assumptions:

(a) $\zeta : [0,1]\times[0,1] \to \mathbb{R}$ is strongly measurable and $\int_0^1 \zeta(\cdot,s)y(s)ds \in L^1[0,1]$ whenever $y \in L^1[0,1]$ and there exists a function $\theta : [0,1] \to \mathbb{R}$ belonging to $L^\infty[0,1]$ such that $0 \leq \zeta(t,s) \leq \theta(t)$ for all $(t,s) \in [0,1]\times[0,1]$. The function $f : [0,1]\times\mathbb{R} \to \mathbb{R}$ is a Carathéodory function and there exist a constant $b > 0$ and a function $a(\cdot) \in L^1_+[0,1]$ such that $|f(t,u)| \leq a(t) + b|u|$ for all $t \in [0,1]$ and $u \in \mathbb{R}$. Moreover, $f(t,x(t)) \geq 0$ whenever $x \in L^1_+[0,1]$;

(b) the function $p : [0,1]\times\mathbb{R} \to \mathbb{R}$ is nonexpansive with respect to the second variable, that is, $|p(t,u) - p(t,v)| \leq |u-v|$ for all $t \in [0,1]$ and $u, v \in \mathbb{R}$;

(c) for all $r > 0$ and for all $x \in B_r^+$ we have

$$0 \leq \theta(t)\|a\| + b\theta(t)\int_0^1 p(s,x(s))ds \leq p(t,x(t)) \leq x(t), \tag{2.3}$$

where $B_r^+ = \{x \in L^1[0,1] : 0 \leq x(t) \leq r \text{ for all } t \in [0,1]\}$,

(d) $b\|\theta\|_\infty < 1$.

Remark 2.4. If $p(t,0) = 0$, then from assumption (b) it follows that $p(t,x(t)) \leq x(t)$ whenever $x \in L^1_+[0,1]$.

Remark 2.5. From assumption (b) it follows that

$$\left|p(t,x(t))\right| \leq \left|p(t,0)\right| + |x(t)|. \tag{2.4}$$

Using Lemma 2.2 together with Lemma 2.1 we infer that $\mathcal{N}_p$ is ww-compact.

Theorem 2.6. *Assume that the conditions (a–d) are satisfied. Then the implicit integral equation* (2.1) *has at least one nonnegative solution in $L^1[0,1]$.*

Proof. The problem (2.1) may be written in the shape:

$$Tx = Sx, \tag{2.5}$$

where T and S are the nonlinear operators given by

$$(Tx)(t) = p(t, x(t)), \qquad (Sx)(t) = \int_0^1 \zeta(t,s) f(s, x(s)). \tag{2.6}$$

Note that T and S may be written in the shape:

$$T = \mathcal{N}_p, \qquad S = K\mathcal{N}_f. \tag{2.7}$$

Here $\mathcal{N}_p$ and $\mathcal{N}_f$ are the superposition operators associated to the functions p and f, respectively, and K denotes the linear integral defined by

$$K : L^1[0,1] \longrightarrow L^1[0,1] : u(t) \longmapsto Ku(t) := \mu \int_0^1 \zeta(t,s) u(s) ds. \tag{2.8}$$

Let r_0 be the real defined by

$$r_0 = \frac{\|a\| \, \|\theta\|}{1 - b\|\theta\|} \tag{2.9}$$

and M be the closed convex subset of $L^1[0,1]$ defined by

$$M = B^+_{r_0} = \left\{ x \in L^1[0,1] : 0 \le x(t) \le r_0 \ \forall t \in [0,1] \right\}. \tag{2.10}$$

Note that, for any $x \in L^1[0,1]$, the functions Tx and Sx belong to $L^1[0,1]$ which is a consequence of the assumptions (a, b). Moreover, from assumption (c) it follows that for each $x \in M$ we have

$$0 \le p(t, x(t)) \le x(t) \le r_0. \tag{2.11}$$

This implies that $T(M) \subseteq M$. Also, in view of assumption (b) we have for all $x \in M$ and for all $s \in [0,1]$

$$0 \le f(s, x(s)) \le a(s) + bx(s) \le a(s) + br_0. \tag{2.12}$$

Hence

$$0 \le \zeta(t,s) f(s, x(s)) \le \theta(t)(a(s) + br_0). \tag{2.13}$$

Thus

$$0 \leq \int_0^1 \zeta(t,s) f(s,x(s)) ds \leq \theta(t)(\|a\| + br_0) \leq \|\theta\|_\infty (\|a\| + br_0) = r_0. \tag{2.14}$$

Therefore, $S(M) \subseteq M$. □

Our strategy consists in applying Corollary 1.29 to find a nonnegative common fixed point for S and T in M. For the sake of simplicity, the proof will be displayed into four steps.

Step 1. S and T are continuous. Indeed, the assumption (a) and Lemma 2.1 guarantee that N_p and $\mathcal{N}_f$ map continuously $L^1[0,1]$ into itself. To complete the proof it remains only to show that K is continuous. This follows immediately from the hypothesis (b).

Step 2. S is ws-compact. Indeed, let (ρ_n) be a weakly convergent sequence of $L^1[0,1]$. Using Lemma 2.2, the sequence $(\mathcal{N}_f(\rho_n))$ has a weakly convergent subsequence, say $(\mathcal{N}_f(\rho_{n_k}))$. Let ρ be the weak limit of $(\mathcal{N}_f(\rho_{n_k}))$. Accordingly, keeping in mind the boundedness of the mapping $\zeta(t,\cdot)$ we get

$$\int_0^1 \zeta(t,s) f(s,\rho_{n_k}(s)) ds \longrightarrow \int_0^1 \zeta(t,s) f(s,\rho(s)) ds. \tag{2.15}$$

The use of the dominated convergence theorem allows us to conclude that the sequence $(S\rho_{n_k})$ converges in $L^1[0,1]$.

Step 3. S maps bounded sets of $L^1[0,1]$ into weakly compact sets. To see this, let $\mathcal{O}$ be a bounded subset of $L_1[0,1]$ and let $C > 0$ such that $\|x\| \leq C$ for all $x \in \mathcal{O}$. For $x \in \mathcal{O}$ we have

$$\begin{aligned} |(Sx)(t)|dt &\leq \int_0^1 |\zeta(t,s)| \big| f(s,x(s)) \big| ds \\ &\leq \int_0^1 \theta(t)(a(s) + b|x(s)|) ds \\ &\leq \theta(t)(\|a\| + bC). \end{aligned} \tag{2.16}$$

Consequently,

$$\int_E |(Sx)(t)|dt \leq (\|a\| + bC) \int_E \theta(t) dt \tag{2.17}$$

for all measurable subsets E of $[0,1]$. Taking into account the fact that any set consisting of one element is weakly compact and using Corollary 11 in [24, page 294] we get $\lim_{|E|\to 0} \int_E |\theta(t)|dt = 0$, where $|E|$ is the Lebesgue measure of E. Applying Corollary 11 in [24, page 294] once again we infer that the set $S(\mathcal{O})$ is weakly compact.

Step 4. S and T are weakly isotone decreasing. Indeed, using assumption (c) we get

$$\int_0^1 \zeta(t,s)f(s,x(s))ds \geq p\left(t, \int_0^1 \zeta(t,s)f(s,x(s))ds\right), \tag{2.18}$$

and from our assumptions we have

$$\begin{aligned}\int_0^1 \zeta(t,s)f(s,p(s,x(s)))ds &\leq \int_0^1 \zeta(t,s)a(s)ds + b\int_0^1 \zeta(t,s)p(s,x(s))ds \\ &\leq \theta(t)\|a\| + b\theta(t)\int_0^1 p(s,x(s))ds \\ &\leq p(t,x(t)).\end{aligned} \tag{2.19}$$

Hence $Sx \geq TSx$ and $Tx \geq STx$ and therefore S and T are weakly isotone decreasing.

Note that the *Steps* 1–4 show that the hypotheses of Corollary 1.29 are satisfied. Thus S and T have a common fixed point and therefore the problem (2.1) has at least one nonnegative solution in M.

Acknowledgment

The first and second authors gratefully acknowledge the support from the Deanship of Scientific Research (DSR) at King Abdulaziz University (KAU) during this research.

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