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Fixed point theorems of contractive mappings in cone b-metric spaces and applications

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Abstract

In this paper we present some new examples in cone b-metric spaces and prove some fixed point theorems of contractive mappings without the assumption of normality in cone b-metric spaces. The results not only directly improve and generalize some fixed point results in metric spaces and b-metric spaces, but also expand and complement some previous results in cone metric spaces. In addition, we use our results to obtain the existence and uniqueness of a solution for an ordinary differential equation with a periodic boundary condition.

Keywords: cone b-metric space; fixed point; periodic boundary problem

1 Introduction

Fixed point theory plays a basic role in applications of many branches of mathematics. Finding a fixed point of contractive mappings becomes the center of strong research activity. There are many works about the fixed point of contractive maps (see, for example, [1, 2]). In [2], Polish mathematician Banach proved a very important result regarding a contraction mapping, known as the Banach contraction principle, in 1922. In [3], Bakhtin introduced b-metric spaces as a generalization of metric spaces. He proved the contraction mapping principle in b-metric spaces that generalized the famous Banach contraction principle in metric spaces. Since then, several papers have dealt with fixed point theory or the variational principle for single-valued and multi-valued operators in b-metric spaces (see [4–6] and the references therein). In recent investigations, the fixed point in non-convex analysis, especially in an ordered normed space, occupies a prominent place in many aspects (see [7–10]). The authors define an ordering by using a cone, which naturally induces a partial ordering in Banach spaces. In [7], Huang and Zhang introduced cone metric spaces as a generalization of metric spaces. Moreover, they proved some fixed point theorems for contractive mappings that expanded certain results of fixed points in metric spaces. In [10], Hussain and Shah introduced cone b-metric spaces as a generalization of b-metric spaces and cone metric spaces. They established some topological properties in such spaces and improved some recent results about KKM mappings in the setting of a cone b-metric space. Throughout this paper, we firstly offer some new examples in cone b-metric spaces, then obtain some fixed point theorems of contractive mappings without the assumption of normality in cone b-metric spaces. Furthermore, we support our results by an example. The results greatly generalize and improve the work of [3, 4, 7, 8] and [10].

As some applications, we show the existence and uniqueness of a solution for a first-order ordinary differential equation with a periodic boundary condition.

Consistent with Huang and Zhang [7], the following definitions and results will be needed in the sequel.

Let E be a real Banach space and P be a subset of E. By θ we denote the zero element of E and by $\operatorname{int} P$ the interior of P. The subset P is called a cone if and only if:

(i) P is closed, nonempty, and $P \neq \{\theta\}$;
(ii) $a, b \in \mathbb{R}$, $a, b \geq 0$, $x, y \in P \Rightarrow ax + by \in P$;
(iii) $P \cap (-P) = \{\theta\}$.

On this basis, we define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. We shall write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in \operatorname{int} P$. Write $\|\cdot\|$ as the norm on E. The cone P is called normal if there is a number $K > 0$ such that for all $x, y \in E$, $\theta \leq x \leq y$ implies $\|x\| \leq K\|y\|$. The least positive number satisfying the above is called the normal constant of P. It is well known that $K \geq 1$.

In the following, we always suppose that E is a Banach space, P is a cone in E with $\operatorname{int} P \neq \emptyset$ and $\leq$ is a partial ordering with respect to P.

Definition 1.1 ([7]) Let X be a nonempty set. Suppose that the mapping $d : X \times X \to E$ satisfies:

(d1) $\theta < d(x, y)$ for all $x, y \in X$ with $x \neq y$ and $d(x, y) = \theta$ if and only if $x = y$;
(d2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(d3) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a cone metric on X and (X, d) is called a cone metric space.

Definition 1.2 ([10]) Let X be a nonempty set and $s \geq 1$ be a given real number. A mapping $d : X \times X \to E$ is said to be cone b-metric if and only if, for all $x, y, z \in X$, the following conditions are satisfied:

(i) $\theta < d(x, y)$ with $x \neq y$ and $d(x, y) = \theta$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$;
(iii) $d(x, y) \leq s[d(x, z) + d(z, y)]$.

The pair (X, d) is called a cone b-metric space.

Remark 1.3 The class of cone b-metric spaces is larger than the class of cone metric spaces since any cone metric space must be a cone b-metric space. Therefore, it is obvious that cone b-metric spaces generalize b-metric spaces and cone metric spaces.

We can present a number of examples, as follows, which show that introducing a cone b-metric space instead of a cone metric space is meaningful since there exist cone b-metric spaces which are not cone metric spaces.

Example 1.4 Let $E = \mathbb{R}^2$, $P = \{(x, y) \in E : x, y \geq 0\} \subset E$, $X = \mathbb{R}$ and $d : X \times X \to E$ such that $d(x, y) = (|x - y|^p, \alpha|x - y|^p)$, where $\alpha \geq 0$ and $p > 1$ are two constants. Then (X, d) is a cone b-metric space, but not a cone metric space. In fact, we only need to prove (iii) in Definition 1.2 as follows:

Let $x, y, z \in X$. Set $u = x - z$, $v = z - y$, so $x - y = u + v$. From the inequality

$$(a + b)^p \leq \left(2 \max\{a, b\}\right)^p \leq 2^p \left(a^p + b^p\right) \quad \text{for all } a, b \geq 0,$$

we have

$$|x-y|^p = |u+v|^p \le \big(|u|+|v|\big)^p \le 2^p\big(|u|^p+|v|^p\big) = 2^p\big(|x-z|^p+|z-y|^p\big),$$

which implies that $d(x,y) \le s[d(x,z)+d(z,y)]$ with $s = 2^p > 1$. But

$$|x-y|^p \le |x-z|^p + |z-y|^p$$

is impossible for all $x > z > y$. Indeed, taking account of the inequality

$$(a+b)^p > a^p + b^p \quad \text{for all } a, b > 0,$$

we arrive at

$$|x-y|^p = |u+v|^p = (u+v)^p > u^p + v^p = (x-z)^p + (z-y)^p = |x-z|^p + |z-y|^p,$$

for all $x > z > y$. Thus, (d3) in Definition 1.1 is not satisfied, *i.e.*, (X,d) is not a cone metric space.

Example 1.5 Let $X = l^p$ with $0 < p < 1$, where $l^p = \{\{x_n\} \subset \mathbb{R} : \sum_{n=1}^{\infty}|x_n|^p < \infty\}$. Let $d : X \times X \to \mathbb{R}_+$,

$$d(x,y) = \left(\sum_{n=1}^{\infty}|x_n - y_n|^p\right)^{\frac{1}{p}},$$

where $x = \{x_n\}, y = \{y_n\} \in l^p$. Then (X,d) is a b-metric space (see [5]). Put $E = l^1$, $P = \{\{x_n\} \in E : x_n \ge 0, \text{ for all } n \ge 1\}$. Letting the mapping $\tilde{d} : X \times X \to E$ be defined by $\tilde{d}(x,y) = \{\frac{d(x,y)}{2^n}\}_{n\ge 1}$, we conclude that $(X,\tilde{d})$ is a cone b-metric space with the coefficient $s = 2^{\frac{1}{p}} > 1$, but it is not a cone metric space.

Example 1.6 Let $X = \{1,2,3,4\}$, $E = \mathbb{R}^2$, $P = \{(x,y) \in E : x \ge 0, y \ge 0\}$. Define $d : X \times X \to E$ by

$$d(x,y) = \begin{cases} (|x-y|^{-1}, |x-y|^{-1}), & \text{if } x \ne y, \\ \theta, & \text{if } x = y. \end{cases}$$

Then (X,d) is a cone b-metric space with the coefficient $s = \frac{6}{5}$. But it is not a cone metric space since the triangle inequality is not satisfied. Indeed,

$$d(1,2) > d(1,4) + d(4,2), \qquad d(3,4) > d(3,1) + d(1,4).$$

Definition 1.7 ([10]) Let (X,d) be a cone b-metric space, $x \in X$ and $\{x_n\}$ be a sequence in X. Then

(i) $\{x_n\}$ converges to x whenever, for every $c \in E$ with $\theta \ll c$, there is a natural number N such that $d(x_n, x) \ll c$ for all $n \ge N$. We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ $(n \to \infty)$.

(ii) $\{x_n\}$ is a Cauchy sequence whenever, for every $c \in E$ with $\theta \ll c$, there is a natural number N such that $d(x_n, x_m) \ll c$ for all $n, m \geq N$.
(iii) (X, d) is a complete cone b-metric space if every Cauchy sequence is convergent.

The following lemmas are often used (in particular when dealing with cone metric spaces in which the cone need not be normal).

Lemma 1.8 ([9]) *Let P be a cone and $\{a_n\}$ be a sequence in E. If $c \in \operatorname{int} P$ and $\theta \leq a_n \to \theta$ (as $n \to \infty$), then there exists N such that for all $n > N$, we have $a_n \ll c$.*

Lemma 1.9 ([9]) *Let $x, y, z \in E$, if $x \leq y$ and $y \ll z$, then $x \ll z$.*

Lemma 1.10 ([10]) *Let P be a cone and $\theta \leq u \ll c$ for each $c \in \operatorname{int} P$, then $u = \theta$.*

Lemma 1.11 ([11]) *Let P be a cone. If $u \in P$ and $u \leq ku$ for some $0 \leq k < 1$, then $u = \theta$.*

Lemma 1.12 ([9]) *Let P be a cone and $a \leq b + c$ for each $c \in \operatorname{int} P$, then $a \leq b$.*

2 Main results

In this section, we will present some fixed point theorems for contractive mappings in the setting of cone b-metric spaces. Furthermore, we will give examples to support our main results.

Theorem 2.1 *Let (X, d) be a complete cone b-metric space with the coefficient $s \geq 1$. Suppose the mapping $T : X \to X$ satisfies the contractive condition*

$$d(Tx, Ty) \leq \lambda d(x, y), \quad \text{for } x, y \in X,$$

where $\lambda \in [0, 1)$ is a constant. Then T has a unique fixed point in X. Furthermore, the iterative sequence $\{T^n x\}$ converges to the fixed point.

Proof Choose $x_0 \in X$. We construct the iterative sequence $\{x_n\}$, where $x_n = Tx_{n-1}$, $n \geq 1$, *i.e.*, $x_{n+1} = Tx_n = T^{n+1}x_0$. We have

$$d(x_{n+1}, x_n) = d(Tx_n, Tx_{n-1}) \leq \lambda d(x_n, x_{n-1}) \leq \cdots \leq \lambda^n d(x_1, x_0).$$

For any $m \geq 1$, $p \geq 1$, it follows that

$$\begin{aligned}
d(x_{m+p}, x_m) &\leq s\big[d(x_{m+p}, x_{m+p-1}) + d(x_{m+p-1}, x_m)\big] \\
&= sd(x_{m+p}, x_{m+p-1}) + sd(x_{m+p-1}, x_m) \\
&\leq sd(x_{m+p}, x_{m+p-1}) + s^2\big[d(x_{m+p-1}, x_{m+p-2}) + d(x_{m+p-2}, x_m)\big] \\
&= sd(x_{m+p}, x_{m+p-1}) + s^2 d(x_{m+p-1}, x_{m+p-2}) + s^2 d(x_{m+p-2}, x_m) \\
&\leq sd(x_{m+p}, x_{m+p-1}) + s^2 d(x_{m+p-1}, x_{m+p-2}) + s^3 d(x_{m+p-2}, x_{m+p-3}) + \cdots \\
&\quad + s^{p-1} d(x_{m+2}, x_{m+1}) + s^{p-1} d(x_{m+1}, x_m) \\
&\leq s\lambda^{m+p-1} d(x_1, x_0) + s^2\lambda^{m+p-2} d(x_1, x_0) + s^3\lambda^{m+p-3} d(x_1, x_0) + \cdots
\end{aligned}$$

$$
\begin{aligned}
&+ s^{p-1}\lambda^{m+1}d(x_1,x_0) + s^{p-1}\lambda^{m}d(x_1,x_0)\\
&= \bigl(s\lambda^{m+p-1} + s^2\lambda^{m+p-2} + s^3\lambda^{m+p-3} + \cdots + s^{p-1}\lambda^{m+1}\bigr)d(x_1,x_0)\\
&\quad + s^{p-1}\lambda^{m}d(x_1,x_0)\\
&= \frac{s\lambda^{m+p}[(s\lambda^{-1})^{p-1} - 1]}{s-\lambda}d(x_1,x_0) + s^{p-1}\lambda^{m}d(x_1,x_0)\\
&\leq \frac{s^p\lambda^{m+1}}{s-\lambda}d(x_1,x_0) + s^{p-1}\lambda^{m}d(x_1,x_0).
\end{aligned}
$$

Let $\theta \ll c$ be given. Notice that $\frac{s^p\lambda^{m+1}}{s-\lambda}d(x_1,x_0) + s^{p-1}\lambda^{m}d(x_1,x_0) \to \theta$ as $m \to \infty$ for any k. Making full use of Lemma 1.8, we find $m_0 \in \mathbb{N}$ such that

$$\frac{s^p\lambda^{m+1}}{s-\lambda}d(x_1,x_0) + s^{p-1}\lambda^{m}d(x_1,x_0) \ll c,$$

for each $m > m_0$. Thus,

$$d(x_{m+p},x_m) \leq \frac{s^p\lambda^{m+1}}{s-\lambda}d(x_1,x_0) + s^{p-1}\lambda^{m}d(x_1,x_0) \ll c$$

for all $m > m_0$ and any p. So, by Lemma 1.9, $\{x_n\}$ is a Cauchy sequence in (X,d). Since (X,d) is a complete cone b-metric space, there exists $x^* \in X$ such that $x_n \to x^*$. Take $n_0 \in \mathbb{N}$ such that $d(x_n,x^*) \ll \frac{c}{s(\lambda+1)}$ for all $n > n_0$. Hence,

$$d\bigl(Tx^*,x^*\bigr) \leq s\bigl[d\bigl(Tx^*,Tx_n\bigr) + d\bigl(Tx_n,x^*\bigr)\bigr] \leq s\bigl[\lambda d\bigl(x^*,x_n\bigr) + d\bigl(x_{n+1},x^*\bigr)\bigr] \ll c,$$

for each $n > n_0$. Then, by Lemma 1.10, we deduce that $d(Tx^*,x^*) = \theta$, *i.e.*, $Tx^* = x^*$. That is, x^* is a fixed point of T.

Now we show that the fixed point is unique. If there is another fixed point y^*, by the given condition,

$$d\bigl(x^*,y^*\bigr) = d\bigl(Tx^*,Ty^*\bigr) \leq \lambda d\bigl(x^*,y^*\bigr).$$

By Lemma 1.11, $x^* = y^*$. The proof is completed. □

Example 2.2 Let $X = [0,1]$, $E = \mathbb{R}^2$ and $p > 1$ be a constant. Take $P = \{(x,y) \in E : x,y \geq 0\}$. We define $d : X \times X \to E$ as

$$d(x,y) = \bigl(|x-y|^p, |x-y|^p\bigr) \quad \text{for all } x,y \in X.$$

Then (X,d) is a complete cone b-metric space. Let us define $T : X \to X$ as

$$Tx = \frac{1}{2}x - \frac{1}{4}x^2 \quad \text{for all } x \in X.$$

Therefore,

$$
\begin{aligned}
d(Tx,Ty) &= \bigl(|Tx-Ty|^p, |Tx-Ty|^p\bigr)\\
&= \left(\left|\frac{1}{2}(x-y) - \frac{1}{4}(x-y)(x+y)\right|^p, \left|\frac{1}{2}(x-y) - \frac{1}{4}(x-y)(x+y)\right|^p\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \left(|x-y|^p \cdot \left| \frac{1}{2} - \frac{1}{4}(x+y) \right|^p, |x-y|^p \cdot \left| \frac{1}{2} - \frac{1}{4}(x+y) \right|^p \right) \\
&\leq \frac{1}{2^p} \left(|x-y|^p, |x-y|^p \right) \\
&= \frac{1}{2^p} d(x,y).
\end{aligned}
$$

Here $0 \in X$ is the unique fixed point of T.

Theorem 2.3 *Let (X,d) be a complete cone b-metric space with the coefficient $s \geq 1$. Suppose the mapping $T: X \to X$ satisfies the contractive condition*

$$
d(Tx,Ty) \leq \lambda_1 d(x,Tx) + \lambda_2 d(y,Ty) + \lambda_3 d(x,Ty) + \lambda_4 d(y,Tx), \quad \textit{for } x,y \in X,
$$

where the constant $\lambda_i \in [0,1)$ and $\lambda_1 + \lambda_2 + s(\lambda_3 + \lambda_4) < \min\{1, \frac{2}{s}\}$, $i = 1,2,3,4$. Then T has a unique fixed point in X. Moreover, the iterative sequence $\{T^n x\}$ converges to the fixed point.

Proof Fix $x_0 \in X$ and set $x_1 = Tx_0$ and $x_{n+1} = Tx_n = T^{n+1}x_0$. Firstly, we see

$$
\begin{aligned}
d(x_{n+1},x_n) &= d(Tx_n, Tx_{n-1}) \\
&\leq \lambda_1 d(x_n, Tx_n) + \lambda_2 d(x_{n-1}, Tx_{n-1}) + \lambda_3 d(x_n, Tx_{n-1}) + \lambda_4 d(x_{n-1}, Tx_n) \\
&\leq \lambda_1 d(x_n, x_{n+1}) + \lambda_2 d(x_{n-1}, x_n) + s\lambda_4 \left[d(x_{n-1}, x_n) + d(x_n, x_{n+1}) \right] \\
&= (\lambda_1 + s\lambda_4) d(x_n, x_{n+1}) + (\lambda_2 + s\lambda_4) d(x_n, x_{n-1}).
\end{aligned}
$$

It follows that

$$
(1 - \lambda_1 - s\lambda_4) d(x_{n+1}, x_n) \leq (\lambda_2 + s\lambda_4) d(x_n, x_{n-1}). \tag{2.1}
$$

Secondly,

$$
\begin{aligned}
d(x_{n+1},x_n) &= d(Tx_n, Tx_{n-1}) = d(Tx_{n-1}, Tx_n) \\
&\leq \lambda_1 d(x_{n-1}, Tx_{n-1}) + \lambda_2 d(x_n, Tx_n) + \lambda_3 d(x_{n-1}, Tx_n) + \lambda_4 d(x_n, Tx_{n-1}) \\
&\leq \lambda_1 d(x_{n-1}, x_n) + \lambda_2 d(x_n, x_{n+1}) + s\lambda_3 \left[d(x_{n-1}, x_n) + d(x_n, x_{n+1}) \right] \\
&= (\lambda_2 + s\lambda_3) d(x_n, x_{n+1}) + (\lambda_1 + s\lambda_3) d(x_n, x_{n-1}).
\end{aligned}
$$

This establishes that

$$
(1 - \lambda_2 - s\lambda_3) d(x_{n+1}, x_n) \leq (\lambda_1 + s\lambda_3) d(x_n, x_{n-1}). \tag{2.2}
$$

Adding up (2.1) and (2.2) yields

$$
d(x_{n+1}, x_n) \leq \frac{\lambda_1 + \lambda_2 + s(\lambda_3 + \lambda_4)}{2 - \lambda_1 - \lambda_2 - s(\lambda_3 + \lambda_4)} d(x_n, x_{n-1}).
$$

Put $\lambda = \frac{\lambda_1+\lambda_2+s(\lambda_3+\lambda_4)}{2-\lambda_1-\lambda_2-s(\lambda_3+\lambda_4)}$, it is easy to see that $0 \leq \lambda < 1$. Thus,

$$
d(x_{n+1}, x_n) \leq \lambda d(x_n, x_{n-1}) \leq \cdots \leq \lambda^n d(x_1, x_0).
$$

Following an argument similar to that given in Theorem 2.1, there exists $x^* \in X$ such that $x_n \to x^*$. Let $c \gg \theta$ be arbitrary. Since $x_n \to x^*$, there exists N such that

$$d(x_n, x^*) \ll \frac{2 - s\lambda_1 - s\lambda_2 - s^2\lambda_3 - s^2\lambda_4}{2s^2 + 2s} c \quad \text{for all } n > N.$$

Next we claim that x^* is a fixed point of T. Actually, on the one hand,

$$\begin{aligned} d(Tx^*, x^*) &\leq s[d(Tx^*, Tx_n) + d(Tx_n, x^*)] = sd(Tx^*, Tx_n) + sd(x_{n+1}, x^*) \\ &\leq s[\lambda_1 d(x^*, Tx^*) + \lambda_2 d(x_n, Tx_n) + \lambda_3 d(x^*, Tx_n) + \lambda_4 d(x_n, Tx^*)] + sd(x_{n+1}, x^*) \\ &= s[\lambda_1 d(x^*, Tx^*) + \lambda_2 d(x_n, x_{n+1}) + \lambda_3 d(x^*, x_{n+1}) + \lambda_4 d(x_n, Tx^*)] + sd(x_{n+1}, x^*) \\ &\leq s\lambda_1 d(x^*, Tx^*) + s^2\lambda_2 d(x_n, x^*) + s^2\lambda_2 d(x^*, x_{n+1}) + s\lambda_3 d(x^*, x_{n+1}) \\ &\quad + s^2\lambda_4 d(x_n, x^*) + s^2\lambda_4 d(x^*, Tx^*) + sd(x_{n+1}, x^*) \\ &= (s\lambda_1 + s^2\lambda_4) d(x^*, Tx^*) + (s^2\lambda_2 + s^2\lambda_4) d(x_n, x^*) \\ &\quad + (s^2\lambda_2 + s\lambda_3 + s) d(x^*, x_{n+1}), \end{aligned}$$

which implies that

$$(1 - s\lambda_1 - s^2\lambda_4) d(x^*, Tx^*) \leq (s^2\lambda_2 + s^2\lambda_4) d(x_n, x^*) + (s^2\lambda_2 + s\lambda_3 + s) d(x^*, x_{n+1}). \tag{2.3}$$

On the other hand,

$$\begin{aligned} d(x^*, Tx^*) &\leq s[d(x^*, Tx_n) + d(Tx_n, Tx^*)] = sd(x^*, x_{n+1}) + sd(Tx_n, Tx^*) \\ &\leq sd(x^*, x_{n+1}) + s[\lambda_1 d(x_n, Tx_n) + \lambda_2 d(x^*, Tx^*) + \lambda_3 d(x_n, Tx^*) + \lambda_4 d(x^*, Tx_n)] \\ &= sd(x^*, x_{n+1}) + s[\lambda_1 d(x_n, x_{n+1}) + \lambda_2 d(x^*, Tx^*) + \lambda_3 d(x_n, Tx^*) + \lambda_4 d(x^*, x_{n+1})] \\ &\leq sd(x^*, x_{n+1}) + s^2\lambda_1 d(x_n, x^*) + s^2\lambda_1 d(x^*, x_{n+1}) + s\lambda_2 d(x^*, Tx^*) \\ &\quad + s^2\lambda_3 d(x_n, x^*) + s^2\lambda_3 d(x^*, Tx^*) + s\lambda_4 d(x^*, x_{n+1}) \\ &= (s\lambda_2 + s^2\lambda_3) d(x^*, Tx^*) + (s^2\lambda_1 + s^2\lambda_3) d(x_n, x^*) \\ &\quad + (s^2\lambda_1 + s\lambda_4 + s) d(x^*, x_{n+1}), \end{aligned}$$

which means that

$$(1 - s\lambda_2 - s^2\lambda_3) d(x^*, Tx^*) \leq (s^2\lambda_1 + s^2\lambda_3) d(x_n, x^*) + (s^2\lambda_1 + s\lambda_4 + s) d(x^*, x_{n+1}). \tag{2.4}$$

Combining (2.3) and (2.4) yields

$$\begin{aligned} &(2 - s\lambda_1 - s\lambda_2 - s^2\lambda_3 - s^2\lambda_4) d(x^*, Tx^*) \\ &\quad \leq s^2(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4) d(x_n, x^*) + (s^2\lambda_1 + s^2\lambda_2 + s\lambda_3 + s\lambda_4 + 2s) d(x^*, x_{n+1}) \\ &\quad \leq s^2 d(x_n, x^*) + (s^2 + 2s) d(x^*, x_{n+1}). \end{aligned}$$

Simple calculations ensure that

$$d(x^*, Tx^*) \le \frac{s^2 d(x_n, x^*) + (s^2 + 2s) d(x^*, x_{n+1})}{2 - s\lambda_1 - s\lambda_2 - s^2\lambda_3 - s^2\lambda_4} \ll c.$$

It is easy to see from Lemma 1.10 that $d(x^*, Tx^*) = \theta$, *i.e.*, x^* is a fixed point of T. Finally, we show the uniqueness of the fixed point. Indeed, if there is another fixed point y^*, then

$$\begin{aligned} d(x^*, y^*) &= d(Tx^*, Ty^*) \\ &\le \lambda_1 d(x^*, Tx^*) + \lambda_2 d(y^*, Ty^*) + \lambda_3 d(x^*, Ty^*) + \lambda_4 d(y^*, Tx^*) \\ &\le s\lambda_3 [d(x^*, y^*) + d(y^*, Ty^*)] + s\lambda_4 [d(y^*, x^*) + d(x^*, Tx^*)] \\ &= s(\lambda_3 + \lambda_4) d(x^*, y^*). \end{aligned}$$

Owing to $0 \le s(\lambda_3 + \lambda_4) < 1$, we deduce from Lemma 1.11 that $x^* = y^*$. Therefore, we complete the proof. □

Remark 2.4 Theorem 2.1 extends the famous Banach contraction principle to that in the setting of cone b-metric spaces.

Remark 2.5 Any fixed point theorem in the setting of a metric space, a b-metric space or a cone metric space cannot cope with Example 2.2. So, Example 2.2 shows that the fixed point theory of cone b-metric spaces offers independently a strong tool for studying the positive fixed points of some nonlinear operators and the positive solutions of some operator equations.

Remark 2.6 The main results are some valuable additions to the available references regarding cone b-metric spaces since we have known few fixed point theorems of contractive mappings in the setting of cone b-metric spaces.

3 Applications

In this section we shall apply Theorem 2.1 to the first-order periodic boundary problem

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} = F(t, x(t)), \\ x(0) = \xi, \end{cases} \tag{3.1}$$

where $F : [-h, h] \times [\xi - \delta, \xi + \delta]$ is a continuous function.

Example 3.1 Consider the boundary problem (3.1) with the continuous function F and suppose $F(x, y)$ satisfies the local Lipschitz condition, *i.e.*, if $|x| \le h$, $y_1, y_2 \in [\xi - \delta, \xi + \delta]$, it induces

$$|F(x, y_1) - F(x, y_2)| \le L|y_1 - y_2|.$$

Set $M = \max_{[-h,h]\times[\xi-\delta,\xi+\delta]} |F(x, y)|$ such that $h^2 < \min\{\delta/M^2, 1/L^2\}$, then there exists a unique solution of (3.1).

Proof Let $X = E = C([-h,h])$ and $P = \{u \in E : u \geq 0\}$. Put $d : X \times X \to E$ as $d(x,y) = f(t) \max_{-h\leq t\leq h} |x(t) - y(t)|^2$ with $f : [-h,h] \to \mathbb{R}$ such that $f(t) = e^t$. It is clear that (X,d) is a complete cone b-metric space.

Note that (3.1) is equivalent to the integral equation

$$x(t) = \xi + \int_0^t F(\tau, x(\tau))\, d\tau.$$

Define a mapping $T : C([-h,h]) \to \mathbb{R}$ by $Tx(t) = \xi + \int_0^t F(\tau,x(\tau))\, d\tau$. If

$$x(t), y(t) \in B(\xi, \delta f) \triangleq \{\varphi(t) \in C([-h,h]) : d(\xi,\varphi) \leq \delta f\},$$

then from

$$\begin{aligned} d(Tx,Ty) &= f(t) \max_{-h\leq t\leq h} \left| \int_0^t F(\tau,x(\tau))\, d\tau - \int_0^t F(\tau,y(\tau))\, d\tau \right|^2 \\ &= f(t) \max_{-h\leq t\leq h} \left| \int_0^t \left[F(\tau,x(\tau)) - F(\tau,y(\tau))\right] d\tau \right|^2 \\ &\leq h^2 f(t) \max_{-h\leq \tau\leq h} \left|F(\tau,x(\tau)) - F(\tau,y(\tau))\right|^2 \\ &\leq h^2 L^2 f(t) \max_{-h\leq \tau\leq h} \left|x(\tau) - y(\tau)\right|^2 \\ &= h^2 L^2 d(x,y), \end{aligned}$$

and

$$d(Tx,\xi) = f(t) \max_{-h\leq t\leq h} \left| \int_0^t F(\tau,x(\tau))\, d\tau \right|^2 \leq h^2 f \max_{-h\leq \tau\leq h} \left|F(\tau,x(\tau))\right|^2 \leq h^2 M^2 f \leq \delta f,$$

we speculate $T : B(\xi,\delta f) \to B(\xi,\delta f)$ is a contractive mapping.

Finally, we prove that $(B(\xi,\delta f), d)$ is complete. In fact, suppose $\{x_n\}$ is a Cauchy sequence in $B(\xi,\delta f)$. Then $\{x_n\}$ is also a Cauchy sequence in X. Since (X,d) is complete, there is $x \in X$ such that $x_n \to x$ $(n \to \infty)$. So, for each $c \in \operatorname{int} P$, there exists N, whenever $n > N$, we obtain $d(x_n,x) \ll c$. Thus, it follows from

$$d(\xi,x) \leq d(x_n,\xi) + d(x_n,x) \leq \delta f + c$$

and Lemma 1.12 that $d(\xi,x) \leq \delta f$, which means $x \in B(\xi,\delta f)$, that is, $(B(\xi,\delta f),d)$ is complete.

Owing to the above statement, all the conditions of Theorem 2.1 are satisfied. Hence, T has a unique fixed point $x(t) \in B(\xi,\delta f)$. That is to say, there exists a unique solution of (3.1). □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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