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FIXED POINT THEOREMS ON GENERALIZED FUZZY METRIC SPACES

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Abstract

In this paper, a generalized fuzzy metric space is defined and shown to be a proper generalization of a fuzzy metric space. Besides, the results corresponding to Banach's and Ciric's fixed point theorems are obtained under our postulates.

Keywords: Fuzzy generalized metric space, Fuzzy contraction maps, Fuzzy almost contraction, f-orbitally complete generalized fuzzy metric space.

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1. Introduction

The concept of fuzzy set was introduced by Zadeh [5] in 1965. Since its initiation, several mathematicians have worked with fuzzy sets in different branches of Mathematics. The notion of fuzzy metric evolved in two different perspectives. One group of mathematicians consider a fuzzy metric to be a non-negative real-valued function on the collection of all fuzzy points on a set X, satisfying a list of axioms similar to those of a general metric; while another group imposes the fuzziness on the metric itself rather than the points of the space. Our approach in this paper is along the first line of development, initiated by Hu [2].

Following is the existing definition of a fuzzy metric space [4], that we have considered in this article:-

1.1. Definition. Let X be a crisp set and χ the collection of all fuzzy points on X. A function $d : \chi \times \chi \to \mathbb{R}^+$ is said to be a fuzzy metric on X, if the following conditions are satisfied :

(1) $\forall x_\alpha, y_\beta \in \chi$ with $x_\alpha \leq y_\beta$ we have $d(y_\beta, x_\alpha) = 0$,
(2) $\forall x_\alpha, y_\beta \in \chi$ with $x_\alpha \nleq y_\beta$ we have $d(y_\beta, x_\alpha) > 0$,

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(3) $\forall x_\alpha, y_\beta, z_\gamma \in \chi$,

$$d(x_\alpha, y_\beta) \leq d(x_\alpha, z_\gamma) + d(z_\gamma, y_\beta),$$

(4) $\forall x_\alpha, y_\beta \in \chi$ we have $d(x_\alpha, y_\beta) = d(y_{1-\beta}, x_{1-\alpha})$,

(5) d is continuous for membership grade, i.e. $\forall x_\alpha \in \chi$ with $\alpha \in (0,1)$, and given any $\varepsilon > 0, \exists\, \delta > 0$ such that for all β with $|\beta - \alpha| < \delta$ we have $d(x_\alpha, y_\beta) < \varepsilon$.

Fixed point theorems similar to those of Banach and Ciric were fruitfully achieved [1], [3] in generalized metric spaces. In this paper, our prime objective is to prove similar fixed point theorems in the setting of generalized fuzzy metric spaces.

In this context, we state here the definition of generalized metric space.

1.2. Definition. [1] Let X be a set and $d : X \times X \to \mathbb{R}^+$ a mapping such that for all $x, y \in X$ and for all distinct points ξ, η, each of them different from x and y, one has

- $d(x, y) = 0 \iff x = y$,
- $d(x, y) = d(y, x)$,
- $d(x, y) \leq d(x, \xi) + d(\xi, \eta) + d(\eta, y)$.

Then (X, d) is called a *generalized metric space*.

As in the case of a metric, such spaces (X, d) become topological spaces with a neighbourhood basis given by

$$\mathbb{B} = \{B(x, r) | x \in X, r \in \mathbb{R}^+ - 0\}.$$

Also, in [1], an example is cited to establish that a generalized metric space need not be a metric space in general.

The first section of our paper introduces the notion of a generalized fuzzy metric space (in short, GFMS), and shows that all fuzzy metric spaces are generalized fuzzy metric spaces but that the converse need not hold. Besides, a few terms concerning GFMS are also defined that we require subsequently. The second and third sections are devoted to establishing Banach's fixed point theorem for fuzzy contraction maps and Ciric's fixed point theorem for fuzzy almost contraction maps, respectively.

In what follows, x_α, $y_\alpha, \ldots$ always denote fuzzy points on a crisp set X, defined as

$$x_\alpha(z) = \begin{cases} \alpha, & \text{for } z = x \\ 0, & \text{otherwise.} \end{cases}$$

2. Generalized fuzzy metric spaces

2.1. Definition. Let X be a crisp set and χ the collection of all fuzzy points on X. A function $d : \chi \times \chi \to \mathbb{R}^+$ is said to be a *generalized fuzzy metric* on X, if the following conditions are satisfied:

(1) $\forall x_\alpha, y_\beta \in \chi$ with $x_\alpha \leq y_\beta$ we have $d(y_\beta, x_\alpha) = 0$,

(2) $\forall x_\alpha, y_\beta \in \chi$ with $x_\alpha \nleq y_\beta$ we have $d(y_\beta, x_\alpha) > 0$,

(3) $\forall x_\alpha, y_\beta, z_\gamma, t_\delta \in \chi$ where none of z_γ or t_δ equals x_α or y_β,

$$d(x_\alpha, y_\beta) \leq d(x_\alpha, z_\gamma) + d(z_\gamma, t_\delta) + d(t_\delta, y_\beta),$$

(4) $\forall x_\alpha, y_\beta \in \chi$ we have $d(x_\alpha, y_\beta) = d(y_{1-\beta}, x_{1-\alpha})$,

(5) d is continuous for membership grade, i.e. $\forall x_\alpha \in \chi$ with $\alpha \in (0,1)$, and given any $\varepsilon > 0, \exists \delta > 0$ such that for all β with $|\beta - \alpha| < \delta$ we have $d(x_\alpha, y_\beta) < \varepsilon$.

Then (X, d) is called a *generalized fuzzy metric space*. In what follows, we shall always refer to a generalized fuzzy metric space as a GFMS, and the collection of all fuzzy points on X by χ.

2.2. Remark. $t_\delta = z_\gamma$ in Condition 3 of the above definition shows that a fuzzy metric space is a generalized fuzzy metric space.

The following example shows that a GFMS is not necessarily a fuzzy metric space.

2.3. Example. Consider $X = \{a, b, c, e\}$. Define $d_1 : X \times X \to \mathbb{R}^+$ by

$$d_1(a, b) = d_1(b, a) = 3,$$
$$d_1(a, c) = d_1(c, a) = d_1(b, c) = d_1(c, b) = 1,$$
$$d_1(a, e) = d_1(e, a) = d_1(b, e) = d_1(e, b) = d_1(c, e) = d_1(e, c) = 2, \text{ and}$$
$$d_1(x, x) = 0, \forall x \in X.$$

It is shown by Branciari in [1] that d_1 is a generalized metric without being a metric. Now, we define $d : \chi \times \chi \to \mathbb{R}^+$ as follows:

$$d(x_\alpha, y_\beta) = \begin{cases} \max[d_1(x, y),\ |\beta - \alpha|\,], & \text{if } (x \neq y) \text{ or } (\beta > \alpha) \\ 0, & \text{otherwise.} \end{cases}$$

This d is a generalized fuzzy metric. But $d(x_\alpha, y_\beta) = 3 > 2 = d(x_\alpha, z_\gamma) + d(z_\gamma, y_\beta)$ shows that d is not a fuzzy metric on X.

We define a few terms concerning a GFMS:

2.4. Definition. Let (X, d) be a GFMS and $\alpha \in (0, 1)$.

(1) A sequence $\{(x_n)_\alpha\}$ of fuzzy points on X is said to *α-converge to a fuzzy point* x_α, if for any $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $d((x_n)_\alpha, x_\alpha) < \epsilon$.

(2) A sequence $\{(x_n)_\alpha\}$ of fuzzy points on X is said to *α-dually converge to a fuzzy point* x_α, if for any $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $d(x_\alpha, (x_n)_\alpha) < \epsilon$.

In other words, $\{(x_n)_\alpha\}$, α-dually converges to x_α if and only if $\{(x_n)_{1-\alpha}\}$, $(1 - \alpha)$-converges to $x_{1-\alpha}$.

2.5. Definition. Let (X, d) be a GFMS and $\alpha \in (0, 1)$. A sequence $\{(x_n)_\alpha\}$ of fuzzy points on X is said to be a *Cauchy sequence* if for any $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and $i = 1, 2, \ldots$, $d((x_n)_\alpha, (x_{n+i})_\alpha) < \epsilon$.

2.6. Definition. A GFMS is said to be *complete*, if every Cauchy sequence $\{(x_n)_\alpha\}$ of fuzzy points on X, α-converges to some x_α.

2.7. Definition. Let (X, d) be a GFMS and $\alpha \in [0, 1]$. A function $f : X \to X$ is said to be

(1) A *fuzzy contraction* if there exists some $c \in (0, 1)$ such that

$$d(f(x_\alpha), f(y_\alpha)) \leq c \cdot d(x_\alpha, y_\alpha).$$

The constant c is called the *constant of contraction.*

(2) A *fuzzy almost quasi-contraction* if there exists some $c \in (0, 1)$ such that

$$d(f(x_\alpha), f(y_\alpha)) \leq c \cdot \max[\, d(x_\alpha, f(y_\alpha)),\ d(x_\alpha, y_\alpha)\,].$$

The constant c is called the *constant of fuzzy almost contraction.*

2.8. Definition. If (X, d) is a GFMS and $f : X \to X$ a fuzzy almost quasi-contraction, then for all $n \in \mathbb{N}$,

$$O(x_\alpha, n) = \{x_\alpha, f(x_\alpha), f^2(x_\alpha), ..., f^n(x_\alpha)\}$$

and

$$O(x_\alpha, \infty) = \{x_\alpha, f(x_\alpha), f^2(x_\alpha), ..., f^n(x_\alpha), ...\}$$

Further,

$$\delta(O(x_\alpha, n)) = \sup_{0 \le i \le n} d(x_\alpha, f^i(x_\alpha)).$$

2.9. Remark. Since for each $n \in \mathbb{N}$, $\{1, 2, \ldots, n\}$ is a finite set, it is easy to see that $\delta(O(x_\alpha, n)) = d(x_\alpha, f^k(x_\alpha))$ for some $k \in \{1, 2, \ldots, n\}$. Also, $\delta(O(x_\alpha, 1)) \le \delta(O(x_\alpha, 2)) \le \cdots$.

2.10. Definition. Let (X, d) be a GFMS and $f : X \to X$ be a fuzzy almost quasi contraction on X. Then the GFMS (X, d) is said to be f*-orbitally complete*, if, for each $x_\alpha \in \chi$, every Cauchy sequence in $O(x_\alpha, \infty)$, α-converges in X.

2.11. Remark. It is easy to see that in an f-orbitally complete GFMS (X, d), $\forall x_\alpha \in \chi$, if $\{f^n(x_\alpha)\}$ is a Cauchy sequence then $\{f^n(x_\alpha)\}$, α-dually converges in X.

3. Banach's fixed point theorem for fuzzy contraction maps on a GFMS

The well known Banach fixed point theorem for contraction maps on metric spaces was successfully generalized by Branciari [1] for contraction maps on generalized metric spaces. In this section, we further expand its scope by proving a fixed point theorem for fuzzy contraction maps on generalized fuzzy metric spaces. To establish such result, we require the following theorem.

3.1. Theorem. *In a GFMS (X, d), a sequence of fuzzy points $\{(x_n)_\alpha\}$ on X with $0 \le \alpha \le 1/2$, converges to at most one fuzzy point.*

Proof. If possible, let $\{(x_n)_\alpha\}$ converges to x_α and y_α with $x_\alpha \ne y_\alpha$, for $0 \le \alpha \le 1/2$. As $x_\alpha \ne y_\alpha$, $x \ne y$.

For any $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $\forall n \ge n_0$,

$$d((x_n)_\alpha, x_\alpha) < \epsilon/2 \tag{3.1}$$

and

$$d((x_n)_\alpha, y_\alpha) < \epsilon/2. \tag{3.2}$$

Now,

$$\begin{aligned} 0 \le \alpha \le 1/2 &\implies \alpha \le (1-\alpha) \\ &\implies z_\alpha \le z_{1-\alpha},\ \forall z \in X \\ &\implies d(z_{1-\alpha}, z_\alpha) = 0. \end{aligned} \tag{3.3}$$

So, by using Equations (3.1), (3.2) and (3.3),

$$\begin{aligned} d(x_{1-\alpha}, y_\alpha) &\le d(x_{1-\alpha}, (x_n)_{1-\alpha}) + d((x_n)_{1-\alpha}, (x_n)_\alpha)) + d((x_n)_\alpha, y_\alpha) \\ &= d((x_n)_\alpha, x_\alpha) + d((x_n)_{1-\alpha}, (x_n)_\alpha)) + d((x_n)_\alpha, y_\alpha) \\ &< \epsilon/2 + 0 + \epsilon/2 = \epsilon. \end{aligned}$$

Since $x \ne y$, $d(x_{1-\alpha}, y_\alpha) > 0$. The arbitrariness of ϵ gives rise to a contradiction. Hence, $x_\alpha = y_\alpha$. □

3.2. Corollary. *In a GFMS, a sequence of fuzzy points $\{(x_n)_\alpha\}$ on X with $1/2 \le \alpha \le 1$, α-dually converges to at most one fuzzy point.*

Proof. Suppose $\{(x_n)_\alpha\}$ is a sequence of fuzzy points on a GFMS (X, d) that α-dually converges to two distinct points x_α and y_α. Then, by α-dual convergence we get that $\{(x_n)_{1-\alpha}\}$, $(1-\alpha)$-converges to $x_{1-\alpha}$ and $y_{1-\alpha}$. Since $1/2 \le \alpha \le 1 \implies 0 \le (1-\alpha) \le 1/2$, in view of the above Theorem this is not possible. □

3.3. Theorem. *Let (X,d) be a complete GFMS and $f : X \to X$ a fuzzy contraction. Then for any α with $0 \le \alpha \le 1/2$,*

(1) *There exists a fuzzy point a_α such that for all $x_\alpha \in \chi$, $\{f(x_n)_\alpha\}$, α-converges to a_α.*
(2) *$f(a_\alpha) = a_\alpha$ and for each e_α with $f(e_\alpha) = e_\alpha$ we have $e_\alpha = a_\alpha$.*
(3) *$\forall n \in \mathbb{N}$,*

$$d(f^n(x_\alpha), a_\alpha) \le \frac{c^n}{(1-c)} \max[\, d(x_\alpha, f(x_\alpha)),\ d(x_\alpha, f^2(x_\alpha))\,],$$

where $0 < c < 1$ is the constant of fuzzy contraction of f.

Proof. Let x_α be any fuzzy point on X with $0 \le \alpha \le 1/2$. Consider the sequence of fuzzy points $\{f(x_n)_\alpha\}$. If x_α is a periodic point then there exists some $\gamma \in \mathbb{N}$ such that $f^\gamma(x_\alpha) = x_\alpha$. So,

$$d(x_\alpha, f(x_\alpha)) = d(f^\gamma(x_\alpha), f^{\gamma+1}(x_\alpha)) \le c^\gamma d(x_\alpha, f(x_\alpha))$$

(as f is a fuzzy contraction). Thus, $(1-c^\gamma)d(x_\alpha, f(x_\alpha)) \le 0$. Since $c \le 1$, $d(x_\alpha, f(x_\alpha)) = 0$ and hence,

$$f(x_\alpha) \le x_\alpha. \tag{3.4}$$

Now,

$$f(x_\alpha)(z) = \begin{cases} \sup_{y \in f^{-1}(z)} x_\alpha(y), & \text{for } f^{-1}(z) \ne \phi \\ 0, & \text{otherwise,} \end{cases}$$

So, for $y = x$, i.e., $x \in f^{-1}(z)$, $f(x_\alpha)(z) = \alpha$ and for $y \ne x$ or $f^{-1}(z) = \phi$, $f(x_\alpha)(z) = 0$. That is,

$$f(x_\alpha)(z) = \begin{cases} \alpha, & \text{for } x \in f^{-1}(z) \\ 0, & \text{otherwise,} \end{cases}$$

i.e.,

$$f(x_\alpha) = (f(x))_\alpha.$$

Hence, $\forall z \in X$,

$$f(x_\alpha)(z) \le x_\alpha(z) \implies (f(x))_\alpha(z) \le x_\alpha(z) \implies f(x) = x \text{ (as } \alpha > 0) \implies f(x_\alpha) = x_\alpha.$$

Thus we may assume in the sequel that $f^n(x_\alpha) \ne f^m(x_\alpha)$ for all $n \ne m$.

The following two inequalities are easy to obtain: $\forall\, y_\beta \in \chi$, and for $k = 2, 3, 4, \ldots$,

$$d(y_\beta, f^{2k}(y_\beta)) \le \sum_{i=1}^{2k-3} c_i d(y_\beta, f(y_\beta)) + c^{2k-2} d(y_\beta, f^2(y_\beta)), \tag{3.5}$$

and for $k = 0, 1, 2, \ldots$,

$$d(y_\beta, f^{2k+1}(y_\beta)) \le \sum_{i=0}^{2k} c_i d(y_\beta, f(y_\beta)). \tag{3.6}$$

Using these equations we obtain,

$$d(f^n(x_\alpha), f^{n+m}(x_\alpha)) \le \frac{c^n}{(1-c)} \max[\, d(x_\alpha, f(x_\alpha)),\ d(x_\alpha, f^2(x_\alpha))\,]. \tag{3.7}$$

As $n \to \infty$, RHS $\to 0$ and consequently, $\{f^n(x_\alpha)\}$ becomes a Cauchy sequence of fuzzy points on X. By the completeness of X, there is a fuzzy point a_α such that $\{f^n(x_\alpha)\}$, α-converges to a_α. Now, $d(f^{n+1}(x_\alpha), f(a_\alpha)) \le c \cdot d(f^n(x_\alpha), a_\alpha) \to 0$ as $n \to \infty$. Hence,

$f(a_\alpha)$ is another fuzzy point to which $\{f^n(x_\alpha)\}$, α-converges. In view of Theorem 3.1, $f(a_\alpha) = a_\alpha$.

Now suppose, $f(e_\alpha) = e_\alpha$ for some $e_\alpha \in \chi$. Then

$$\begin{aligned} d(a_\alpha, e_\alpha) = d(f(a_\alpha), f(e_\alpha)) \le c \cdot d(a_\alpha, e_\alpha) &\implies (1-c)\cdot d(a_\alpha, e_\alpha) \le 0 \\ &\implies d(a_\alpha, e_\alpha) = 0 \text{ as } 0 \le c < 1 \\ &\implies e_\alpha \le a_\alpha. \end{aligned}$$

Similarly, $a_\alpha \le e_\alpha$. To get the third part of the theorem, we simply take $m \to \infty$ in Equation (3.7). □

3.4. Corollary. *Let (X, d) be a complete GFMS and $f : X \to X$ a fuzzy contraction. Then for any α with $1/2 \le \alpha < 1$,*

(1) *There exists a fuzzy point a_α such that for all $x_\alpha \in \chi$, $f^n(x_\alpha)$, α-dual converges to a_α.*
(2) *$f(a_\alpha) = a_\alpha$ and for each e_α with $f(e_\alpha) = e_\alpha$ we have $a_\alpha = e_\alpha$.*
(3) *$\forall n \in \mathbb{N}$,*

$$d(f^n(x_\alpha), a_\alpha) \le \frac{c^n}{(1-c)} \max[\, d(x_\alpha, f(x_\alpha)),\ d(x_\alpha, f^2(x_\alpha))\,],$$

where $0 < c < 1$ is the constant of fuzzy contraction of f.

Proof. Since $1/2 \le \alpha < 1$, choose $\beta = 1 - \alpha$ so that $0 < \beta \le 1/2$. So, by Theorem 3.3, $\{f^n(x_n)_\beta\}$, β-converges to a_β and hence $\{f^n(x_n)_\alpha\}$, α-dually converges to $f(a_\alpha)$. In view of Corollary 3.2, $f(a_\alpha) = a_\alpha$.

The rest of the proof follows the same line as that of Theorem 3.3. □

Combining Theorem 3.3 and Corollary 3.4, we get the following:

3.5. Theorem. *Let (X, d) be a GFMS and $f : X \to X$ a fuzzy contraction. Then for every α with $0 < \alpha < 1$, f has a unique fixed point.*

4. Ciric's fixed point theorem for fuzzy almost contraction maps on a GFMS

Ljubomir Ciric's fixed point theorem for quasi-contractions on metric spaces was successfully generalized by Lahiri and Das [3] for such maps on generalized metric spaces. Here we present a similar type of fixed point theorem for fuzzy almost quasi-contractions on a GFMS.

To obtain the main result of this section, we require the following couple of lemmas :

4.1. Lemma. *If (X, d) is GFMS and $f : X \to X$ is fuzzy almost quasi-contraction with constant c, then $\forall\, m \in \mathbb{N}$ and $\forall\, \alpha \in [0, 1]$,*

$$d(f(x_\alpha), f^m(x_\alpha)) \le c \cdot \delta(O(x_\alpha, m)).$$

Proof. We have,

$$\begin{aligned} d(f(x_\alpha), f^m(x_\alpha)) &\le c \cdot \max[\, d(x_\alpha, f^m(x_\alpha)),\ d(x_\alpha, f^{m-1}(x_\alpha))\,] \\ &\le c \cdot \max\{\, d(x_\alpha, f^i(x_\alpha)) : i = 1, 2, \dots, m\} \\ &= c \cdot \delta(O(x_\alpha, m)), \end{aligned}$$

$\forall\, m \in \mathbb{N}$. □

4.2. Lemma. *If (X,d) is GFMS and $f : X \to X$ a fuzzy almost quasi-contraction with constant c, then $\forall\, n \in \mathbb{N}$, $\forall\, \alpha \in [0,1]$ and $\forall\, x_\alpha \in \chi$,*

$$\delta(O(x_\alpha, n)) \le \frac{1}{(1-c)} \max[\, d(x_\alpha, f(x_\alpha)),\ d(x_\alpha, f^2(x_\alpha))\,].$$

Proof. By the definition of $\delta(O(x_\alpha, n))$ we have $\delta(O(x_\alpha, n)) = d(x_\alpha, f^k(x_\alpha))$ for some k with $1 \le k \le n$.

If $k = 1$ or 2, then

$$\begin{aligned}(1-c)\delta(O(x_\alpha, n)) &= (1-c) \cdot d(x_\alpha, f^k(x_\alpha))\\ &\le d(x_\alpha, f^k(x_\alpha))\\ &\le \max_{k=1,2}\{d(x_\alpha, f^k(x_\alpha))\}.\end{aligned}$$

Therefore,

$$\delta(O(x_\alpha, n)) \le \frac{1}{(1-c)} \max[\, d(x_\alpha, f(x_\alpha)),\ d(x_\alpha, f^2(x_\alpha))\,].$$

Let k be a positive integer such that $3 \le k \le n$. If $x_\alpha = f(x_\alpha)$, $x_\alpha = f^2(x_\alpha)$ or $f(x_\alpha) = f^2(x_\alpha)$, then the result follows trivially. So we assume, x_α, $f(x_\alpha)$ and $f^2(x_\alpha)$ are all distinct.

Let $f^k(x_\alpha)$ be a fuzzy point other than $f(x_\alpha)$ and $f^2(x_\alpha)$.

$$\begin{aligned}d(x_\alpha, f^k(x_\alpha)) &\le d(x_\alpha, f(x_\alpha)) + d(f(x_\alpha), f^2(x_\alpha)) + d(f^2(x_\alpha), f^k(x_\alpha))\\ &\le d(x_\alpha, f(x_\alpha)) + c \cdot \max\{d(x_\alpha, f^2(x_\alpha)), d(x_\alpha, f(x_\alpha))\}\\ &\qquad + d(f(f(x_\alpha)), f^{k-1}(f(x_\alpha)))\\ &\le d(x_\alpha, f(x_\alpha)) + c \cdot \max\{d(x_\alpha, f^2(x_\alpha)), d(x_\alpha, f(x_\alpha))\}\\ &\qquad + c \cdot \delta(O(f(x_\alpha), k-1)) \text{ (by Lemma 4.1)}\\ &= (1+c) \cdot \max\{d(x_\alpha, f^2(x_\alpha)), d(x_\alpha, f(x_\alpha))\}\\ &\qquad + c \cdot \delta(O(f(x_\alpha), k-1))\\ &= (1+c) \cdot \max\{d(x_\alpha, f^2(x_\alpha)), d(x_\alpha, f(x_\alpha))\}\\ &\qquad + c \cdot d(f(x_\alpha), f^m(f(x_\alpha)) \text{ (for some } m \le (k-1))\\ &\le (1+c) \cdot \max\{d(x_\alpha, f^2(x_\alpha)), d(x_\alpha, f(x_\alpha))\\ &\qquad + c \cdot c \cdot \delta(O(x_\alpha, m+1)) \text{ (by Lemma 4.1)}\\ &\le (1+c) \cdot \max\{d(x_\alpha, f^2(x_\alpha)), d(x_\alpha, f(x_\alpha)) + c^2 \cdot \delta(O(x_\alpha, n))\\ &\qquad \text{(as } m+1 \le k \le n).\end{aligned}$$

Therefore,

$$\begin{aligned}\delta(O(x_\alpha, n)) &= d(x_\alpha, f^k(x_\alpha))\\ &\le (1+c) \cdot \max\{d(x_\alpha, f(x_\alpha)), d(x_\alpha, f^2(x_\alpha))\} + c^2 \cdot \delta(O(x_\alpha, n)),\end{aligned}$$

so $(1-c^2) \cdot \delta(O(x_\alpha, n)) \le (1+c) \cdot \max\{d(x_\alpha, f(x_\alpha)), d(x_\alpha, f^2(x_\alpha))\}$. Hence,

$$\delta(O(x_\alpha, n)) \le \tfrac{1}{(1-c)} \max\{d(x_\alpha, f(x_\alpha)), d(x_\alpha, f^2(x_\alpha))\}. \qquad \square$$

Since the above lemma guarantees the boundedness of $\{\delta(O(x_\alpha, n)) : n \in \mathbb{N}\}$, we define:

4.3. Definition. For each $x_\alpha \in \chi$, $(\alpha \in [0,1])$,

$$\delta(O(x_\alpha,\infty) = \sup_{n\in\mathbb{N}} \delta(O(x_\alpha,n)).$$

In view of Lemma 4.2 and Definition 4.3, we have,

4.4. Lemma. *If (X,d) is a GFMS and $f : X \to X$ a fuzzy almost quasi-contraction with constant c, then $\forall\, \alpha \in [0,1]$ and $\forall\, x_\alpha \in \chi$,*

$$\delta(O(x_\alpha,\infty)) \le \frac{1}{(1-c)} \max[d(x_\alpha, f(x_\alpha)), d(x_\alpha, f^2(x_\alpha))].$$

4.5. Theorem. *Let (X,d) be an f-orbitally complete GFMS and $f : X \to X$ a fuzzy almost quasi contraction with constant c. Then*

(1) *For each fuzzy point x_α on X, $(0 \le \alpha \le 1)$, the sequence $\{f^n(x_\alpha)\}$, α-converges to some a_α.*
(2) *$f(a_\alpha) = a_\alpha$, and for any e_α on X with $f(e_\alpha) = e_\alpha$ we have $a_\alpha = e_\alpha$.*
(3) $d(f^n(x_\alpha), x_\alpha) \le \frac{c^n}{(1-c)} \max\{\, d(x_\alpha, f(x_\alpha)),\ d(x_\alpha, f^2(x_\alpha))\,\}$.

Proof. For each x_α, consider the sequence $\{f^n(x_\alpha)\}$. Now,

$$\begin{aligned} d(f^n(x_\alpha), f^{n+k}(x_\alpha)) &= d(f(f^{n-1}(x_\alpha)), f^{k+1}(fn-1(x_\alpha))) \\ &\le c\cdot\delta(f^{n-1}(x_\alpha), k+1) \\ &= c\cdot d(f^{n-1}(x_\alpha), f^{n-1+m}(x_\alpha)) \text{ (for some } m \le k+1) \\ &\le c^2\cdot\delta(O(f^{n-2}(x_\alpha), m+1)) \text{ (by Lemma 4.1)} \\ &\le c^2\cdot\delta(O(f^{n-2}(x_\alpha), k+2)) \end{aligned}$$

Proceeding in this way, we get

$$\begin{aligned} d(f^n(x_\alpha, f^{n+k}(x_\alpha)) &\le c^n\cdot\delta(O(x_\alpha, k+n)) \\ &\le \frac{c^n}{(1-c)} \max\{d(x_\alpha, f(x_\alpha)),\ d(x_\alpha, f^2(x_\alpha))\} \end{aligned} \tag{4.1}$$

As $n \to \infty$, $c^n \to 0$ and hence RHS $\to 0$, proving $\{f^n(x_\alpha)\}$ to be a Cauchy sequence. Since $\{f^n(x_\alpha)\}$ is a Cauchy sequence contained in $O(x_\alpha,\infty)$, the f-orbital completeness of X enures that $\{f^n(x_\alpha)\}$, α-converges to some a_α.

As $k \to \infty$ in Equation (4.1), we get

$$d(f^n(x_\alpha), a_\alpha) \le \frac{c^n}{(1-c)} \max\{d(x_\alpha, f(x_\alpha)), d(x_\alpha, f^2(x_\alpha))\}.$$

Now we shall show that $f(a_\alpha) = a_\alpha$, and that such an a_α is unique. Now,

$$d(f(a_\alpha), f^{n+1}(x_\alpha)) \le c\cdot\max\{d(a_\alpha, f^n(x_\alpha)), d(a_\alpha, f^{n+1}(x_\alpha))\}.$$

As $n \to \infty$,

$$\begin{aligned} d(f(a_\alpha), a_\alpha) \le c\cdot d(a_\alpha, a_\alpha) = 0 &\implies d(f(a_\alpha), a_\alpha) = 0 \\ &\implies a_\alpha \le f(a_\alpha). \end{aligned} \tag{4.2}$$

Again,

$$d(f^{n+1}(a_\alpha), f(a_\alpha)) \le c\cdot\max\{d(f^n(a_\alpha), f(a_\alpha)),\ d(f^n(a_\alpha), a_\alpha)\}.$$

As $n \to \infty$ and $f^n(x_\alpha)$, α-dual converges to a_α, we get

$$\begin{aligned} d(a_\alpha, f(a_\alpha)) \le c\cdot d(a_\alpha, f(a_\alpha)) &\implies (1-c)\cdot d(a_\alpha, f(a_\alpha)) \le 0 \\ &\implies d(a_\alpha, f(a_\alpha)) = 0 \\ &\implies f(a_\alpha) \le a_\alpha. \end{aligned} \tag{4.3}$$

Hence, Equations (4.2) and (4.3) imply $f(a_\alpha) = a_\alpha$. If e_α is such that $f(e_\alpha) = e_\alpha$, then

$$d(f(a_\alpha), f(e_\alpha)) \leq c \cdot \max\{d(a_\alpha, f(a_\alpha)),\ d(a_\alpha, e_\alpha)\} = c \cdot d(a_\alpha, e_\alpha),$$

i.e.,

$$(1 - c) \cdot d(a_\alpha, e_\alpha) \leq 0 \implies d(a_\alpha, e_\alpha) = 0 \implies e_\alpha \leq a_\alpha.$$

Changing the role of a_α and e_α we get the reverse inclusion. Hence, $a_\alpha = e_\alpha$, completing the proof. □

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References

[1] Branciari, A. *A fixed point theorem of Banach-Caccioppoli type on a class of generalized metric spaces*, Publ. Math. Debrecen **57** (1-2),31–37, 2000.
[2] Hu, C.-M. *Fuzzy metrics*, Nature J. **5** (12), 947, 1982.
[3] Lahiri, B. K. and Das, P. *Fixed point of a Ljubomir Ciric quasi-contraction mapping in a generalized metric space*, Publ. Math. Debrecen **61** (3-4), 589–594, 2002.
[4] Palaniappan, N. *Fuzzy Topology (Second Edition)* (Narosa Publishing House, India).
[5] Zadeh, L. A. *Fuzzy Sets*, Information and Control **8**, 338–353, 1965.