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Review Article

Fixed Point Theory and the Ulam Stability

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The fixed point method has been applied for the first time, in proving the stability results for functional equations, by Baker (1991); he used a variant of Banach's fixed point theorem to obtain the stability of a functional equation in a single variable. However, most authors follow the approaches involving a theorem of Diaz and Margolis. The main aim of this survey is to present applications of different fixed point theorems to the theory of stability of functional equations, motivated by a problem raised by Ulam in 1940.

1. Introduction

Speaking of the stability of a functional equation we follow a question raised in 1940 by Ulam, concerning approximate homomorphisms of groups (see [1]). The first partial answer (in the case of Cauchy's functional equation in Banach spaces) to Ulam's question was given by Hyers (see [2]). After his result a great number of papers (see for instance monographs [3–5], survey articles [6–14], and the references given there) on the subject have been published, generalizing Ulam's problem and Hyers's theorem in various directions and to other (not necessarily functional) equations.

The method used by Hyers in [2] (quite often called the *direct method*) has been successfully applied for study of the stability of large variety of equations, but unfortunately, as it was shown in [15], it does not work in numerous significant cases. Apart from it, there are also several other efficient approaches to the Hyers-Ulam stability, using different tools, for example, the method of *invariant means* (introduced in [16]), the method based on *sandwich theorems* (see [17]), and the method using the concept of *shadowing* (see [18]).

It this paper we discuss the *fixed point method*, which is the second most popular technique of proving the stability of functional equations. It was used for the first time by Baker (see [19]) who applied a variant of Banach's fixed point theorem to obtain the Hyers-Ulam stability of a functional equation in a single variable. At present, numerous authors follow Radu's approach (see [20]) and make use of a theorem of Diaz and Margolis. Our aim is to show connections between different fixed point theorems and the theory of stability, inspired by the problem of Ulam (see [5, 7, 9]).

The paper contains both classical and more recent results. In Section 2 we present applications of some classical fixed point theorems. Section 3 shows a somewhat different (but still fixed point) approach, when the results on the stability are simple consequences of the proved (new) fixed point theorems. In Section 4 we deal with the stability of the fixed point equation and its generalization. Section 5 contains final remarks.

In the paper $\mathbb{N}$ denotes the set of positive integers and we put $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $\mathbb{R}_+ := [0, \infty)$.

2. Applications of Known Fixed Point Theorems

2.1. (Some Variants of) Banach's Theorem. The fixed point method was used for the investigation of the Hyers-Ulam stability of functional equations for the first time by Baker in [19], where actually he applied the following variant of Banach's fixed point theorem.

Theorem 1 ([19], Theorem 1). *Assume that (Y,ρ) is a complete metric space and $T: Y \to Y$ is a contraction (i.e., there is a $\lambda \in [0,1)$ such that $\rho(T(x),T(y)) \le \lambda\rho(x,y)$ for all $x,y \in Y$). Then T has a unique fixed point $p \in Y$. Moreover,*

$$\rho(u,p) \le \frac{\rho(u,T(u))}{1-\lambda}, \quad u \in Y. \tag{1}$$

He obtained in this way the subsequent result concerning the stability of a quite general functional equation in a single variable.

Theorem 2 ([19], Theorem 2). *Let S be a nonempty set, (X,d) be a complete metric space, $\varphi: S \to S$, $F: S \times X \to X$, $\lambda \in [0,1)$ and*

$$d(F(t,u),F(t,v)) \le \lambda d(u,v), \quad t \in S,\ u,v \in X. \tag{2}$$

If $g: S \to X$, $\delta > 0$ and

$$d(g(t),F(t,g(\varphi(t)))) \le \delta, \quad t \in S, \tag{3}$$

then there is a unique function $f: S \to X$ such that

$$f(t) = F(t,f(\varphi(t))), \quad t \in S, \tag{4}$$

$$d(f(t),g(t)) \le \frac{\delta}{1-\lambda}, \quad t \in S. \tag{5}$$

Theorem 2 with

$$F(t,x) := \alpha(t) + \beta(t)x, \quad t \in S,\ x \in E \tag{6}$$

gives the following.

Corollary 3 ([19], Theorem 3). *Let S be a nonempty set, E a real (or complex) Banach space, $\varphi: S \to S$, $\alpha: S \to E$, $\beta: S \to \mathbb{R}$ (or $\mathbb{C}$), $\lambda \in [0,1)$ and*

$$|\beta(t)| \le \lambda, \quad t \in S. \tag{7}$$

If $g: S \to E$, $\delta > 0$ and

$$\|g(t) - (\alpha(t) + \beta(t)g(\varphi(t)))\| \le \delta, \quad t \in S, \tag{8}$$

then there exists a unique function $f: S \to E$ such that

$$f(t) = \alpha(t) + \beta(t)f(\varphi(t)), \quad t \in S, \tag{9}$$

$$\|f(t) - g(t)\| \le \frac{\delta}{1-\lambda}, \quad t \in S. \tag{10}$$

The following stability result for a more general functional equation has been deduced in [21] from Theorem 1.

Theorem 4 ([21], Theorem 2.2). *Let S be a nonempty set, (X,d) be a complete metric space, $\varphi: S \to S$, $F: X \times X \to X$, $\lambda,\mu \in \mathbb{R}_+$ and*

$$d(F(s,u),F(t,v)) \le \mu d(s,t) + \lambda d(u,v), \quad s,t,u,v \in X. \tag{11}$$

Assume also that $g: S \to X$, $\Phi: S \to \mathbb{R}_+$ are such that

$$d(g(t),F(g(t),g(\varphi(t)))) \le \Phi(t), \quad t \in S, \tag{12}$$

and there exists an $L \in [0,1)$ with

$$\lambda\Phi(\varphi(t)) + \mu\Phi(t) \le L\Phi(t), \quad t \in S. \tag{13}$$

Then there is a unique function $f: S \to X$ such that

$$\begin{gathered} f(t) = F(f(t),f(\varphi(t))), \quad t \in S \\ d(f(t),g(t)) \le \frac{\Phi(t)}{1-L}, \quad t \in S. \end{gathered} \tag{14}$$

In some recent papers the authors applied the *weighted space method* to prove the generalized Hyers-Ulam stability properties of several nonlinear functional equations. We recall that the weighted space method uses the classical mathematical results in spaces endowed with weighted distances. In those papers, the classical mathematical result is just the Banach fixed point theorem. This new method is used to prove a stability result for (4), described in the following theorem.

Theorem 5 ([22], Theorem 2.1). *Let S be a nonempty set, (X,d) a complete metric space, and the functions $\varphi: S \to S$, $F: S \times X \to X$, $\alpha: S \to (0,\infty)$ satisfy*

$$\begin{gathered} \alpha(\varphi(t))\, d(F(t,u(\varphi(t))),F(t,v(\varphi(t)))) \\ \le \lambda\alpha(t)\, d(u(\varphi(t)),v(\varphi(t))), \end{gathered} \tag{15}$$

for any $t \in S$, $u,v \in X^S$ and $\lambda \in [0,1)$.

If $g: S \to X$ satisfies the inequality

$$d(g(t),F(t,g(\varphi(t)))) \le \alpha(t), \quad t \in S, \tag{16}$$

then there exists a solution $f: S \to X$ of (4) such that

$$d(f(t),g(t)) \le \frac{\alpha(t)}{1-\lambda}, \quad t \in S. \tag{17}$$

The results in Theorems 2, 4, and 5 have been extended in [23], where a result on the generalized Hyers-Ulam stability of the nonlinear equation

$$y(x) = F(x,y(x),y(\eta(x))) \tag{18}$$

has been obtained, also by the weighted space method. Here, S is a nonempty set, (X,d) is a complete metric space, $F: S \times X \times X \to X$ and $\eta: S \to S$ are given mappings (the unknown function in (18) is $y: S \to X$).

Theorem 6 ([23], Theorem 2). *Suppose that $L \in [0,1)$ and $\lambda,\mu: S \to \mathbb{R}_+$ satisfy*

$$\lambda(x)\varphi(x) + \mu(x)\varphi(\eta(x)) \le L\varphi(x), \quad x \in S, \tag{19}$$

for some given function $\varphi: S \to (0,\infty)$. Suppose also that $F: S \times X \times X \to X$ fulfils the inequality

$$\begin{gathered} d(F(x,u(x),u(\eta(x))),F(x,v(x),v(\eta(x)))) \\ \le \lambda(x)\, d(u(x),v(x)) + \mu(x)\, d(u(\eta(x)),v(\eta(x))), \end{gathered} \tag{20}$$

for all $x \in S$ and for all $u,v: S \to X$.

If $y : S \to X$ is a mapping with the property

$$d(y(x), F(x, y(x), y(\eta(x)))) \le \varphi(x), \quad x \in S, \tag{21}$$

then there exists a unique $y_0 : S \to X$ such that

$$\begin{aligned} y_0(x) &= F(x, y_0(x), y_0(\eta(x))), \quad x \in S, \\ d(y(x), y_0(x)) &\le \frac{\varphi(x)}{1-L}, \quad x \in S. \end{aligned} \tag{22}$$

For the proof it is enough to show that the set

$$\mathscr{Y} := \left\{ u : S \to X : \sup_{x \in S} \frac{d(u(x), y(x))}{\varphi(x)} < \infty \right\} \tag{23}$$

is a complete metric space with the weighted metric

$$\rho(u, v) = \sup_{x \in S} \frac{d(u(x), v(x))}{\varphi(x)}; \tag{24}$$

moreover, it can be proved that the nonlinear operator $T : \mathscr{Y} \to \mathscr{Y}$ given by

$$(Tu)(x) := F(x, u(x), u(\eta(x))), \tag{25}$$

is a strictly contractive self-mapping of $\mathscr{Y}$, with the Lipschitz constant $L < 1$.

On the other hand, if

$$F(x, y(x), y(\eta(x))) := g(x) \cdot y(\eta(x)) + h(x), \tag{26}$$

then (18) becomes

$$y(x) = g(x) \cdot y(\eta(x)) + h(x), \tag{27}$$

where g, η, h are given mappings and y is the unknown function. The above equation is called a *linear functional equation* and was intensively investigated by a lot of authors (e.g., Kuczma et al. in [24] obtained some results concerning monotonic, regular, and convex solutions of (27)). The following theorem contains a generalized Hyers-Ulam stability result for the above linear functional equation, obtained as a particular case of Theorem 6.

Let us consider a nonempty set S, a real (or complex) Banach space X, endowed with the norm $\|\cdot\|$ and the given functions $\eta : S \to S$, $g : S \to \mathbb{R}$ (or $\mathbb{C}$) and $h : S \to X$.

Theorem 7 ([23], Theorem 5). *Let $L \in [0, 1)$ and $\lambda, \mu : S \to \mathbb{R}_+$ satisfy*

$$\lambda(x)\varphi(x) + \mu(x)\varphi(\eta(x)) \le L\varphi(x), \quad x \in S, \tag{28}$$

for some fixed mapping $\varphi : S \to (0, \infty)$. Let $F : S \times X \times X \to X$ fulfil

$$\begin{aligned} &(|g(x)| - \mu(x)) \|u(\eta(x)) - v(\eta(x))\| \\ &\quad \le \lambda(x) \|u(x) - v(x)\|, \end{aligned} \tag{29}$$

for all $x \in S$ and for all $u, v : S \to X$. If $y : S \to X$ has the property

$$\|y(x) - g(x) y(\eta(x)) - h(x)\| \le \varphi(x), \quad x \in S, \tag{30}$$

then there exists a unique mapping $y_0 : S \to X$, defined by

$$\begin{aligned} y_0(x) &= h(x) \\ &+ \lim_{n \to \infty} \Bigg(y(\eta^n(x)) \prod_{i=0}^{n-1} g(\eta^i(x)) \\ &\qquad + \sum_{j=0}^{n-2} h(\eta^{j+1}(x)) \prod_{i=0}^{j} g(\eta^i(x)) \Bigg), \end{aligned} \tag{31}$$

for $x \in S$, such that

$$\begin{aligned} y_0(x) &= g(x) y_0(\eta(x)) + h(x), \quad x \in S, \\ \|y(x) - y_0(x)\| &\le \frac{\varphi(x)}{1-L}, \quad x \in S. \end{aligned} \tag{32}$$

The following outcome proved by the weighted space method concerns a generalized Hyers-Ulam stability for a general class of the Volterra nonlinear integral equations, in Banach spaces.

Let us consider a Banach space X over the (real or complex) field $\mathbb{K}$, an interval $I = [a, b]$ $(a < b)$ and the continuous given functions $L : I \times I \to \mathbb{R}_+$ and $\varphi : I \to (0, \infty)$. We write

$$\mathscr{C}(I, X) := \{ f : I \to X : f \text{ is continuous} \} \tag{33}$$

and denote by $\|\cdot\|$ the norm in X.

The result on stability of the nonlinear Volterra integral equation

$$y(x) = h(x) + \lambda \int_a^x G(x, t, y(t))\, dt, \quad x \in I \tag{34}$$

($y : I \to X$ is an unknown function, $h : I \to X$ and $G : I \times I \times X \to X$ are continuous given mappings and $\lambda \in \mathbb{K}$ is a fixed nonzero scalar), reads as follows.

Theorem 8 ([23], Theorem 8). *Suppose that there exists a positive constant α such that*

$$\int_a^x L(x, t)\varphi(t)\, dt \le \alpha\varphi(x), \quad x \in I. \tag{35}$$

Suppose also that $G : I \times I \times X \to X$ is a continuous function, which satisfies

$$\begin{aligned} &\|G(x, t, u(t)) - G(x, t, v(t))\| \\ &\quad \le L(x, t) \|u(t) - v(t)\|, \quad x, t \in I,\ u, v \in \mathscr{C}(I, X). \end{aligned} \tag{36}$$

If $y : I \to X$ is continuous and has the property

$$\left\| y(x) - h(x) - \lambda \int_a^x G(x, t, y(t))\, dt \right\| \le \varphi(x), \quad x \in I \tag{37}$$

and if

$$|\lambda| < \frac{1}{\alpha}, \quad (38)$$

then there exists a unique $y_0 \in \mathscr{C}(I, X)$ *such that*

$$y_0(x) = h(x) + \lambda \int_a^x G(x, t, y_0(t))\, dt, \quad x \in I,$$
$$\|y(x) - y_0(x)\| \le \frac{\varphi(x)}{1 - |\lambda|\alpha}, \quad x \in I. \quad (39)$$

Note that, if we replace in Theorem 8 the functions G and L by λG and $|\lambda| L$, respectively, and write $\mu := |\lambda|\alpha$, then the theorem takes the subsequent equivalent, and a bit simpler, form.

Theorem 9. *Suppose that there is a positive constant* $\mu < 1$ *such that*

$$\int_a^x L(x, t)\varphi(t)\, dt \le \mu\varphi(x), \quad x \in I. \quad (40)$$

Suppose also that $G : I \times I \times X \to X$ *is a continuous function, which satisfies*

$$\|G(x, t, u(t)) - G(x, t, v(t))\| \le L(x, t)\|u(t) - v(t)\|, \quad x, t \in I,\ u, v \in \mathscr{C}(I, X). \quad (41)$$

If $y : I \to X$ *is continuous and has the property*

$$\left\| y(x) - h(x) - \int_a^x G(x, t, y(t))\, dt \right\| \le \varphi(x), \quad x \in I, \quad (42)$$

then there exists a unique $y_0 \in \mathscr{C}(I, X)$ *such that*

$$y_0(x) = h(x) + \int_a^x G(x, t, y_0(t))\, dt, \quad x \in I,$$
$$\|y(x) - y_0(x)\| \le \frac{\varphi(x)}{1 - \mu}, \quad x \in I. \quad (43)$$

Next, following [25], we recall some notations.

Let X be a nonempty set, (Y, d) a complete metric space, $f : X \to Y$ and $\varphi : X \to \mathbb{R}_+$. Then $(\Delta_{f,\varphi}, \rho_{f,\varphi})$ is a complete metric space and $f \in \Delta_{f,\varphi}$, where $\Delta_{f,\varphi}$ denotes the set of all $u : X \to Y$ such that there is a real constant $K_u \ge 0$ with

$$d(u(x), f(x)) \le K_u\varphi(x), \quad x \in X, \quad (44)$$

$$\rho_{f,\varphi}(u, v) := \inf\{K \ge 0 : d(u(x), v(x)) \le K\varphi(x), x \in X\}, \quad u, v \in \Delta_{f,\varphi}. \quad (45)$$

Let $\sigma : X \to X$, $\tau : Y \to Y$, $u : X \to Y$ and $\varepsilon : X^2 \to \mathbb{R}_+$. Put

$$(T_{\sigma,\tau}u)(x) := \tau(u(\sigma(x))), \quad x \in X,$$
$$\alpha_{\sigma,\varepsilon} := \inf\{K \ge 0 : \varepsilon(\sigma(x), \sigma(x)) \le K\varepsilon(x, x), x \in X\},$$
$$\beta_{\sigma,\varphi} := \inf\{K \ge 0 : \varphi(\sigma(x)) \le K\varphi(x), x \in X\},$$
$$\gamma_\tau := \inf\{K \ge 0 : d(\tau(x), \tau(y)) \le Kd(x, y), x, y \in Y\}. \quad (46)$$

In [25], the authors used the contraction principle to get the following fixed point result.

Theorem 10 ([25], Proposition 1.1). *Let* X *be a nonempty set,* (Y, d) *be a complete metric space,* $f : X \to Y$, $\varphi : X \to \mathbb{R}_+$, $\sigma : X \to X$ *and* $\tau : Y \to Y$. *If* $T_{\sigma,\tau}f \in \Delta_{f,\varphi}$, $\beta_{\sigma,\varphi} < \infty$, $\gamma_\tau < \infty$ *and* $\beta_{\sigma,\varphi}\gamma_\tau < 1$, *then* $T_{\sigma,\tau}(\Delta_{f,\varphi}) \subset \Delta_{f,\varphi}$ *and* $T_{\sigma,\tau}$ *has a unique fixed point* f_∞ *in* $\Delta_{f,\varphi}$. *Moreover,*

$$\lim_{n\to\infty} d\left(\left(T^n_{\sigma,\tau}f\right)(x), f_\infty(x)\right) = 0, \quad x \in X,$$
$$d(f(x), f_\infty(x)) = \frac{\rho_{f,\varphi}(T_{\sigma,\tau}f, f)}{1 - \beta_{\sigma,\varphi}\gamma_\tau}\varphi(x), \quad x \in X. \quad (47)$$

Next, applying Theorem 10, they have proved the following theorem.

Theorem 11 ([25], Theorem 2.1). *Let* X *be a nonempty set,* $\delta : X \to \mathbb{R}_+$, $\varepsilon : X^2 \to \mathbb{R}_+$, $\circ : X^2 \to X$ *and* $\widehat{\sigma} : x \mapsto x \circ x$ *an automorphism of* $(X, \circ)$. *Assume also that* (Y, d) *is a complete metric space,* $\diamond : Y^2 \to Y$ *is continuous and* $\widehat{\tau} : y \mapsto y \diamond y$ *is an endomorphism of* $(Y, \diamond)$. *If* $\alpha_{\widehat{\sigma}^{-1},\varepsilon} < \infty$, $\beta_{\widehat{\sigma}^{-1},\delta} < \infty$, $\gamma_{\widehat{\tau}} < \infty$, $\gamma_{\widehat{\tau}} \max\{\alpha_{\widehat{\sigma}^{-1},\varepsilon}, \beta_{\widehat{\sigma}^{-1},\delta}\} < 1$ *and mappings* $f, g : X \to Y$ *satisfy*

$$d(f(x \circ y), g(x) \diamond g(y)) \le \varepsilon(x, y), \quad x, y \in X, \quad (48)$$

$$d(f(x), g(x)) \le \delta(x), \quad x \in X, \quad (49)$$

then there exists a unique mapping $f_\infty : X \to Y$ *such that*

$$f_\infty(x \circ y) = f_\infty(x) \diamond f_\infty(y), \quad x, y \in X, \quad (50)$$

$$d(f(x), f_\infty(x)) \le \frac{\alpha_{\widehat{\sigma}^{-1},\varepsilon}\varepsilon(x, x) + \beta_{\widehat{\sigma}^{-1},\delta}\gamma_{\widehat{\tau}}\delta(x)}{1 - \gamma_{\widehat{\tau}} \max\{\alpha_{\widehat{\sigma}^{-1},\varepsilon}, \beta_{\widehat{\sigma}^{-1},\delta}\}}, \quad x \in X,$$
$$d(g(x), f_\infty(x)) \le \frac{\alpha_{\widehat{\sigma}^{-1},\varepsilon}\varepsilon(x, x) + \delta(x)}{1 - \gamma_{\widehat{\tau}} \max\{\alpha_{\widehat{\sigma}^{-1},\varepsilon}, \beta_{\widehat{\sigma}^{-1},\delta}\}}, \quad x \in X. \quad (51)$$

Theorem 11 with $g = f$ and $\delta \equiv 0$ gives the following corollary, which corresponds to the results in [26–31].

Corollary 12 ([25], Corollary 2.1). *Let* X *be a nonempty set,* $\varepsilon : X^2 \to \mathbb{R}_+$, $\circ : X^2 \to X$ *and* $\widehat{\sigma} : x \mapsto x \circ x$ *be an automorphism of* $(X, \circ)$. *Assume also that* (Y, d) *is a complete metric space,* $\diamond : Y^2 \to Y$ *is continuous and* $\widehat{\tau} : y \mapsto y \diamond y$ *is an endomorphism of* $(Y, \diamond)$. *If* $\alpha_{\widehat{\sigma}^{-1},\varepsilon} < \infty$, $\gamma_{\widehat{\tau}} < \infty$, $\gamma_{\widehat{\tau}}\alpha_{\widehat{\sigma}^{-1},\varepsilon} < 1$ *and a mapping* $f : X \to Y$ *satisfies*

$$d(f(x \circ y), f(x) \diamond f(y)) \le \varepsilon(x, y), \quad x, y \in X, \quad (52)$$

then there exists a unique mapping $f_\infty : X \to Y$ *fulfilling* (50) *and*

$$d(f(x), f_\infty(x)) \le \frac{\alpha_{\widehat{\sigma}^{-1},\varepsilon}\varepsilon(x, x)}{1 - \gamma_{\widehat{\tau}}\alpha_{\widehat{\sigma}^{-1},\varepsilon}}, \quad x \in X. \quad (53)$$

Another consequence of Theorem 11 is the following.

Theorem 13 ([25], Theorem 3.1). *Let X be a nonempty set, $\delta, \varepsilon : X \to \mathbb{R}_+$ and $\sigma : X \to X$ be a bijection. Assume also that (Y, d) is a complete metric space and $\tau : Y \to Y$ is continuous. If $\beta_{\sigma^{-1},\varepsilon} < \infty$, $\beta_{\sigma^{-1},\delta} < \infty$, $\gamma_\tau < \infty$, $\gamma_\tau \max\{\beta_{\sigma^{-1},\varepsilon}, \beta_{\sigma^{-1},\delta}\} < 1$ and mappings $f, g : X \to Y$ satisfy (49) and*

$$d(f(\sigma(x)), \tau(g(x))) \le \varepsilon(x), \quad x \in X, \tag{54}$$

then there exists a unique mapping $f_\infty : X \to Y$ such that

$$f_\infty(\sigma(x)) = \tau(f_\infty(x)), \quad x \in X, \tag{55}$$

$$\begin{aligned} d(f(x), f_\infty(x)) &\le \frac{\beta_{\sigma^{-1},\varepsilon}\varepsilon(x) + \beta_{\sigma^{-1},\delta}\gamma_\tau\delta(x)}{1 - \gamma_\tau \max\{\beta_{\sigma^{-1},\varepsilon}, \beta_{\sigma^{-1},\delta}\}}, \quad x \in X, \\ d(g(x), f_\infty(x)) &\le \frac{\beta_{\sigma^{-1},\varepsilon}\varepsilon(x) + \delta(x)}{1 - \gamma_\tau \max\{\beta_{\sigma^{-1},\varepsilon}, \beta_{\sigma^{-1},\delta}\}}, \quad x \in X. \end{aligned} \tag{56}$$

Theorem 13 with $g = f$ and $\delta \equiv 0$ implies the following.

Corollary 14 ([25], Corollary 3.1). *Let X be a nonempty set, $\varepsilon : X \to \mathbb{R}_+$ and $\sigma : X \to X$ a bijection. Assume also that (Y, d) is a complete metric space and $\tau : Y \to Y$ is continuous. If $\beta_{\sigma^{-1},\varepsilon} < \infty$, $\gamma_\tau < \infty$, $\gamma_\tau\beta_{\sigma^{-1},\varepsilon} < 1$ and a mapping $f : X \to Y$ satisfies the inequality*

$$d(f(\sigma(x)), \tau(f(x))) \le \varepsilon(x), \quad x \in X, \tag{57}$$

then there exists a unique mapping $f_\infty : X \to Y$ fulfilling (55) and

$$d(f(x), f_\infty(x)) \le \frac{\beta_{\sigma^{-1},\varepsilon}\varepsilon(x)}{1 - \gamma_\tau\beta_{\sigma^{-1},\varepsilon}}, \quad x \in X. \tag{58}$$

Theorem 15 ([25], Theorem 2.2). *Let X be a nonempty set, $\delta : X \to \mathbb{R}_+$, $\varepsilon : X^2 \to \mathbb{R}_+$, $\circ : X^2 \to X$ and $\widehat{\sigma} : x \mapsto x \circ x$ an endomorphism of $(X, \circ)$. Assume also that (Y, d) is a complete metric space, $\diamond : Y^2 \to Y$ is continuous and $\widehat{\tau} : y \mapsto y \diamond y$ is an automorphism of $(Y, \diamond)$. If $\alpha_{\widehat{\sigma},\varepsilon} < \infty$, $\beta_{\widehat{\sigma},\delta} < \infty$, $\gamma_{\widehat{\tau}^{-1}} < \infty$, $\gamma_{\widehat{\tau}^{-1}} \max\{\alpha_{\widehat{\sigma},\varepsilon}, \beta_{\widehat{\sigma},\delta}\} < 1$ and mappings $f, g : X \to Y$ satisfy inequalities (48) and (49), then there exists a unique mapping $f_\infty : X \to Y$ such that (50) holds,*

$$\begin{aligned} d(f(x), f_\infty(x)) &\le \frac{\gamma_{\widehat{\tau}^{-1}}\varepsilon(x, x) + \delta(x)}{1 - \gamma_{\widehat{\tau}^{-1}} \max\{\alpha_{\widehat{\sigma},\varepsilon}, \beta_{\widehat{\sigma},\delta}\}}, \quad x \in X, \\ d(g(x), f_\infty(x)) &\le \frac{\gamma_{\widehat{\tau}^{-1}}(\varepsilon(x, x) + \beta_{\widehat{\sigma},\delta}\delta(x))}{1 - \gamma_{\widehat{\tau}^{-1}} \max\{\alpha_{\widehat{\sigma},\varepsilon}, \beta_{\widehat{\sigma},\delta}\}}, \quad x \in X. \end{aligned} \tag{59}$$

Theorem 15 with $g = f$ and $\delta \equiv 0$ yields the next corollary.

Corollary 16 ([25], Corollary 2.3). *Let X be a nonempty set, $\varepsilon : X^2 \to \mathbb{R}_+$, $\circ : X^2 \to X$ and $\widehat{\sigma} : x \mapsto x \circ x$ an endomorphism of X. Assume also that (Y, d) is a complete metric space, $\diamond : Y^2 \to Y$ is continuous and $\widehat{\tau} : y \mapsto y \diamond y$ is an automorphism of Y. If $\alpha_{\widehat{\sigma},\varepsilon} < \infty$, $\gamma_{\widehat{\tau}^{-1}} < \infty$, $\gamma_{\widehat{\tau}^{-1}}\alpha_{\widehat{\sigma},\varepsilon} < 1$ and a mapping $f : X \to Y$ satisfies inequality (52), then there exists a unique mapping $f_\infty : X \to Y$ such that (50) holds and*

$$d(f(x), f_\infty(x)) \le \frac{\gamma_{\widehat{\tau}^{-1}}\varepsilon(x, x)}{1 - \gamma_{\widehat{\tau}^{-1}}\alpha_{\widehat{\sigma},\varepsilon}}, \quad x \in X. \tag{60}$$

Another consequence of Theorem 15 is the following.

Theorem 17 ([25], Theorem 3.2). *Let X be a nonempty set, $\delta, \varepsilon : X \to \mathbb{R}_+$ and $\sigma : X \to X$. Assume also that (Y, d) is a complete metric space and $\tau : Y \to Y$ is a continuous bijection. If $\beta_{\sigma,\varepsilon} < \infty$, $\beta_{\sigma,\delta} < \infty$, $\gamma_{\tau^{-1}} < \infty$, $\gamma_{\tau^{-1}} \max\{\beta_{\sigma,\varepsilon}, \beta_{\sigma,\delta}\} < 1$ and mappings $f, g : X \to Y$ satisfy (49) and (48), then there exists a unique mapping $f_\infty : X \to Y$ such that (55) holds,*

$$\begin{aligned} d(f(x), f_\infty(x)) &\le \frac{\gamma_{\tau^{-1}}\varepsilon(x) + \delta(x)}{1 - \gamma_{\tau^{-1}} \max\{\beta_{\sigma,\varepsilon}, \beta_{\sigma,\delta}\}}, \quad x \in X, \\ d(g(x), f_\infty(x)) &\le \frac{\gamma_{\tau^{-1}}(\varepsilon(x) + \beta_{\sigma,\delta}\delta(x))}{1 - \gamma_{\tau^{-1}} \max\{\beta_{\sigma,\varepsilon}, \beta_{\sigma,\delta}\}}, \quad x \in X. \end{aligned} \tag{61}$$

Theorem 17 with $g = f$ and $\delta \equiv 0$ implies the following.

Corollary 18 ([25], Corollary 3.2). *Let X be a nonempty set, $\varepsilon : X \to \mathbb{R}_+$ and $\sigma : X \to X$. Assume also that (Y, d) is a complete metric space and $\tau : Y \to Y$ is a continuous bijection. If $\beta_{\sigma,\varepsilon} < \infty$, $\gamma_{\tau^{-1}} < \infty$, $\gamma_{\tau^{-1}}\beta_{\sigma,\varepsilon} < 1$ and a mapping $f : X \to Y$ satisfies (57), then there exists a unique mapping $f_\infty : X \to Y$ such that (55) holds and*

$$d(f(x), f_\infty(x)) \le \frac{\gamma_{\tau^{-1}}\varepsilon(x)}{1 - \gamma_{\tau^{-1}}\beta_{\sigma,\varepsilon}}, \quad x \in X. \tag{62}$$

Let us finally mention that it is also shown in [25] that the above results imply some classical outcomes on the generalized stability of the Cauchy functional equation.

2.2. Other Classical Theorems. In this section, we present applications of three other fixed point theorems. To formulate the first of them we need two more definitions.

A nondecreasing function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is called a *comparison function* [32, 33] or *Matkowski gauge function* [34, 35] if

$$\lim_{n\to\infty} \gamma^n(t) = 0, \quad t \in (0, \infty). \tag{63}$$

Given such a mapping γ and a metric space (X, d), we say that a function $\psi : X \to X$ is a *Matkowski γ-contraction* if

$$d(\psi(x), \psi(y)) \le \gamma(d(x, y)), \quad x, y \in X. \tag{64}$$

We can now recall Matkowski's fixed point theorem from [36].

Theorem 19. *If (X, d) is a complete metric space and $T : X \to X$ is a Matkowski contraction, then T has a unique fixed point $p \in X$ and the sequence $(T^n(x))_{n\in\mathbb{N}}$ converges to p for every $x \in X$.*

In [37], this theorem was applied to prove the following generalization of Theorem 2.

Theorem 20 ([37], Theorem 2.2). *Let S be a nonempty set, (X,d) be a complete metric space, $\varphi : S \to S$, $F : S\times X \to X$. Assume also that*

$$d(F(t,u),F(t,v)) \le \gamma(d(u,v)), \quad t\in S,\ u,v\in X, \tag{65}$$

where $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is a comparison function, and let $g : S \to X$, $\delta > 0$ be such that (3) *holds. Then there is a unique function $f : S \to X$ satisfying* (4) *and*

$$\rho(f,g) := \sup\{d(f(t),g(t)) : t\in S\} < \infty. \tag{66}$$

Moreover,

$$\rho(f,g) - \gamma(\rho(f,g)) \le \delta. \tag{67}$$

Theorem 20 with

$$F(t,x) := \psi(x), \quad t\in S,\ x\in X \tag{68}$$

gives the subsequent result.

Corollary 21 ([37], Corollary 2.3). *Let S be a nonempty set, (X,d) a complete metric space, $\varphi : S \to S$. Assume also that $\psi : X \to X$ is a Matkowski γ-contraction and let $g : S \to X$, $\delta > 0$ be such that*

$$d((\psi\circ g\circ\varphi)(t),g(t)) \le \delta, \quad t\in S. \tag{69}$$

Then there is a unique function $f : S \to X$ satisfying the equation

$$\psi\circ f\circ\varphi = f \tag{70}$$

and condition (66). *The function f is given by*

$$f(t) = \lim_{n\to\infty}\psi^n(g(\varphi^n(t))), \quad t\in S. \tag{71}$$

On the other hand, in [38], the following variant of Ćirić's fixed point theorem was proved.

Theorem 22 ([38], Theorem 2.1). *Assume that (Y,ρ) is a complete metric space and $T : Y \to Y$ is a mapping such that*

$$\begin{aligned}\rho(T(x),T(y)) &\le \alpha_1(x,y)\rho(x,y) + \alpha_2(x,y)\rho(x,T(x)) \\ &\quad + \alpha_3(x,y)\rho(y,T(y)) + \alpha_4(x,y)\rho(x,T(y)) \\ &\quad + \alpha_5(x,y)\rho(y,T(x)), \quad x,y\in Y,\end{aligned} \tag{72}$$

where $\alpha_1,\dots,\alpha_5 : Y\times Y \to \mathbb{R}_+$ satisfy

$$\sum_{i=1}^{5}\alpha_i(x,y) \le \lambda, \tag{73}$$

for all $x,y\in Y$ and some fixed $\lambda\in[0,1)$. Then T has a unique fixed point $p\in Y$ and

$$\rho(u,p) \le \frac{(2+\lambda)\rho(u,T(u))}{2(1-\lambda)}, \quad u\in Y. \tag{74}$$

Next, Baker's idea and Theorem 22 were used to obtain the following result concerning the stability of (4).

Theorem 23 ([38], Theorem 2.2). *Let S be a nonempty set, (X,d) a complete metric space, $\varphi : S \to S$, $F : S\times X \to X$ and*

$$\begin{aligned}d(F(t,x),F(t,y)) &\le \alpha_1(x,y)d(x,y) + \alpha_2(x,y)d(x,F(t,x)) \\ &\quad + \alpha_3(x,y)d(y,F(t,y)) + \alpha_4(x,y)d(x,F(t,y)) \\ &\quad + \alpha_5(x,y)d(y,F(t,x)), \quad t\in S,\ x,y\in X,\end{aligned} \tag{75}$$

where $\alpha_1,\dots,\alpha_5 : X\times X \to \mathbb{R}_+$ satisfy (73) *for all $x,y\in X$ and some $\lambda\in[0,1)$. If $g : S \to X$, $\delta > 0$ and* (3) *holds, then there is a unique function $f : S \to X$ satisfying* (4) *and*

$$d(f(t),g(t)) \le \frac{(2+\lambda)\delta}{2(1-\lambda)}, \quad t\in S. \tag{76}$$

A consequence of Theorem 23 is the following.

Corollary 24 ([38], Theorem 2.3). *Let S be a nonempty set, E a real or complex Banach space, $\varphi : S \to S$, $\alpha : S \to E$, $B : S \to \mathfrak{L}(E)$ (here $\mathfrak{L}(E)$ denotes the Banach algebra of all bounded linear operators on E), $\lambda\in[0,1)$ and*

$$\|B(t)\| \le \lambda, \quad t\in S. \tag{77}$$

If $g : S \to E$, $\delta > 0$ and

$$\|g(t) - (\alpha(t) + B(t)(g(\varphi(t))))\| \le \delta, \quad t\in S, \tag{78}$$

then there exists a unique function $f : S \to E$ satisfying the equation

$$f(t) = \alpha(t) + B(t)(f(\varphi(t))), \quad t\in S \tag{79}$$

and condition (10).

Now, let us recall the Markov-Kakutani theorem (see [39, 40]).

Theorem 25. *Let Y be a linear topological space and let $K\subset Y$ be a nonempty convex compact subset of Y. Let $\mathcal{F}$ be a family of affine continuous self-mappings of K such that*

$$F\circ G = G\circ F, \quad F,G\in\mathcal{F}. \tag{80}$$

Then there is a common fixed point $y\in K$ of family $\mathcal{F}$; that is,

$$F(y) = y, \quad F\in\mathcal{F}. \tag{81}$$

Theorem 25 has been applied in [41] to provide an alternative (quite involved) proof of the following classical stability result due to Hyers [2].

Theorem 26. *Let S be an abelian semigroup, $\varepsilon \ge 0$, $\mathbb{K}\in\{\mathbb{R},\mathbb{C}\}$, $f : S \to \mathbb{K}$ and*

$$|f(x+y) - f(x) - f(y)| \le \varepsilon, \quad x,y\in S. \tag{82}$$

Then there exists an additive function $a : S \to \mathbb{K}$ such that

$$|a(s) - f(s)| \le \varepsilon, \quad s\in S. \tag{83}$$

2.3. Fixed Point Alternatives Theorems on Generalized Metric Space. In this part of the paper, we show how several fixed points alternatives can be used to get some Hyers-Ulam stability results.

In order to do this let us first recall (see [42, 43]) that $d : X^2 \to [0,\infty]$ is said to be a *generalized metric* on a nonempty set X if and only if for any $x, y, z \in X$ we have

$$\begin{gathered} d(x,y) = 0 \quad \text{if and only if } x = y, \\ d(x,y) = d(y,x), \\ d(x,z) \le d(x,y) + d(y,z). \end{gathered} \tag{84}$$

In 2002, at the 14th *European Conference on Iteration Theory* (ECIT 2002 - Evora, Portugal), L. Cădariu and V. Radu delivered a lecture titled "On the stability of the Cauchy functional equation: a fixed points approach." They presented a generalized Hyers-Ulam stability result for the Cauchy functional equation, in the case when the equation perturbation is controlled by a given mapping φ, with a simple property of contractive type. Their idea was to obtain the existence of the exact solution and the error estimations by using the following fixed point alternative theorem of Diaz and Margolis [44].

Theorem 27. *Let (X,d) be a complete generalized metric space. Assume that $T : X \to X$ is a strictly contractive operator with the Lipschitz constant $L < 1$. Then, for each given element $x \in X$, either*

(A_1) $d(T^n x, T^{n+1}x) = +\infty$, *for all $n \ge 0$, or*

(A_2) *there exists $n_0 \in \mathbb{N}$ such that $d(T^n x, T^{n+1}x) < +\infty$, for all $n \ge n_0$. Actually, if (A_2) holds, then the following three conditions are valid.*

(A_{21}) *The sequence $(T^n(x))_{n\in\mathbb{N}}$ converges to a fixed point y^* of T.*

(A_{22}) *y^* is the unique fixed point of T in the set*

$$Z := \{y \in X : d(T^{n_0}(x), y) < \infty\}. \tag{85}$$

(A_{23}) *If $y \in Z$, then*

$$d(y, y^*) \le \frac{1}{1-L} d(T(y), y). \tag{86}$$

Remark 28. If the fixed point y^* exists, it is not necessarily unique in the whole space X; this may depend on the starting approximation. It is worth noting that, in case (A_2), the pair (Z,d) is a complete metric space and $A(Z) \subset Z$. Therefore, properties (A_{21})–(A_{23}) follow from Banach's Contraction Principle.

This method has been next used in [20] (for the additive Cauchy equation) and in [45] (for Jensen's equation).

Now, let us remind one of the most classical results, which was first proved by the direct method: for $p \in [0,1)$ in [46] (see also [47], where a similar result has been obtained under some regularity assumptions), and for $p \in (1,\infty)$ in [48] (see also for instance [5] and [20, Theorem]; information and recent results on the case $p < 0$ can be found in [49, 50]).

Theorem 29. *Let E be a real normed space, F a real Banach space, $\theta \in [0,\infty)$, $p \in [0,\infty)\backslash\{1\}$ and $f : E \to F$ be such that*

$$\|f(x+y) - f(x) - f(y)\| \le \theta\left(\|x\|^p + \|y\|^p\right), \quad x, y \in E. \tag{87}$$

Then there exists a unique additive mapping $a : E \to F$ such that

$$\|f(x) - a(x)\| \le \frac{2\theta}{|2-2^p|}\|x\|^p, \quad x \in E. \tag{88}$$

The lecture from ECIT 2002 was materialized in [51] in the following extension of Theorem 29.

Theorem 30 ([51], Theorem 2.5). *Let E_1, E_2 be two linear spaces over the same (real or complex) field, E_2 a complete β-normed space, $q_0 = 2$, $q_1 = 1/2$, and $\varphi : E_1 \times E_1 \to \mathbb{R}_+$. Assume that $f : E_1 \to E_2$ with $f(0) = 0$ satisfies*

$$\|f(x+y) - f(x) - f(y)\|_\beta \le \varphi(x,y), \quad x, y \in E_1. \tag{89}$$

If there exist an $i \in \{0,1\}$ and a positive constant $L < 1$ such that the mapping

$$x \longrightarrow \psi(x) = \varphi\left(\frac{x}{2}, \frac{x}{2}\right) \tag{90}$$

has the property

$$\psi(x) \le L \cdot q_i^\beta \cdot \psi\left(\frac{x}{q_i}\right), \quad x \in E_1, \tag{91}$$

and φ satisfies the condition

$$\lim_{n\to\infty} \frac{\varphi(q_i^n x, q_i^n y)}{q_i^{n\beta}} = 0, \quad x, y \in E_1, \tag{92}$$

then there exists a unique additive mapping $a : E_1 \to E_2$ such that

$$\|f(x) - a(x)\|_\beta \le \frac{L^{1-i}}{1-L}\psi(x), \quad x \in E_1. \tag{93}$$

Let E be a linear space over the field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Recall that a mapping $\|\cdot\|_\beta : E \to \mathbb{R}_+$ is called a *β-norm* provided it has the following properties:

n_β^I : $\|x\|_\beta = 0$ if and only if $x = 0$;

n_β^{II} : $\|\lambda \cdot x\|_\beta = |\lambda|^\beta \cdot \|x\|_\beta$, $x \in E$, $\lambda \in \mathbb{K}$;

n_β^{III} : $\|x + y\|_\beta \le \|x\|_\beta + \|y\|_\beta$, $x, y \in E$.

The idea emphasized in [20, 45, 51] has been subsequently used for the quadratic equation in [52], the cubic equation in [53], the quartic equation in [54], equations (2) and (5) in [55], and the monomial equation in [56]. We present that last result below. To this end, let us recall that a function f (mapping an abelian group $(X,+)$ into a real vector space Y)

is called a *monomial function of degree N* (N is a fixed positive integer) if it is a solution of the monomial functional equation

$$\Delta_y^N f(x) - N! f(y) = 0, \quad x, y \in X, \tag{94}$$

where the difference operator Δ_y^N is defined in the following manner:

$$\Delta_y^1 g(x) := g(x+y) - g(x), \quad x, y \in X, \; g \in Y^X, \tag{95}$$

and, inductively,

$$\Delta_y^{n+1} = \Delta_y^1 \circ \Delta_y^n, \quad n \in \mathbb{N}, \; y \in X. \tag{96}$$

Theorem 31 ([56], Theorem 2.1). *Let X be a group that is uniquely divisible by 2 (i.e., for any $x \in X$ there exists a unique $a \in X$ with $x = 2a$), Y a (real or complex) complete β-normed space, and $\varphi : X \times X \to \mathbb{R}_+$ fulfil the following property:*

$$\lim_{m\to\infty} \frac{\varphi(r^m x, r^m y)}{r^{mN\beta}} = 0, \quad x, y \in X, \; r \in \left\{2, \frac{1}{2}\right\}. \tag{97}$$

Suppose $f : X \to Y$, with $f(0) = 0$, satisfies the condition

$$\left\| \Delta_y^N f(x) - N! f(y) \right\|_\beta \le \varphi(x, y), \quad x, y \in X. \tag{98}$$

If there exists a positive constant $L < 1$ such that the mapping

$$x \longrightarrow \psi(x) = \frac{1}{(N!)^\beta} \left(\varphi(0, x) + \sum_{i=0}^{N} \binom{N}{N-i} \varphi\left(\frac{ix}{2}, \frac{x}{2}\right) \right), \quad x \in X, \tag{99}$$

satisfies the inequality

$$\psi(r_j x) \le L r_j^{N\beta} \psi(x), \quad x \in X, \tag{100}$$

then there exists a unique monomial mapping $g : X \to Y$ of degree N with

$$\| f(x) - g(x) \|_\beta \le \frac{L^{1-j}}{1-L} \psi(x), \quad x \in X. \tag{101}$$

In [43], Mihet has given one more generalization of Theorem 2; he obtained it by proving another fixed point alternative.

Recall that a mapping $\gamma : [0, \infty] \to [0, \infty]$ is called a *generalized strict comparison function* if it is nondecreasing, $\gamma(\infty) = \infty$,

$$\lim_{n\to\infty} \gamma^n(t) = 0, \quad t \in (0, \infty) \tag{102}$$

and $\lim_{t\to\infty}(t - \gamma(t)) = \infty$. Given such a mapping γ and a generalized metric space (X, d), we say that a function $\psi : X \to X$ is a *strict γ-contraction* if it satisfies inequality (64).

Now, we can formulate the fixed point result from [43].

Theorem 32 ([43], Theorem 2.2). *Let (X, d) be a complete generalized metric space and $T : X \to X$ a strict γ-contraction such that $d(x, T(x)) < \infty$ for an $x \in X$. Then T has a unique fixed point p in the set*

$$Z := \{ y \in X : d(x, y) < \infty \}, \tag{103}$$

and the sequence $(T^n(y))_{n\in\mathbb{N}}$ converges to p for every $y \in Z$. Moreover,

$$d(p, x) \le \sup \{ s > 0 : s - \gamma(s) \le d(x, T(x)) \}, \quad x \in X. \tag{104}$$

Using this theorem we can get the following generalization of Theorem 2.

Theorem 33 ([43], Theorem 3.1). *Let S be a nonempty set, (X, d) a complete metric space, $\varphi : S \to S$, $F : S \times X \to X$. Assume also that*

$$d(F(t, u), F(t, v)) \le \gamma(d(u, v)), \quad t \in S, \; u, v \in X, \tag{105}$$

where $\gamma : [0, \infty] \to [0, \infty]$ is a generalized strict comparison function, and let $g : S \to X$, $\delta > 0$ be such that (3) *holds. Then there is a unique function $f : S \to X$ satisfying* (4) *and*

$$d(f(t), g(t)) \le \sup \{ s > 0 : s - \gamma(s) \le \delta \}, \quad t \in S. \tag{106}$$

We also have the following.

Theorem 34 ([43], Theorem 4.1). *Let S be a nonempty set, (X, d) a complete metric space, $\varphi : S \to S$, $\psi : X \to X$. Assume also that $g : S \to X$ and $\delta > 0$ are such that* (69) *holds. If $\gamma : [0, \infty] \to [0, \infty]$ is a generalized strict comparison function satisfying inequality* (64)*, then there is a unique mapping $f : S \to X$ such that* (70) *and* (106) *hold. The function f is given by formula* (71).

We end this section with some applications of the fixed point alternatives of the Bianchini-Grandolfi and Matkowski types. To this end, we introduce some notations and definitions.

A nondecreasing function $c : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be a *c-comparison function* [32] or *Bianchini-Grandolfi gauge function* [35, 57] if for each $t \in (0, \infty)$ the series $\sum_{k=0}^{\infty} c^k(t)$ is convergent (here c^k denotes the kth iteration of c).

A self-mapping A of the metric space (X, d) for which there exists a c-comparison function c such that

$$d(Ax, Ay) \le c(d(x, y)), \quad x, y \in X \tag{107}$$

is called a *Bianchini-Grandolfi contraction* [34] (see also the notion of a Matkowski contraction in Section 2.2).

The following result is the fixed point alternative of Bianchini and Grandolfi [57] (see [58, Lemma 2.1]).

Theorem 35. *Let (X, d) be a complete generalized metric space, that is, one for which d may assume infinite values, and $A : X \to X$ a Bianchini-Grandolfi contraction. Then, for each $x \in X$, either*

(A_1) *$d(A^n x, A^{n+1} x) = +\infty$, for all $n \ge 0$, or*

(A_2) *there exists* $n_0 \in \mathbb{N}$ *such that* $d(A^n x, A^{n+1} x) < +\infty$ *for* $n \geq n_0$.

If (A_2) *holds, then the sequence* $(A^n(x))_{n\in\mathbb{N}}$ *converges to a fixed point* y^* *of* A, y^* *is the unique fixed point of* A *in the set*

$$\mathscr{X} := \{y \in X : d(A^{n_0}(x), y) < \infty\},$$
$$d(y, y^*) \leq \sum_{k=0}^{\infty} c^k (d(y, A(y))), \quad y \in \mathscr{X}. \tag{108}$$

Next, we introduce the notion of an *admissible pair*. Let us consider a 2-divisible group X and denote by $\mathscr{M}$ the family of all mappings $m : X \times \mathbb{R}_+ \to \mathbb{R}_+$ with the properties:

(i) m_x is continuous at 0, for each $x \in X$;

(ii) m_x is superadditive; that is

$$m_x(t+s) \geq m_x(t) + m_x(s), \quad t, s \in \mathbb{R}_+, \tag{109}$$

where $m_x := m(x, \cdot)$ for each $x \in X$.

Let us consider a comparison function c, an element $m \in \mathscr{M}$ and numbers $r_0 = 2$ and $r_1 = 1/2$.

Given $0 < \beta \leq 1$ and $N \in \mathbb{N}$, we say that (m, c) is a *j-admissible pair of order* $N\beta$ if

$$m(r_j x, t) \leq r_j^{N\beta} m(x, c(t)), \quad t \in \mathbb{R}_+, \; x \in X. \tag{A_j}$$

Now, we are in a position to present a stability theorem for monomial functional equation (94) of degree N.

Theorem 36 ([58], Theorem 3.1). *Let* $(G, +)$ *be a 2-divisible group,* Y *a (real or complex) complete* β*-normed space,* c *be a c-comparison function, and* (m, c) *be a j-admissible pair of order* $N\beta$*, with a* $j \in \{0, 1\}$. *Suppose that* $\varphi : G \times G \to \mathbb{R}_+$ *and* $f : G \to Y$ *with* $f(0) = 0$ *satisfy the inequality*

$$\left\|\Delta_y^N f(x) - N! f(y)\right\|_\beta \leq \varphi(x, y), \quad x, y \in G. \tag{110}$$

If there exist $j \in \{0, 1\}$ *and* $\delta > 0$ *such that*

$$\frac{1}{(N!)^\beta} \left(\varphi(0, x) + \sum_{i=0}^{N} \binom{N}{N-i} \varphi\left(\frac{ix}{2}, \frac{x}{2}\right) \right) \leq m(x, \delta), \quad x \in G,$$
$$\lim_{m\to\infty} \frac{\varphi(r_j^m x, r_j^m y)}{r_j^{mN\beta}} = 0, \quad x, y \in G, \tag{111}$$

then there exists a unique monomial function $g : G \to Y$ *of degree* N *such that*

$$\|f(x) - g(x)\|_\beta \leq m\left(x, \sum_{k=0}^{\infty} c^{k+1-j}(\delta)\right), \quad x \in G. \tag{112}$$

Remark 37. If (in Theorem 36) $m(x, t) = \mu(t) \cdot (\gamma(t) + \psi(x))$, with γ and μ having suitable properties, then we obtain [58, Corollary 4.1]; if $\mu(t) = t$, $\gamma \equiv 0$ and $c(t) = Lt$ with an $L < 1$, then we get Theorem 31.

The fixed point alternative of Bianchini and Grandolfi has been used in [59] to prove a generalized Hyers-Ulam stability result for the additive Cauchy functional equation. In what follows, we yet recall the fixed point alternative of Matkowski and the corresponding outcome on stability of the Cauchy equation.

Theorem 38 ([59], Lemma 3.2). *Let* (X, d) *be a complete generalized metric space and* $A : X \to X$ *a* γ*-Matkowski contraction. Then, for each* $x \in X$*, either*

(A_1) $d(A^n x, A^{n+1} x) = +\infty$, *for all* $n \geq 0$, *or*

(A_2) *there exists an* $n_0 \in \mathbb{N}$ *such that* $d(A^n x, A^{n+1} x) < +\infty$, *for all* $n \geq n_0$.

If (A_2) *holds, then the sequence* $(A^n(x))_{n\in\mathbb{N}}$ *converges to a fixed point* y^* *of* A *and* y^* *is the unique fixed point of* A *in the set*

$$\mathscr{X} := \{y \in X : d(A^{n_0}(x), y) < \infty\}; \tag{113}$$

moreover, if additionally the mapping $\gamma_0 : t \mapsto t - \gamma(t)$ *is a bijection, then*

$$d(y, y^*) \leq \gamma_0^{-1}(d(y, A(y))), \quad y \in \mathscr{X}. \tag{114}$$

The above fixed point result has been used to prove the following stability result for the Cauchy equation.

Theorem 39 ([59], Theorem 2.5). *Let us consider a real linear space* X*, a complete* β*-normed space* Y*, a comparison function* γ*, and a j-admissible pair* (m, γ) *of order* β *with a* $j \in \{0, 1\}$. *Let us suppose that the mapping* $\gamma_0 : t \mapsto t - \gamma(t)$ *is an increasing bijection and that a mapping* $f : X \to Y$ *with* $f(0) = 0$ *satisfies the inequality*

$$\|f(x+y) - f(x) - f(y)\|_\beta \leq \varphi(x, y), \quad x, y \in X, \tag{115}$$

where $\varphi : X \times X \to \mathbb{R}_+$ *is a given function such that*

$$\lim_{n\to\infty} \frac{\varphi(r_j^n x, r_j^n y)}{r_j^{n\beta}} = 0, \quad x, y \in X. \tag{116}$$

If there exists a $\delta > 0$ *such that*

$$\varphi\left(\frac{x}{2}, \frac{x}{2}\right) \leq m(x, \delta), \quad x \in X, \tag{117}$$

then there exists a unique additive mapping $a : X \to Y$ *such that*

$$\|f(x) - a(x)\|_\beta \leq m\left(x, \gamma_0^{-1}\left(\gamma^{1-j}(\delta)\right)\right), \quad x \in X. \tag{118}$$

3. New Fixed Point Theorems and Their Applications

In this section, we present a somewhat different fixed point approach to the stability of functional equations, in which the stability results are simple consequences of some new fixed point theorems.

Given a nonempty set S and a metric space (X, d), we define a mapping $\Delta : (X^S)^2 \to \mathbb{R}_+{}^S$ (A^B denotes the family of all functions mapping a set B into a set A) by

$$\Delta(\xi, \mu)(t) := d(\xi(t), \mu(t)), \quad \xi, \mu \in X^S,\ t \in S. \tag{119}$$

With this notation, we have the following.

Theorem 40 ([60], Theorem 1). *Let S be a nonempty set, (X, d) a complete metric space, $k \in \mathbb{N}$, $f_1, \ldots, f_k : S \to S$, $L_1, \ldots, L_k : S \to \mathbb{R}_+$ and $\Lambda : \mathbb{R}_+{}^S \to \mathbb{R}_+{}^S$ be given by*

$$(\Lambda\delta)(t) := \sum_{i=1}^{k} L_i(t)\,\delta(f_i(t)), \quad \delta \in \mathbb{R}_+{}^S,\ t \in S. \tag{120}$$

If $\mathcal{T} : X^S \to X^S$ is an operator satisfying the inequality

$$\begin{aligned}&\Delta(\mathcal{T}\xi, \mathcal{T}\mu)(t)\\ &\quad \le \Lambda(\Delta(\xi, \mu))(t), \quad \xi, \mu \in X^S,\ t \in S\end{aligned} \tag{121}$$

and functions $\varepsilon : S \to \mathbb{R}_+$ and $g : S \to X$ are such that

$$\Delta(\mathcal{T}g, g)(t) \le \varepsilon(t), \quad t \in S, \tag{122}$$

$$\sum_{n=0}^{\infty} (\Lambda^n \varepsilon)(t) =: \sigma(t) < \infty, \quad t \in S, \tag{123}$$

then for every $t \in S$ the limit

$$\lim_{n\to\infty} (\mathcal{T}^n g)(t) =: f(t) \tag{124}$$

exists and the function $f : S \to X$, defined in this way, is a unique fixed point of $\mathcal{T}$ with

$$\Delta(g, f)(t) \le \sigma(t), \quad t \in S. \tag{125}$$

A consequence of Theorem 40 is the following result on the stability of a quite wide class of functional equations in a single variable.

Corollary 41 ([60], Corollary 3). *Let S be a nonempty set, (X, d) be a complete metric space, $k \in \mathbb{N}$, $f_1, \ldots, f_k : S \to S$, $L_1, \ldots, L_k : S \to \mathbb{R}_+$, a function $\Phi : S \times X^k \to X$ satisfy the inequality*

$$d(\Phi(t, y_1, \ldots, y_k), \Phi(t, z_1, \ldots, z_k)) \le \sum_{i=1}^{k} L_i(t)\, d(y_i, z_i) \tag{126}$$

for any $(y_1, \ldots, y_k), (z_1, \ldots, z_k) \in X^k$ and $t \in S$, and $\mathcal{T} : X^S \to X^S$ be an operator defined by

$$(\mathcal{T}\varphi)(t) := \Phi(t, \varphi(f_1(t)), \ldots, \varphi(f_k(t))), \quad \varphi \in X^S,\ t \in S. \tag{127}$$

Assume also that Λ is given by (120) *and functions $g : S \to X$ and $\varepsilon : S \to \mathbb{R}_+$ are such that*

$$d(g(t), \Phi(t, g(f_1(t)), \ldots, g(f_k(t)))) \le \varepsilon(t), \quad t \in S \tag{128}$$

and (123) *holds. Then for every $t \in S$ limit* (124) *exists and the function $f : S \to X$ is a unique solution of the functional equation*

$$\Phi(t, f(f_1(t)), \ldots, f(f_k(t))) = f(t), \quad t \in S \tag{129}$$

satisfying inequality (125).

Another application of Theorem 40 has been given in [61]; it concerns stability of the polynomial equation (for a survey on related results see [62]).

Next, following [63], we consider the case of non-Archimedean metric spaces (let us mention here that the first paper dealing with the Hyers-Ulam stability of functional equations in non-Archimedean normed spaces was [64], whereas [65] seems to be the first one in which the stability problem in a particular type of these spaces was considered). In order to do this, we introduce some notations and definitions.

Let S be a nonempty set. For any $\delta_1, \delta_2 \in \mathbb{R}_+^S$ we write $\delta_1 \le \delta_2$ provided

$$\delta_1(t) \le \delta_2(t), \quad t \in S, \tag{130}$$

and we say that an operator $\Lambda : \mathbb{R}_+^S \to \mathbb{R}_+^S$ is *nondecreasing* if it satisfies the condition

$$\Lambda\delta_1 \le \Lambda\delta_2, \quad \delta_1, \delta_2 \in \mathbb{R}_+^S,\ \delta_1 \le \delta_2. \tag{131}$$

Moreover, given a sequence $(g_n)_{n\in\mathbb{N}}$ in $\mathbb{R}_+^S$, we write $\lim_{n\to\infty} g_n = 0$ provided

$$\lim_{n\to\infty} g_n(t) = 0, \quad t \in S. \tag{132}$$

We will also use the following hypothesis concerning operators $\Lambda : \mathbb{R}_+^S \to \mathbb{R}_+^S$:

($\mathcal{C}$) $\lim_{n\to\infty} \Lambda\delta_n = 0$ for every sequence $(\delta_n)_{n\in\mathbb{N}}$ in $\mathbb{R}_+^S$ with $\lim_{n\to\infty} \delta_n = 0$.

Finally, let us recall that a metric d on a nonempty set X is called *non-Archimedean* (or an *ultrametric*) provided

$$d(x, z) \le \max\{d(x, y), d(y, z)\}, \quad x, y, z \in X. \tag{133}$$

We can now formulate the following fixed point theorem.

Theorem 42 ([63], Theorem 1). *Let S be a nonempty set, (X, d) a complete non-Archimedean metric space and $\Lambda : \mathbb{R}_+{}^S \to \mathbb{R}_+{}^S$ a nondecreasing operator satisfying hypothesis ($\mathcal{C}$). If $\mathcal{T} : X^S \to X^S$ is an operator satisfying inequality* (121) *and functions $\varepsilon : S \to \mathbb{R}_+$ and $g : S \to X$ are such that*

$$\lim_{n\to\infty} \Lambda^n \varepsilon = 0 \tag{134}$$

and (122) *holds, then for each $t \in S$ limit* (124) *exists and the function $f : S \to X$, defined in this way, is a fixed point of $\mathcal{T}$ with*

$$\Delta(g, f)(t) \le \sup_{n\in\mathbb{N}_0} (\Lambda^n \varepsilon)(t) =: \sigma(t), \quad t \in S. \tag{135}$$

If, moreover,

$$(\Lambda\sigma)(t) \le \sup_{n\in\mathbb{N}} (\Lambda^n \varepsilon)(t), \quad t \in S, \tag{136}$$

then f is the unique fixed point of $\mathcal{T}$ satisfying (135).

An immediate consequence of Theorem 42 is the following result on the stability of (129) in the complete non-Archimedean metric spaces.

Corollary 43 (see [63]). *Let S be a nonempty set, (X, d) a complete non-Archimedean metric space, $k \in \mathbb{N}$, $f_1, \dots, f_k : S \to S$, $L_1, \dots, L_k : S \to \mathbb{R}_+$, a function $\Phi : S \times X^k \to X$ satisfy the inequality*

$$d(\Phi(t, y_1, \dots, y_k), \Phi(t, z_1, \dots, z_k)) \le \max_{i\in\{1,\dots,k\}} L_i(t)\, d(y_i, z_i) \tag{137}$$

for any $(y_1, \dots, y_k), (z_1, \dots, z_k) \in X^k$ and $t \in S$, and $\mathcal{T} : X^S \to X^S$ an operator defined by (127). *Assume also that Λ is given by*

$$(\Lambda\delta)(t) := \max_{i\in\{1,\dots,k\}} L_i(t)\, \delta(f_i(t)), \quad \delta \in {\mathbb{R}_+}^S,\ t \in S \tag{138}$$

and functions $g : S \to X$ and $\varepsilon : S \to \mathbb{R}_+$ are such that (128) *and* (134) *hold. Then for every $t \in S$ limit* (124) *exists and the function $f : S \to X$ is a solution of functional equation* (129) *satisfying inequality* (135).

A variant of Theorem 42 in arbitrary complete metric spaces was also proved in [63, Theorem 2]. A slightly improved version of this outcome was obtained in [66] in response to an open problem concerning the uniqueness of the mapping ψ, defined below. Namely, we have the following result.

Theorem 44 ([66], Corollary 2.2). *Let X be a nonempty set, (Y, d) a complete metric space and $\Lambda : \mathbb{R}_+^X \to \mathbb{R}_+^X$ a nondecreasing operator satisfying the hypothesis*

$$\lim_{n\to\infty} \delta_n(t) = 0 \Longrightarrow \lim_{n\to\infty} (\Lambda\delta_n)(t) = 0, \tag{$\mathscr{C}_1$}$$

for every sequence $(\delta_n)_{n\in\mathbb{N}}$ of elements of $\mathbb{R}_+^X$ and every $t \in X$. If $\mathcal{T} : Y^X \to Y^X$ is an operator satisfying the inequality

$$d((\mathcal{T}\xi)(x), (\mathcal{T}\mu)(x)) \le \Lambda(\Delta(\xi, \mu))(x), \quad \xi, \mu \in Y^X,\ x \in X, \tag{139}$$

and the functions $\varepsilon : X \to \mathbb{R}_+$, $\varphi : X \to Y$ are such that

$$d((\mathcal{T}\varphi)(x), \varphi(x)) \le \varepsilon(x), \quad x \in X, \tag{140}$$

$$\varepsilon^*(x) := \sum_{k=0}^{\infty} (\Lambda^k \varepsilon)(x) < \infty, \quad x \in X, \tag{$\mathscr{C}_2$}$$

then, for every $x \in X$, the limit

$$\psi(x) := \lim_{n\to\infty} (\mathcal{T}^n \varphi)(x) \tag{141}$$

exists and the function $\psi \in Y^X$, defined in this way, is a fixed point of $\mathcal{T}$ with

$$d(\varphi(x), \psi(x)) \le \varepsilon^*(x), \quad x \in X. \tag{142}$$

Moreover, if the condition

$$\lim_{n\to\infty} (\Lambda^n \varepsilon^*)(t) = 0, \quad t \in X, \tag{$\mathscr{C}_3$}$$

holds, then the mapping ψ is the unique fixed point of $\mathcal{T}$ with the property

$$d(\varphi(x), \psi(x)) \le \varepsilon^*(x), \quad x \in X. \tag{143}$$

Theorem 44 is a consequence of the following fixed point theorem for a class of operators satisfying some very general conditions (recall that given $\Lambda : \mathbb{R}_+^X \to \mathbb{R}_+^X$, we say that $\mathcal{T} : Y^X \to Y^X$ is Λ-*contractive* if for any $u, v : X \to Y$ and $\delta \in \mathbb{R}_+^X$ with $d(u(t), v(t)) \le \delta(t)$ for $t \in X$, it follows that $d((\mathcal{T}u)(t), (\mathcal{T}v)(t)) \le (\Lambda\delta)(t)$ for $t \in X$).

Theorem 45 ([66], Theorem 2.1). *Let X be a nonempty set, (Y, d) a complete metric space and $\Lambda : \mathbb{R}_+^X \to \mathbb{R}_+^X$. Suppose that $\mathcal{T} : Y^X \to Y^X$ is Λ-contractive, hypotheses $(\mathscr{C}_1)$ and $(\mathscr{C}_2)$ hold, and $f \in Y^X$ fufils*

$$d((\mathcal{T}f)(t), f(t)) \le \varepsilon(t), \quad t \in X. \tag{144}$$

Then, for every $t \in X$, the limit

$$g(t) := \lim_{n\to\infty} (\mathcal{T}^n f)(t), \tag{145}$$

exists and the mapping g is the unique fixed point of $\mathcal{T}$ with the property

$$d((\mathcal{T}^m f)(t), g(t)) \le \sum_{k=m}^{\infty} (\Lambda^k \varepsilon)(t), \quad t \in X,\ m \in \mathbb{N}. \tag{146}$$

Moreover, if we have

$$\lim_{n\to\infty} (\Lambda^n \varepsilon^*)(t) = 0, \quad t \in X, \tag{147}$$

then g is the unique fixed point of $\mathcal{T}$ with the property

$$d(f(t), g(t)) \le \varepsilon^*(t), \quad t \in X. \tag{148}$$

It is also worth noting that Theorem 40 can be directly obtained from Theorem 45 (see [66] for details).

Given nonempty sets S, Z and functions $\varphi : S \to S$, $F : S \times Z \to Z$, we define an operator $\mathscr{L}_\varphi^F : Z^S \to Z^S$ by

$$\mathscr{L}_\varphi^F(g)(t) := F(t, g(\varphi(t))), \quad g \in Z^S,\ t \in S, \tag{149}$$

and we say that $\mathscr{U} : Z^S \to Z^S$ is an *operator of substitution* provided $\mathscr{U} = \mathscr{L}_\psi^G$ with some $\psi : S \to S$ and $G : S \times Z \to Z$. Moreover, if $G(t, \cdot)$ is continuous for each $t \in S$ (with respect to a topology in Z), then we say that $\mathscr{U}$ is *continuous*.

Theorem 46 ([67], Theorem 2.1). *Let S be a nonempty set, (X, d) a complete metric space, $\Lambda : S \times \mathbb{R}_+ \to \mathbb{R}_+$, $\mathcal{T} : X^S \to X^S$, $\varphi : S \to S$ and*

$$\Delta(\mathcal{T}\alpha, \mathcal{T}\beta)(t) \leq \Lambda(t, \Delta(\alpha \circ \varphi, \beta \circ \varphi)(t)), \quad \alpha, \beta \in X^S, \quad t \in S. \tag{150}$$

Assume also that for every $t \in S$, $\Lambda_t := \Lambda(t, \cdot)$ is nondecreasing, $\varepsilon : S \to \mathbb{R}_+$, $g : S \to X$,

$$\sum_{n=0}^{\infty} \left(\left(\mathcal{L}_\varphi^\Lambda \right)^n \varepsilon \right)(t) =: \sigma(t) < \infty, \quad t \in S \tag{151}$$

and (122) *holds. Then for every $t \in S$ limit* (124) *exists and inequality* (125) *is satisfied. Moreover, the following two statements are true.*

(i) *If $\mathcal{T}$ is a continuous operator of substitution or Λ_t is continuous at 0 for each $t \in S$, then f is a fixed point of $\mathcal{T}$.*

(ii) *If Λ_t is subadditive; that is,*

$$\Lambda_t(a + b) \leq \Lambda_t(a) + \Lambda_t(b) \tag{152}$$

for all $a, b \in \mathbb{R}_+$, $t \in S$, then $\mathcal{T}$ has at most one fixed point $f \in X^S$ such that there exists $M \in \mathbb{N}$ with

$$\Delta(g, f)(t) \leq M\sigma(t), \quad t \in S. \tag{153}$$

Actually, it can be deduced from the proof of [67, Theorem 2.1] that Theorem 46 can be derived from Theorem 44.

Theorem 46 with $\mathcal{T} = \mathcal{L}_\varphi^F$ immediately gives the following generalization of Theorem 2.

Corollary 47 ([67], Corollary 2.1). *Let S be a nonempty set, (X, d) a complete metric space, $F : S \times X \to X$, $\Lambda : S \times \mathbb{R}_+ \to \mathbb{R}_+$ and*

$$d(F(t, x), F(t, y)) \leq \Lambda(t, d(x, y)), \quad t \in S, \ x, y \in X. \tag{154}$$

Assume also that $\varphi : S \to S$, $\varepsilon : S \to \mathbb{R}_+$, (151) *holds, $g : S \to X$, for every $t \in S$, $\Lambda_t := \Lambda(t, \cdot)$ is nondecreasing, $F(t, \cdot)$ is continuous and*

$$d(g(t), F(t, g(\varphi(t)))) \leq \varepsilon(t), \quad t \in S. \tag{155}$$

Then for every $t \in S$ the limit

$$f(t) := \lim_{n \to \infty} \left(\mathcal{L}_\varphi^F \right)^n (g)(t) \tag{156}$$

exists, (125) *holds and f is a solution of* (4)*. Moreover, if Λ_t is subadditive for every $t \in S$ and $M \in \mathbb{N}$, then $f : S \to X$ is the unique solution of* (4) *fulfilling* (153).

Some results related to those presented above, proved for functions taking values in Riesz spaces, can be found in [68].

Further applications of Theorem 40 have been proposed in [49, 50] (in solving a problem of Th. M. Rassias concerning optimality of estimations in Theorem 29) and in [69] (in proving stability of the equation of p-Wright affine functions); in particular, it has been discovered in this way in [49] that the property of hyperstability for the additive Cauchy equation appears quite often in a natural way (see [8, 70] for more information on this issue and related results); generalizations of that approach have been presented in [71, 72]. Similar methods (also involving Theorem 42) have been applied for some other equations in [73–78].

4. Stability of the Fixed Point Equation and Its Generalization

In [79] one can find the following definition (as well as some related notions concerning the generalized Ulam-Hyers stability).

Let (X, d) be a metric space and $f : X \to X$. We say that the fixed point equation

$$x = f(x) \tag{157}$$

is *Ulam-Hyers stable* if there is a $s > 0$ such that for any $\varepsilon > 0$ and $y \in X$ with

$$d(y, f(y)) \leq \varepsilon \tag{158}$$

there exists an $x \in X$ satisfying (157) and

$$d(y, x) \leq s\varepsilon. \tag{159}$$

Let us recall (see [79, 80]) that $f : X \to X$ is called a *weakly Picard operator* if for every $x \in X$ the sequence $(f^n(x))_{n \in \mathbb{N}}$ is convergent and its limit, denoted by $f^\infty(x)$, is a fixed point of f. Given such an operator f and a $c > 0$, we say that f is a *c-weakly Picard operator* if

$$d(x, f^\infty(x)) \leq cd(x, f(x)), \quad x \in X. \tag{160}$$

The following two results comes from [79].

Theorem 48 ([79], Remark 2.1). *Let (X, d) be a metric space and $c > 0$. If $f : X \to X$ is a c-weakly Picard operator, then* (157) *is Ulam-Hyers stable.*

Theorem 49 ([79], Remark 2.2). *Let (X, d) be a metric space, $I \neq \emptyset$ a set and $X = \bigcup_{i \in I} X_i$. Assume also that $f : X \to X$ satisfies $f(X_i) \subset X_i$ for $i \in I$. If for every $i \in I$, the equation*

$$x = f|_{X_i}(x) \tag{161}$$

is Ulam-Hyers stable, then (157) *is also Ulam-Hyers stable.*

Let us mention here that in [79] some applications of these outcomes (e.g., to the stability of an integral equation) are also presented.

Now, let (X, d) and (Y, ρ) be metric spaces, and $f, g : X \to Y$. The coincidence equation

$$f(x) = g(x) \tag{162}$$

is called *Ulam-Hyers stable* if there is a $s > 0$ such that for any $\varepsilon > 0$ and $y \in X$ with

$$\rho(g(y), f(y)) \le \varepsilon \tag{163}$$

there exists an $x \in X$ satisfying (162) and (159).

Let $c > 0$ and $f, g : X \to Y$. We say that (f, g) is a *c-weakly Picard pair* if there exists a weakly Picard operator $h : X \to X$ such that

$$\begin{aligned} &\{x \in X : h(x) = x\} = \{x \in X : f(x) = g(x)\}, \\ &d(x, h^{\infty}(x)) \le c\rho(f(x), g(x)), \quad x \in X. \end{aligned} \tag{164}$$

Theorem 50 ([79], Remark 6.1). *Let (X, d), (Y, ρ) be metric spaces, $f, g : X \to Y$ and $c > 0$. If (f, g) is a c-weakly Picard pair, then* (162) *is Ulam-Hyers stable.*

More results on these as well as related problems can be found in [81–95].

5. Final Remarks

Applications of different fixed point theorems to the Ulam type stability have been presented in this survey. On the other hand, some fixed point theorems can be derived from such stability results; we refer to [96–98] for suitable examples. Below we show how to get another such an example.

Let (Y, d) be a metric space. We denote by $n(Y)$ the family of all nonempty subsets of Y. The convergence of subsets of Y is with respect to the Hausdorff metric derived from the metric d. The number $\delta(A) := \sup\{d(x, y) : x, y \in A\}$ is said to be the *diameter* of $A \subset Y$. The next theorem has been obtained in [99, Theorem 2] (cf. [100, Theorem 1]; we refer the reader to [99–102] for further similar results and to [103] for a survey on the subject).

Theorem 51 ([99], Theorem 2). *Let K be a nonempty set, $a : K \to K$, $\Psi : Y \to Y$, $\lambda \in (0, \infty)$ and $F : K \to n(Y)$ satisfy*

$$\begin{aligned} &d(\Psi(x), \Psi(y)) \le \lambda d(x, y), \quad x, y \in Y, \\ &\lim_{n\to\infty} \lambda^n \delta(F(a^n(x))) = 0, \quad x \in K. \end{aligned} \tag{165}$$

Then one of the following two statements is valid.

(i) *If Y is complete and*

$$\Psi(F(a(x))) \subset F(x), \quad x \in K, \tag{166}$$

then, for each $x \in K$, the limit

$$\lim_{n\to\infty} cl(\Psi^n(F(a^n(x)))) =: \hat{f}(x) \tag{167}$$

exists, the function $\hat{f} : K \to n(Y)$ is single-valued and it is the unique function (which maps K into $n(Y)$) such that $\Psi(\hat{f}(a(x))) = \hat{f}(x)$ and $\hat{f}(x) \subset clF(x)$ for $x \in K$.

(ii) *If*

$$F(x) \subset \Psi(F(a(x))), \quad x \in K, \tag{168}$$

then F is single-valued and $\Psi(F(a(x))) = F(x)$ for $x \in K$.

It is easily seen that Theorem 51 yields the following fixed point result.

Corollary 52. *Let K be a nonempty set, $a : K \to K$, $\Psi : Y \to Y$, $\lambda \in (0, \infty)$, and $F : K \to n(Y)$ satisfy* (165). *Write $\Phi G(x) := \Psi(G(a(x)))$ for $G : K \to n(Y)$ and $x \in K$. Then one of the following two statements is valid.*

(a) *If Y is complete and*

$$\Phi F(x) \subset F(x), \quad x \in K, \tag{169}$$

then, for each $x \in K$, the limit

$$\lim_{n\to\infty} \mathrm{cl}(\Phi^n F(x)) =: \hat{f}(x) \tag{170}$$

exists, $\hat{f}$ is single-valued and it is the unique fixed point of Φ such that $\hat{f}(x) \subset clF(x)$ for $x \in K$.

(b) *If*

$$F(x) \subset \Phi F(x), \quad x \in K, \tag{171}$$

then F is a single-valued fixed point of Φ.

Analogously, a fixed point result can be deduced from the main outcome in [104], concerning stability of a generalization of the Volterra integral equation.

We end the paper with an example of stability result for the Cauchy additive equation, proved in [26, Corollary 1] through a modified fixed point approach (somewhat analogous to that in [105]); it corresponds to Theorem 29.

Theorem 53 ([26], Corollary 1). *Let X be a normed space, $c_1, c_2, p, q, r \in \mathbb{R}_+$ with*

$$(p-1)(q+r-1) > 0, \tag{172}$$

a nonempty $S \subset X$ such that $2S = S$, and $h : S \to \mathbb{R}$ satisfy the inequality

$$\begin{aligned} -c_1\|x\|^q\|y\|^r &\le h(x+y) - h(x) - h(y) \\ &\le c_2(\|x\|^p + \|y\|^p), \quad x, y \in S, \ x + y \in S. \end{aligned} \tag{173}$$

Then there exists a unique function $F : S \to \mathbb{R}$ such that

$$\begin{aligned} &F(x+y) = F(x) + F(y), \quad x, y \in S, x + y \in S, \\ &-\frac{c_1\|x\|^{q+r}}{|1 - 2^{q+r-1}|} \le F(x) - h(x) \le \frac{c_2\|x\|^p}{|1 - 2^{p-1}|}, \quad x \in S. \end{aligned} \tag{174}$$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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