

FIXED POINT THEORY FOR MULTIMAPS IN EXTENSION TYPE SPACES

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ABSTRACT. New fixed point results for the \mathfrak{A}_c^κ selfmaps are given. The analysis relies on a factorization idea. The notion of an essential map is also introduced for a wide class of maps. Finally, from a new fixed point theorem of ours, we deduce some equilibrium theorems.

1. Introduction

This paper presents new fixed point results for multivalued selfmaps, in particular the $\mathfrak{A}_c^\kappa(X, X)$ maps. The most general result in the literature [12] assumes X is convex and admissible (in the sense of Klee), but here we will show that it is enough to assume X is an extension space (so it could be an absolute retract), or an approximate extension space, or indeed a neighborhood extension space under some restrictions. In Section 3 we present the notion of an essential map and discuss some of its properties. Section 4 presents some quasi-equilibrium theorems.

For the remainder of this section we present some definitions and known results which will be needed throughout this paper. We will follow mainly [1, 2, 3, 12].

Let Y be a convex subset of a Hausdorff topological vector space E . Recall a *polytope* P in Y is any convex hull of a nonempty finite subset of Y . A nonempty subset X of E is said to be *admissible* (in the sense of Klee) if for every compact subset K of X and every neighborhood V of 0, there exists a continuous function $h : K \rightarrow X$ such that $x - h(x) \in V$ for all $x \in K$ and $h(K)$ is contained in a finite dimensional subspace of E . For example, every convex subset of a Hausdorff locally convex topological vector space is admissible. For other examples, see [12] and references therein.

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Of particular importance in this paper will be the class \mathfrak{A}_c^κ due to Park. Suppose X and Y are topological spaces. Given a class \mathfrak{X} of maps, $\mathfrak{X}(X, Y)$ denotes the set of maps $F : X \rightarrow 2^Y$ (the set of nonempty subsets of Y) belonging to \mathfrak{X} , and \mathfrak{X}_c the set of finite compositions of maps in \mathfrak{X} . We let

$$\mathcal{F}(\mathfrak{X}) = \{X : \text{Fix } F \neq \emptyset \text{ for all } F \in \mathfrak{X}(X, X)\},$$

where $\text{Fix } F$ denotes the set of fixed points of $F : X \rightarrow 2^X$.

A class \mathfrak{A} of maps is defined by the following properties:

- (i) \mathfrak{A} contains the class \mathbb{C} of single-valued continuous functions;
- (ii) each $F \in \mathfrak{A}_c$ is upper semicontinuous (u.s.c.) and compact-valued; and
- (iii) for any polytope P , $F \in \mathfrak{A}_c(P, P)$ has a fixed point, where the intermediate spaces of compositions are suitably chosen for each \mathfrak{A} .

An *admissible* class $\mathfrak{A}_c^\kappa(X, Y)$ of maps $F : X \rightarrow 2^Y$ is one such that, for each F and each nonempty compact subset K of X there exists a map $G \in \mathfrak{A}_c(X, Y)$ satisfying $G(x) \subset F(x)$ for all $x \in K$.

Examples of \mathfrak{A}_c^κ are classes of continuous functions \mathbb{C} , the Kakutani maps \mathbb{K} (u.s.c. with nonempty compact convex values and codomains are convex spaces), the Aronszajn maps \mathbb{M} (u.s.c. with R_δ values), the acyclic maps \mathbb{V} (u.s.c. with compact acyclic values), the Powers maps \mathbb{V}_c (finite compositions of acyclic maps), the O'Neill maps \mathbb{N} (continuous with values of one or m acyclic components, where m is fixed), the approachable maps \mathbb{A} (whose domains and codomains are uniform spaces), admissible maps of Górniewicz, σ -selectional maps of Haddad and Lasry, permissible maps of Dzedzej, the class \mathbb{K}_c^+ of Lassonde, the class \mathbb{V}_c^+ of Park *et al.*, and approximable maps of Ben-El-Mechaiekh and Idzik, and others. For details on the admissible classes, see [12].

In [12] Park gave an elementary proof of the following result.

THEOREM 1.1. *Let E be a Hausdorff topological vector space and X an admissible, convex, compact subset of E . Then any map $F \in \mathfrak{A}_c^\kappa(X, X)$ has a fixed point.*

A class of maps $\mathcal{R}(X, Y)$ is said to be *admissible* (in the sense of Ben-El-Mechaiekh and Deguire [3]) if

- (i) \mathcal{R} contains the class \mathbb{C} ; and
- (ii) each $F \in \mathcal{R}_c$ is upper semicontinuous and closed-valued.

The following result is given in [3, Proposition 2.2].

THEOREM 1.2. *Let \mathcal{R} be an admissible class of maps. Then the Hilbert cube I^∞ (subset of l^2 consisting of points (x_1, x_2, \dots) with $|x_i| \leq 1/i$ for all i) and the Tychonoff cube T (cartesian product of copies of the unit interval imbedded in a normed space) are in $\mathcal{F}(\mathcal{R}_c)$ provided the closed unit ball $\mathbf{B}^n = \{x \in \mathbf{R}^n : \|x\| \leq 1\}$ is in $\mathcal{F}(\mathcal{R}_c)$ for all $n \geq 1$.*

From Theorem 1.1 or 1.2, we immediately have the following.

THEOREM 1.3. *I^∞ and T are in $\mathcal{F}(\mathfrak{A}_c^\kappa)$.*

REMARK 1.1. It is worth remarking that we do not need to introduce the class \mathcal{R} (we did so to give credit to the authors in [1, 3]) since if we assume $\mathbf{B}^n \in \mathcal{F}(\mathcal{R}_c)$, then since \mathbf{B}^n is a homeomorphic image of a polytope, we have for any polytope P , that $F \in \mathcal{R}_c(P, P)$ has a fixed point, where the intermediate spaces of compositions are suitably chosen for each \mathcal{R} . Thus Theorem 1.3 follows immediately from [3] and the fact that $\mathbf{B}^n \in \mathcal{F}(\mathfrak{A}_c^\kappa)$.

REMARK 1.2. Since I^∞ and T are compact, Theorem 1.3 is actually equivalent to

(1) I^∞ and T are in $\mathcal{F}(\mathfrak{A}_c)$.

However, considering a Browder type map F (having nonempty convex values and open fibers), we notice that $F \notin \mathfrak{A}_c$ but $F \in \mathfrak{A}_c^\kappa$.

For a subset K of a topological space X , we denote by $Cov_X(K)$ the directed set of all coverings of K by open sets in X (usually we write $Cov(K) = Cov_X(K)$). Given a map $F : X \rightarrow 2^X$ and $\alpha \in Cov(X)$, a point $x \in X$ is said to be an α -fixed point of F if there exists a member $U \in \alpha$ such that $x \in U$ and $F(x) \cap U \neq \emptyset$.

Given two maps $F, G : X \rightarrow 2^Y$ and $\alpha \in Cov(Y)$, F and G are said to be α -close, if for any $x \in X$ there exist $U_x \in \alpha$, $y \in F(x) \cap U_x$, and $w \in G(x) \cap U_x$.

2. Extension type spaces and fixed points

In this section, we show that various extension type spaces have the fixed point property with respect to the \mathfrak{A}_c^κ selfmaps. For details and examples of such extension type spaces, see [1, 3] and references therein.

In the definitions in this section by a space we mean a Hausdorff topological space.

Let Q be a class of topological spaces. A space Y is an *extension space* for Q (written $Y \in ES(Q)$) if for any pair (X, K) in Q with $K \subset X$ closed, any continuous function $f_0 : K \rightarrow Y$ extends to a continuous function $f : X \rightarrow Y$.

We now present a new fixed point result for the \mathfrak{A}_c^κ maps.

THEOREM 2.1. *Let $X \in ES(\text{compact})$ and $F \in \mathfrak{A}_c^\kappa(X, X)$ a compact map. Then F has a fixed point.*

Proof. It is known [8] that every compact space is homeomorphic to a closed subset of the Tychonoff cube T , so as a result $K = \overline{F(X)}$ can be embedded as a closed subset K^* of T ; let $s : K \rightarrow K^*$ be a homeomorphism. Also let $i : K \hookrightarrow X$ and $j : K^* \hookrightarrow T$ be inclusions. Now since $X \in ES(\text{compact})$ and $is^{-1} : K^* \rightarrow X$, then is^{-1} extends to a continuous function $h : T \rightarrow X$. Let $G = jsFh$ and notice $G \in \mathfrak{A}_c^\kappa(T, T)$. Hence, Theorem 1.3 guarantees that there exists $x \in T$ with $x \in G(x)$. Let $y = h(x)$, so

$$y \in hjsF(y) \quad \text{i.e.} \quad y = hjs(q) \quad \text{for some} \quad q \in F(y).$$

Since $hj(z) = is^{-1}(z)$ for $z \in K^*$, we have $hjs(q) = (hj)s(q) = i(q) = q$, and so $y \in F(y)$. \square

REMARK 2.1. If $X \in AR$ (an absolute retract as defined in [5]) then of course $X \in ES(\text{compact})$ [We know from the Arens–Eells theorem that X is r -dominated by a normed space E so there exist maps $r : E \rightarrow X$ and $s : X \rightarrow E$ with $rs = 1$. Now since any normed space is $ES(\text{compact})$, it follows immediately that $X \in ES(\text{compact})$]. So a special case of Theorem 2.1 occurs if $X \in AR$.

A space Y is an *approximate extension space* for Q (written $Y \in AES(Q)$) if for any $\alpha \in Cov(Y)$ and any pair (X, K) in Q with $K \subset X$ closed, and any continuous function $f_0 : K \rightarrow Y$, there exists a continuous function $f : X \rightarrow Y$ such that $f|_K$ is α -close to f_0 .

We now extend Theorem 2.1 to approximate extension spaces. To prove this we need the following elementary result for α -fixed points (see [1, Lemma 1.2]).

LEMMA 2.2. *Let X be a regular topological space and $F : X \rightarrow 2^X$ an upper semicontinuous map with closed values. Suppose there exists a cofinal family of coverings $\theta \subset Cov(X)$ such that F has an α -fixed point for every $\alpha \in \theta$. Then F has a fixed point.*

THEOREM 2.3. *Let $X \in AES(\text{compact})$ and $F \in \mathfrak{A}_c^\kappa(X, X)$ a compact map. Then F has a fixed point.*

Proof. Let K, K^*, s, i and j be as in the proof of Theorem 2.1. Let $\alpha \in \text{Cov}(X)$ and let $h : T \rightarrow X$ be such that h and is^{-1} are α -close on K^* (guaranteed since $X \in AES(\text{compact})$). Let $G = jsFh$ and notice $G \in \mathfrak{A}_c^\kappa(T, T)$. Now Theorem 1.3 guarantees that there exists $x \in T$ with $x \in G(x)$. Let $y = h(x)$, so

$$y \in hjsF(y) \quad \text{i.e.} \quad y = hjs(q) \quad \text{for some} \quad q \in F(y).$$

Since is^{-1} and h are α -close on K^* there exists $U \in \alpha$ with $is^{-1}(s(q)) \in U$ and $hjs(q) \in U$ i.e. $q \in U$ and $y \in U$. Thus

$$y \in U \quad \text{and} \quad F(y) \cap U \neq \emptyset \quad \text{since} \quad q \in F(y).$$

As a result F has an α -fixed point. Since α is arbitrary, Lemma 2.2 guarantees that F has a fixed point. \square

DEFINITION 2.1. Let V be a subset of a Hausdorff topological vector space E . Then we say V is *Schauder admissible* if for every compact subset K of V and every covering $\alpha \in \text{Cov}_V(K)$, there exists a continuous function (called the Schauder projection) $\pi_\alpha : K \rightarrow V$ such that

- (i) π_α and $i : K \hookrightarrow V$ are α -close;
- (ii) $\pi_\alpha(K)$ is contained in a subset $C \subset V$ with $C \in AES(\text{compact})$.

If $V \in AES(\text{compact})$ then V is trivially Schauder admissible. If V is an open convex subset of a Hausdorff locally convex topological vector space E , then it is well known [1, Lemma 4.8] that V is Schauder admissible.

We next present a result of Himmelberg type [9].

THEOREM 2.4. Let V be a Schauder admissible subset of a Hausdorff topological vector space E and $F \in \mathfrak{A}_c^\kappa(V, V)$ a compact map. Then F has a fixed point.

Proof. Since $F(V) \subset K$, K compact, for each $\alpha \in \text{Cov}_V(K)$ there exist $\pi_\alpha : K \rightarrow V$ (as described in Definition 2.1) and a subset $C \subset V$ with $C \in AES(\text{compact})$ such that, by putting $F_\alpha \equiv \pi_\alpha F$,

$$F_\alpha(V) = \pi_\alpha F(V) \subset C.$$

Notice $F_\alpha \in \mathfrak{A}_c^\kappa(C, C)$ so Theorem 2.3 guarantees that there exists $x \in C$ with $x \in \pi_\alpha F(x)$ i.e. $x = \pi_\alpha(q)$ for some $q \in F(x)$. Now Definition 2.1 (i) guarantees that there exists $U \in \alpha$ with $\pi_\alpha(q) \in U$ and $i(q) \in U$ i.e. $x \in U$ and $q \in U$. Thus

$$x \in U \quad \text{and} \quad F(x) \cap U \neq \emptyset \quad \text{since} \quad q \in F(x).$$

As a result F has an α -fixed point. Since α is arbitrary, Lemma 2.2 guarantees that F has a fixed point. \square

A space Y is a *neighborhood extension space* for Q (written $Y \in NES(Q)$) if for any pair (X, K) in Q with $K \subset X$ closed and any continuous function $f_0 : K \rightarrow Y$ there is a continuous extension $f : U \rightarrow Y$ of f_0 over a neighborhood U of K in X .

We would like to extend Theorem 2.3 to neighborhood extension spaces. However even in the case when F is admissible in the sense of Górniewicz [6] extra conditions need to be added (recall that maps admissible in the sense of Górniewicz are in the class \mathfrak{A}_c^κ).

Recall the following well known result [1, Lemma 4.7].

LEMMA 2.5. *Let T be a Tychonoff cube contained in a Hausdorff topological vector space. Then T is a retract of $\text{span}(T)$.*

Let $X \in NES(\text{compact})$ and $F \in \mathfrak{A}_c^\kappa(X, X)$ a compact map.

Let K, K^*, s and i be as in the proof of Theorem 2.1. Let U be an open neighborhood of K^* in T and $h : U \rightarrow X$ be a continuous extension of $is^{-1} : K^* \rightarrow X$ on U (guaranteed since $X \in NES(\text{compact})$). Let $j : K^* \hookrightarrow U$ be the natural embedding so $hj = is^{-1}$. Now consider $\text{span}(T)$ in a Hausdorff locally convex topological vector space containing T . Now Lemma 2.5 guarantees that there exists a retraction $r : \text{span}(T) \rightarrow T$. Let $i^* : U \hookrightarrow r^{-1}(U)$ be an inclusion and consider $G = i^*jsFhr$. Notice $G \in \mathfrak{A}_c^\kappa(r^{-1}(U), r^{-1}(U))$. Assume

$$(2.1) \quad G \in \mathfrak{A}_c^\kappa(r^{-1}(U), r^{-1}(U)) \text{ has a fixed point.}$$

If (2.1) is true then there exists $x \in r^{-1}(U)$ with $x \in Gx$. Let $y = hr(x)$, so

$$y \in hri^*jsF(y) \text{ i.e. } y = hri^*js(q) \text{ for some } q \in F(y).$$

Since $h(z) = is^{-1}(z)$ for $z \in K^*$, we have $hri^*js(q) = (hri^*j)s(q) = i(q)$, and so $y \in F(y)$.

Thus existence of a fixed point of F is guaranteed if (2.1) is satisfied; recall $G = i^*jsFhr$ and $r^{-1}(U)$ is an open subset of a Hausdorff locally convex topological vector space.

For specific classes of maps (2.1) is known to be true. For example, if F is admissible in the sense of Górniewicz [6] and the Lefschetz set $\Lambda(F) \neq \{0\}$ then we know [6] that (2.1) holds. More generally, if F is permissible in the sense of Dzedzej [7] and $\Lambda(F) \neq \{0\}$ then (2.1) holds. It would be of interest to know other examples.

3. Essential maps

Throughout this section Y will be a completely regular topological space with $Y \in AES(\text{compact})$, so in particular the results in this section will hold if $Y \in ES(\text{compact})$ or $Y \in AR$. [Of course $Y \in AES(\text{compact})$ could be replaced by Y Schauder admissible in this section]. Also U will be an open subset of Y . In this section we consider a subclass \mathcal{A} of \mathcal{A}_c^k . The subclass must have the following property: for subsets X_1, X_2 and X_3 of Hausdorff topological vector spaces

$$\text{if } F \in \mathcal{A}(X_2, X_3) \text{ and } f \in \mathbb{C}(X_1, X_2), \text{ then } Ff \in \mathcal{A}(X_1, X_3).$$

The theory in this section will work for any class of maps \mathcal{A} which satisfy a normalization property. In particular one can view the class \mathcal{A} as any class where we can get a Leray–Schauder type result. For example we could take \mathcal{A} to be \mathbb{V} since clearly (3.3) (and (3.4), (3.5)) hold.

DEFINITION 3.1. $F \in \mathcal{A}_{\partial U}(\bar{U}, Y)$ if $F \in \mathcal{A}(\bar{U}, Y)$ with F compact and $x \notin F(x)$ for $x \in \partial U$.

DEFINITION 3.2. A map $F \in \mathcal{A}_{\partial U}(\bar{U}, Y)$ is *essential* if for every $G \in \mathcal{A}_{\partial U}(\bar{U}, Y)$ with $G|_{\partial U} = F|_{\partial U}$ there exists $x \in U$ with $x \in G(x)$.

THEOREM 3.1. (Homotopy Invariance) *Let Y and U be as above. Suppose $F \in \mathcal{A}_{\partial U}(\bar{U}, Y)$ is an essential map and $H \in \mathcal{A}(\bar{U} \times [0, 1], Y)$ is a compact map. Also assume the following two properties hold:*

$$(3.1) \quad H(x, 0) = F(x) \text{ for } x \in \bar{U}$$

and

$$(3.2) \quad x \notin H_t(x) \text{ for any } x \in \partial U \text{ and } t \in (0, 1] \text{ (here } H_t(x) = H(x, t)).$$

Then H_1 has a fixed point in U .

Proof. Let

$$B = \{x \in \bar{U} : x \in H_t(x) \text{ for some } t \in [0, 1]\}.$$

When $t = 0$, $H_t = F$ and since $F \in \mathcal{A}_{\partial U}(\bar{U}, Y)$ is essential there exists an $x \in U$ with $x \in F(x)$. Thus $B \neq \emptyset$. Since H is upper semicontinuous and compact, it is immediate that B is closed and compact. In addition (3.2) (together with $F \in \mathcal{A}_{\partial U}(\bar{U}, Y)$) implies $B \cap \partial U = \emptyset$. Thus (since Y is completely regular) there exists a continuous function $\mu : \bar{U} \rightarrow [0, 1]$ with $\mu(\partial U) = 0$ and $\mu(B) = 1$. Define a map R by $R(x) = H(x, \mu(x))$ for $x \in \bar{U}$. Let $j : \bar{U} \rightarrow \bar{U} \times [0, 1]$ be given

by $j(x) = (x, \mu(x))$. Note j is continuous so $R = H j \in \mathcal{A}(\bar{U}, Y)$. In addition, R is compact since H is. Also notice for $x \in \partial U$ that $R(x) = H_0(x) = F(x)$, and so $R \in \mathcal{A}_{\partial U}(\bar{U}, Y)$. Now $R|_{\partial U} = F|_{\partial U}$ and $F \in \mathcal{A}_{\partial U}(\bar{U}, Y)$ essential implies that there exists $x \in U$ with $x \in R(x)$ (i.e. $x \in H_{\mu(x)}(x)$). Thus $x \in B$ and so $\mu(x) = 1$. Consequently $x \in H_1(x)$. \square

Next we give an example of an essential map.

THEOREM 3.2. (Normalization) *Let Y and U be as above with $0 \in U$. Suppose the following condition is satisfied:*

$$(3.3) \quad \begin{cases} \text{for any map } \theta \in \mathcal{A}_{\partial U}(\bar{U}, Y) \text{ with } \theta|_{\partial U} = \{0\}, \\ \text{the map } J \text{ is in } \mathfrak{A}_c^\kappa(Y, Y); \text{ here} \\ J(x) = \begin{cases} \theta(x), & x \in \bar{U} \\ \{0\}, & x \in Y \setminus \bar{U}. \end{cases} \end{cases}$$

Then the zero map is essential in $\mathcal{A}_{\partial U}(\bar{U}, Y)$.

Proof. Let $\theta \in \mathcal{A}_{\partial U}(\bar{U}, Y)$ with $\theta|_{\partial U} = \{0\}$. We must show that there exists $x \in U$ with $x \in \theta(x)$. Define a map J by

$$J(x) = \begin{cases} \theta(x), & x \in \bar{U} \\ \{0\}, & x \in Y \setminus \bar{U}. \end{cases}$$

From (3.3) we know $J \in \mathfrak{A}_c^\kappa(Y, Y)$. Clearly J is compact since θ is. Hence, Theorem 2.3 implies that there exists $x \in Y$ with $x \in J(x)$. Now if $x \notin U$ we have $x \in J(x) = \{0\}$, which is a contradiction since $0 \in U$. Thus $x \in U$ so $x \in J(x) = \theta(x)$. \square

Next we present another version of the normalization property when we are in the topological vector space setting. Let $Y \in AES(\text{compact})$ be a convex subset of a topological vector space E and let U be an open subset of Y with $0 \in U$. In addition assume there exists a continuous retraction $r : Y \rightarrow \bar{U}$.

THEOREM 3.3. (Normalization) *Let E, Y, U and r be as above and suppose the following condition is satisfied:*

$$(3.4) \quad \begin{cases} \text{for any continuous function } \mu : Y \rightarrow [0, 1] \text{ and} \\ \text{any map } G \in \mathcal{A}(Y, Y) \text{ we have } \mu G \in \mathfrak{A}_c^\kappa(Y, Y). \end{cases}$$

Then the zero map is essential in $\mathcal{A}_{\partial U}(\bar{U}, Y)$.

Proof. Let $\theta \in \mathcal{A}_{\partial U}(\bar{U}, Y)$ with $\theta|_{\partial U} = \{0\}$. Let

$$A = \{x \in \bar{U} : x \in \lambda \theta(x) \text{ for some } \lambda \in [0, 1]\}.$$

Now $A \neq \emptyset$ is compact and $A \subset U$ (this is clear since $0 \in U$ and $\theta|_{\partial U} = \{0\}$). Thus there exists a continuous function $\mu : Y \rightarrow [0, 1]$ with $\mu(A) = 1$ and $\mu(Y \setminus U) = 0$. Define a map J_0 by

$$J_0(x) = \mu(x)\theta(r(x)) \quad \text{for } x \in Y.$$

Note $\theta r \in \mathcal{A}(Y, Y)$ so $J_0 \in \mathfrak{A}_c^k(Y, Y)$ from (3.4). Theorem 2.3 implies that there exists $x \in Y$ with $x \in \mu(x)\theta(r(x))$. If $x \in Y \setminus U$ then $\mu(x) = 0$, a contradiction since $0 \in U$. Thus $x \in U$ and so $x \in \mu(x)\theta(x)$. As a result $x \in A$, so $\mu(x) = 1$. Consequently $x \in \theta(x)$. \square

Of course we can obtain a nonlinear alternative of Leray–Schauder type by combining Theorems 3.1 and 3.3. In fact, we can obtain a more general result.

THEOREM 3.4. *Let E, Y, U and r be as above. Suppose $F \in \mathcal{A}(\bar{U}, Y)$ satisfies (3.4) and assume the following conditions hold:*

$$(3.5) \quad \begin{cases} \text{for any continuous function } \mu : \bar{U} \rightarrow [0, 1] \text{ and} \\ \text{any map } G \in \mathcal{A}(\bar{U}, Y) \text{ we have } \mu G \in \mathcal{A}(\bar{U}, Y) \end{cases}$$

and

$$(3.6) \quad x \notin \lambda F(x) \quad \text{for every } x \in \partial U \text{ and } \lambda \in (0, 1].$$

Then F is essential in $\mathcal{A}_{\partial U}(\bar{U}, Y)$.

Proof. Let $\Phi \in \mathcal{A}_{\partial U}(\bar{U}, Y)$ with $\Phi|_{\partial U} = F|_{\partial U}$. We must show Φ has a fixed point in U . Let

$$D = \{x \in \bar{U} : x \in \lambda \Phi(x) \text{ for some } \lambda \in [0, 1]\}.$$

Now $D \neq \emptyset$ is compact and $D \cap \partial U = \emptyset$ (note (3.6) with $\Phi|_{\partial U} = F|_{\partial U}$ and that $0 \in U$). Thus there exists a continuous function $\mu : \bar{U} \rightarrow [0, 1]$ with $\mu(\partial U) = 0$ and $\mu(D) = 1$. Define a map R by $R(x) = \mu(x)\Phi(x)$. Now (3.5) guarantees that $R \in \mathcal{A}(\bar{U}, Y)$. Also R is compact with $R|_{\partial U} = \{0\}$. Now since $R \in \mathcal{A}_{\partial U}(\bar{U}, Y)$ and since the zero map is essential in $\mathcal{A}_{\partial U}(\bar{U}, Y)$ (Theorem 3.3) there exists $x \in U$ with $x \in R(x)$. Thus $x \in D$ and so $\mu(x) = 1$, i.e., $x \in \Phi(x)$. \square

4. Quasi–equilibrium theorem

We begin this section by expressing Theorem 2.4 as an equilibrium theorem. Then a general result will be deduced from our main theorem.

THEOREM 4.1. *Let E and Y be Hausdorff topological vector spaces, Q a subset of E , $G : Q \rightarrow k(Q)$ (nonempty compact subsets of Q) and*

$T : Q \rightarrow 2^C$ where C is a subset of Y . In addition assume the following conditions hold:

$$(4.1) \quad f : Q \times C \times Q \rightarrow \mathbf{R} \text{ is a upper semicontinuous function,}$$

$$(4.2) \quad G \text{ and } T \text{ are compact maps,}$$

$$(4.3) \quad Q \times C \text{ is an Schauder admissible subset of } E \times Y,$$

and

$$(4.4) \quad F \in \mathfrak{A}_c^\kappa(Q \times C, Q \times C);$$

here $F(x, y) = \Phi(x, y) \times T(x)$ for $(x, y) \in Q \times C$ with

$$\Phi(x, y) = \{w \in G(x) : f(x, y, w) = M(x, y)\}$$

and $M(x, y) = \max_{w \in G(x)} f(x, y, w)$. Then there exist $(x_0, y_0) \in Q \times C$, $x_0 \in G(x_0)$, and $y_0 \in T(x_0)$ with

$$f(x_0, y_0, z) \leq f(x_0, y_0, x_0) \text{ for all } z \in G(x_0).$$

If in addition

$$(4.5) \quad f(x, y, x) \leq 0 \text{ for all } (x, y) \in Q \times C,$$

then there exists $(x_0, y_0) \in Q \times C$ such that $x_0 \in G(x_0)$, $y_0 \in T(x_0)$, and

$$f(x_0, y_0, z) \leq 0 \text{ for all } z \in G(x_0).$$

Proof. Notice $\Phi(x, y)$ is nonempty (and compact) for each $(x, y) \in Q \times C$. As a result $F : Q \times C \rightarrow 2^{Q \times C}$ and also F is compact since $F(Q \times C) \subseteq G(Q) \times T(Q)$. Now Theorem 2.4 guarantees that there exists $(x_0, y_0) \in Q \times C$ with $(x_0, y_0) \in \Phi(x_0, y_0) \times T(x_0)$. That is, there exists $(x_0, y_0) \in Q \times C$ with $x_0 \in G(x_0)$, $y_0 \in T(x_0)$ and $f(x_0, y_0, x_0) = M(x_0, y_0)$ (i.e., $f(x_0, y_0, z) \leq f(x_0, y_0, x_0)$ for all $z \in G(x_0)$), so we are finished the first part. For the second part assume (4.5) holds, and so the result is immediate from the first part. \square

Next we consider a subclass \mathcal{D} of \mathfrak{A}_c^κ . If X and Y are subsets of Hausdorff topological vector spaces then we say $F \in \mathcal{D}(X, Y)$ if $F \in \mathfrak{A}_c^\kappa(X, Y)$ and is upper semicontinuous with nonempty compact values and satisfies Property (C) (to be specified in the examples considered). Also we assume for subsets X_1 and X_2 of Hausdorff topological vector spaces

$$(4.6) \quad \begin{cases} \text{if } F_1 \in \mathcal{D}(X_1 \times X_2, X_1) \text{ and } F_2 \in \mathcal{D}(X_1, X_2) \\ \text{then } F_3 \in \mathfrak{A}_c^\kappa(X_1 \times X_2, X_1 \times X_2); \end{cases}$$

here $F_3(x, y) = F_1(x, y) \times F_2(x)$. A typical example of a class \mathcal{D} is the acyclic maps \mathbb{V} (i.e., Property (C) means the map is acyclic valued).

THEOREM 4.2. *Let E and Y be Hausdorff topological vector spaces, Q a subset of E , $G : Q \rightarrow k(Q)$ and $T : Q \rightarrow k(C)$ where C is a subset of Y . Suppose (4.1), (4.2), (4.3), (4.6) hold and in addition assume the following conditions are satisfied:*

$$(4.7) \quad G : Q \rightarrow 2^Q \text{ is upper semicontinuous}$$

$$(4.8) \quad \begin{cases} M : Q \times C \rightarrow Q \text{ is lower semicontinuous} \\ \text{(here } M(x, y) = \max_{w \in G(x)} f(x, y, w) \text{)} \end{cases}$$

$$(4.9) \quad T \in \mathfrak{A}_c^k(Q, C) \text{ is upper semicontinuous and satisfies Property (C)}$$

and

$$(4.10) \quad \Phi \in \mathfrak{A}_c^k(Q \times C, Q) \text{ and satisfies Property (C);}$$

here

$$\Phi(x, y) = \{w \in G(x) : f(x, y, w) = M(x, y)\}.$$

Then there exists $(x_0, y_0) \in Q \times C$, $x_0 \in G(x_0)$ and $y_0 \in T(x_0)$ with

$$f(x_0, y_0, z) \leq f(x_0, y_0, x_0) \text{ for all } z \in G(x_0).$$

If in addition (4.5) holds, then there exists $(x_0, y_0) \in Q \times C$, $x_0 \in G(x_0)$ and $y_0 \in T(x_0)$ with $f(x_0, y_0, z) \leq 0$ for all $z \in G(x_0)$.

Proof. The result follows from Theorem 4.1 once we show (4.4) holds. First we show Φ is upper semicontinuous. To show this it suffices (note Φ is compact) to show Φ is closed. Let $\{(x_\alpha, y_\alpha, w_\alpha)\}$ be a net in $graph(\Phi)$ with $(x_\alpha, y_\alpha, w_\alpha) \rightarrow (x, y, w)$. From (4.8) it follows that

$$f(x, y, w) \geq \limsup f(x_\alpha, y_\alpha, w_\alpha) \geq \liminf M(x_\alpha, y_\alpha) \geq M(x, y).$$

Also $w_\alpha \in G(x_\alpha)$ together with $x_\alpha \rightarrow x$, $w_\alpha \rightarrow w$ and G upper semicontinuous (so G is closed) implies $w \in G(x)$ and $f(x, y, w) \geq M(x, y)$. Consequently $f(x, y, w) = M(x, y)$, so $(x, y, w) \in graph(\Phi)$. Thus Φ is upper semicontinuous with nonempty, compact values, so this together with (4.10) implies $\Phi \in \mathcal{D}(Q \times C, Q)$. Also (4.2) and (4.9) guarantees that $T \in \mathcal{D}(Q, C)$. As a result $F \in \mathfrak{A}_c^k(Q \times C, Q \times C)$ from (4.6); here $F(x, y) = \Phi(x, y) \times T(x)$. Thus (4.4) holds. \square

For the motivation and some related results in this section, the reader can refer to [4, 10, 11, 13, 14, 16].

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