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Research Article

Fixed Points for Multivalued Mappings in b-Metric Spaces

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In 2012, Samet et al. introduced the notion of α-ψ-contractive mapping and gave sufficient conditions for the existence of fixed points for this class of mappings. The purpose of our paper is to study the existence of fixed points for multivalued mappings, under an α-ψ-contractive condition of Ćirić type, in the setting of complete b-metric spaces. An application to integral equation is given.

1. Introduction

The theory of multivalued mappings has an important role in various branches of pure and applied mathematics because of its many applications, for instance, in real and complex analysis as well as in optimal control problems. Over the years, this theory has increased its significance and hence in the literature there are many papers focusing on the discussion of abstract and practical problems involving multivalued mappings. As a matter of fact, amongst the various approaches utilized to develop this theory, one of the most interesting is based on methods of fixed point theory, often in view of the constructive character of fixed point theorems, especially in its metric branch (see, for instance, [1]). Thus, Nadler [2] was the first author who combined the notion of contraction (see condition (1) below) with multivalued mapping by establishing the following fixed point theorem.

Theorem 1 (see [2]). *Let (X, d) be a complete metric space and let $T : X \to CL(X)$ be a multivalued mapping satisfying*

$$H(Tx, Ty) \le kd(x, y), \tag{1}$$

for all $x, y \in X$, where k is a constant such that $k \in (0, 1)$ and $CL(X)$ denotes the family of nonempty closed subsets of X. Then T has a fixed point; that is, there exists a point $z \in X$ such that $z \in Tz$.

Later on, many authors discussed this result and gave their generalizations, extensions, and applications; see, for instance, [3–7].

On the other hand, the concept of metric space has been generalized in different directions to better cover much more general situations, arising in computer science and others (see, for instance, [3, 8]). Here, we deal with the notion of b-metric space, which is a metric space satisfying a relaxed form of triangle inequality; see Bakhtin [9] and Czerwick [10]. Many researchers followed this idea and proved various results in the b-metric setting [11–14].

In 2012, Samet et al. [15] introduced the notion of α-ψ-contractive mapping and gave sufficient conditions for the existence of fixed points for this class of mappings. In this paper, we study the existence of fixed points for multivalued mappings, under an α-ψ-contractive condition of Ćirić type, in the setting of complete b-metric spaces. Also, we consider a complete b-metric space endowed with a partial ordering. An inspiration for the paper is the recent work of Mohammadi et al. [16] in which some good ideas are suggested to the reader. Finally, an application to integral equation is given.

2. Preliminaries

Let $\mathbb{R}^+$ denote the set of all nonnegative real numbers and let $\mathbb{N}$ denote the set of positive integers. From [9, 10, 17, 18] we get some basic definitions, lemmas, and notations concerning the b-metric space.

Definition 2. Let X be a nonempty set and let $s \geq 1$ be a given real number. A function $d : X \times X \to \mathbb{R}^+$ is said to be a b-metric if and only if for all $x, y, z \in X$ the following conditions are satisfied:

(1) $d(x, y) = 0$ if and only if $x = y$;

(2) $d(x, y) = d(y, x)$;

(3) $d(x, z) \leq s[d(x, y) + d(y, z)]$.

Then, the triplet (X, d, s) is called a b-metric space.

It is an obvious fact that a metric space is also a b-metric space with $s = 1$, but the converse is not generally true. To support this fact, we have the following example.

Example 3. Consider the set $X = [0, 1]$ endowed with the function $d : X \times X \to \mathbb{R}^+$ defined by $d(x, y) = |x - y|^2$ for all $x, y \in X$. Clearly, $(X, d, 2)$ is a b-metric space but it is not a metric space.

Let (X, d, s) be a b-metric space. The following notions are natural deductions from their metric counterparts.

(i) A sequence $\{x_n\} \subseteq X$ converges to $x \in X$ if $\lim_{n \to +\infty} d(x_n, x) = 0$.

(ii) A sequence $\{x_n\} \subseteq X$ is said to be a Cauchy sequence if, for every given $\varepsilon > 0$, there exists $n(\varepsilon) \in \mathbb{N}$ such that $d(x_m, x_n) < \varepsilon$ for all $m, n \geq n(\varepsilon)$.

(iii) A b-metric space (X, d, s) is said to be complete if and only if each Cauchy sequence converges to some $x \in X$.

From the literature on b-metric spaces, we choose the following significant example.

Example 4 (see [11]). Let $p \in (0, 1)$. Consider the space $L^p([0, 1])$ of all real functions $f : [0, 1] \to \mathbb{R}$ such that $\int_0^1 |f(t)|^p dt < +\infty$, endowed with the functional $d : L^p([0, 1]) \times L^p([0, 1]) \to \mathbb{R}$ defined by

$$d(f, g) = \left(\int_0^1 |f(t) - g(t)|^p dt\right)^{1/p} \quad \forall f, g \in L^p([0, 1]). \tag{2}$$

Then, $(X, d, 2^{1/p})$ is a b-metric space.

Next, we collect some lemmas and notions concerning the theory of multivalued mappings on b-metric spaces. We recall that $CB(X)$ denotes the class of nonempty closed and bounded subsets of X. For $A, B \in CB(X)$, define the function $H : CB(X) \times CB(X) \to \mathbb{R}^+$ by

$$H(A, B) = \max\{\delta(A, B), \delta(B, A)\}, \tag{3}$$

where

$$\delta(A, B) = \sup\{d(a, B), a \in A\}, \qquad \delta(B, A) = \sup\{d(b, A), b \in B\} \tag{4}$$

with

$$d(a, C) = \inf\{d(a, x), x \in C\}. \tag{5}$$

Note that H is called the Hausdorff b-metric induced by the b-metric d.

We recall the following properties from [10, 13, 18]; see also [12] and the references therein.

Lemma 5. *Let (X, d, s) be a b-metric space. For any $A, B, C \in CB(X)$ and any $x, y \in X$, one has the following:*

(i) $d(x, B) \leq d(x, b)$, *for any* $b \in B$;

(ii) $\delta(A, B) \leq H(A, B)$;

(iii) $d(x, B) \leq H(A, B)$, *for any* $x \in A$;

(iv) $H(A, A) = 0$;

(v) $H(A, B) = H(B, A)$;

(vi) $H(A, C) \leq s(H(A, B) + H(B, C))$;

(vii) $d(x, A) \leq s(d(x, y) + d(y, A))$.

Remark 6. The function $H : CL(X) \times CL(X) \to \mathbb{R}^+$ is a generalized Hausdorff b-metric; that is, $H(A, B) = +\infty$ if $\max\{\delta(A, B), \delta(B, A)\}$ does not exist.

Lemma 7. *Let (X, d, s) be a b-metric space. For $A \in CL(X)$ and $x \in X$, one has*

$$d(x, A) = 0 \Longleftrightarrow x \in \overline{A} = A, \tag{6}$$

where $\overline{A}$ denotes the closure of the set A.

Lemma 8. *Let (X, d, s) be a b-metric space and $A, B \in CL(X)$. Then, for each $h > 1$ and for each $a \in A$ there exists $b(a) \in B$ such that $d(a, b(a)) < h\, H(A, B)$ if $H(A, B) > 0$.*

Finally, to prove our results we need the following class of functions.

Let $s \geq 1$ be a real number; we denote by Ψ_s the family of strictly increasing functions $\psi : [0, +\infty) \to [0, +\infty)$ such that

$$\sum_{n=1}^{+\infty} s^n \psi^n(t) < +\infty \quad \text{for each } t > 0, \tag{7}$$

where ψ^n denotes nth iterate of the function ψ. It is well known that $\psi(t) < t$ for all $t > 0$. An example of function $\psi \in \Psi_s$ is given by $\psi(t) = ct/s$ for all $t \geq 0$, where $c \in (0, 1)$.

Definition 9. A multivalued mapping $T : X \to CL(X)$ is said to be α-admissible, with respect to a function $\alpha : X \times X \to [0, +\infty)$, if for each $x \in X$ and $y \in Tx$ with $\alpha(x, y) \geq 1$, we have $\alpha(y, z) \geq 1$ for all $z \in Ty$.

Definition 10. Let (X, d, s) be a b-metric space and let $\delta(\cdot, \cdot)$ be as in (4). Then, a multivalued mapping $F : X \to CL(X)$ is said to be h-upper semicontinuous at $x_0 \in X$, if the function

$$\delta(Fx, Fx_0) := \sup\{d(y, Fx_0) : y \in Fx\} \tag{8}$$

is continuous at x_0. Clearly, F is said to be h-upper semicontinuous, whenever F is h-upper semicontinuous at every $x_0 \in X$.

3. Fixed Point Theory in b-Metric Spaces

We study the existence of fixed points for multivalued mappings, by adapting the ideas in [16] to the b-metric setting.

Definition 11. Let (X, d, s) be a b-metric space. A multivalued mapping $T : X \to CL(X)$ is said to be an α-ψ-contraction of Ćirić type if there exist a function $\alpha : X \times X \to [0, +\infty)$ and a function $\psi \in \Psi_s$ such that, for all $x, y \in X$ with $\alpha(x, y) \geq 1$, the following condition holds

$$H(Tx, Ty) \leq \psi(M(x, y)), \tag{9}$$

where

$$M(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2s}\left[d(x, Ty) + d(y, Tx)\right]\right\}. \tag{10}$$

Now, we are ready to state and prove our first main theorem.

Theorem 12. *Let (X, d, s) be a complete b-metric space and let $T : X \to CL(X)$. Assume that there exist two functions $\alpha : X \times X \to [0, +\infty)$ and $\psi \in \Psi_s$ such that T is an α-ψ-contraction of Ćirić type. Also, suppose that the following conditions are satisfied:*

(i) *T is an α-admissible multivalued mapping;*

(ii) *there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$;*

(iii) *T is h-upper semicontinuous.*

Then T has a fixed point.

Proof. By condition (ii) there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$. Clearly, if $x_0 = x_1$ or $x_1 \in Tx_1$, we deduce that x_1 is a fixed point of T and so we can conclude the proof. Now, we assume that $x_0 \neq x_1$ and $x_1 \notin Tx_1$ and hence $d(x_1, Tx_1) > 0$. First, from (9), we deduce

$$\begin{aligned} 0 &< d(x_1, Tx_1) \\ &\leq H(Tx_0, Tx_1) \\ &\leq \psi\left(\max\left\{d(x_0, x_1), d(x_0, Tx_0), d(x_1, Tx_1), \frac{1}{2s}\left[d(x_0, Tx_1) + d(x_1, Tx_0)\right]\right\}\right) \\ &\leq \psi\left(\max\left\{d(x_0, x_1), d(x_0, x_1), d(x_1, Tx_1), \frac{1}{2}\left[d(x_0, x_1) + d(x_1, Tx_1)\right]\right\}\right) \\ &= \psi(\max\{d(x_0, x_1), d(x_1, Tx_1)\}). \end{aligned} \tag{11}$$

If $\max\{d(x_0, x_1), d(x_1, Tx_1)\} = d(x_1, Tx_1)$, then we have

$$0 < d(x_1, Tx_1) \leq \psi(d(x_1, Tx_1)) < d(x_1, Tx_1), \tag{12}$$

which is a contradiction. Thus,

$$\max\{d(x_0, x_1), d(x_1, Tx_1)\} = d(x_0, x_1), \tag{13}$$

and since ψ is strictly increasing, we have

$$0 < d(x_1, Tx_1) \leq \psi(d(x_0, x_1)) < \psi(\tau d(x_0, x_1)), \tag{14}$$

where $\tau > 1$ is a real number. This ensures that there exists $x_2 \in Tx_1$ (obviously, $x_2 \neq x_1$) such that

$$0 < d(x_1, x_2) < \psi(\tau d(x_0, x_1)). \tag{15}$$

Since T is α-admissible, from condition (ii) and $x_2 \in Tx_1$, we have $\alpha(x_1, x_2) \geq 1$. If $x_2 \in Tx_2$, then x_2 is a fixed point. Assume that $x_2 \notin Tx_2$; that is, $d(x_2, Tx_2) > 0$.

Next, from (9), we deduce

$$\begin{aligned} 0 &< d(x_2, Tx_2) \\ &\leq H(Tx_1, Tx_2) \\ &\leq \psi\left(\max\left\{d(x_1, x_2), d(x_1, Tx_1), d(x_2, Tx_2), \frac{1}{2s}\left[d(x_1, Tx_2) + d(x_2, Tx_1)\right]\right\}\right) \\ &= \psi(\max\{d(x_1, x_2), d(x_2, Tx_2)\}). \end{aligned} \tag{16}$$

If $\max\{d(x_1, x_2), d(x_2, Tx_2)\} = d(x_2, Tx_2)$, then we have

$$0 < d(x_2, Tx_2) \leq \psi(d(x_2, Tx_2)) < d(x_2, Tx_2), \tag{17}$$

which is a contradiction. Thus,

$$\max\{d(x_1, x_2), d(x_2, Tx_2)\} = d(x_1, x_2), \tag{18}$$

and since ψ is strictly increasing, we have

$$0 < d(x_2, Tx_2) \leq \psi(d(x_1, x_2)) < \psi^2(\tau d(x_0, x_1)). \tag{19}$$

This ensures that there exists $x_3 \in Tx_2$ (obviously, $x_3 \neq x_2$) such that

$$0 < d(x_2, x_3) < \psi^2(\tau d(x_0, x_1)). \tag{20}$$

Iterating this procedure, we construct a sequence $\{x_n\} \subset X$ such that

$$\begin{gathered} x_n \notin Tx_n, x_{n+1} \in Tx_n, \alpha(x_n, x_{n+1}) \geq 1, \\ 0 < d(x_n, Tx_n) \leq d(x_n, x_{n+1}) < \psi^n(\tau d(x_0, x_1)), \quad \forall n \in \mathbb{N}. \end{gathered} \tag{21}$$

Let $m > n$. Then

$$\begin{aligned} d(x_n, x_m) &\leq \sum_{k=n}^{m-1} s^{k-n+1} d(x_k, x_{k+1}) \\ &\leq \sum_{k=n}^{m-1} s^k \psi^k(\tau d(x_0, x_1)), \end{aligned} \tag{22}$$

and so $\{x_n\}$ is a Cauchy sequence in X. Hence, there exists $z \in X$ such that $x_n \to z$.

From

$$\begin{aligned} d(z, Tz) &\leq s[d(z, x_{n+1}) + d(x_{n+1}, Tz)] \\ &\leq sd(z, x_{n+1}) + s\delta(Tx_n, Tz), \end{aligned} \tag{23}$$

since T is h-upper semicontinuous, passing to limit as $n \to +\infty$, we get

$$d(z, Tz) \leq 0, \tag{24}$$

which implies $d(z, Tz) = 0$. Finally, since Tz is closed we obtain that $z \in Tz$; that is, z is a fixed point of T. □

In view of Theorem 12, we have the following corollary.

Corollary 13. *Let (X, d, s) be a complete b-metric space and let $T : X \to CL(X)$. Assume that there exist two functions $\alpha : X \times X \to [0, +\infty)$ and $\psi \in \Psi_s$ such that*

$$\alpha(x, y) H(Tx, Ty) \leq \psi(M(x, y)) \quad \forall x, y \in X. \tag{25}$$

Also, suppose that the following conditions are satisfied:

(i) *T is an α-admissible multivalued mapping;*

(ii) *there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$;*

(iii) *T is h-upper semicontinuous.*

Then T has a fixed point.

Proof. Condition (25) ensures that condition (9) holds for all $x, y \in X$ with $\alpha(x, y) \geq 1$. Thus, T is an α-ψ-contraction of Ćirić type and by Theorem 12 the multivalued mapping T has a fixed point. □

Notice that one can relax the h-upper semicontinuity hypothesis on T, by introducing another regularity condition as shown in the next theorem.

Theorem 14. *Let (X, d, s) be a complete b-metric space and let $T : X \to CL(X)$. Assume that there exist two functions $\alpha : X \times X \to [0, +\infty)$ and $\psi \in \Psi_s$ such that T is an α-ψ-contraction of Ćirić type. Also, suppose that the following conditions are satisfied:*

(i) *T is an α-admissible multivalued mapping;*

(ii) *there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$;*

(iii) *for a sequence $\{x_n\}$ in X with $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x \in X$, then $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$.*

If $\psi(t) < t/s$ for all $t > 0$, then T has a fixed point.

Proof. By condition (ii) there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$. Proceeding as in the proof of Theorem 12, we obtain a sequence $\{x_n\}$ that converges to some $z \in X$ such that $x_n \notin Tx_n$, $x_{n+1} \in Tx_n$ and $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$. By condition (iii), we get $\alpha(x_n, z) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$. If $z \in Tz$, then the proof is concluded. Assume $d(z, Tz) > 0$. From $x_n \to z$, we deduce that

(i) the sequences $\{d(x_n, z)\}$, $\{d(x_n, Tx_n)\}$, and $\{d(z, Tx_n)\}$ converge to 0;

(ii) $\limsup_{n \to +\infty} d(x_n, Tz) \leq sd(z, Tz)$.

These facts ensure that there exists $N \in \mathbb{N}$ such that

$$\begin{aligned} \max \Big\{ &d(x_n, z), d(x_n, Tx_n), d(z, Tz), \\ &\frac{1}{2s}[d(x_n, Tz) + d(z, Tx_n)] \Big\} = d(z, Tz), \end{aligned} \tag{26}$$

for all $n \in \mathbb{N}$ with $n \geq N$. Since T is an α-ψ-contraction of Ćirić type, for all $n \geq N$, we have

$$\begin{aligned} d(z, Tz) &\leq s[d(z, x_{n+1}) + d(x_{n+1}, Tz)] \\ &\leq sd(z, x_{n+1}) + sH(Tx_n, Tz) \\ &\leq sd(z, x_{n+1}) + s\psi(d(z, Tz)). \end{aligned} \tag{27}$$

From $\psi(t) < t/s$, letting $n \to +\infty$, we get

$$d(z, Tz) \leq s\psi(d(z, Tz)) < d(z, Tz), \tag{28}$$

which implies $d(z, Tz) = 0$. Finally, since Tz is closed we obtain that $z \in Tz$; that is, z is a fixed point of T. □

In view of Theorem 14, we have the following corollary.

Corollary 15. *Let (X, d, s) be a complete b-metric space and let $T : X \to CL(X)$. Assume that there exist two functions $\alpha : X \times X \to [0, +\infty)$ and $\psi \in \Psi_s$ such that*

$$\alpha(x, y) H(Tx, Ty) \leq \psi(M(x, y)) \quad \forall x, y \in X. \tag{29}$$

Also, suppose that the following conditions are satisfied:

(i) *T is an α-admissible multivalued mapping;*

(ii) *there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$;*

(iii) *for a sequence $\{x_n\}$ in X with $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x \in X$, then $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$.*

If $\psi(t) < t/s$ for all $t > 0$, then T has a fixed point.

All results in the paper may be stated with respect to a self-mapping $T : X \to X$. For instance, and for our further use, we consider the following version of Theorem 14.

Corollary 16. *Let (X, d, s) be a complete b-metric space and let $T : X \to X$. Assume that there exist two functions $\alpha : X \times X \to [0, +\infty)$ and $\psi \in \Psi_s$ such that, for all $x, y \in X$ with $\alpha(x, y) \geq 1$, the following condition holds*

$$d(Tx, Ty) \leq \psi(M(x, y)). \tag{30}$$

Also, suppose that the following conditions are satisfied:

(i) *$x, y \in X$, $\alpha(x, y) \geq 1$ implies $\alpha(Tx, Ty) \geq 1$;*

(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*

(iii) *for a sequence $\{x_n\}$ in X with $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x \in X$, then $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$.*

If $\psi(t) < t/s$ for all $t > 0$, then T has a fixed point.

4. Fixed Point Theory in Ordered b-Metric Spaces

The study of fixed points in partially ordered sets has been developed in [15, 19–21] as a useful tool for applications on matrix equations and boundary value problems. In this section, we give some results of fixed point for multivalued mappings in the setting of ordered b-metric spaces. In fact, a b-metric space (X, d, s) may be naturally endowed with a partial ordering; that is, if $(X, \preceq)$ is a partially ordered set, then $(X, d, s, \preceq)$ is called an ordered b-metric space. We say that $x, y \in X$ are comparable if $x \preceq y$ or $y \preceq x$ holds. Also, let $A, B \subseteq X$; then $A \preceq B$ whenever for each $a \in A$ there exists $b \in B$ such that $a \preceq b$.

Theorem 17. *Let $(X, d, s, \preceq)$ be a complete ordered b-metric space and let $T : X \to CL(X)$. Assume that there exists a function $\psi \in \Psi_s$ such that*

$$H(Tx, Ty) \leq \psi(M(x, y)), \quad (31)$$

for all $x, y \in X$ with $Tx \preceq Ty$. Also, suppose that the following conditions are satisfied:

(i) *there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $Tx_0 \preceq Tx_1$;*

(ii) *for each $x \in X$ and $y \in Tx$ with $Tx \preceq Ty$, we have $Ty \preceq Tz$ for all $z \in Ty$;*

(iii) *T is h-upper semicontinuous.*

Then T has a fixed point.

Proof. Define the function $\alpha : X \times X \to [0, +\infty)$ by

$$\alpha(x, y) = \begin{cases} 1 & \text{if } Tx \preceq Ty \\ 0 & \text{otherwise.} \end{cases} \quad (32)$$

Clearly, the multivalued mapping T is α-admissible. In fact, for each $x \in X$ and $y \in Tx$ with $\alpha(x, y) \geq 1$, we have $Tx \preceq Ty$ and by condition (ii) we obtain that $Ty \preceq Tz$ for all $z \in Ty$. This implies that $\alpha(y, z) \geq 1$ for all $z \in Ty$. Also, by condition (31), T is an α-ψ-contraction of Ćirić type. Thus all the hypotheses of Theorem 12 are satisfied and T has a fixed point. □

Also in this case, one can relax the h-upper semicontinuity hypothesis on T, by using condition (iii) in Theorem 14. Precisely we state the following result.

Theorem 18. *Let $(X, d, s, \preceq)$ be a complete ordered b-metric space and let $T : X \to CL(X)$. Assume that there exists a function $\psi \in \Psi_s$ such that*

$$H(Tx, Ty) \leq \psi(M(x, y)), \quad (33)$$

for all $x, y \in X$ with $Tx \preceq Ty$. Also, suppose that the following conditions are satisfied:

(i) *there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $Tx_0 \preceq Tx_1$;*

(ii) *for each $x \in X$ and $y \in Tx$ with $Tx \preceq Ty$, we have $Ty \preceq Tz$ for all $z \in Ty$;*

(iii) *for a sequence $\{x_n\}$ in X with $Tx_n \preceq Tx_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x \in X$, then $Tx_n \preceq Tx$ for all $n \in \mathbb{N} \cup \{0\}$.*

If $\psi(t) < t/s$ for all $t > 0$, then T has a fixed point.

Following the same ideas in [16], we propose the following results, which provide an interesting alternative to partial ordering.

Theorem 19. *Let (X, d, s) be a complete b-metric space, $x^* \in X$, and let $T : X \to CL(X)$. Assume that there exists a function $\psi \in \Psi_s$ such that*

$$H(Tx, Ty) \leq \psi(M(x, y)), \quad (34)$$

for all $x, y \in X$ with $x^* \in Tx \cap Ty$. Also, suppose that the following conditions are satisfied:

(i) *there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $x^* \in Tx_0 \cap Tx_1$;*

(ii) *for each $x \in X$ and $y \in Tx$ with $x^* \in Tx \cap Ty$, we have $x^* \in Ty \cap Tz$ for all $z \in Ty$;*

(iii) *T is h-upper semicontinuous.*

Then T has a fixed point.

Proof. Define the function $\alpha : X \times X \to [0, +\infty)$ by

$$\alpha(x, y) = \begin{cases} 1 & \text{if } x^* \in Tx \cap Ty \\ 0 & \text{otherwise.} \end{cases} \quad (35)$$

Clearly, the multivalued mapping T is α-admissible. In fact, for each $x \in X$ and $y \in Tx$ with $\alpha(x, y) \geq 1$, we have $x^* \in Tx \cap Ty$ and by condition (ii) we obtain that $x^* \in Ty \cap Tz$ for all $z \in Ty$. This implies that $\alpha(y, z) \geq 1$ for all $z \in Ty$. Also, by condition (34), T is an α-ψ-contraction of Ćirić type. Thus all the hypotheses of Theorem 12 are satisfied and T has a fixed point. □

The following result is a consequence of Theorem 14; in order to avoid repetition we omit the proof that is similar to the one of Theorem 19.

Theorem 20. *Let (X, d, s) be a complete b-metric space, $x^* \in X$, and let $T : X \to CL(X)$. Assume that there exists a function $\psi \in \Psi_s$ such that*

$$H(Tx, Ty) \leq \psi(M(x, y)), \quad (36)$$

for all $x, y \in X$ with $x^* \in Tx \cap Ty$. Also, suppose that the following conditions are satisfied:

(i) *there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $x^* \in Tx_0 \cap Tx_1$;*

(ii) *for each $x \in X$ and $y \in Tx$ with $x^* \in Tx \cap Ty$, we have $x^* \in Ty \cap Tz$ for all $z \in Ty$;*

(iii) *for a sequence $\{x_n\}$ in X with $x^* \in Tx_n \cap Tx_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x \in X$, then $x^* \in Tx_n \cap Tx$ for all $n \in \mathbb{N} \cup \{0\}$.*

If $\psi(t) < t/s$ for all $t > 0$, then T has a fixed point.

5. Application to Integral Equation

In this section, inspired by Cosentino et al. [22] we give a typical application of fixed point methods to the study of existence of solutions for integral equations. Briefly, we give the background and notation. Let $X = C([0, I], \mathbb{R})$ be the set of real continuous functions defined on $[0, I]$, where $I > 0$, and let $d : X \times X \to [0, +\infty)$ be given by

$$d(x, y) = \left\| (x - y)^2 \right\|_\infty = \sup_{t \in [0, I]} (x(t) - y(t))^2, \tag{37}$$

for all $x, y \in X$. Then $(X, d, 2)$ is a complete b-metric space.

Consider the integral equation

$$x(t) = p(t) + \int_0^I S(t, u) f(u, x(u)) \, du, \tag{38}$$

where $f : [0, I] \times \mathbb{R} \to \mathbb{R}$ and $p : [0, I] \to \mathbb{R}$ are two continuous functions and $S : [0, I] \times [0, I] \to [0, +\infty)$ is a function such that $S(t, \cdot) \in L^1([0, I])$ for all $t \in [0, I]$.

Consider the operator $T : X \to X$ defined by

$$T(x)(t) = p(t) + \int_0^I S(t, u) f(u, x(u)) \, du. \tag{39}$$

Then we prove the following existence result.

Theorem 21. *Let $X = C([0, I], \mathbb{R})$. Suppose that the following conditions are satisfied:*

(i) *there exist $\eta : X \times X \to [0, +\infty)$ and $\alpha : X \times X \to [0, +\infty)$ such that if $\alpha(x, y) \geq 1$ for $x, y \in X$, then, for every $u \in [0, I]$ and some $\lambda > 0$, one has*

$$\begin{gathered} 0 \leq \left| f(u, x(u)) - f(u, y(u)) \right| \leq \eta(x, y) \left| x(u) - y(u) \right|, \\ \left\| \int_0^I S(t, u) \eta(x, y) \, du \right\|_\infty \leq \frac{1}{\sqrt{2 + \lambda}}; \end{gathered} \tag{40}$$

(ii) *$x, y \in X$, $\alpha(x, y) \geq 1$ implies $\alpha(Tx, Ty) \geq 1$;*

(iii) *there exists $x_0 \in X$ such that $\alpha(x_0, T(x_0)) \geq 1$;*

(iv) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$ as $n \to +\infty$, then $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$.*

Then the integral equation (38) has a solution in X.

Proof. Clearly, any fixed point of (39) is a solution of (38). By condition (i), we obtain

$$\begin{aligned} &\left| T(x)(t) - T(y)(t) \right|^2 \\ &\quad = \left[\left| \int_0^I S(t, u) \left[f(u, x(u)) - f(u, y(u)) \right] du \right| \right]^2 \\ &\quad \leq \left[\int_0^I S(t, u) \left| f(u, x(u)) - f(u, y(u)) \right| du \right]^2 \\ &\quad \leq \left[\int_0^I S(t, u) \eta(x, y) \sqrt{\left| x(u) - y(u) \right|^2} du \right]^2 \\ &\quad \leq \left[\int_0^I S(t, u) \eta(x, y) \sqrt{\left\| (x - y)^2 \right\|_\infty} du \right]^2 \\ &\quad = \left\| (x - y)^2 \right\|_\infty \left[\int_0^I S(t, u) \eta(x, y) \, du \right]^2. \end{aligned} \tag{41}$$

Then we have

$$\begin{aligned} &\left\| (T(x) - T(y))^2 \right\|_\infty \\ &\qquad \leq \left\| (x - y)^2 \right\|_\infty \left\| \int_0^I S(t, u) \eta(x, y) \, du \right\|_\infty^2, \end{aligned} \tag{42}$$

and hence, for all $x, y \in X$, we obtain

$$d(T(x), T(y)) \leq \frac{d(x, y)}{2 + \lambda}, \tag{43}$$

which implies that (30) holds true with $\psi \in \Psi_2$ given by $\psi(t) = t/(2 + \lambda)$ for all $t \geq 0$. The other conditions of Corollary 16 are immediately satisfied and hence the operator T has a fixed point, that is, a solution of the integral equation (38) in X. □

Remark 22. Notice that $\alpha : X \times X \to [0, +\infty)$ defined by

$$\alpha(x, y) = \begin{cases} 1 & \text{if } x \preceq y \\ 0 & \text{otherwise} \end{cases} \tag{44}$$

is an easy example of function suitable for Theorem 21. Clearly, as $X = C([0, I], \mathbb{R})$, then we can say that $x \preceq y$ if and only if $x(t) \leq y(t)$ for all $t \in [0, I]$, where $\leq$ denotes the usual order of real numbers. In this case, condition (ii) is satisfied by assuming that f is nondecreasing with respect to its second variable.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Authors' Contribution

All authors contributed equally and significantly in writing this paper. All authors read and approved the paper.

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