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Fixed points of a new class of pseudononspreading mappings

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Dedicated to the memory of Professor Ştefan Măruşter

Abstract. We extend the notion of *k-strictly pseudononspreading mappings* introduced in Nonlinear Analysis 74 (2011) 1814-1822 to the notion of the more general *pseudononspreading mappings.* It is shown with example that the class of pseudononspreading mappings is more general than the class of k-strictly pseudononspreading mappings. Furthermore, it is shown with explicit examples that the class of pseudononspreading mappings and the important class of *pseudocontractive mappings* are independent. Some fundamental properties of the class of pseudononspreading mappings are proved. In particular, it is proved that the fixed point set of certain class of pseudononspsreading selfmappings of a nonempty closed and convex subset of a real Hilbert space is closed and convex. *Demiclosedness property* of such class of pseudononspreading mappings is proved. Certain weak and strong convergence theorems are then proved for the iterative approximation of fixed points of the class of pseudononspreading mappings.

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1 Introduction

Let H be a real Hilbert space and C a nonempty closed convex subset of H. In [11] Kurokawa and Takahashi studied the class of *nonspreading mappings* in real Hilbert spaces. They called a mapping $T : C \to C$ nonspreading if

$$||Tx - Ty||^2 \leq ||x - y||^2 + 2\langle x - Tx, y - Ty\rangle, \ \forall x, y \in C. \tag{1.1}$$

This class of mappings contains the important class of *firmly nonexpansive mappings*, that is, mappings $T : D(T) \subset H \to H$ such that

$$||Tx - Ty||^2 \leq \langle Tx - Ty, x - y\rangle, \ \forall x, y \in D(T).$$

Firmly nonexpansive mappings have intimate connection with *maximal monotone operators* on Hilbert spaces, where an operator $T : D(T) \subset H \to 2^H$ with effective domain $D(T) = \{x \in H : Tx \neq \emptyset\}$ is maximal monotone if

$$\langle u - v, x - y\rangle \geq 0, \ \forall x, y \in D(T), u \in Tx, v \in Ty;$$

and its graph $G(T) = \{(x, u) : x \in D(T), u \in Tx\}$ is not properly contained in the graph of any other monotone operator on H. It is well-known that if T is maximal mononotone then the *resolvent* $J_r = (I + rT)^{-1}, r > 0$ is single-valued and firmly nonexpansive, and $F(J_r) = \{x \in D(T) : J_r x = x\} = T^{-1}0 = \{x \in D(T) : 0 \in Tx\}$.

Kurokawa and Takahashi [11] obtained a weak mean ergodic theorem of Bailon's type [1] for nonspreading mappings in Hilbert spaces. They further proved a strong convergence theorem somewhat related to Halpern's type [6] for this class of mappings using an idea of mean convergence in Hilbert spaces. Other authors that have studied nonspreading mappings and related operators include ([4],[7],[8],[9],[11],[13],[19]).

In [17] Osilike and Isiogugu introduced a new class of nonspreading type mappings which is more general than the class studied by Kurokawa and Takahashi [11]. Following the terminology of Browder and Petryshyn [3], they called a mapping $T : C \to C$ *k-strictly pseudononspreading* if there exists $k \in [0, 1)$ such that

$$\begin{aligned} ||Tx - Ty||^2 \ \leq \ & ||x - y||^2 + k||x - Tx - (y - Ty)||^2 \\ & +2\langle x - Tx, y - Ty\rangle, \ \forall x, y \in C. \end{aligned} \tag{1.2}$$

They exhibited examples of this class of mappings and showed with example that it is more general than the class of nonspeading mappings of Kurokawa and Takahashi [11]. They further exhibited some important properties of the class of k-strictly pseudononspreading mappings; and extended the results of Kurokawa and Takahashi [11] to this class of mappings. Osilike and Isiogugu [17] used an auxiliary mapping to obtain a strong convergence theorem of Halpern's type [6] for the class of k-strictly pseudononspreading mappings and for the case where T is averaged, this resolved in the affirmative an open problem posed by Kurokawa and Takahashi [11] in their final remark.

In this work we further extend the notion of k-strictly pseudononspreading mappings to the notion of *pseudononspreading mappings.*
We show with an example that this class of mappings is more general than the class of k-strictly pseudononspreading mappings. Furthermore, it is shown with explicit examples that this class of mappings and the important class of *pseudocontractive* mappinng (i.e., mappings T satisfying the inequality $||Tx-Ty||^2 \leq ||x-y||^2 + ||x-Tx-(y-Ty)||^2,\ \forall x, y \in C$) are independent. Some fundamental properties of certain subclass of pseudononspreading mappings are proved. In particular, for the *Lipschitz* pseudononspreading mappings $T : C \to C$, it is proved that $F(T) = \{x \in C : Tx = x\}$ is closed and convex. We also proved a *demiclosedness principle* which established that if $T : C \to C$ is a Lipschitz pseudononspreading mapping, then $(I - T)$ is demiclosed at zero. Certain weak and strong convergence theorems are then proved for the iterative approximation of fixed points of the class of pseudononspreading mappings T when $F(T) \neq \emptyset$.

2 Preliminaries

In what follows, we shall need the following results.

Lemma 2.1. *(*[20]*) Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\lambda_n)a_n + \lambda_n\gamma_n + \sigma_n,\ n \geq 1,$$

where $\{\lambda_n\} \subseteq (0,1), \{\gamma_n\} \subseteq \Re$, $\{\sigma_n\}$ is a sequence of nonnegative real numbers and

(i) $\Sigma_{n=0}^{\infty}\lambda_n = \infty,$ or equivalently, $\Pi_{n=0}^{\infty}(1-\lambda_n) = 0,$

(ii) $\limsup_{n\to\infty} \gamma_n \leq 0,$ and

(iii) $\Sigma_{n=0}^{\infty}\sigma_n < \infty.$

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.2. *([14], Lemma 3.1). Let* $\{\Gamma_n\}_{n=0}^{\infty}$ *be a sequence of real numbers that does not decrease at infinity, in the sense that there exists a subsequence* $\{\Gamma_{n_j}\}_{j=0}^{\infty}$ *of* $\{\Gamma_n\}_{n=0}^{\infty}$ *such that*

$$\Gamma_{n_j} < \Gamma_{n_j+1},\ \forall j \geq 0.$$

Also consider the sequence of integers $\{\tau(n)\}_{n\geq N_0}$ *defined by*

$$\tau(n) = \max\{k \leq n : \Gamma_k < \Gamma_{k+1}\}.$$

Then $\{\tau(n)\}_{n\geq N_0}$ *is a nondecreasing sequence which satisfies*

$$\lim_{n\to\infty} \tau(n) = \infty,$$

and for all $n \geq N_0$*, the following two estimates hold:*

$$\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$$
$$\Gamma_n \leq \Gamma_{\tau(n)+1}.$$

Let C be a closed convex subset of a real Hilbert space H. Let $P_C : H \to C$ denote the metric projection (the proximity map) which assigns to each point $x \in H$ the unique nearest point in C, denoted by $P_C(x)$. It is well known that

$$z = P_C(x) \text{ if and only if } \langle x - z, z - y\rangle \geq 0,\ \forall y \in C,$$

and that P_C is nonexpansive.

It is also well known that in real Hilbert spaces H, we have the following:

$$||(1-\lambda)x + \lambda y||^2 = (1-\lambda)||x||^2 + \lambda||y||^2 - \lambda(1-\lambda)||x-y||^2, \tag{2.1}$$

$\forall x, y \in H,$ and $\forall \lambda \in [0,1]$,

$$\|x+y\|^2 \leq \|y\|^2 + 2\langle x, x+y\rangle,\ \forall x, y \in H. \tag{2.2}$$

3 Main results

Let H be a real Hilbert space. Following the terminology of Browder-Petryshyn [3], we define the following new class of mappings.

Definition 3.1. *A mapping* $T : D(T) \subseteq H \to H$ *is* pseudononspreading *if*

$$\begin{aligned} ||Tx - Ty||^2 \;\; \leq \;\; & ||x - y||^2 + ||x - Tx - (y - Ty)||^2 \\ & +2\langle x - Tx, y - Ty\rangle, \; \forall x, y \in D(T). \end{aligned} \tag{3.1}$$

Clearly, every strictly pseudononspreading mapping is pseudononspreading. The following example shows that class of pseudononspreading mapping is more general than the class of strictly pseudononspreading mappings.

Example 3.1. Let $\Re$ denote the reals with the usual norm and define $T : \Re :\to \Re$ by $Tx = -3x$.

Then

$$\begin{aligned} & |Tx - Ty|^2 = 9|x - y|^2, \\ & |x - Tx - (y - Ty)|^2 = 16|x - y|^2, \\ & 2\langle x - Tx, y - Ty\rangle = 32xy. \end{aligned}$$

Thus

$$\begin{aligned} & |x - y|^2 + |x - Tx - (y - Ty)|^2 + 2\langle x - Tx, y - Ty\rangle \\ & = |x - y|^2 + 16|x - y|^2 + 32xy \\ & = 9|x - y|^2 + 8(x^2 - 2xy + y^2) + 32xy \\ & = 9|x - y|^2 + 8(x + y)^2 \\ & \geq 9|x - y|^2 \\ & = |Tx - Ty|^2. \end{aligned}$$

Hence T is pseudononspreading. T is not strictly pseudononspreading since for $x = \frac{1}{2}$, $y = \frac{-1}{2}$, we have

$$|x-y|^2 = 1, \; |Tx-Ty|^2 = 9, \; |x-Tx-(y-Ty)|^2 = 16, \; 2\langle x-Tx, y-Ty\rangle = -8.$$

Thus

$$\begin{aligned} |Tx - Ty|^2 = 9 \;\; = \;\; & |x - y|^2 + |x - Tx - (y - Ty)|^2 + 2\langle x - Tx, y - Ty\rangle \\ > \;\; & |x - y|^2 + \beta|x - Tx - (y - Ty)|^2 \\ & +2\langle x - Tx, y - Ty\rangle, \; \forall \beta \in [0, 1) \text{ and } \forall x, y \in \Re. \end{aligned}$$

It is also interesting to note that the class of pseudononspreading mappings and the class of pseudocontractive mappings are independent. This follows from the following examples:

Example 3.2. Define $T : \Re \to \Re$ by

$$Tx = \begin{cases} 0, & x \in (-\infty, 2] \\ 1, & x \in (2, \infty) \end{cases}$$

Then for all $x, y \in (-\infty, 2]$ and $\beta \in [0, 1)$, we obtain

$$\begin{aligned} &|x-y|^2 + \beta|x - Tx - (y - Ty)|^2 + 2\langle x - Tx, y - Ty\rangle \\ &= x^2 + y^2 + \beta|x - Tx - (y - Ty)|^2 \geq 0 = |Tx - Ty|^2. \end{aligned}$$

Furthermore, for all $x, y \in (2, \infty)$, we obtain

$$\begin{aligned} &|x-y|^2 + \beta|x - Tx - (y - Ty)|^2 + 2\langle x - Tx, y - Ty\rangle \\ &= |x-y|^2 + \beta|x - Tx - (y - Ty)|^2 + 2(x-1)(y-1) \\ &\geq 0 = |Tx - Ty|^2. \end{aligned}$$

For all $x \in (-\infty, 2]$ and $y \in (2, \infty)$, we obtain

$$\begin{aligned} &|x-y|^2 + \beta|x - Tx - (y - Ty)|^2 + 2\langle x - Tx, y - Ty\rangle \\ &= x^2 + y^2 - 2xy + \beta|x - Tx - (y - Ty)|^2 + 2x(y-1) \\ &= x^2 - 2x + y^2 + \beta|x - Tx - (y - Ty)|^2 \\ &= (x-1)^2 + y^2 - 1 + \beta|x - Tx - (y - Ty)|^2 \\ &\geq y^2 - 1 > 3 > 1 = |Tx - Ty|^2. \end{aligned}$$

Thus for all $x, y \in \Re$ and $\beta \in [0, 1)$ we have

$$||Tx - Ty||^2 \leq ||x-y||^2 + \beta||x - Tx - (y - Ty)||^2 + 2\langle x - Tx, y - Ty\rangle,$$

and hence T is strictly pseudononspreading. T is not pseudocontractive since for $x = 2, y = \frac{5}{2}$ we obtain $|x-y|^2 = \frac{1}{4}$, $|Tx - Ty|^2 = 1$, $|x - Tx - (y - Ty)|^2 = \frac{1}{4}$. Thus

$$|Tx - Ty|^2 = 1 > \frac{1}{2} = |x-y|^2 + |x - Tx - (y - Ty)|^2.$$

Example 3.3. Let $\Re^2$ denote the 2-dimensional Euclidean plane. Define $T : \Re^2 \to \Re^2$ by

$$Tx = T((x_1, x_2)) = (x_1, x_2) + (x_2, -x_1) = (x_1 + x_2, x_2 - x_1),$$

for each $x = (x_1, x_2) \in \Re^2$.

Then for all $x = (x_1, x_2), y = (y_1, y_2) \in \Re^2$, we obtain

$$\begin{aligned} ||Tx - Ty||^2 &= ||(x_1 + x_2, x_2 - x_1) - (y_1 + y_2, y_2 - y_1)||^2 \\ &= ||(x_1 - y_1 + x_2 - y_2, x_2 - y_2 - (x_1 - y_1))||^2 \\ &= (x_1 - y_1 + x_2 - y_2)^2 + (x_2 - y_2 - (x_1 - y_1))^2 \\ &= 2[(x_1 - y_1)^2 + (x_2 - y_2)^2] \\ &= 2||x - y||^2. \end{aligned}$$

We observe that T is $2-$Lipschitzian. Furthermore, $x - Tx = -(x_2, -x_1)$, $y - Ty = -(y_2, -y_1)$, and

$$\begin{aligned} ||x - Tx - (y - Ty)||^2 &= ||(x_2 - y_2, -(x_1 - y_1))||^2 \\ &= (x_1 - y_1)^2 + (x_2 - y_2)^2 \\ &= ||x - y||^2. \end{aligned}$$

Thus

$$||x - y||^2 + ||x - Tx - (y - Ty)||^2 = 2||x - y||^2, \text{ and}$$

$$||Tx - Ty||^2 = 2||x - y||^2 = ||x - y||^2 + ||x - Tx - (y - Ty)||^2 = 2||x - y||^2.$$

Hence T is Lipschitz pseudocontractive.
T is not pseudononspreading since for $x = (1, 1),\ y = (-1, -1) \in \Re^2$, we obtain $||Tx - Ty||^2 = 16,\ ||x - y||^2 = 8,\ ||x - Tx - (y - Ty)||^2 = 8$, $2\langle x - Tx, y - Ty\rangle = -4$. Thus

$$||x-y||^2+||x-Tx-(y-Ty)||^2+2\langle x-Tx, y-Ty\rangle = 12 < 16 = ||Tx-Ty||^2.$$

It is also easy to verify that any translation $T : \Re \to \Re$, given by $Tx = x + a$ for a fixed nonzero $a \in \Re$ is a Lipschitz pseudononspreading mapping without a fixed point.

Lemma 3.1. *Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. Let $T : C \to C$ be an L-Lipschitzian pseudononspreading mapping. Then $F(T) = \{x \in C : Tx = x\}$ is closed and convex.*

Proof. If $F(T) = \emptyset$, then it is closed. If $F(T) \neq \emptyset$, let $\{p_n\}_{n=1}^{\infty} \subseteq F(T)$ be such that $p_n \to p$. We prove that $p \in F(T)$.

$$\begin{aligned} ||p - Tp|| &\leq ||p - Tp_n|| + ||Tp_n - Tp|| \\ &\leq ||p - p_n|| + L||p_n - p|| = (1 + L)||p_n - p|| \to 0 \text{ as } n \to \infty. \end{aligned}$$

Thus $p \in F(T)$, and $F(T)$ is closed.

To prove convexity, we first note that if $F(T)$ is empty or singleton, then it is convex. Thus we may assume that $F(T)$ has more than one element. Let $p_1, p_2 \in F(T)$ be arbitrary. Let $\lambda \in [0,1]$ be arbitrary, and $p = \lambda p_1 + (1-\lambda)p_2$. We prove that $p \in F(T)$. Observe that

$$||p - p_1|| = (1-\lambda)||p_1 - p_2||; \; ||p - p_2|| = \lambda||p_1 - p_2||. \quad (3.2)$$

For $\beta \in (0, \frac{1}{1+\sqrt{1+L^2}})$, set $G_\beta x = T[(1-\beta)x + \beta Tx]$.
Then $G_\beta p_1 = p_1$, and $G_\beta p_2 = p_2$. Furthermore,

$$\begin{aligned} ||p - G_\beta p||^2 &= ||\lambda p_1 + (1-\lambda)p_2 - G_\beta p||^2 \\ &= ||\lambda(p_1 - G_\beta p) + (1-\lambda)(p_2 - G_\beta p)||^2 \\ &= \lambda||p_1 - G_\beta p||^2 + (1-\lambda)||p_2 - G_\beta p||^2 \\ &\quad -\lambda(1-\lambda)||p_1 - p_2||^2, \end{aligned} \quad (3.3)$$

$$\begin{aligned} ||G_\beta p - p_1||^2 &= ||T[(1-\beta)p + \beta Tp] - Tp_1||^2 \\ &\leq ||(1-\beta)p + \beta Tp - p_1||^2 \\ &\quad +||(1-\beta)p + \beta Tp - G_\beta p - (p_1 - Tp_1)||^2 \\ &\quad +2\langle (1-\beta)p + \beta Tp - T[(1-\beta)p + \beta Tp], p_1 - Tp_1\rangle \\ &= ||(1-\beta)(p - p_1) + \beta(Tp - p_1)||^2 \\ &\quad +||(1-\beta)(p - G_\beta p) + \beta(Tp - G_\beta p)||^2 \\ &= (1-\beta)||p - p_1||^2 + \beta||Tp - Tp_1||^2 \\ &\quad -\beta(1-\beta)||p - Tp||^2 + (1-\beta)||p - G_\beta p||^2 \\ &\quad +\beta||Tp - G_\beta p||^2 - \beta(1-\beta)||p - Tp||^2 \\ &\leq (1-\beta)||p - p_1||^2 + \beta[||p - p_1||^2 + ||p - Tp||^2] \\ &\quad -\beta(1-\beta)||p - Tp||^2 + (1-\beta)||p - G_\beta p||^2 \\ &\quad +\beta^3 L^2||p - \beta Tp||^2 - \beta(1-\beta)||p - Tp||^2 \\ &= ||p - p_1||^2 - \beta[1 - 2\beta - \beta^2 L^2]||p - Tp||^2 \\ &\quad +(1-\beta)||p - G_\beta p||^2. \end{aligned}$$

Thus,

$$||G_\beta p - p_1||^2 \leq ||p - p_1||^2 + (1-\beta)||p - G_\beta p||^2. \quad (3.4)$$

Similarly,

$$||G_\beta p - p_2||^2 \leq ||p - p_2||^2 + (1-\beta)||p - G_\beta p||^2. \quad (3.5)$$

Using (3.4) and (3.5) in (3.3) we obtain

$$\begin{aligned} ||p - G_\beta p||^2 &\leq \lambda[||p - p_1||^2 + (1-\beta)||p - G_\beta p||^2] \\ &\quad +(1-\lambda)[||p - p_2||^2 + (1-\beta)||p - G_\beta p||^2] \\ &\quad -\lambda(1-\lambda)||p_1 - p_2||^2 \\ &\leq (1-\beta)||p - G_\beta p||^2. \end{aligned}$$

It follows that $0 \leq \beta||p - G_\beta p||^2 \leq 0$, and hence $G_\beta p = p$.
Observe that

$$\begin{aligned} ||Tp - p|| &\leq ||p - G_\beta p|| + ||G_\beta p - Tp|| \\ &\leq ||p - G_\beta p|| + L||(1-\beta)p + \beta Tp - p|| \\ &= ||p - G_\beta p|| + L\beta||p - Tp||. \end{aligned}$$

Thus $0 \leq (1 - L\beta)||Tp - p|| \leq 0$, and hence $Tp = p$. □

Lemma 3.2. *Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. Let $T : C \to C$ be an L-Lipschitzian pseudononspreading mapping.Then $(I - T)$ is demiclosed at zero.*

Proof. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in C such that $x_n \rightharpoonup p$ and $x_n - Tx_n \to 0$. We prove that $p - Tp = 0$. Define $f : H \to \Re^+$ for each $x \in H$ by

$$f(x) := \limsup_{n\to\infty} ||x_n - x||^2. \tag{3.6}$$

Thus

$$f(x) = \limsup_{n\to\infty} ||x_n - p||^2 + ||p - x||^2, \ \forall x \in H. \tag{3.7}$$

It follows that

$$f(x) = f(p) + ||p - x||^2, \ \forall x \in H. \tag{3.8}$$

Hence

$$f(G_\beta p) = f(p) + ||p - G_\beta p||^2. \tag{3.9}$$

Observe that

$$\begin{aligned} ||G_\beta x_n - x_n|| &\leq ||G_\beta x_n - Tx_n|| + ||x_n - Tx_n|| \\ &\leq L||(1-\beta)x_n + \beta Tx_n - x_n|| + ||x_n - Tx_n|| \\ &= (L\beta + 1)||x_n - Tx_n|| \to 0 \text{ as } n \to \infty. \end{aligned}$$

Furthermore,

$$||Tx_n - G_\beta x_n|| \leq ||Tx_n - x_n|| + ||x_n - G_\beta x_n|| \to 0 \text{ as } n \to \infty.$$

Thus

$$\begin{aligned}
f(G_\beta p) &= \limsup_{n\to\infty} ||x_n - G_\beta p||^2 \\
&= \limsup_{n\to\infty} ||x_n - G_\beta x_n + G_\beta x_n - G_\beta p||^2. \\
&= \limsup_{n\to\infty} ||G_\beta x_n - G_\beta p||^2 \\
&\le \limsup_{n\to\infty} \Big[||T[(1-\beta)x_n + \beta T x_n] - T[(1-\beta p) + \beta T p]||^2 \Big] \\
&\le \limsup_{n\to\infty} \Big[||(1-\beta)x_n + \beta T x_n - [(1-\beta)p + \beta T p]||^2 \\
&\quad + ||(1-\beta)x_n + \beta T x_n - G_\beta x_n - [(1-\beta)p + \beta T p - G_\beta p]||^2 \\
&\quad + 2\langle (1-\beta)x_n + \beta T x_n - G_\beta x_n, (1-\beta)p + \beta T p - G_\beta p\rangle \Big] \\
&= \limsup_{n\to\infty} \Big[||(1-\beta)(x_n - p) + \beta(T x_n - T p)||^2 \\
&\quad + ||(1-\beta)(x_n - G_\beta x_n) + \beta(T x_n - G_\beta x_n) \\
&\qquad - [(1-\beta)(p - G_\beta p) + \beta(T p - G_\beta p)||^2 \Big] \\
&= \limsup_{n\to\infty} \Big[||(1-\beta)(x_n - p) + \beta(T x_n - T p)||^2 \\
&\quad + ||(1-\beta)(p - G_\beta p) + \beta(T p - G_\beta p)||^2 \Big] \\
&= \limsup_{n\to\infty} \Big[(1-\beta)||x_n - p||^2 + \beta||T x_n - T p||^2 \\
&\quad - \beta(1-\beta)||x_n - T x_n - (p - T p)||^2 \\
&\quad + (1-\beta)||p - G_\beta p||^2 + \beta||T p - G_\beta p||^2 - \beta(1-\beta)||p - T p||^2 \Big] \\
&\le \limsup_{n\to\infty} \Big[(1-\beta)||x_n - p||^2 + \beta(||x_n - p||^2 + ||x_n - T x_n - (p - T p)||^2 \\
&\quad + 2\langle x_n - T x_n, p - T p\rangle) - \beta(1-\beta)||p - T p||^2 \\
&\quad + (1-\beta)||p - G_\beta p||^2 + \beta^3 L^2||p - T p||^2 - \beta(1-\beta)||p - T p||^2 \Big] \\
&= \limsup_{n\to\infty} \Big[||x_n - p||^2 + (1-\beta)||p - G_\beta p||^2 \\
&\quad - \beta[1 - 2\beta - \beta^2 L^2]||p - T p||^2 \Big] \\
&\le \limsup_{n\to\infty} \Big[||x_n - p||^2 + (1-\beta)||p - G_\beta p||^2 \Big].
\end{aligned}$$

Thus

$$f(G_\beta p) \le f(p) + (1-\beta)||p - G_\beta p||^2. \tag{3.10}$$

From (3.9) and (3.10) we obtain

$$f(G_\beta p) = f(p) + ||p - G_\beta p||^2 \leq f(p) + (1-\beta)||p - G_\beta p||^2.$$

Thus $0 \leq \beta||p - G_\beta p||^2 \leq 0$, and we obtain $||p - G_\beta p|| = 0$. Observe that

$$0 \leq ||p - Tp|| \leq ||p - G_\beta p|| + ||G_\beta p - Tp|| \leq ||p - G_\beta p|| + L\beta||p - Tp||.$$

Thus $0 \leq (1 - L\beta)||p - Tp|| \leq ||p - G_\beta p|| = 0$, and we obtain $Tp = p$. □

In what follows, we prove convergence theorems for iterative approximation of fixed points of Lipschitz pseudononspreading mappings in real Hilbert spaces. For more details on the major iteration schemes and the schemes related to the scheme we introduce below, see for example ([2],[5],[12],[15],[16]).

Theorem 3.3. *Let H be a real Hilbert space and let $T : H \to H$ be a an L-Lipschitzian pseudononspreading mapping with $F(T) = \{x \in H : Tx = x\} \neq \emptyset$. Let $\{x_n\}_{n=1}^\infty$ be the sequence generated from an arbitrary $x_1 \in H$ by*

$$x_{n+1} = (1 - \alpha_n - \beta_n)x_n + \beta_n T[(1-\lambda_n)x_n + \lambda_n Tx_n],\ n \geq 1, \tag{3.11}$$

where $\{\alpha_n\}_{n=1}^\infty$, $\{\beta_n\}_{n=1}^\infty$, and $\{\lambda_n\}_{n=1}^\infty$ are real sequences in $(0,1)$ which satisfy the following conditions:
(i) $\lim\limits_{n\to\infty} \alpha_n = 0,\ \forall n \geq 1$
(ii) $0 < \epsilon \leq \beta_n \leq \lambda_n(1-\alpha_n),\ \forall n \geq 1$
(iii) $\lambda_n \in (0, \frac{1}{2(1+\sqrt{1+L^2})})$.
Then $\{x_n\}_{n=1}^\infty$ converges strongly to a fixed point of T.

Proof. Let $y_n = (1-\lambda_n)x_n + \lambda_n Tx_n$, and let $p \in F(T)$ be arbitrary. Then

$$\begin{aligned} ||x_{n+1} - p|| &= ||(1 - \alpha_n - \beta_n)x_n + \beta_n Ty_n - p|| \\ &= ||(1 - \alpha_n - \beta_n)(x_n - p) + \beta_n(Ty_n - p) - \alpha_n p|| \\ &\leq ||(1 - \alpha_n - \beta_n)(x_n - p) + \beta_n(Ty_n - p)|| \\ &\quad + \alpha_n||p||. \end{aligned} \tag{3.12}$$

Observe that

$$
\begin{aligned}
&||(1-\alpha_n-\beta_n)(x_n-p)+\beta_n(Ty_n-p)||^2\\
&=||(1-\beta_n)(x_n-p)+\beta_n(Ty_n-p)-\alpha_n(x_n-p)||^2\\
&=||(1-\beta_n)(x_n-p)+\beta_n(Ty_n-p)||^2+\alpha_n^2||x_n-p||^2\\
&\quad-2\alpha_n\langle(1-\beta_n)(x_n-p)+\beta_n(Ty_n-p),x_n-p\rangle\\
&=(1-\beta_n)||x_n-p||^2+\beta_n||Ty_n-p||^2\\
&\quad-\beta_n(1-\beta_n)||x_n-Ty_n||^2+\alpha_n^2||x_n-p||^2\\
&\quad-2\alpha_n(1-\beta_n)||x_n-p||^2-2\alpha_n\beta_n\langle Ty_n-p,x_n-p\rangle\\
&=[(1-\beta_n)+\alpha_n^2-2\alpha_n(1-\beta_n)]||x_n-p||^2\\
&\quad-2\alpha_n\beta_n\langle Ty_n-p,x_n-p\rangle\\
&\quad+\beta_n||Ty_n-p||^2-\beta_n(1-\beta_n)||x_n-Ty_n||^2. \qquad (3.13)
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
||Ty_n-p||^2 &= ||Ty_n-x_n+x_n-p||^2\\
&= ||Ty_n-x_n||^2+||x_n-p||^2+2\langle Ty_n-x_n,x_n-p\rangle.
\end{aligned}
$$

$$
2\langle Ty_n-x_n,x_n-p\rangle=||Ty_n-p||^2-||Ty_n-x_n||^2-||x_n-p||^2. \qquad (3.14)
$$

Using (3.14) we obtain

$$
\begin{aligned}
2\langle Ty_n-p,x_n-p\rangle &= 2[\langle Ty_n-x_n+x_n-p,x_n-p\rangle]\\
&= 2[\langle Ty_n-x_n,x_n-p\rangle+||x_n-p||^2]\\
&= ||Ty_n-p||^2-||Ty_n-x_n||^2+||x_n-p||^2. (3.15)
\end{aligned}
$$

Using (3.15) in (3.13) we obtain

$$
\begin{aligned}
&||(1-\alpha_n-\beta_n)(x_n-p)+\beta_n(Ty_n-p)||^2\\
&=[(1-\beta_n)+\alpha_n^2-2\alpha_n(1-\beta_n)]||x_n-p||^2\\
&\quad-\alpha_n\beta_n[||Ty_n-p||^2-||Ty_n-x_n||^2+||x_n-p||^2]\\
&\quad+\beta_n||Ty_n-p||^2-\beta_n(1-\beta_n)||x_n-Ty_n||^2\\
&=[(1-\beta_n)+\alpha_n^2-2\alpha_n(1-\beta_n)-\alpha_n\beta_n]||x_n-p||^2\\
&\quad+\beta_n(1-\alpha_n)||Ty_n-p||^2\\
&\quad+[\alpha_n\beta_n-\beta_n(1-\beta_n)]||x_n-Ty_n||^2. \qquad (3.16)
\end{aligned}
$$

Since T is pseudononspreading, we also obtain

$$\begin{aligned}
||Ty_n - p||^2 &\leq ||y_n - p||^2 + ||y_n - Ty_n||^2 \\
&= ||(1-\lambda_n)(x_n - p) + \lambda_n(Tx_n - p)||^2 \\
&\quad +||(1-\lambda_n)(x_n - Ty_n) + \lambda_n(Tx_n - Ty_n)||^2 \\
&= (1-\lambda_n)||x_n - p||^2 + \lambda_n||Tx_n - p||^2 \\
&\quad -\lambda_n(1-\lambda_n)||x_n - Tx_n||^2 \\
&\quad +(1-\lambda_n)||x_n - Ty_n||^2 + \lambda_n||Tx_n - Ty_n||^2 \\
&\quad -\lambda_n(1-\lambda_n)||x_n - Tx_n||^2 \\
&\leq ||x_n - p||^2 - \lambda_n[1 - 2\lambda_n - \lambda_n^2 L^2]||x_n - Tx_n||^2 \\
&\quad +(1-\lambda_n)||x_n - Ty_n||^2. \qquad (3.17)
\end{aligned}$$

Substituting (3.17) in (3.16) and using the conditions on the real sequences we obtain

$$\begin{aligned}
&||(1-\alpha_n-\beta_n)(x_n - p) + \beta_n(Ty_n - p)||^2 \\
&\leq [(1-\beta_n) + \alpha_n^2 - 2\alpha_n(1-\beta_n) \\
&\quad -\alpha_n\beta_n + \beta_n(1-\alpha_n)]||x_n - p||^2 \\
&\quad +[\alpha_n\beta_n - \beta_n(1-\beta_n) \\
&\quad +\beta_n(1-\alpha_n)(1-\lambda_n)]||x_n - Ty_n||^2 \\
&\quad -\beta_n\lambda_n(1-\alpha_n)[1 - 2\lambda_n - \lambda_n^2 L^2]||x_n - Tx_n||^2 \\
&= (1-\alpha_n)^2||x_n - p||^2 \\
&\quad -\beta_n[\lambda_n(1-\alpha_n) - \beta_n]||x_n - Ty_n||^2 \\
&\quad -\beta_n\lambda_n(1-\alpha_n)[1 - 2\lambda_n - \lambda_n^2 L^2]||x_n - Tx_n||^2 \\
&\leq (1-\alpha_n)^2||x_n - p||^2. \qquad (3.18)
\end{aligned}$$

Using (3.18) in (3.12) yields

$$\begin{aligned}
||x_{n+1} - p|| &\leq (1-\alpha_n)||x_n - p|| + \alpha_n||p|| \\
&\leq \max\{||x_n - p||, ||p||\}. \qquad (3.19)
\end{aligned}$$

It follows from (3.19) and a simple induction that $\{x_n\}_{n=1}^{\infty}$ is bounded. Using

(2.2) and (3.14) we obtain

$$\begin{aligned}
||x_{n+1}-p||^2 &= ||(1-\alpha_n-\beta_n)x_n+\beta_n Ty_n-p||^2\\
&= ||(x_n-p)-\beta_n(x_n-Ty_n)-\alpha_n x_n||^2\\
&\le ||(x_n-p)-\beta_n(x_n-Ty_n)||^2-2\alpha_n\langle x_n,x_{n+1}-p\rangle\\
&= ||x_n-p||^2+\beta_n^2||x_n-Ty_n||^2-2\beta_n\langle x_n-Ty_n,x_n-p\rangle\\
&\quad -2\alpha_n\langle x_n,x_{n+1}-p\rangle\\
&= ||x_n-p||^2+\beta_n^2||x_n-Ty_n||^2\\
&\quad -\beta_n[||Ty_n-x_n||^2+||x_n-p||^2-||Ty_n-p||^2]\\
&\quad -2\alpha_n\langle x_n,x_{n+1}-p\rangle\\
&= (1-\beta_n)||x_n-p||^2-\beta_n(1-\beta_n)||x_n-Ty_n||^2\\
&\quad +\beta_n||Ty_n-p||^2-2\alpha_n\langle x_n,x_{n+1}-p\rangle\\
&\le (1-\beta_n)||x_n-p||^2-\beta_n(1-\beta_n)||x_n-Ty_n||^2\\
&\quad +\beta_n[||y_n-p||^2+||y_n-Ty_n||^2]-2\alpha_n\langle x_n,x_{n+1}-p\rangle\\
&= (1-\beta_n)||x_n-p||^2-\beta_n(1-\beta_n)||x_n-Ty_n||^2\\
&\quad +\beta_n\Big[(1-\lambda_n)||x_n-p||^2+\lambda_n||Tx_n-p||^2\\
&\quad -\lambda_n(1-\lambda_n)||x_n-Tx_n||^2+(1-\lambda_n)||x_n-Ty_n||^2\\
&\quad +\lambda_n||Tx_n-Ty_n||^2-\lambda_n(1-\lambda_n)||x_n-Tx_n||^2\Big]\\
&\quad -2\alpha_n\langle x_n,x_{n+1}-p\rangle\\
&\le [(1-\beta_n)+\beta_n]||x_n-p||^2-\beta_n(\lambda_n-\beta_n)||x_n-Ty_n||^2\\
&\quad -\beta_n\lambda_n[1-2\lambda_n-\lambda_n^2L^2]||x_n-Tx_n||^2\\
&\quad -2\alpha_n\langle x_n,x_{n+1}-p\rangle\\
&\le ||x_n-p||^2-\beta_n\lambda_n[1-2\lambda_n-\lambda_n^2L^2]||x_n-Tx_n||^2\\
&\quad -2\alpha_n\langle x_n,x_{n+1}-p\rangle. \qquad (3.20)
\end{aligned}$$

Since $\{x_n\}_{n=1}^\infty$ is bounded, there exists $D>0$ such that

$$-2\langle x_n,x_{n+1}-p\rangle\le D,\ \forall n\ge 1. \qquad (3.21)$$

It thus follows from (3.20) that

$$||x_{n+1}-p||^2-||x_n-p||^2+\beta_n\lambda_n[1-2\lambda_n-\lambda_n^2L^2]||x_n-Tx_n||^2\le D\alpha_n. \qquad (3.22)$$

To complete the proof, we consider the following two cases:

Case I. suppose $\{||x_n-p||\}_{n=1}^\infty$ is monotone decreasing. Then $\{||x_n-p||\}_{n=1}^\infty$ converges and since

$$\beta_n\lambda_n[1-2\lambda_n-\lambda_n^2L^2]\ge\epsilon^2\frac{1}{4(1+\sqrt{1+L^2})}[4\sqrt{1+L^2}+3L^2+4]>0,$$

we obtain

$$\lim ||x_n - Tx_n|| = 0. \tag{3.23}$$

Since $\{x_n\}_{n=1}^{\infty}$ is bounded and $(I - T)$ is demiclosed at zero, it follows from standard argument that $\{x_n\}_{n=1}^{\infty}$ converges weakly to a fixed point x^* of T. We now prove that $\{x_n\}_{n=1}^{\infty}$ converges strongly to x^*. Observe that

$$\begin{aligned} x_{n+1} &= (1-\alpha_n-\beta_n)x_n + \beta_n Ty_n \\ &= (1-\beta_n)x_n + \beta_n Ty_n - \alpha_n x_n \\ &= z_n - \alpha_n x_n, \text{ where } z_n = (1-\beta_n)x_n + \beta_n Ty_n \\ &= (1-\alpha_n)z_n - \alpha_n(x_n - z_n) \\ &= (1-\alpha_n)z_n - \alpha_n\beta_n(x_n - Ty_n). \end{aligned} \tag{3.24}$$

Furthermore

$$\begin{aligned} ||z_n - x^*||^2 &= ||(1-\beta_n)(x_n - x^*) + \beta_n(Ty_n - x^*)||^2 \\ &= (1-\beta_n)||x_n - x^*||^2 + \beta_n||Ty_n - x^*||^2 \\ &\quad -\beta_n(\beta_n)||x_n - Ty_n||^2 \\ &\le (1-\beta_n)||x_n - x^*||^2 + \beta_n[||y_n - x^*||^2 \\ &\quad +||y_n - Ty_n||^2] - \beta_n(1-\beta_n)||x_n - Ty_n||^2 \\ &= (1-\beta_n)||x_n - x^*||^2 + \beta_n\Big[(1-\lambda_n)||x_n - x^*||^2 \\ &\quad +\lambda_n||Tx_n - x^*||^2 - \lambda_n(1-\lambda_n)||x_n - Tx_n||^2 \\ &\quad +(1-\lambda_n)||x_n - Ty_n||^2 + \lambda_n||Tx_n - Ty_n||^2 \\ &\quad -\lambda_n(1-\lambda_n)||x_n - Tx_n||^2\Big] \\ &\quad -\beta_n(1-\beta_n)||x_n - Ty_n||^2 \\ &\le ||x_n - x^*||^2 - \beta_n(\lambda_n - \beta_n)||x_n - Ty_n||^2 \\ &\quad -\beta_n\lambda_n[1 - 2\lambda_n - \lambda_n^2 L^2]||x_n - Tx_n||^2 \\ &\le ||x_n - x^*||^2. \end{aligned} \tag{3.25}$$

From (3.24) and (3.25) we obtain

$$\begin{aligned} ||x_{n+1} - x^*||^2 &= ||(1-\alpha_n)z_n - \alpha_n\beta_n(x_n - Ty_n) - x^*||^2 \\ &= ||(1-\alpha_n)(z_n - x^*) - \alpha_n\beta_n(x_n - Ty_n) - \alpha_n x^*||^2 \\ &\le (1-\alpha_n)^2||z_n - x^*||^2 \\ &\quad -2\alpha_n\langle\beta_n(x_n - Ty_n) + x^*, x_{n+1} - x^*\rangle \\ &\le (1-\alpha_n)||x_n - x^*||^2 - 2\alpha_n\beta_n\langle x_n - Ty_n, x_{n+1} - x^*\rangle \\ &\quad -2\alpha_n\langle x^*, x_{n+1} - x^*\rangle. \end{aligned} \tag{3.26}$$

Since $||x_n - Ty_n|| \leq ||x_n - Tx_n|| + L\lambda_n||x_n - Tx_n|| \to 0$ as $n \to \infty$, and $\{x_n\}_{n=1}^{\infty}$ converges weakly to x^*, we have
$\lim_{n\to\infty} \langle x_n - Ty_n, x_{n+1} - x^*\rangle = \lim_{n\to\infty} \langle x^*, x_{n+1} - x^*\rangle = 0.$
It now follows from (3.26) and Lemma 2.1 that $\{x_n\}_{n=1}^{\infty}$ converges strongly to x^*.

Case II. Suppose $\{||x_n - p||\}_{n=1}^{\infty}$ is not monotone decreasing, then there exists a subsequence $\{||x_{n_j} - p||\}$ of $\{||x_n - p||\}$ such that

$$||x_{n_j} - p|| < ||x_{n_j+1} - p||, \ \forall j \in N.$$

Then Lemma 2.2 implies that there exists a nondecreasing sequence $\{\tau(n)\}_{n\geq N_0}$ such that $\tau(n) \to \infty$, and

$$||x_{\tau(n)} - p|| \leq ||x_{\tau(n)+1} - p|| \text{ and } ||x_n - p|| \leq ||x_{\tau(n)+1} - p||, \ \forall n \geq N_0. \quad (3.27)$$

From (3.22), we obtain

$$||x_{\tau(n)} - Tx_{\tau(n)}||^2 \leq \frac{D\alpha_{\tau(n)}}{\beta_{\tau(n)}\lambda_{\tau(n)}[1 - 2\lambda_{\tau(n)} - \lambda_{\tau(n)}^2 L^2]} \to 0 \text{ as } n \to \infty.$$

As in Case I we obtain that $\{x_{\tau(n)}\}$ converges weakly to $x^* \in F(T)$ as $\tau(n) \to \infty$. It follows from (3.26) that for all $n \geq N_0$ we have

$$\begin{aligned} ||x_{\tau(n)} - x^*||^2 \ \leq \ & 2\beta_{\tau(n)}\langle x_{\tau(n)} - Ty_{\tau(n)}, x^* - x_{\tau(n)+1}\rangle \\ & +2\langle x^*, x^* - x_{\tau(n)n+1}\rangle. \end{aligned} \quad (3.28)$$

It now follows from (3.28) that

$$\lim_{n\to\infty} ||x_{\tau(n)} - x^*|| = 0.$$

From (3.26) we obtain
$\lim_{n\to\infty} ||x_{\tau(n)+1} - x^*|| = 0$. Since $||x_n - x^*|| \leq ||x_{\tau(n)+1} - x^*||, \ \forall n \in N$, we obtain $x_n \to x^*$ as $n \to \infty$. From Cases I and II, we conclude that $\{x_n\}$ converges strongly $x^* \in F(T)$. □

Remark 3.1. If C is a nonempty closed convex subset of H, then we can modify the iteration scheme accordingly and obtain the following:

Theorem 3.4. *Let C be a nonempty closed convex subset of a real Hilbert space H and let $T : C \to C$ be an L-Lipschitzian pseudononspreading mapping with $F(T) = \{x \in C : Tx = x\} \neq \emptyset$. Let $\{x_n\}_{n=1}^{\infty}$ be the sequence generated from an arbitrary $x_1 \in C$ by*

$$x_{n+1} = (1 - \alpha_n - \beta_n)x_n + \beta_n T[(1 - \lambda_n)x_n + \lambda_n Tx_n] + \alpha_n x_1, \ n \geq 1, \quad (3.29)$$

where $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$, *and* $\{\lambda_n\}_{n=1}^{\infty}$ *are real sequences in* $(0,1)$ *which satisfy the following conditions:*
(i) $\lim_{n\to\infty} \alpha_n = 0,\ \forall n \geq 1$
(ii) $0 < \epsilon \leq \beta_n \leq \lambda_n(1-\alpha_n),\ \forall n \geq 1$
(iii) $\lambda_n \in (0, \frac{1}{2(1+\sqrt{1+L^2})})$.
Then $\{x_n\}_{n=1}^{\infty}$ *converges strongly to a fixed point of* T.

Thus if $0 \in C$, one may generate the sequence $\{x_n\}$ from $x_1 = 0$ and (3.29) reduces to (3.11). Alternatively, one can use the proximity map $P_C : H \to C$ to generate the sequence from arbitrary $x_1 \in C$ by

$$x_{n+1} = P_C\Big((1-\alpha_n-\beta_n)x_n + \beta_n T[(1-\lambda_n)x_n + \lambda_n T x_n]\Big),\ \forall n \geq 1.$$

Remark 3.2. In [18] the authors proved the following:

Theorem 3.5. [18] *Let* H *be a real Hilbert space and let* $T : H \to H$ *be an* L*-Lipschitzian hemicontractive map such that* $(I-T)$ *is demiclosed at* 0. *Let* $\{t_n\}_{n=1}^{\infty}$, $\{\alpha_n\}_{n=1}^{\infty}$ *and* $\{\beta_n\}$ *be sequences in* $(0,1)$ *satisfying the conditions*
(c1) $\lim_{n\to\infty} t_n = 0$;
(c2) $\sum_{n=1}^{\infty} t_n = \infty$;
(c3) $\alpha_n \leq \beta_n,\ n \geq 1$; *and* $0 < \epsilon \leq \beta_n \leq b < 1$ *for some* $\epsilon > 0$ *and for some* $b \in \left(0, \frac{1}{\sqrt{1+L^2}+1}\right)$;
(c4) $\lim_{n\to\infty} \frac{t_n}{\alpha_n} = 0$.
Then the modified Ishikawa iteration sequence $\{x_n\}_{n=1}^{\infty}$ *generated from an* $x_1 \in H$ *by*

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)(1-t_n)x_n \\ &\quad +\alpha_n T\Big[(1-\beta_n)(1-t_n)x_n + \beta_n T[(1-t_n)x_n]\Big],\ n \geq 1 \end{aligned} \tag{3.30}$$

converges strongly to $P_{F(T)}(0)$, *the least norm element of* $F(T)$.

Since every pseudononspreading mapping T with a nonempty fixed point set is hemicontractive and $(I-T)$ is demiclosed at zero, Theorem 3.5 remains true if we replace L-Lipschitzian hemicontractive mapping T for which $(I-T)$ is demiclosed at 0 with L-Lipschitzian pseudononspreading mapping with $F(T) \neq \emptyset$.

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