

Fixed points of two-degree of freedom systems

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The presence of fixed points in a frequency response of vibrating systems can greatly complicate the vibration reduction if these points are not recognized.

In this paper, the fixed points of two-degree of freedom systems are studied. The frequencies at which fixed points occur and their amplitudes are determined analytically.

1. Introduction

In general, vibrations are undesirable. Their effects on mechanical systems are injurious and can cause costly failures. Therefore, vibrations and their effects need to be suppressed.

A phenomenon which can strongly complicate vibration reduction is the occurrence of fixed points in a frequency response. These points are only recognized if the system parameters are varied. There are three types of fixed points: damping, mass, and stiffness. Fixed points, which occur in a frequency response during a parametric variation of damping, are called damping fixed points. At the frequencies where these points occur, the vibration amplitudes remain constant regardless of the damping values.

Mass and stiffness fixed points are defined in a similar manner. These points occur perfectly in a frequency response only in the absence of damping. Therefore, an undamped system is assumed by their determination.

Damping fixed points were treated by Den Hartog [2], Klotter [5], and Dimarogonas and Haddad [3] in connection with vibration isolation and vibration absorption of systems with a single degree of freedom. In vibration isolation, one fixed point occurs at the frequency $\omega = \sqrt{\omega_n}$, where ω_n is the natural frequency

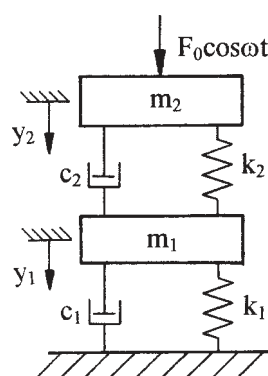


Fig. 1. Two-degree of freedom system.

of the system. In vibration absorption, two fixed points occur in the frequency response of the primary mass and three fixed points occur in the frequency response of the absorber. Also, Bogy and Paslay [1] and Henny and Raney [4] have used the damping fixed points to obtain optimal damping. To our knowledge, however, mass and stiffness fixed points have not been explored previously.

In this paper, a linear system with two degrees of freedom, as shown in Fig. 1, is presented. It is excited by a harmonic force $F(t) = F_0 \cos \omega t$, that acts on a mass, m_2 . All fixed points, which occur in this system in the frequency responses of the vibrating masses m_1 and m_2 , and the frequency response of the force transmitted to the base will be discussed in this paper. The frequencies at which these fixed points occur and their amplitudes will be determined analytically. Also, their effect on vibration reduction will be discussed.

2. Equations of motion

The equations of motion of the system shown in Fig. 1 are:

$$m_1 \ddot{y}_1 + c_1 \dot{y}_1 + c_2(\dot{y}_1 - \dot{y}_2) + k_1 y_1 + k_2(y_1 - y_2) = 0, \quad (1)$$

$$m_2 \ddot{y}_2 + c_2(\dot{y}_2 - \dot{y}_1) + k_2(y_2 - y_1) = F_0 \cos \omega t. \quad (2)$$

Substituting the steady-state solution

$$y_i(t) = Y_i \cos(\omega t - \phi_i), \quad i = 1, 2 \quad (3)$$

into Eqs (1) and (2), and solving for the amplitudes Y_i , yields

$$Y_1 = \frac{F_0}{\Delta} \sqrt{k_2^2 + (c_2\omega)^2}, \quad (4)$$

$$Y_2 = \frac{F_0}{\Delta} \sqrt{(k_1 + k_2 - m_1\omega^2)^2 + (c_1 + c_2)^2\omega^2}, \quad (5)$$

where

$$\begin{aligned} \Delta = & [\{m_1 m_2 \omega^4 - [k_1 m_2 + k_2(m_1 + m_2) + c_1 c_2] \omega^2 \\ & + k_1 k_2\}^2 + \{(k_1 c_2 + k_2 c_1) \omega \\ & - [m_1 c_2 + m_2(c_1 + c_2)] \omega^3\}^2]^{1/2}, \quad (6) \end{aligned}$$

Y_1 and Y_2 are the vibration amplitudes of the masses m_1 and m_2 , respectively. The force transmitted F_T to the base is determined by

$$F_T = c_1 \dot{y} + k_1 y_1 = F_T \cos(\omega t - \psi), \quad (7)$$

where the amplitude of the force transmitted is given by

$$F_T = \frac{F_0}{\Delta} \left((k_1 k_2 - c_1 c_2 \omega^2)^2 + [(k_1 c_2 + k_2 c_1) \omega]^2 \right)^{1/2}. \quad (8)$$

3. Mass fixed points

By varying the values of m_1 while all other parameters of the undamped system remain constant, all curves of the amplitudes Y_1 and F_T pass through a mass fixed point, independent of the values of m_1 . This fixed point is determined by equating Y_1 or F_T to two different values of m_1 . Equating Y_1 for the values $m_1 = 0$ and $m_1 = 1$ yields that this fixed point occurs at the frequency

$$\omega_{m,1} = \sqrt{k_2/m_2}, \quad (9)$$

where the values of Y_1 and F_T at the aforementioned frequency are obtained from Eqs (4) and (8), respectively, as:

$$Y_1(\omega_{m,1}) = F_0/k_2, \quad (10)$$

$$F_T(\omega_{m,1}) = (k_1/k_2)F_0. \quad (11)$$

Since these values are independent of m_1 , then by their determination values of m_1 , which simplify Eqs (4) and (8) are selected. Usually, the values 0 and/or ∞ are selected. The control of the amplitudes Y_1 and F_T at the working frequencies close to $\omega_{m,1}$ can not be succeeded by varying the value of m_1 .

On the other hand, by varying the values of m_2 , all curves of the amplitudes Y_1 and F_T pass through a mass fixed point, independent of the values of m_2 . This fixed point can be determined as previously described and is located at the frequency

$$\omega_{m,2} = \sqrt{(k_1 + k_2)/m_1}, \quad (12)$$

where the amplitudes Y_1 and F_T at this frequency are determined from Eqs (4) and (8), respectively, as:

$$Y_1(\omega_{m,2}) = F_0/k_2, \quad (13)$$

$$F_T(\omega_{m,2}) = (k_1/k_2)F_0. \quad (14)$$

These amplitudes, which are independent of m_2 , cannot be controlled by varying the values of m_2 .

The frequency response of the mass m_2 does not possess mass fixed points neither by varying the values of m_1 nor the values of m_2 . However, all curves of the amplitude Y_2 become zero at the absorber frequency $\omega_a = \sqrt{(k_1 + k_2)/m_1}$, regardless of the values of m_2 .

4. Stiffness fixed points

By varying the values of k_1 of the undamped system, the following outcomes are achieved:

1. The curves of Y_1 possess a stiffness fixed point at the frequency

$$\omega_{k,1} = \sqrt{k_2/m_2}, \quad (15)$$

where the amplitude Y_1 at this frequency is obtained from Eq. (4) as:

$$Y_1(\omega_{k,1}) = F_0/k_2. \quad (16)$$

2. The curves of F_T possess a stiffness fixed point at the frequency

$$\omega_{k,2} = \sqrt{\frac{k_2(m_1 + m_2)}{m_1 m_2}}, \quad (17)$$

where the amplitude F_T at this frequency is obtained from Eq. (8) as:

$$F_T(\omega_{k,2}) = (m_1/m_2)F_0. \quad (18)$$

3. The reduction of $Y_1(\omega_{k,1})$ and $F_T(\omega_{k,2})$ will not succeed by varying the values of k_1 , since these amplitudes are independent of the values of k_1 .
4. The curves of the amplitude Y_2 possess no fixed points.

On the other hand, by varying the values of k_2 of the undamped system, the curves of all three amplitudes, namely, Y_1 , Y_2 , and F_T possess the same stiffness fixed point. This fixed point occurs at the frequency

$$\omega_{k,3} = \sqrt{k_1/m_1}, \quad (19)$$

where the amplitudes Y_1 , Y_2 , and F_T at this frequency are obtained from Eqs (4), (5), and (8), respectively, as:

$$Y_1(\omega_{k,3}) = \frac{m_1}{k_1 m_2} F_0, \quad (20)$$

$$Y_2(\omega_{k,3}) = \frac{m_1}{k_1 m_2} F_0, \quad (21)$$

$$F_T(\omega_{k,3}) = \frac{m_1}{m_2} F_0. \quad (22)$$

These amplitudes cannot be reduced by varying the values of k_2 .

5. Damping fixed points

By varying the values of c_1 in the absence of c_2 , the following outcomes are achieved:

1. The curves of Y_1 possess only one damping fixed point, which occurs at the frequency

$$\omega_{c,1} = \sqrt{k_2/m_2}, \quad (23)$$

where Y_1 at this frequency is obtained from Eq. (4) as:

$$Y_1(\omega_{c,1}) = F_0/k_2. \quad (24)$$

2. The curves of Y_2 possess two damping fixed points which are located at the frequencies

$$\omega_{c,2,3} = \frac{1}{\sqrt{2m_1 m_2}} [k_1 m_2 + k_2(m_1 + m_2) \mp ((k_1 m_2 - k_2 m_1)^2 + m_2^2 k_2(2k_1 + k_2))^{1/2}]^{1/2}, \quad (25)$$

where Y_2 at these frequencies is obtained from Eq. (5) as:

$$Y_2(\omega_{c,i}) = \frac{F_0}{k_2 - m_2 \omega_{c,i}^2}, \quad i = 2, 3. \quad (26)$$

3. The curves of F_T possess three damping fixed points. These points are located at the frequencies

$$\omega_{c,4,5} = \frac{1}{\sqrt{2m_1 m_2}} [2k_1 m_2 + k_2(m_1 + m_2) \mp ([2k_1 m_2 + k_2(m_1 + m_2)]^2 - 8k_1 k_2 m_1 m_2)^{1/2}]^{1/2}, \quad (27)$$

$$\omega_{c,6} = \sqrt{\frac{k_2(m_1 + m_2)}{m_1 m_2}}, \quad (28)$$

where F_T at these frequencies is obtained from Eq. (8) as:

$$F_T(\omega_{c,i}) = \frac{F_0 k_2}{k_2 - m_2 \omega_{c,i}^2}, \quad i = 4, 5, 6. \quad (29)$$

By varying the values of c_2 in the absence of c_1 yields that the curves of Y_1 , Y_2 , and F_T possess three damping fixed points. The fixed points of Y_1 and F_T are located at the frequencies

$$\omega_{c,7,8} = \frac{1}{\sqrt{2m_1 m_2}} [k_1 m_2 + 2k_2(m_1 + m_2) \mp ([k_1 m_2 + 2k_2(m_1 + m_2)]^2 - 8k_1 k_2 m_1 m_2)^{1/2}]^{1/2}, \quad (30)$$

$$\omega_{c,9} = \sqrt{k_1/m_1}, \quad (31)$$

where Y_1 and F_T at these frequencies are determined by using Eqs (4) and (8), respectively, as:

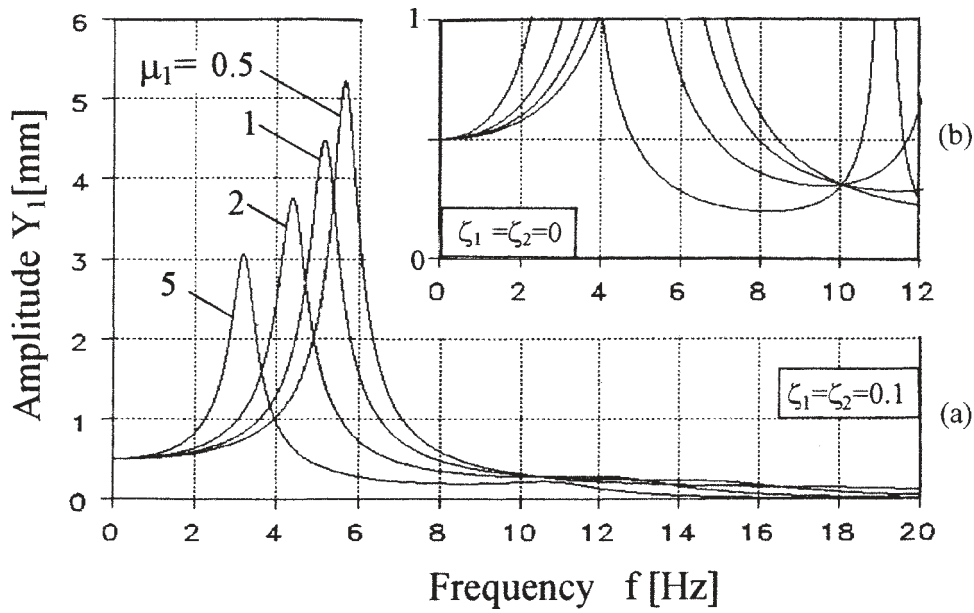


Fig. 2. Frequency response for different values of $\mu_1 = m_1/(80 \text{ kg})$: (a) damped, (b) undamped.

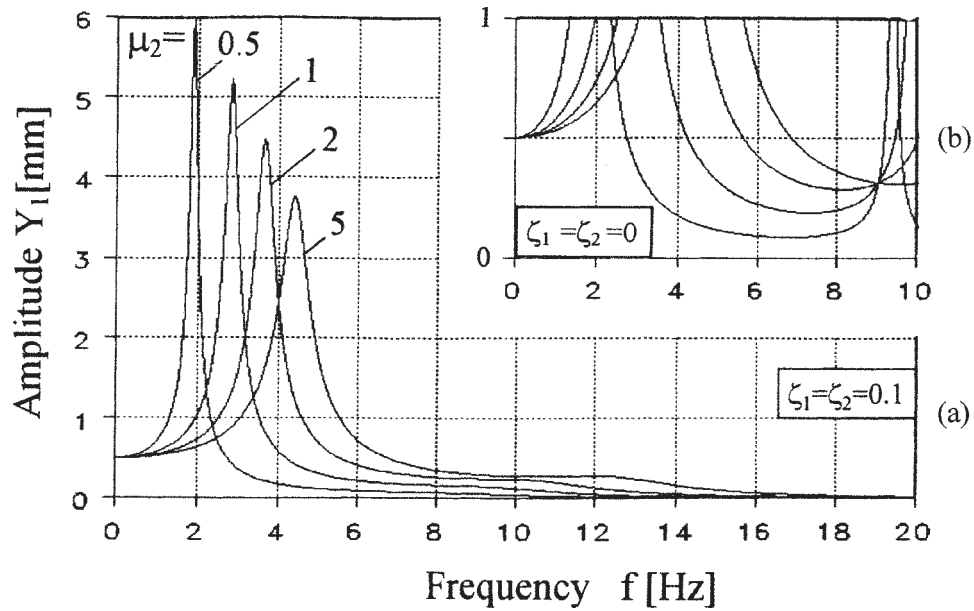


Fig. 3. Frequency response for different values of $\mu_2 = m_2/(160 \text{ kg})$: (a) damped, (b) undamped.

$$Y_1(\omega_{c,i}) = \frac{F_0}{k_1 - (m_1 + m_2)\omega_{c,i}^2}, \quad i = 7, 8, 9, \quad (32)$$

$$F_T(\omega_{c,i}) = \frac{k_1 F_0}{k_1 - (m_1 + m_2)\omega_{c,i}^2}, \quad i = 7, 8, 9. \quad (33)$$

$$\begin{aligned} \omega_{c,10,11} &= \frac{1}{\sqrt{m_1^2 + 2m_1m_2}} [(k_1 + k_2)(m_1 + m_2) \\ &\mp [(k_1 + k_2)(m_1 + m_2)]^2 \\ &- (2k_1k_2 + k_1^2)(m_1^2 + 2m_1m_2)]^{1/2}]^{1/2}, \end{aligned} \quad (34)$$

The fixed points of Y_2 are located at the frequencies

$$\omega_{c,12} = \sqrt{k_1/m_1}, \quad (35)$$

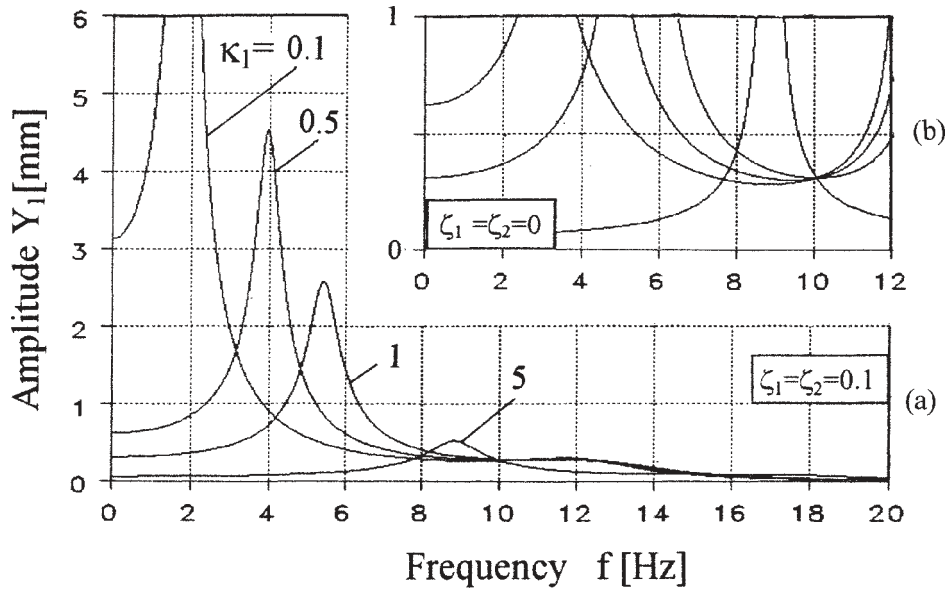


Fig. 4. Frequency response for different values of $\kappa_1 = k_1/(320 \text{ kN/m})$: (a) damped, (b) undamped.

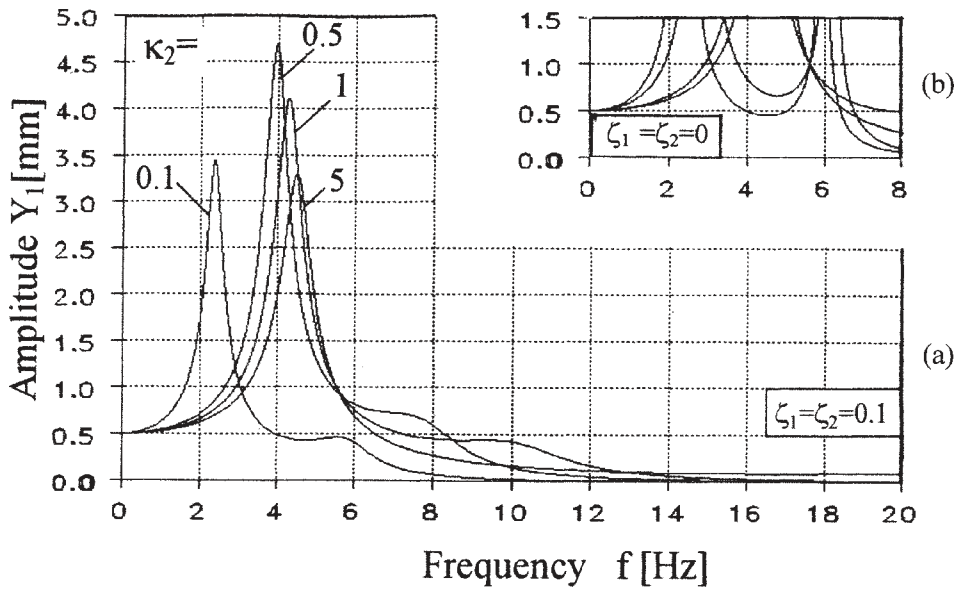


Fig. 5. Frequency response for different values of $\kappa_2 = k_2/(200 \text{ kN/m})$: (a) damped, (b) undamped.

where Y_2 at these frequencies is obtained from Eq. (5) as:

$$Y_2(\omega_{c,i}) = \frac{F_0}{k_1 - (m_1 + m_2)\omega_{c,i}^2}, \quad i = 10, 11, 12. \quad (36)$$

When the working frequencies are near damping fixed points, the vibration amplitudes can only be slightly influenced by varying the values of c_i ($i = 1, 2$). In this

case, isolators and/or absorbers can be used for vibration reduction.

6. Numerical example

To verify the above model, a numerical example will be presented. It is based on the following data: $m_1 =$

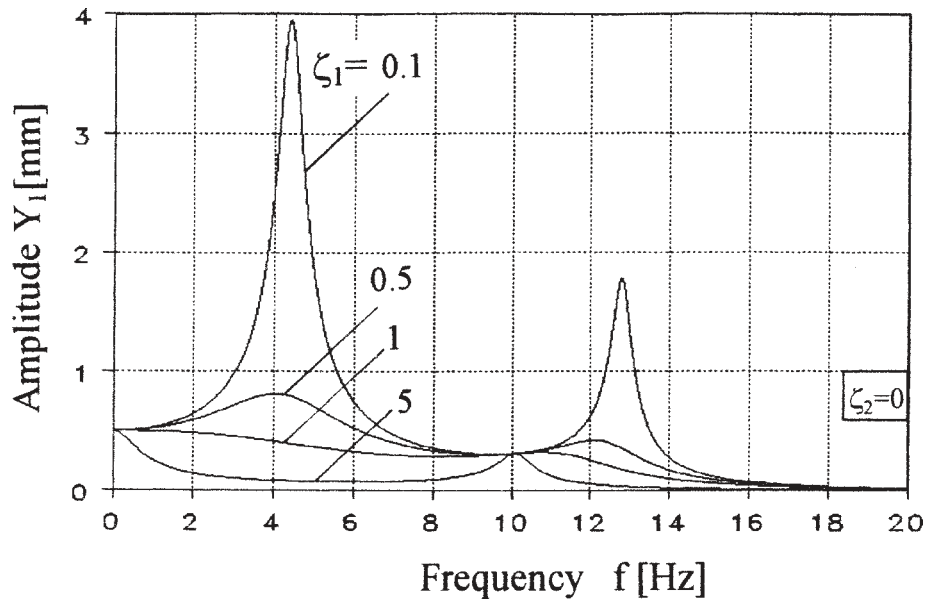


Fig. 6. Frequency response for different values of $\zeta_1 = c_1/(11314 \text{ N s/m})$.

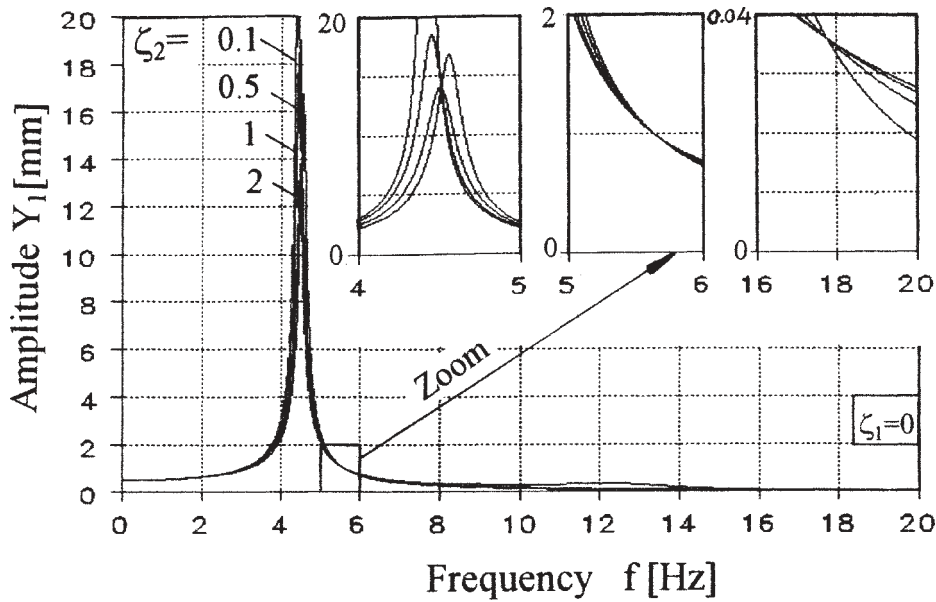


Fig. 7. Frequency response for different values of $\zeta_2 = c_2/(10119 \text{ N s/m})$.

160 kg, $m_2 = 80 \text{ kg}$, $k_1 = 200 \text{ kN/m}$, $k_2 = 320 \text{ kN/m}$, $c_1 = 1131 \text{ N s/m}$, $c_2 = 1012 \text{ N s/m}$, $F_0 = 100 \text{ N}$.

For convenience, the following nondimensional parameters are used: $\mu_1 = m_1/m_2$, $\mu_2 = m_2/m_1$, $\kappa_1 = k_1/k_2$, $\kappa_2 = k_2/k_1$, $\zeta_1 = c_1/2\sqrt{k_1 m_1}$, $\zeta_2 = c_2/2\sqrt{k_2 m_2}$.

Representatively, only the frequency responses of m_1 will be discussed. On the other hand, the frequency

responses of m_2 and the force transmitted will not be presented due to the similarity in the discussion.

Figures 2–5 present the frequency responses for the damped (i.e., $\zeta_1 = \zeta_2 = 0.1$), and the undamped (i.e., $\zeta_1 = \zeta_2 = 0$) systems. Figures 2 and 3 present frequency responses for different values of μ_1 and μ_2 , respectively. While Figs 4 and 5 present frequency responses for different values of κ_1 and κ_2 , respectively.

The frequency responses of the undamped systems enable us to determine the fixed points precisely. However, these points can only be estimated from the frequency responses of the damped systems.

Figures 6 and 7 show the frequency responses for different values of ζ_1 and ζ_2 , respectively. In order to make the fixed points of Fig. 7 clear, the regions of these points are plotted in zoom-windows.

From the previous analysis it is obtained for Y_1 :

a) Mass fixed points:

$$f_{m,1} = \frac{\omega_{m,1}}{2\pi} = 10.07 \text{ Hz,}$$

$$Y_1(f_{m,1}) = 0.313 \text{ mm,}$$

$$f_{m,2} = 9.07 \text{ Hz, } Y_1(f_{m,2}) = 0.313 \text{ mm.}$$

b) Stiffness fixed points:

$$f_{k,1} = 10.07 \text{ Hz, } Y_1(f_{k,1}) = 0.313 \text{ mm,}$$

$$f_{k,3} = 5.63 \text{ Hz, } Y_1(f_{k,3}) = 1 \text{ mm.}$$

c) Damping fixed points:

$$f_{c,1} = 10.07 \text{ Hz, } Y_1(f_{c,1}) = 0.313 \text{ mm,}$$

$$f_{c,7} = 4.51 \text{ Hz, } Y_1(f_{c,7}) = 13.95 \text{ mm,}$$

$$f_{c,8} = 17.76 \text{ Hz, } Y_1(f_{c,8}) = 0.036 \text{ mm,}$$

$$f_{c,9} = 5.63 \text{ Hz, } Y_1(f_{c,9}) = 1 \text{ mm.}$$

Comparison of these values with the plots (i.e., Figs 2–7) yields that they coincide with each other.

7. Conclusions

In this paper, different types of fixed points, which can occur for a two-degree of freedom system, are pre-

sented. The frequencies at which fixed points occur and their amplitudes are determined analytically. The following can be concluded:

- The presence of fixed points may complicate the reduction of vibrations.
- The presence of fixed points cannot be recognized unless a parametric study is performed.
- When the operating frequency is near a mass fixed point, the amplitudes of vibration cannot be effectively reduced by varying the values of masses.
- When the operating frequency is near a stiffness fixed point, the amplitudes of vibration cannot be effectively reduced by varying the stiffness values.
- When the operating frequency is near a damping fixed point, the amplitudes of vibrations cannot be effectively reduced by varying the damping values.

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