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FIXED-WIDTH CONFIDENCE INTERVALS FOR CONTRASTS IN THE MEANS

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Abstract. The sequential procedure developed by Bhargava and Srivastava (1973, *J. Roy. Statist. Soc. Ser. B*, **35**, 147–152) to construct fixed-width confidence intervals for contrasts in the means is further analyzed. Second-order approximations for the first two moments of the stopping time and the coverage probability associated with the sequential procedure, are obtained. A lower bound for the number of "additional" observations after stopping is derived, which ensures the "exact" probability of coverage. Moreover, two-stage, three-stage and "modified" sequential procedures are proposed for the same estimation problem. Relative advantages and disadvantages of these sampling schemes are discussed and their properties are studied.

Key words and phrases: Confidence intervals, contrasts in the means, fixed-width, multi-stage estimation.

1. Introduction and preliminaries

Let us consider a sequence $\{\boldsymbol{X}_i = (X_{i1}, \dots, X_{ip})'\}$, $i = 1, 2, \dots$ of i.i.d. r.v.'s from a p-variate normal population $N_p(\boldsymbol{\mu}, \Sigma)$, where $\boldsymbol{\mu} = (\mu_1, \dots, \mu_p)'$ is $p \times 1$ mean vector and Σ is $p \times p$ covariance matrix of intraclass correlation form $\Sigma = \sigma^2[(1-\rho)I_p + \rho \boldsymbol{e}\boldsymbol{e}']$. Here, I_p stands for a $p \times p$ identity matrix, $\sigma \in (0, \infty)$ and $\rho \in (-1, 1)$. We denote by R_p—the p-dimensional Euclidean space. For the case when σ is unknown but ρ is known, Scheffé (1959) and Miller (1966) developed fixed sample size procedures to construct Tukey's confidence intervals for all contrasts $\{\boldsymbol{b}'\boldsymbol{\mu} : \boldsymbol{b} \in R_p, \boldsymbol{b}'\boldsymbol{e} = 0\}$ having prescribed confidence coefficient $\alpha \in (0,1)$. Bhargava and Srivastava (1973) extended the results to the case when σ and ρ both are unknown. They also proved the non-existence of fixed sample size procedures to construct Tukey's confidence intervals having "pre-assigned width and coverage probability" when σ and/or ρ are unknown. To meet these requirements, they proposed a sequential procedure which can be described as follows.

Let $\boldsymbol{C} = \{\boldsymbol{c} : \boldsymbol{c} \in R_p, \boldsymbol{c}'\boldsymbol{e} = 0, \sum_{j=1}^{p} |c_j| = 2\}$, $\boldsymbol{c} = (c_1, \dots, c_p)'$. Given a random sample $\boldsymbol{X}_1, \dots, \boldsymbol{X}_n$ of size n (≥ 2) and for specified d, α $(d > 0,\ 0 < \alpha < 1)$, suppose one wishes to construct a confidence interval $R_n(\boldsymbol{c})$ of width $2d$ for $\boldsymbol{c}'\boldsymbol{\mu}$, such that $P\{\boldsymbol{c}'\boldsymbol{\mu} \in R_n(\boldsymbol{c}) \text{ for all } \boldsymbol{c} \in \boldsymbol{C}\} \geq \alpha$. Let us define $R_n(\boldsymbol{c}) = (\boldsymbol{c}'\bar{\boldsymbol{X}}_n - d, \boldsymbol{c}'\bar{\boldsymbol{X}}_n + d)$, where $\bar{\boldsymbol{X}}_n = n^{-1}\sum_{i=1}^{n} \boldsymbol{X}_i$. When σ and ρ are known, the (fixed) sample size needed to achieve the goals is the smallest positive integer $n \geq n_o$, where $n_o = (q^2\sigma_w^2/d^2)$, $\sigma_w^2 = \sigma^2(1-\rho)$, $P(R \leq q) = \alpha$, and R is a r.v. following the distribution of range of p independently distributed standard normal variates. However, in the absence of any knowledge about σ and/or ρ, no fixed sample size procedure can achieve the goals of "pre-assigned width and coverage probability" simultaneously and the following sequential procedure is proposed.

Let us define, for $n \geq 2$, $\bar{X}_{i\cdot} = p^{-1}\sum_{j=1}^{p} X_{ij}$, $\bar{X}_{\cdot j} = n^{-1}\sum_{i=1}^{n} X_{ij}$, $\bar{X}_{\cdot\cdot} = (np)^{-1}\sum_{i=1}^{n}\sum_{j=1}^{p} X_{ij}$, $\sigma_w^2(n) = \{(p-1)(n-1)\}^{-1}\sum_{i=1}^{n}\sum_{j=1}^{p}(X_{ij} - \bar{X}_{i\cdot} - \bar{X}_{\cdot j} + \bar{X}_{\cdot\cdot})^2$. Start by taking an initial sample of size m (≥ 2). Then take observations one-by-one sequentially and stop sampling when sample size becomes N, where $N = N(d)$ is defined by

$$N = \inf\{n \geq m : n \geq (q/d)^2\sigma_w^2(n)\}. \tag{1.1}$$

When stop, construct $R_N(\boldsymbol{c}) = (\boldsymbol{c}'\bar{\boldsymbol{X}}_N - d, \boldsymbol{c}'\bar{\boldsymbol{X}}_N + d)$ for $\boldsymbol{c}'\boldsymbol{\mu}$.

Bhargava and Srivastava (1973) studied asymptotic (as $d \to 0$) properties of the sequential procedure (1.1) and proved that "asymptotic efficiency and consistency" (see, Chow and Robbins (1965) for definitions) hold for it. They also derived upper bounds for the first two moments of the stopping time and along the lines of Simons (1968), they proved that if after stopping, an "additional" number of observations, say K, is taken, then $P\{|(\boldsymbol{c}'\bar{\boldsymbol{X}}_{(N+K)} - \boldsymbol{c}'\boldsymbol{\mu})| \leq d\} \geq \alpha$ for all $\boldsymbol{\mu}$, σ and ρ.

Wald (1947) pointed out that due to changing nature of sample space at each stage of sampling, the purely sequential procedure is complicated in nature to apply. However, this variability can be reduced by sampling in "bulks". The sampling scheme which requires only two stages has been proposed by Stein (1945) to construct fixed-width confidence interval for a normal mean when variance is unknown. Of course, Stein's two-stage procedure is easy to apply and achieves the exact probability of coverage (see, Ruben (1961)), it is "asymptotically inefficient" and, in many situations, leads to considerable over sampling (see, Remark 1). For estimating the mean of a univariate normal population, Hall (1981) developed a three-stage procedure. In a later publication, Hall (1983) developed a sampling scheme in which the sampling stages required by purely sequential procedure can be reduced by a pre-determined factor at the cost of only finite number of observations. Both the procedures were shown to be strongly competitive with purely sequential procedure. In fact, the sampling schemes developed by Hall (1981, 1983) combine the advantages of both the two-stage and purely sequential procedures. For multivariate extensions of Stein's and Hall's procedures, one may refer to Chatterjee (1959*a*, 1959*b*, 1960, 1962), Chaturvedi (1988*a*) and Singh and Chaturvedi (1988*a*).

The purpose of this note is many-fold. In Section 2, improving the results obtained by Bhargava and Srivastava (1973), we obtain second-order approximations for the first two moments of the stopping time and coverage probability of the sequential procedure (1.1). Moreover, a lower bound for the number of "additional" observations "K" is derived, which guarantees the exact probability of coverage. In Section 3, a two-stage procedure is proposed and its properties are studied. In Sections 4 and 5, respectively, three-stage and "modified" sequential procedures are developed and their asymptotic properties are established. Finally, in Section 6, the moderate sample size performance of these procedures is studied to demonstrate their practical applicability.

2. Second-order approximations and a lower bound for "K"

The following theorem provides second-order approximations for the first two moments of the stopping time N, defined at (1.1). The second-order approximations for $E(N^2)$ are obtained by using technique similar to that adopted by Chaturvedi (1988*b*). Since the asymptotic distribution of N is normal, these first two moments specify the asymptotic distribution of N completely.

THEOREM 2.1. *For all $m > 1 + 2(p-1)^{-1}$, as $d \to 0$,*

$$E(N) = n_o + \nu - (p+1)(p-1)^{-1} + o(1), \tag{2.1}$$

$$E(N^2) = n_o^2 + 2n_o[\nu - p(p-1)^{-1}] + o(d^{-2}), \tag{2.2}$$

where ν is specified.

PROOF. Using the fact that $(n-1)\sigma_w^2(n)/\sigma_w^2 = (p-1)^{-1}\sum_{j=1}^{n} Y_j = S_{(n-1)}$, say, with $Y_j \sim \chi^2_{(p-1)}$, the stopping rule (1.1) can be rewritten as

$$N = \inf\{n \geq m : S_{(n-1)} \leq (n-1)(n/n_o)\}.$$

Let us define a new stopping variable N^* by

$$N^* = \inf\{n \geq m - 1 : S_n \leq n^2(1+n^{-1})n_o^{-1}\}. \tag{2.3}$$

Following the methods similar to those applied by Swanepoel and van Wyk (1982) in the proof of Lemma 1, it can be shown that the stopping rules N and N^* follow the same probability distribution. Comparing (2.3) with equation (1.1) of Woodroofe (1977), we obtain in his notations, $c = n_o^{-1}$, $\alpha = 2$, $\beta = 1$, $\mu = 1$, $\tau^2 = 2(p-1)^{-1}$, $\lambda = n_o$, $a = (p-1)/2$ and ν given by equation (2.4) using these values of β, μ, α, τ^2 and S_n. Hence, from Theorem 2.4 of Woodroofe (1977), we obtain for all $m > 1 + 2(p-1)^{-1}$, as $d \to 0$,

$$E(N^*) = n_o + \nu - 1 - 2(p-1)^{-1} + o(1),$$

and (2.1) follows.

To prove (2.2), let us write

$$E(N^2) = n_o^2 + 2n_oE(N - n_o) + n_oE\{(N-n_o)^2/n_o\}. \tag{2.4}$$

It follows from a result of Bhattacharya and Mallik (1973) that the asymptotic distribution of $(N-n_o)/(n_o)^{1/2}$ is $N(0, 2(p-1)^{-1})$. Moreover, from Theorem 2.3 of

Woodroofe (1977), $(N-n_o)^2/(n_o)$ is uniformly integrable for all $m > 1+2(p-1)^{-1}$. Utilizing these results and (2.1), we obtain from (2.4), for all $m > 1+2(p-1)^{-1}$, as $d \to 0$,

$$E(N^2) = n_o^2 + 2n_o\{\nu - (p+1)(p-1)^{-1} + o(1)\} + 2(p-1)^{-1}n_o,$$

and (2.2) holds.

The following theorem provides second-order approximations for the coverage probability associated with the sequential procedure (1.1).

THEOREM 2.2. *For all $m > 1+2(p-1)^{-1}$, as $d \to 0$, $P\{\boldsymbol{c}'\boldsymbol{\mu} \in R_N(\boldsymbol{c})$ for all $\boldsymbol{c} \in \boldsymbol{C}\} = \alpha + n_o^{-1}q^2\{(\nu - (p+1)(p-1)^{-1})G'(q^2) + q^2(p-1)^{-1}G''(q^2)\} + o(d^2)$, where $G(x) = F(x^{1/2})$, $F(x) = p\int_{-\infty}^{\infty}\phi(y)[\Phi(y) - \Phi(y-x)]^{p-1}dy$ and $\phi(\cdot)$ and $\Phi(\cdot)$ denote, respectively, the p.d.f. and c.d.f. of a standard normal variate.*

PROOF. From equation (4.3) of Bhargava and Srivastava (1973),

$$\begin{aligned} P\{\boldsymbol{c}'\boldsymbol{\mu} \in R_N(\boldsymbol{c}) \text{ for all } \boldsymbol{c} \in C\} &= E\{F(q(N/n_o)^{1/2})\} \\ &= E\{G(q^2 n_o^{-1}N)\}. \end{aligned}$$

Applying Taylor series expansion, we obtain for $|q^2 - W| \le q^2|n_o^{-1}N - 1|$,

$$\begin{aligned} E\{G(q^2n_o^{-1}N)\} &= G(q^2) + n_o^{-1}q^2G'(q^2)E(N-n_o) \\ &\quad + (1/2)n_o^{-1}q^4E\{n_o^{-1}(N-n_o)^2G''(W)\}. \end{aligned}$$

Utilizing (2.1) and the fact that the asymptotic distribution of $n_o^{-1}(N-n_o)^2G''(W)$ is $2(p-1)^{-1}\chi^2_{(1)}G''(q^2)$, we obtain for all $m > 1+2(p-1)^{-1}$, as $d \to 0$,

$$\begin{aligned} E\{G(q^2n_o^{-1}N)\} &= \alpha + n_o^{-1}G'(q^2)\{\nu - (p-1)(p+1)^{-1} + o(1)\} \\ &\quad + n_o^{-1}q^4(p-1)^{-1}G''(q^2), \end{aligned}$$

and the theorem follows.

In the next theorem, we obtain a lower bound for "K". But, before proving the main theorem, we state a lemma, the proof of which can be obtained along the lines of that of Lemma 1 in Singh and Chaturvedi (1988*a*, 1988*b*) after necessary modifications at various places. We omit the details for brevity.

LEMMA 2.1. *Let $M = [(q/d)^2\sigma_w^2(N)] + K$, where N is determined by the rule (1.1) and y denotes the smallest positive integer $\ge y$. Then, for all $m > 1+2(p-1)^{-1}$, as $d \to 0$,*

$$\begin{aligned} &E(M) = (q/d)^2\sigma_w^2 - (p+1)(p-1)^{-1} + K + o(1), \\ &\mathrm{Var}(M) = 2(p-1)^{-1}(q/d)^2\sigma_w^2 + o(d^{-2}) \end{aligned}$$

and, for all $r > 0$, $E|M - E(M)|^r = O(d^{-r})$.

THEOREM 2.3. *For all* $\boldsymbol{\mu}$, σ^2, ρ *and sufficiently small* d,

$$P\{|\boldsymbol{c}'\bar{\boldsymbol{X}}_{(N+K)} - \boldsymbol{c}'\boldsymbol{\mu}| \leq d \text{ for all } \boldsymbol{c} \in \boldsymbol{C}\} \geq \alpha,$$

if $K \geq (p-1)^{-1}\{(p+1) + (3/4)q^2\}$.

PROOF. For $F(x)$ and $G(x)$ defined in Theorem 2.2, let $f(x) = F'(x)$. We note that

$$\begin{aligned} G'(x) &= (1/2x^{1/2})f(x^{1/2}), \\ G''(x) &= (1/4)\{x^{-1}f'(x^{1/2}) - x^{-3/4}f(x^{1/2})\}. \end{aligned} \tag{2.5}$$

Using Taylor series expansion, we obtain

$$\begin{aligned} &P\{|\boldsymbol{c}'\bar{\boldsymbol{X}}_{(N+K)} - \boldsymbol{c}'\boldsymbol{\mu} \leq d \text{ for all } \boldsymbol{c} \in \boldsymbol{C}\} \\ &\quad = E\{G(d^2M/\sigma_w^2)\} \\ &\quad = G(q^2) + \sigma_w^{-2}d^2G'(q^2)E\{M - (q/d)^2\sigma_w^2\} + (1/2)\sigma_w^{-4}d^4 \\ &\qquad \cdot G''(q^2)E\{M - (q/d)^2\sigma_w^2\}^2 + O(d^6E|M - E(M)|^3) \\ &\quad = \alpha + (\sigma_w^{-2}d^2/2q)f(q)E\{M - (q/d)^2\sigma_w^2\} + (1/2)\sigma_w^{-4}d^4(4q)^{-1} \\ &\qquad \cdot f(q)\{q^{-1}f'(q)/f(q) - 1\}E\{M - (q/d)^2\sigma_w^2\}^2 \\ &\qquad + O(d^6E|M - E(M)|^3). \end{aligned} \tag{2.6}$$

It has been shown in Bhargava and Srivastava (1973, p. 152) that $f'(q)/f(q) \geq -q/2$. Utilizing this result and Lemma 2.1, we obtain from (2.6),

$$\begin{aligned} &P\{|\boldsymbol{c}'\bar{\boldsymbol{X}}_{(N+K)} - \boldsymbol{c}'\boldsymbol{\mu}| \leq d \text{ for all } \boldsymbol{c} \in \boldsymbol{C}\} \\ &\quad \geq \alpha + (\sigma_w^{-2}d^2/2q)f(q)\{K - (p+1)(p-1)^{-1}\} - (3/16q)(\sigma_w^{-4}d^4) \\ &\qquad \cdot f(q)\{2(p-1)^{-1}(q/d)^2\sigma_w^2\} + o(d^2) \\ &\quad = \alpha + (\sigma_w^{-2}d^2/2q)f(q)\{K - (p-1)^{-1}((p+1) + (3/4)q^2)\} + o(d^2), \end{aligned}$$

and the theorem follows.

3. The two-stage procedure

Start with a sample of size m (≥ 2). Then, the second stage sample size is given by

$$N = \max\{m, [q_\gamma^2\sigma_w^2(m)/d^2] + 1\}, \tag{3.1}$$

where q_γ is the upper $100(1-\alpha)\%$ point of Studentized range on $\gamma = (p-1)(m-1)$ d.f. Construct $R_N(\boldsymbol{c}) = (\boldsymbol{c}'\bar{\boldsymbol{X}}_N - d, \boldsymbol{c}'\bar{\boldsymbol{X}}_N + d)$ for $\boldsymbol{c}'\boldsymbol{\mu}$.

The properties of the two-stage procedure (3.1) are established in the following theorem.

THEOREM 3.1.

$$\lim_{d\to 0} E(N/n_o) = (q_\gamma/q)^2, \tag{3.2}$$

$$P\{\boldsymbol{c}'\boldsymbol{\mu} \in R_N(\boldsymbol{c}) \textit{ for all } \boldsymbol{c} \in \boldsymbol{C}\} \geq \alpha. \tag{3.3}$$

PROOF. Taking expectation throughout, from the basic inequality

$$(q_\gamma/d)^2\sigma_w^2(m) \leq N \leq (q_\gamma/d)^2\sigma_w^2(m) + m,$$

we obtain

$$(q_\gamma/d)^2\sigma_w^2 \leq E(N) \leq (q_\gamma/d)^2\sigma_w^2 + m, \tag{3.4}$$

or,

$$(q_\gamma/q)^2 \leq E(N/n_o) \leq (q_\gamma/q)^2 + (m/n_o).$$

Result (3.2) follows since $\lim_{d\to 0} n_o = \infty$.

By the definition of N, $N^{1/2} \geq q_\gamma\sigma_w/d$. Since $F(x)$ is monotonically increasing in x and $\gamma\sigma_w^2(m)/\sigma_w^2 \sim \chi^2_{(\gamma)}$, we obtain

$$\begin{aligned} &P\{\boldsymbol{c}'\boldsymbol{\mu} \in R_N(\boldsymbol{c}) \text{ for all } \boldsymbol{c} \in \boldsymbol{C}\} \\ &\quad = E\{F(dN^{1/2}/\sigma_w)\} \geq E\{F(q_\gamma)\} = \alpha, \end{aligned}$$

and (3.3) follows.

Remark 1. It can be verified from the table of percentage points of Studentized range that $(q_\gamma/q) > 1$. Thus, from (3.3), $\lim_{d\to 0}E(N/n_o) > 1$, implying that the two-stage procedure (3.1) is "asymptotically inefficient". Moreover, from (3.4),

$$(q_\gamma^2 - q^2)(\sigma_w/d)^2 \leq E(N) - n_o \leq (q_\gamma^2 - q^2)(\sigma_w/d)^2 + m.$$

Thus, $E(N) - n_o \to \infty$ as $d \to 0$, showing that the "cost of ignorance" (see, Simons (1968)) is unbounded for the two-stage procedure. These drawbacks are removed in the sampling schemes discussed in the next two sections.

4. The three-stage procedure

Let us choose and fix a number $\eta \in (0,1)$. Take an initial sample $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_k$ of size k (≥ 2) and define $M = \max\{k, [\eta q^2\sigma_w^2(k)/d^2] + 1\}$. If $M > k$, take the additional observations to complete the sample $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_k, \ldots, \boldsymbol{X}_M$. Let

$$N = \max\{M, [q^2\sigma_w^2(M)/d^2] + 1\}, \tag{4.1}$$

and if $N > M$, we sample the difference to obtain $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_M, \ldots, \boldsymbol{X}_N$. Construct $R_N(\boldsymbol{c}) = (\boldsymbol{c}'\bar{\boldsymbol{X}}_N - d, \boldsymbol{c}'\bar{\boldsymbol{X}}_N + d)$.

Now we state a lemma, the proof of which can be obtained along the lines of that of Theorem 1 in Hall (1981).

LEMMA 4.1. *As* $d \to 0$,

$$E(N) = n_o + 1/2 - 2\eta^{-1} + o(1),$$
$$E|N - n_o|^2 = 2\eta^{-1} n_o + o(d^{-2}),$$
$$E|N - n_o|^3 = o(d^{-4}).$$

The following theorem contains the main result of this section.

THEOREM 4.1. *As* $d \to 0$,

$$P\{c'\mu \in R_N(c) \text{ for all } c \in C\}$$
$$\geq \alpha + \{(1/2 - 2\eta^{-1}) - (3/4)\eta^{-1}q^2\}\{f(q)/2q\}(d/\sigma_w^2) + o(d^2).$$

PROOF. $P\{c'\mu \in R_N(c) \text{ for all } c \in C\} = E\{G(d^2N/\sigma_w^2)\}$. Using the Taylor series expansion given at (2.6), the result that $f'(q)/f(q) \geq -q/2$ and applying Lemma 4.1, we obtain

$$P\{c'\mu \in R_N(c) \text{ for all } c \in C\}$$
$$= \alpha + (d/\sigma_w^2)(1/2q)f(q)\{1/2 - 2\eta^{-1} + o(1)\} + (1/8)(d^2/\sigma_w^2)^2$$
$$\cdot f(q)\{q^{-2}f'(q)/f(q) - q^{-1}\}\{2\eta^{-1}n_o + o(d^{-2})\} + o(d^2)$$
$$\geq \alpha + (1/2 - 2\eta^{-1})\{f(q)/2q\}(d/\sigma_w^2) - (3/4)\eta^{-1}q^2\{f(q)/2q\}$$
$$\cdot (d/\sigma_w^2) + o(d^2),$$

and the theorem follows after some algebra.

5. The modified sequential procedure

Let $\eta \in (0, 1)$ and $K \in (0, \infty)$ be specified. Take the observations sequentially with the stopping time N_1 defined by

$$N_1 = \inf\{n_1 \geq m : n_1 \geq \eta(q/d)^2\sigma_w^2(n_1)\}. \tag{5.1a}$$

Then we jump ahead by collecting N_2 observations, where

$$N_2 = [(q/d)^2\sigma_w^2(N_1) + K] + 1. \tag{5.1b}$$

Let $N = \max(N_1, N_2)$ and construct $R_N(c) = (c'\bar{X}_N - d, c'\bar{X}_N + d)$ for $c'\mu$.
We first establish two lemmas.

LEMMA 5.1. *As* $d \to 0$,

$$N_1 = N_2 = \infty \quad a.s. \tag{5.2}$$
$$(N/n_o) = 1 \quad a.s. \tag{5.3}$$
$$(\eta n_o)^{-1/2}(N_1 - \eta n_o) \xrightarrow{\mathcal{L}} N(0, 2(p-1)^{-1}). \tag{5.4}$$

For all $m > 1 + 2(p-1)^{-1}$,

$$E(N_1) = \eta n_o + \nu - (p+1)(p-1)^{-1} + o(1), \tag{5.5}$$

where ν *is same as that defined in Theorem* 2.1.

PROOF. Result (5.2) is an immediate consequence of the definitions of N_1 and N_2.

We notice the inequality

$$\eta(q/d)^2\sigma_w^2(N_1) \le N_1 \le \eta(q/d)^2\sigma_w^2(N_1) + (m-1)$$

or,

$$\{\sigma_w^2(N_1)/\sigma_w^2\} \le (N_1/\eta n_o) \le \{\sigma_w^2(N_1)/\sigma_w^2\} + (m-1)/(\eta n_o),$$

which, on using (5.2) and the result $\lim_{N_1\to\infty}\sigma_w^2(N_1) = \sigma_w^2$ a.s., leads us to $\lim_{d\to 0}(N_1/\eta n_o) = 1$ a.s. Similarly, we can prove that $\lim_{d\to 0}(N_2/n_o) = 1$ a.s. Result (5.3) now follows from the definition of N.

Result (5.4) follows from Theorem 2 of Bhattacharya and Mallik (1973) and the proof of (5.5) is same as that of (2.1).

LEMMA 5.2. *For all* $m > 1 + 2(p-1)^{-1}$, *as* $d \to 0$,

$$E(N) = n_o + K - \eta^{-1}(p+1)(p-1)^{-1} + o(1), \tag{5.6}$$

$$\text{Var}(N) = 2\eta^{-1}n_o(p-1)^{-1} + o(d^{-2}), \tag{5.7}$$

and, for $r > 0$,

$$E|N - n_o|^r = O(d^{-r}). \tag{5.8}$$

PROOF. Consider the difference

$$R_d = N_1(N_1-1)(\eta n_o)^{-1} - S_{(N_1-1)}.$$

The mean of the asymptotic distribution of R_d is ν. Let $R_d^* = (\eta n_o)(N_1-1)^{-1}R_d$. By the definition of N_1, $\sigma_w^2(N_1-1) \ge \eta^{-1}(d/q)^2(N_1-1)$. Thus,

$$\begin{aligned} 0 \le R_d^* &= N_1 - \eta(q/d)^2\sigma_w^2(N_1) \\ &\le N_1 - \eta(q/d)^2\{(p-1)(N_1-1)\}^{-1} \\ &\quad \cdot \sum_{i=1}^{N_1-1}\sum_{j=1}^{p}(X_{ij} - \bar{\boldsymbol{X}}_{i\cdot} - \bar{\boldsymbol{X}}_{\cdot j} + \bar{\boldsymbol{X}}_{\cdot\cdot})^2 \\ &\le 2, \end{aligned} \tag{5.9}$$

and hence, by dominated convergence theorem, $E(R_d^*) \to \nu$ as $d \to 0$. Now, from (5.5) and (5.9), we obtain for all $m > 1 + 2(p-1)^{-1}$, as $d \to 0$,

$$\begin{aligned} E[(q/d)^2\sigma_w^2(N_1)] &= \eta^{-1}\{E(N_1) - \nu\} \\ &= \eta^{-1}\{\eta n_o - (p+1)(p-1)^{-1} + o(1)\}, \end{aligned}$$

and (5.6) follows.

By the definition of N, $\text{Var}(N) = \eta^{-2}\,\text{Var}(N_1)$. Let $h(N_1) = (\eta n_o)^{-1/2}(N_1 - \eta n_o)$. It follows from Theorem 2.3 of Woodroofe (1977) that $h^2(N_1)$ is uniformly integrable for all $m > 1 + 2(p-1)^{-1}$. Thus,

$$\text{Var}(N) = \eta^{-2}\{2\eta n_o(p-1)^{-1}(1+o(1))\},$$

and (5.7) holds.

The proof of (5.8) is same as that of result (2) in Hall (1983).

Now we prove the main theorem.

THEOREM 5.1. *For all $\boldsymbol{\mu}$, σ^2, ρ and sufficiently small d, say $d \le d_o$,*

$$P\{\boldsymbol{c}'\boldsymbol{\mu} \in R_N(\boldsymbol{c}) \text{ for all } \boldsymbol{c} \in \boldsymbol{C}\} \ge \alpha,$$

if $K \ge \eta^{-1}(p-1)^{-1}\{(p+1) + (3/4)q^2\}$.

PROOF. It can be seen that

$$P\{\boldsymbol{c}'\boldsymbol{\mu} \in R_N(c) \text{ for all } \boldsymbol{c} \in \boldsymbol{C}\} = E\{G(d^2N/\sigma_w^2)\},$$

where N is determined by the rule (5.1a)–(5.1b). Using Taylor series expansion (2.6), the result $f'(x)/f(x) \ge -x/2$, and applying Lemma 5.2, we obtain

$$\begin{aligned}
&P\{\boldsymbol{c}'\boldsymbol{\mu} \in R_N(\boldsymbol{c}) \text{ for all } \boldsymbol{c} \in \boldsymbol{C}\}\\
&\quad= \alpha + (d^2/\sigma_w^2)\{f(q)/2q\}\{K - \eta^{-1}(p-1)^{-1}(p+1) + o(1)\}\\
&\qquad+ (1/8)(d^2/\sigma_w^2)^2 f(q)\{q^{-2}f'(q)/f(q) - q^{-1}\}\{2\eta^{-1}n_o(p-1)^{-1} + o(1)\}\\
&\quad\ge \alpha + (d^2/\sigma_w^2)\{f(q)/2q\}\{K - \eta^{-1}(p-1)^{-1}(p+1)\}\\
&\qquad- (d^2/\sigma_w^2)\{3f(q)/8q\}\eta^{-1}(p-1)^{-1}q^2 + o(d^2)\\
&\quad= \alpha + \{f(q)/2q\}(d^2/\sigma_w^2)\{K - \eta^{-1}(p-1)^{-1}((p+1) + (3/4)q^2)\} + o(d^2),
\end{aligned}$$

and the theorem follows.

Remark 2. It is clearly reflected by the result of Theorem 5.1 that (for given p and q) smaller the value of η, larger we have to take "K"—the observations after stopping sequential sampling. Thus, as expected, making η smaller, that is, terminating sequential sampling at an earlier stage, increases the ASN of the procedure to achieve a given coverage probability.

6. The moderate sample performance

The Tables 1–3 present the results of Monte-Carlo experiment. We fix $p = 2$, $\sigma = 1$, $\rho = .5$ and $\alpha = .95$. For 6 values of d, we conducted 1000 trials. We computed expected sample sizes $\bar{N}$, as well as, the coverage probabilities P that the confidence interval covers the origin for different estimation procedures. The procedures behave quite satisfactorily.

Table 1. Results of 1000 Monte-Carlo trials for purely sequential procedure.

d	n_o	$\bar{N}$	P
.225	108.20	112	.9383
.200	136.95	139	.9449
.175	178.87	192	.9732
.150	243.46	253	.9509
.125	350.59	342	.9479
.100	547.80	551	.9511

Table 2. Results of 1000 Monte-Carlo trials for three-stage procedure.

d	n_o	$k = 5$, $\eta = .5$		$k = 8$, $\eta = .5$		$k = 10$, $\eta = .5$	
		$\bar{N}$	P	$\bar{N}$	P	$\bar{N}$	P
.225	108.20	111	.9505	120	.9554	131	.9561
.200	136.95	139	.9533	143	.9548	152	.9567
.175	178.87	182	.9541	197	.9591	209	.9566
.150	243.46	253	.9512	267	.9531	265	.9555
.125	350.59	361	.9507	372	.9551	385	.9571
.100	547.80	559	.9553	570	.9570	591	.9572

Table 3. Results of 1000 Monte-Carlo trials for modified sequential procedure.

d	n_o	$K = 5$, $\eta = .5$		$K = 8$, $\eta = .5$		$K = 10$, $\eta = .5$	
		$\bar{N}$	P	$\bar{N}$	P	$\bar{N}$	P
.225	108.20	110	.9494	117	.9553	130	.9572
.200	136.95	137	.9499	138	.9511	144	.9565
.175	178.87	181	.9509	194	.9585	200	.9552
.150	243.46	248	.9508	262	.9502	255	.9531
.125	350.59	358	.9499	365	.9541	377	.9567
.100	547.80	552	.9517	566	.9587	588	.9566

Remark 3. It is clearly reflected from Tables 1–3 that both the ASN and coverage probability for the purely sequential procedure are least and maximum for the three-stage procedure. The ASN and coverage probability for both the three-stage and modified sequential procedure increase with increasing values of k and K. These results justify the conclusions drawn by Hall (1981, 1983) that at the cost of a finite number of observations, we can reach the target value of the coverage probability.

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