Fixing an ambiguity in two dimensional string theory using string field theory

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ABSTRACT: In a recent paper, Balthazar, Rodriguez and Yin found some remarkable agreement between the results of $c = 1$ matrix model and D-instanton corrections in two dimensional string theory. Their analysis left undetermined two constants in the string theory computation which had to be fixed by comparing the results with the matrix model results. One of these constants is affected by possible renormalization of the D-instanton action that needs to be computed separately. In this paper we fix the other constant by reformulating the world-sheet analysis in the language of string field theory.

Contents

1 Introduction

In a recent paper [1], Balthazar, Rodriguez and Yin found some remarkable agreement between the results of $c = 1$ matrix model [2–5] and two dimensional string theory. In their analysis on the string theory side they left two constants undetermined. Our goal in this paper will be to fix one of these constants by carefully repeating their analysis in the language of string field theory. We find that the constant c' in the analysis of [1], that was left undermined in the world-sheet analysis, can be determined using the framework of string field theory and takes value:

$$c' = -\ln 4 \simeq -1.38629\,. \tag{1.1}$$

This is within 1% of the numerical value -1.399 that was required for the agreement between the results of world-sheet calculation and the matrix model results. The other constant, called $S_{ZZ}^{(1)}$ in [1], is affected by possible renormalization of the D-instanton action that needs to be computed separately. In particular since the D-instanton contribution to any amplitude is proportional to $\exp[-c/g_s]$ for some positive constant c, a multiplicative renormalization of the D-instanton action c/g_s by $(1 - b\,g_s - a\,g_s^2)$ will generate an overall multiplicative factor of $e^{c(b+a\,g_s)}$. Therefore contributions proportional to g_s multiplying the leading order term will be affected by this renormalization of the D-instanton action.[1]

We shall begin in section 2 by describing the analysis of [1] in the language of string field theory, and the origin of the apparent ambiguity in their analysis. We shall then describe in section 3 why in a careful treatment using the framework of string field theory there is no ambiguity. Finally in section 4 we shall use the machinery of string field theory to fix the undetermined constant c'.

Before we turn to the details, let us first make a general remark on why string field theory is useful for fixing the ambiguity described in [1]. The ambiguity arises from having to cancel the divergences between two diagrams, each of which is separately infrared

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[1]However once we have determined the renormalization factor by comparing one amplitude, the same renormalization must continue to hold for all other amplitudes.

divergent. Therefore we need to regulate the diagrams before combining them, and since *a priori* the regulators can be chosen independently, we can be left with a finite term after the cancellation that depends on the finite parts of each regulated diagram. This is precisely the kind of situation where string field theory is useful. String field theory makes clear the physical origin of the divergences and therefore gives precise relation between the regulators in different diagrams, leaving behind no scope of an ambiguity except those induced by field redefinition.

2   Formulation of the problem in the language of string field theory

In this section we shall interpret the analysis of [1] in the language of string field theory. The main goal of [1] was to compute the contribution of a single D-instanton to the scattering of massless closed strings. Since the fluctuations of D-instanton are described by open strings with ends on the D-instanton, the relevant string field theory is that of open and closed strings [6], with the open strings living on the D-instanton. Since our ultimate goal is to compute closed string scattering amplitudes, it would be natural to integrate out the open string degrees of freedom and write an effective action for closed string fields only. However on the D-instanton world-volume there is a zero mode, describing the freedom of translating the D-instanton along the time direction. This corresponds to a particular mode of the open string with vanishing propagator. For this reason, this mode cannot be integrated out at the beginning. The best we can hope is to integrate out all the open string modes other than the zero mode. The corresponding action still satisfies the BV master equation and was described as the Wilsonian effective action in [7].

Let us denote by ϕ the zero mode of the open string — called x^0 in [1]. It was argued in [1] that the ϕ dependence of the Wilsonian action has a particularly simple form. The sum of the terms in the Wilsonian action with a fixed set of external closed string fields but arbitrary number of ϕ modes is proportional to $\sum_n (i\omega\phi)^n/n! = e^{i\omega\phi}$ were ω is the total energy of all the closed string fields. If we integrate over ϕ *after computing the Green's function of a set of closed string fields in some background ϕ*, we generate a factor proportional to $\delta(\omega_{\rm tot})$, — $\omega_{\rm tot}$ being the total energy of all the closed string states in the Green's function. This restores energy conservation, which is otherwise broken for a fixed D-instanton configuration. From this we can compute the S-matrix of closed string states by restricting the external closed string states to be on-shell.

We are now in a position to describe the origin of the apparent ambiguity encountered in [1] in the language of string field theory. The stems from the fact that the string field theory action is not unique. The construction of the action requires the choice of local coordinates at the punctures where vertex operators are inserted, and different choices lead to apparently different string field theories. As we shall elaborate in section 3, the apparent ambiguity encountered in [1] can be traced to this ambiguity. However, it follows from a general analysis in [8], after suitable generalization to the field theory of open and closed strings, that the ambiguity described above can be regarded as a result of field redefinition. Therefore it is not expected to change any of the physical results. We shall see in section 3 how the ambiguity in the result of [1] is actually resolved.

Figure 1. The figure on the left shows a Feynman diagram obtained by joining the open string states from a pair of open-closed interaction vertices by an open string propagator. The thick lines labelled by the symbol $\mathcal{C}$ denote closed strings and the thin line labelled by the symbol $\mathcal{O}$ denotes open string. A $\times$ denote an interaction vertex. This is the convention we shall follow in all the subsequent figures. The figure on the right gives the representation of the Feynman diagram as a disk amplitude, with the closed string vertex operators, denoted by $\circ$, inserted at 0 and $z_1(1-u)/(1+u)$.

3 String field theory as regulator

We shall begin with some simple examples that illustrate how string field theory can be used to systematically analyze the infrared divergences in string theory, arising as ultraviolet divergences in the world-sheet theory [9, 10]. Let us consider a unit disk with a bulk puncture at the centre and a boundary puncture at the point z_1 on the boundary of the disk. Using rotational symmetry we can fix z_1 to 1, but we shall keep it arbitrary for later use. For defining off-shell amplitudes we need to choose local coordinates at both punctures, with the property that the local coordinate vanishes at the puncture, and that the local coordinate around a boundary puncture must take real values on the boundary. Let us for definiteness choose the local coordinate w at the bulk puncture to be βz and the local coordinate w_1 at the boundary puncture to be

$$w_1 = i\,\lambda\,\frac{(z_1 - z)}{(z_1 + z)}\,, \tag{3.1}$$

for some real positive constants β and λ. Note that (3.1) gives real w_1 as long as z_1 and z are on the boundary of the unit disk. It is convenient to take λ and β to be large — in the language of string field theory this is described as adding long stubs to the vertices [8]. Once a choice of local coordinates is made, we can compute a disk amplitude with an off-shell closed string vertex operator inserted at the center and an off-shell open string vertex operator inserted at the boundary at the point z_1. This corresponds to an interaction vertex of string theory with one open string and one closed string.

Now consider a Feynman diagram where we take two such vertices and join the open string states by a propagator. This has been shown in figure 1 with closed strings represented by thick lines and open strings represented by thin lines. This describes a disk amplitude with two external closed strings. However this does not cover the full moduli space of the disk with two bulk punctures. Instead, it covers part of the moduli space that corresponds to the surface

$$w_1\tilde{w}_1 = -q, \quad 0 \le q \le 1, \quad \Leftrightarrow \quad \frac{(z - z_1)(\tilde{z} - z_1)}{(z + z_1)(\tilde{z} + z_1)} = q/\lambda^2 \equiv u, \quad 0 \le u \le \epsilon, \quad \epsilon \equiv \lambda^{-2}, \tag{3.2}$$
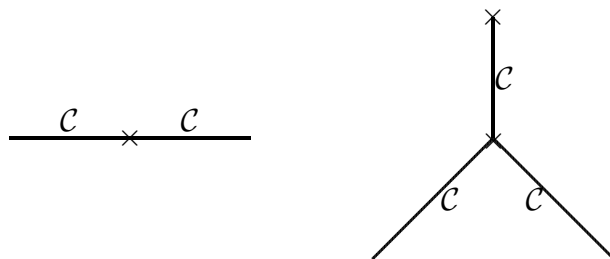
Figure 2. This diagram represents two other contributions, besides the one shown in figure 1, to the disk two point function of a pair of closed strings. The first diagram represents the two point closed string interaction vertex described by a disk amplitude with two closed strings, and the second diagram represents a Feynman diagram that joins the closed string three point vertex to a closed string one point vertex by a closed string propagator. Since in our analysis the second diagram will not give a divergent contribution, we shall include its contribution in the definition of the interaction vertex of two closed strings shown in the first diagram.

where z and $\tilde{z}$ are the coordinates on two unit disks and $w_1 = i\lambda(z_1 - z)/(z_1 + z)$, $\tilde{w}_1 = i\lambda(z_1 - \tilde{z})/(z_1 + \tilde{z})$ are the local coordinates around the boundary punctures on the two disks. The identification (3.2) joins the two disks into a single disk. If we parametrize this by the coordinate z, then the bulk punctures at $z = 0$ and $\tilde{z} = 0$ are situated at

$$z = 0, \qquad z = z_1(1 - u)/(1 + u). \tag{3.3}$$

Therefore as u varies from 0 to ϵ according to (3.2), the location of the second puncture varies between z_1 and $z_1(1 - \epsilon)/(1 + \epsilon)$. For large λ, this covers a small part of the moduli space around the degenerate configuration where the second bulk puncture is close to the boundary point z_1. The rest of the moduli space, where the second puncture lies on the line segment between 0 and $z_1(1 - \epsilon)/(1 + \epsilon)$, needs to be covered by a combination of two other Feynman diagrams shown in figure 2.

In string field theory the parameter q introduced in (3.2) appears as follows. An open string propagator contains a $1/L_0$ factor, which is expressed as

$$(L_0)^{-1} = \int_0^1 dq\, q^{-1+L_0} = \epsilon^{-L_0} \int_0^\epsilon du\, u^{-1+L_0}, \tag{3.4}$$

where u and ϵ are the variables introduced in (3.2). Upon substituting this into the expressions for the Feynman diagram we can recover the geometric picture where connecting a pair of interaction vertices by a propagator corresponds to sewing the surfaces via the relation (3.2). Note however that (3.4) is valid only for states with positive L_0 eigenvalue. For $L_0 < 0$, the right hand side diverges but the left hand side is finite. Therefore while using the right hand side to find a geometric interpretation of the amplitude is problematic, we can always use the left hand side. Equivalently, whenever we encounter an expression of the form given in the right hand side of (3.4), we use the replacement rule:

$$\int_0^\epsilon du\, u^{-1+\alpha} \quad \Rightarrow \quad \epsilon^\alpha\, \alpha^{-1}. \tag{3.5}$$

Note that this does not work for $\alpha = 0$. We shall return to this issue shortly.

As discussed in [10], as long as α is not an integer, the rule (3.5) is invariant under a change of variables $u \to v = a_1 u + a_2 u^2 + \cdots$ that is regular at $u = 0$. Therefore given a divergent integral, we can apply this rule blindly without knowing if the integration variable is actually the sewing parameter u introduced in (3.2). However for negative integer α, application of this rule using a different variable v could yield an expression that differs from the original expression by a constant. Therefore in this case we need to identify the variable u first before applying the rule.

In the case at hand, the intermediate open string in figure 1 has a tachyonic mode that gives a contribution to the integrand proportional to u^{-2}. The region of integration corresponding to $u > \epsilon$ comes from the diagrams in figure 2, but the region of integration corresponding to $u \leq \epsilon$ comes from figure 1. Now suppose we want to integrate out the open string modes to construct the Wilsonian effective action for the closed string fields. This would, in particular, require including the contribution from figure 1 into the definition of the interaction vertex in the Wilsonian effective action. For this we simply add the contribution from figure 1 after making the replacement (3.5) for the $\alpha = -1$ term representing open string tachyon exchange. This procedure cannot be applied to the $\alpha = 0$ term, representing the effect of the zero mode exchange in figure 1. This reflects the fact that in constructing the Wilsonian effective action we cannot integrate out the open string zero mode. We have to keep this unintegrated for now.

Next let us consider an interaction vertex with two external open strings and one external closed string, again associated with the disk amplitude. Using appropriate $SL(2, \mathbb{R})$ transformation we take the bulk puncture at the origin of the disk. The boundary punctures then lie at points z_1 and z_2 on the disk. We now need to choose local coordinates at the punctures. Let $w = \tilde{\beta} z$ describe the local coordinate around the bulk puncture, and

$$ w_a = i\tilde{\lambda}\frac{(z_a - z)}{(z_a + z)}, \quad a = 1, 2, \tag{3.6} $$

be the local coordinates around the boundary punctures. Here $\tilde{\beta}$ and $\tilde{\lambda}$ are some large positive constants that are *a priori* independent of the constants β and λ appearing in (3.1). The choice (3.6), although not unique, is consistent with various symmetry requirements, e.g. symmetry under the cyclic permutation of the two open strings (which in this case is simply implemented by overall rotation of the z-plane). The choice of $\tilde{\beta}$ will not affect any of our results since in our analysis the closed string inserted at the vertex will always be on-shell. The ambiguity mentioned in [1] is related to the freedom of choosing $\tilde{\lambda}$ to be independent of λ. As we shall now explain however, the choice $\tilde{\lambda} \neq \lambda$ is in conflict with another hidden assumption that was used in the analysis of [1].

For this let us consider the result of joining the open-open-closed vertex to an open-closed vertex via an open string propagator, as in figure 3. The Wilsonian effective action will contain contribution from this diagram, except the part where the internal open string state represents the zero mode. Now formally, using the representation (3.4) of the propagator, the total contribution from figure 3 can be represented as an integral over certain region of the moduli space of a disk with two closed string and one open string puncture. In computing its contribution to the Wilsonian effective action, we need to express the in-

Figure 3. A Feynman diagram with two external closed strings and one external open string. The right hand diagram is a representation of this as a disk amplitude, with the ∘'s denoting closed string vertex operator and the ⋆ on the boundary denoting open string vertex operator.

tegrand as a sum over intermediate open string states, and subtract the contribution of the internal ϕ mode from the integrand. Using analysis similar to those given in (3.2), (3.3), it is easy to see that the region of the moduli space associated with figure 3 corresponds to one bulk puncture being located at the origin, the second bulk puncture being located at $z_1(1-u)/(1+u)$ with $0 \le u \le (\lambda\tilde{\lambda})^{-1}$, and the boundary puncture being located at some point z_2. Let us now take the external open string state to be the open string zero mode ϕ. Using the fact that the vertex operator of the state ϕ is proportional to ∂X^0 where X^0 is the world-sheet scalar field associated with the time coordinate, it is easy to see that after integration over z_2, the contribution to the amplitude with an external open string zero mode field ϕ is given by $i\omega\phi$ times the amplitude where the external open string state is removed.[2] Here ω is the total energy carried by the external closed strings. However over certain subspace of the moduli space, where the second bulk puncture is located at $z_1(1-u)/(1+u)$ with $0 \le u \le (\lambda\tilde{\lambda})^{-1}$, the amplitude will still have missing internal ϕ state. On the other hand the contribution to the term in the Wilsonian effective action without any external ϕ insertion is given by a similar term, except that the subspace of the moduli space, over which the internal ϕ contribution is removed, corresponds to the second bulk puncture being located at $z_1(1-u)/(1+u)$ with $0 \le u \le \lambda^{-2}$ (see (3.2), (3.3)). If $\tilde{\lambda} = \lambda$, then these two subspaces are identical, and we can express the sum of the terms with and without external ϕ as $(1+i\omega\phi)$ times the term without external ϕ. This gives the first two terms in the expansion of $e^{i\omega\phi}$. Requiring that the terms with higher powers of ϕ add up to the result $\exp[i\omega\phi]$ would demand *using the same parameter λ in defining the local coordinates at the boundary punctures in the vertices with one closed and multiple open strings.*

Since [1] used $\exp[i\omega\phi]$ as the ϕ dependence of the Wilsonian effective action, we see that we do not have the freedom of choosing $\tilde{\lambda}$ independently of λ in (3.6). Instead we must choose:

$$\tilde{\lambda} = \lambda. \tag{3.7}$$

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[2]When z_2 approaches z_1 in the open-open-closed vertex, there are potential singularities and in the full string field theory we represent the contribution from this region by another Feynman diagram involving open-open-open vertex joined to an open-closed vertex. However for the term that we shall be analyzing, the contribution from this region will be suppressed by positive powers of ϵ. This is reflected in the absence of singularity in the integrand in (4.4) in the $x \to 0$ limit and has been discussed in footnote 5. Therefore for the analysis in this paper, we can include in the definition of open-open-closed vertex the entire moduli space of the disk with two boundary punctures and one bulk puncture, including the region where the boundary punctures are coincident.
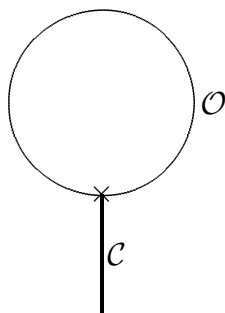
Figure 4. A contribution to the annulus amplitude with one external closed string.

We can now use this open-open-closed vertex to compute the contribution to the Wilsonian effective action with one external closed string, obtained by joining the two open strings of the open-open-closed vertex by a propagator, as shown in figure 4. Geometrically this describes an annulus amplitude with one external closed string. Naively, in order to evaluate this contribution, the u parameter associated with the sewing

$$w_1 w_2 = -q \equiv -u\lambda^2 \,, \tag{3.8}$$

has to be integrated from 0 to ϵ. However intermediate tachyons need special treatment, and we need to remove the contribution from the state ϕ in the internal open string propagator. Following our earlier discussion, this leads to the following prescription for computing the full Wilsonian effective action. In the expression for the full annulus amplitude, expressed as integrals over the independent moduli parameters u and z_2/z_1, we need to impose a lower cut-off $\epsilon = \lambda^{-2}$ on the u integral and add a compensating term using the replacement rule (3.5) for $\alpha = 1$. This would correspond to integrating out the tachyonic mode in figure 4, but leaving the zero mode unintegrated. Integration over the zero mode will have to be taken care of at the end.

To summarize, we need to augment the world-sheet analysis of [1] using the following procedure for regulating divergences:

1. When we encounter a divergence, we need to identify the degeneration responsible for the divergence, and change the integration variables to the analog of the parameter u appearing in sewing relations of type described in (3.2) and the moduli of the Riemann surfaces that are being sewed.

2. If near $u = 0$ the integrand has a term of the form $A \, du \, u^{-2}$ for some u independent A, we replace the lower limit of integration by ϵ, and subtract $A \, \epsilon^{-1}$ from the integral. The subtraction of $A \, \epsilon^{-1}$ represents the effect of 'integrating out' the tachyon field in the Wilsonian effective action.

3. If near $u = 0$ the integrand has a term of the form $B \, du \, u^{-1}$ for some u independent B, we replace the lower limit of u integration by ϵ without adding any compensating term. A term proportional to $du \, u^{-1}$ in the integrand represent the effect of zero mode exchange, and we cannot integrate out these modes.

4. Integration over the open string zero modes is performed at the end *after computing the desired Green's function/S-matrix*. This produces the energy conserving delta function.

4 Cancellation of the extra terms

In this section we shall implement the general principles described in section 3 and determine the constant that was left unfixed in the world-sheet analysis of [1]. We shall begin by describing the strategy that we shall follow.

The analysis of [1] involved addition of some extra terms to the result of the world-sheet computation on the string theory side. These terms were used to regulate the infrared divergences, and are given in the second, fourth and fifth lines of eq. (2.36) of [1]. They take the following form:

$$
e^{-1/g_s} g_s \, \delta(\omega - \omega') \, 8 \, \pi \, \mathcal{N} \left[- \frac{A}{16} \pi^{1/2} 2^{-1/4} \left(\Psi^{ZZ}(\omega/2) \right)^2 \int_0^1 dy \, y^{-2}(1 - 2\,\omega^2 y) \right.
$$
$$
- \frac{A}{4} \, 2^{3/4} \, \pi^{3/2} \left(\Psi^{ZZ}(\omega/2) \right)^2 \int_0^\infty dt \int_0^{1/4} dx \left(\frac{e^{2\pi t} - 1}{\sin^2(2\pi x)} + 2\omega^2 \right)
$$
$$
\left. - S_{ZZ}^{(1)} \sinh^2(\pi\omega) + c'\omega^2 \sinh^2 \pi\omega \right], \tag{4.1}
$$

where $\mathcal{N}$ is a normalization constant,

$$
\Psi^{ZZ}(\omega/2) = 2^{5/4} \sqrt{\pi} \, \sinh(\pi\omega), \quad A = 2^{3/4} \pi^{-3/2}, \tag{4.2}
$$

and $S_{ZZ}^{(1)}$ and c' are constants which were eventually adjusted to make the amplitude agree with the matrix model result. Since our goal will be to show that string theory results agree with the matrix model results without any ad hoc adjustment, we need to show that (4.1) vanishes after we use the regulator implied by string field theory.[3]

The terms inside the square bracket in (4.1) can be divided into two classes: the ones proportional to $\sinh^2(\pi\omega)$ and the ones proportional to $\omega^2 \sinh^2(\pi\omega)$. The leading term (not displayed here) is also proportional to $\sinh^2(\pi\omega)$. Therefore, renormalization of the D-instanton action, mentioned in the first paragraph of the introduction, will change the coefficient of the term proportional to $\sinh^2(\pi\omega)$. The coefficient of the $\omega^2 \sinh^2(\pi\omega)$ term cannot be changed this way, and so we focus on that term. Changing the integration variable t to

$$
v = e^{-2\pi t}, \tag{4.3}
$$

we can express the terms proportional to $\omega^2 \sinh^2(\pi\omega)$ inside the square bracket as:

$$
\left\{ \int_0^1 dy \, y^{-1} - 4 \int_0^{1/4} dx \int_0^1 dv \, v^{-1} + c' \right\} \omega^2 \sinh^2(\pi\omega). \tag{4.4}
$$

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[3]The logic goes as follows. Since the counterterms (4.1) have been chosen to cancel the divergences in the world-sheet result, if we regulate both the original world-sheet results and the counterterms following the prescription given by string field theory, the result remains unchanged, But now the regulated world-sheet integrals give the correct result. Therefore the regulated counterterms must add up to zero.

The first integral is divergent at $y = 0$ while the second integral is divergent at $v = 0$. Our goal now will be to identify the source of these divergences to appropriate degenerations of Riemann surfaces, find the relation between the integration variables in (4.4) and the sewing parameter u in (3.2) or (3.8) and possibly other moduli of the Riemann surfaces, and then translate the cut-off procedure on u prescribed by string field theory to a cut-off on y and v.

We begin with the y integral in the first term in (4.4). This term was chosen to remove the divergences in the two point function on the disk, with the closed string vertex operators inserted at i and iy in the upper half plane [1], which we shall label by w. If we denote by z the coordinate on the unit disk, related to w via $z = i\frac{i-w}{i+w}$, then in the z coordinate the punctures are at $z = 0$ and $z = i(1-y)/(1+y)$. Therefore the $y \to 0$ limit corresponds to one of the closed string vertex operators coming close to the boundary point $z = i$. Comparing this with (3.3) we see that this corresponds to the degeneration of an open string propagator joining two disks associated with the Feynman diagrams shown in figure 1, with the identification:

$$z_1 = i, \quad y = u. \tag{4.5}$$

Therefore the cut-off $u > \epsilon$ translates to $y > \epsilon$.

Next we turn to the divergence of the second term in (4.4) in the $v \to 0$ limit. In the analysis of [1] this comes from the one point function of a closed string state on the annulus, parametrized by coordinate w, satisfying,

$$0 \le Re(w) \le \pi, \quad w \sim w + 2\pi it = w - i \ln v, \tag{4.6}$$

with the bulk puncture located at $w = 2\pi x$. Here $\sim$ denotes identification of points under the given transformation. Our goal will be to compare this configuration with the one associated with the diagram shown in figure 4. The latter is parametrized by the sewing parameter u appearing in (3.8), and the ratio z_2/z_1, with z_1, z_2 labelling the locations of the boundary punctures on the disk in the open-open-closed interaction vertex. Once we have determined the relation between (x, v) and $(z_2/z_1, u)$, we can translate the cut-off $u > \epsilon$ into a cut-off on the (x, v) integration and carry out the integration over x and v.

We begin by recalling the choice of local coordinates around the two boundary punctures in the open-open-closed interaction vertex:[4]

$$w_a = i\lambda\frac{(z_a - z)}{(z_a + z)}, \quad a = 1, 2, \tag{4.7}$$

and the sewing relation

$$w_1 w_2 = -q = -\lambda^2 u \quad \Leftrightarrow \quad \frac{(z_1 - z)}{(z_1 + z)} \sim u\,\frac{(z_2 + z)}{(z_2 - z)}. \tag{4.8}$$

We now introduce the coordinate

$$\tilde{w} = i\frac{z_1 - z}{z_1 + z}, \tag{4.9}$$

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[4]If we had taken $\tilde{\lambda}$ in (3.6) to be different from λ, the λ in (4.7) will change to $\tilde{\lambda}$. Subsequent equations will change accordingly, replacing ϵ by $\tilde{\epsilon} \equiv \tilde{\lambda}^{-2}$ in (4.22).

taking value in the upper half plane, and define

$$e^{2i\theta} \equiv \frac{z_2}{z_1} \,. \tag{4.10}$$

In the $\tilde{w}$ coordinate the sewing relation (4.8) takes the form

$$\tilde{w} \sim \frac{u^{1/2} \sin\theta \, \tilde{w} + u^{1/2} \cos\theta}{-u^{-1/2} \cos\theta \, \tilde{w} + u^{-1/2} \sin\theta} \,. \tag{4.11}$$

For small u, the right hand side of (4.11) is an $\mathrm{SL}(2, \mathbb{R})$ transformation with a hyperbolic element. We can diagonalize this by an $\mathrm{SL}(2, \mathbb{R})$ transformation $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right)$ such that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \begin{pmatrix} u^{1/2} \sin\theta & u^{1/2} \cos\theta \\ -u^{-1/2} \cos\theta & u^{-1/2} \sin\theta \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \beta^{1/2} & 0 \\ 0 & \beta^{-1/2} \end{pmatrix}, \tag{4.12}$$

so that in the new coordinate system $\hat{w}$, related to $\tilde{w}$ via

$$\tilde{w} = \frac{a \, \hat{w} + b}{c \, \hat{w} + d} \,, \tag{4.13}$$

the identification (4.11) takes the form

$$\hat{w} \sim \beta \, \hat{w} \,. \tag{4.14}$$

We shall not give the explicit form of β and $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right)$ for general u, since we only need this for small u. In the small u limit,

$$\beta = u^{-1} \sin^2\theta, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & \cot\theta \end{pmatrix}. \tag{4.15}$$

This gives

$$\tilde{w} = \frac{1}{-\hat{w} + \cot\theta}, \quad \hat{w} = \frac{\cot\theta \, \tilde{w} - 1}{\tilde{w}} \,. \tag{4.16}$$

Finally we map the upper half plane spanned by $\hat{w}$ to the strip spanned by the coordinate w, defined as:

$$w = \frac{1}{i} \ln \hat{w} \,. \tag{4.17}$$

Since $\hat{w}$ takes value in the upper half plane, we get

$$0 \leq \mathrm{Re}(w) \leq \pi \,. \tag{4.18}$$

Furthermore the identification (4.14) with β defined in (4.15) gives

$$w \sim w + i \ln\beta = w + i \ln(u^{-1} \sin^2\theta) \,. \tag{4.19}$$

Comparing this with (4.6) we get

$$v = u / \sin^2\theta \,. \tag{4.20}$$

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It remains to locate the position of the bulk puncture in the w plane since its real part gives the value of $2\pi x$, — due to translational invariance of the annulas, $\mathrm{Im}\, w$ does not carry any physical information. Since in the z-plane the bulk puncture is located at $z = 0$, we see from (4.9) that in the $\tilde{w}$ plane it is at $\tilde{w} = i$. (4.16) now gives its location at $\hat{w} = \cot\theta + i = e^{i\theta}/\sin\theta$ and therefore at $w = \theta + i\ln\sin\theta$. Since $2\pi x$ is the real part of w, we get

$$2\,\pi\,x = \theta\,. \tag{4.21}$$

(4.20) and (4.21) now shows that the cut-off $u \geq \epsilon$ can be translated to[5]

$$v \geq \epsilon/\sin^2(2\pi x)\,. \tag{4.22}$$

Using (4.22), and the earlier result that the cut-off on the y integral is $y \geq \epsilon$, we can express (4.4) as

$$\left\{\int_\epsilon^1 dy\, y^{-1} - 4\int_0^{1/4} dx \int_{\epsilon/\sin^2(2\pi x)}^1 dv\, v^{-1} + c'\right\} \omega^2 \sinh^2(\pi\omega)$$
$$= \left(c' - 4\int_0^{1/4} dx\, \ln\left(\sin^2(2\pi x)\right)\right) \omega^2 \sinh^2(\pi\omega)$$
$$= \left(c' + \ln 4\right)\omega^2 \sinh^2(\pi\omega)\,. \tag{4.23}$$

Demanding that this ad hoc term vanishes, gives

$$c' = -\ln 4 \simeq -1.38629\,. \tag{4.24}$$

This is within 1% of the numerical value -1.399 determined in [1] by comparing the string theory results with the matrix model results.

Acknowledgments

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[5]Eq. (4.22) is valid only for $\sin^2(2\pi x) \gg \epsilon$ since in arriving at (4.20) we have used the approximation $u \ll \sin^2\theta$. Furthermore (4.11) ceases to represent a hyperbolic element of $SL(2, \mathbb{R})$ for $(u^{1/2} + u^{-1/2})\sin\theta \leq 1$. This is related to the issues discussed in footnote 2. However since the contribution to the integral from the region $\sin^2(2\pi x) \sim \epsilon$ is suppressed by powers of ϵ, we ignore this complication.

References

[1] B. Balthazar, V.A. Rodriguez and X. Yin, *ZZ Instantons and the Non-Perturbative Dual of c = 1 String Theory*, arXiv:1907.07688 [INSPIRE].

[2] S.R. Das and A. Jevicki, *String Field Theory and Physical Interpretation of D = 1 Strings*, *Mod. Phys. Lett.* **A 5** (1990) 1639 [INSPIRE].

[3] A.M. Sengupta and S.R. Wadia, *Excitations and interactions in d = 1 string theory*, *Int. J. Mod. Phys.* **A 6** (1991) 1961 [INSPIRE].

[4] D.J. Gross and I.R. Klebanov, *Fermionic string field theory of c = 1 two-dimensional quantum gravity*, *Nucl. Phys.* **B 352** (1991) 671 [INSPIRE].

[5] I.R. Klebanov, *String theory in two-dimensions*, in *Spring School on String Theory and Quantum Gravity (to be followed by Workshop)*, Trieste, Italy, 15–23 April 1991, pp. 30–101 (1991) [hep-th/9108019] [INSPIRE].

[6] B. Zwiebach, *Oriented open-closed string theory revisited*, *Annals Phys.* **267** (1998) 193 [hep-th/9705241] [INSPIRE].

[7] A. Sen, *Wilsonian Effective Action of Superstring Theory*, *JHEP* **01** (2017) 108 [arXiv:1609.00459] [INSPIRE].

[8] H. Hata and B. Zwiebach, *Developing the covariant Batalin-Vilkovisky approach to string theory*, *Annals Phys.* **229** (1994) 177 [hep-th/9301097] [INSPIRE].

[9] P.V. Larocca and C. Maccaferri, *BCFT and OSFT moduli: an exact perturbative comparison*, *Eur. Phys. J.* **C 77** (2017) 806 [arXiv:1702.06489] [INSPIRE].

[10] A. Sen, *String Field Theory as World-sheet UV Regulator*, *JHEP* **10** (2019) 119 [arXiv:1902.00263] [INSPIRE].