Flag-transitive non-symmetric 2-designs with $(r, \lambda) = 1$ and alternating socle

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Abstract

This paper deals with flag-transitive non-symmetric 2-designs with $(r, \lambda) = 1$. We prove that if $\mathcal{D}$ is a non-trivial non-symmetric 2-(v, k, λ) design with $(r, \lambda) = 1$ and $G \leqslant Aut(\mathcal{D})$ is flag-transitive with $Soc(G) = A_n$ for $n \geqslant 5$, then $\mathcal{D}$ is a 2-$(6, 3, 2)$ design, the projective space $PG(3, 2)$, or a 2-$(10, 6, 5)$ design.

Keywords: non-symmetric design; automorphism group; flag-transitive; alternating group

1 Introduction

This paper is inspired by a paper of P. H. Zieschang [17] on flag-transitive 2-designs with $(r, \lambda) = 1$. He proved in [17, Theorem] that if G is a flag-transitive automorphism group of a 2-design with $(r, \lambda) = 1$ and T is a minimal normal subgroup of G, then T is abelian, or simple and $C_G(T) = 1$. From [6, 2.3.7(a)](see also Lemma 3 below) we know that if $G \leqslant Aut(\mathcal{D})$ is flag-transitive with $(r, \lambda) = 1$ then G acts primitively on P. It follows that G is an affine or almost simple group. So it is possible to classify this type of designs by using the classification of finite primitive permutation groups, especially for the case of G is almost simple, with alternating socle. A 2-design with $\lambda = 1$ is also called a finite linear space. In 2001, A. Delandtsheer [5] classified flag-transitive finite linear spaces, with alternating socle, see Lemma 4 below. Recently, in [16], Zhu, Guan and Zhou have classified flag-transitive symmetric designs with $(r, \lambda) = 1$ and alternating socle. This paper is a continuation of [5, 16] and a contribution to the case where $\mathcal{D}$ is a non-symmetric 2-design.

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A 2-(v, k, λ) *design* $\mathcal{D}$ is a pair $(P, \mathcal{B})$ where P is a v-set and $\mathcal{B}$ is a collection of b k-subsets (called blocks) of P such that each point of P is contained in exactly r blocks and any 2-subset of P is contained in exactly λ blocks. The numbers v, b, r, k, λ are parameters of the design. It is well known that

$$bk = vr,$$
$$b \geqslant v,$$

and so

$$r \geqslant k.$$

The complement $\overline{\mathcal{D}}$ of a 2-(v, k, λ) design $\mathcal{D} = (P, \mathcal{B})$ is a 2-$(v, v-k, b-2r+\lambda)$ design $(P, \overline{\mathcal{B}})$, where $\overline{\mathcal{B}} = \{P \setminus B \,|\, B \in \mathcal{B}\}$. A 2-$(v, k, \lambda)$ design is symmetric if $b = v$ (or equivalently, $r = k$), otherwise is non-symmetric. This paper deals only with non-trivial non-symmetric designs, those with $2 < k < v - 1$. So that we have

$$b > v \text{ and } r > k.$$

An *automorphism* of $\mathcal{D}$ is a permutation of P which leaves $\mathcal{B}$ invariant. The full automorphism group of $\mathcal{D}$, denoted by $Aut(\mathcal{D})$, is the group consisting of all automorphisms of $\mathcal{D}$. A $flag$ of $\mathcal{D}$ is a point-block pair (x, B) such that $x \in B$. For $G \leqslant Aut(\mathcal{D})$, G is called *flag-transitive* if G acts transitively on the set of flags, and *point-primitive* if G acts primitively on P. A design $\mathcal{D}$ is antiflag transitive if $G \leqslant Aut(\mathcal{D})$ acts transitively on the set $\{(x, B) \,|\, x \notin B\} \subseteq P \times \mathcal{B}$ of antiflags of $\mathcal{D}$. It is easily known that $G \leqslant Aut(\mathcal{D})$ is flag-transitive on $\mathcal{D}$ if and only if G is antiflag transitive on $\overline{\mathcal{D}}$.

Flag-transitivity is one of many conditions that can be imposed on the automorphism group G of a design $\mathcal{D}$. Lots of work have been done on flag-transitive symmetric 2-designs, see [11, 12, 13, 15], for example. Although there exists large families of non-symmetric 2-designs, less is known when $\mathcal{D}$ is non-symmetric admitting a flag-transitive automorphism group.

The aim of this paper is to classify the flag-transitive non-symmetric 2-designs with $(r, \lambda) = 1$, whose automorphism group is almost simple with an alternating group as socle. This can be viewed as a first step towards a classification of non-symmetric 2-designs with $(r, \lambda) = 1$. The main result of this paper is the following.

Theorem 1. *Let $\mathcal{D}$ be a non-symmetric 2-(v, k, λ) design with $(r, \lambda) = 1$, where r is the number of blocks through a point. If $G \leqslant Aut(\mathcal{D})$ is flag-transitive with alternating socle, then up to isomorphism $(\mathcal{D}, G)$ is one of the following:*

(i) *$\mathcal{D}$ is a unique 2-$(15, 3, 1)$ design and $G = A_7$ or A_8.*

(ii) *$\mathcal{D}$ is a unique 2-$(6, 3, 2)$ design and $G = A_5$.*

(iii) *$\mathcal{D}$ is a unique 2-$(10, 6, 5)$ design and $G = A_6$ or S_6.*

This, together with [16, Theorem 1.1], yields the following.

Corollary 2. *If $\mathcal{D}$ is a 2-(v,k,λ) design with $(r,\lambda)=1$, which admits a flag-transitive automorphism group G with alternating socle, then $\mathcal{D}$ is a 2-$(6,3,2)$ design, a 2-$(10,6,5)$ design, the projective space $PG(3,2)$ or $PG_2(3,2)$.*

The paper is organized as follows. In Section 2, we introduce some preliminary results that are important for the remainder of the paper. In Section 3, we complete the proof of Theorem 1 in three parts.

2 Preliminaries

Lemma 3. ([6, 2.3.7(a)]) *Let $\mathcal{D}$ be a 2-(v,k,λ) design with flag-transitive automorphism group G. If $(r,\lambda)=1$ then G is point-primitive.*

The following result due to A. Delandtsheer [5] gives the classification of flag-transitive finite linear spaces with alternating socle.

Lemma 4. *Let $\mathcal{S}$ be a finite non-trivial linear space having an automorphism group G which acts flag-transitively on $\mathcal{S}$. If $A_n \trianglelefteq G \leqslant Aut(A_n)$ with $n \geqslant 5$, then $\mathcal{S}=PG(3,2)$ and $G \cong A_7$ or $A_8 \cong PSL_4(2)$.*

Lemma 5. *Let $\mathcal{D}$ be a 2-(v,k,λ) design. Then*

(i) *$bk(k-1)=\lambda v(v-1)$.*

(ii) *$r=\frac{\lambda(v-1)}{k-1}$. In particular, if $(r,\lambda)=1$ then $r \mid v-1$ and $(r,v)=1$.*

Proof. Counting in two ways triples (α,β,B), where α and β are distinct points and B is a block incident with both of them, gives (i). Part (ii) follows from the basic equation $bk=vr$ and Part (i). $\square$

Lemma 6. *If $\mathcal{D}$ is a non-symmetric 2-(v,k,λ) design and G is a flag-transitive point-primitive automorphism group of $\mathcal{D}$, then*

(i) *$r^2>\lambda v$, and $|G_x|^3>\lambda|G|$, where $x \in P$;*

(ii) *$r \mid \lambda d_i$, where d_i is any subdegree of G. Furthermore, if $(r,\lambda)=1$ then $r \mid d_i$.*

Proof. (i) The equality $r=\frac{\lambda(v-1)}{k-1}$ implies $\lambda v = r(k-1)+\lambda < r(r-1)+\lambda = r^2-r+\lambda$, and by $r>k>\lambda$ we have $r^2>\lambda v$. Combining this with $v=|G:G_x|$ and $r \leqslant |G_x|$ gives $|G_x|^3>\lambda|G|$. Part (ii) was proved in [4]. $\square$

Lemma 7. ([9, p.366]) *If G is A_n or S_n, acting on a set Ω of size n, and H is any maximal subgroup of G with $H \neq A_n$, then H satisfies one of the following:*

(i) *$H=(S_k \times S_\ell)\cap G$, with $n=k+\ell$ and $k \neq \ell$ (intransitive case);*

(ii) *$H=(S_k \wr S_\ell)\cap G$, with $n=k\ell$, $k>1$ and $\ell>1$ (imprimitive case);*

(iii) *$H = AGL_k(p) \cap G$, with $n = p^k$ and p prime (affine case);*

(iv) *$H = (T^k.(OutT \times S_k)) \cap G$, with T a nonabelian simple group, $k \geqslant 2$ and $n = |T|^{k-1}$ (diagonal case);*

(v) *$H = (S_k \wr S_\ell) \cap G$, with $n = k^\ell$, $k \geqslant 5$ and $\ell > 1$ (wreath case);*

(vi) *$T \trianglelefteq H \leqslant Aut(T)$, with T a nonabelian simple group, $T \neq A_n$ and H acting primitively on Ω (almost simple case).*

Remark 8. This lemma does not deal with the groups M_{10}, $PGL_2(9)$ and $P\Gamma L_2(9)$ that have A_6 as socle. These exceptional cases will be handled in the first part of Section 3.

Lemma 9. [10, Theorem (b)(I)] *Let G be a primitive permutation group of odd degree n on a set Ω with simple socle $X := Soc(G)$, and let $H = G_x$, $x \in \Omega$. If $X \cong A_c$, an alternating group, then one of the following holds:*

(i) *H is intransitive, and $H = (S_a \times S_{c-a}) \cap G$ where $1 \leqslant a < \frac{1}{2}c$;*

(ii) *H is transitive and imprimitive, and $H = (S_a \wr S_{c/a}) \cap G$ where $a > 1$ and $a \mid c$;*

(iii) *H is primitive, $n = 15$ and $G \cong A_7$.*

Lemma 10. [7, Theorem 5.2A] *Let $G := Alt(\Omega)$ where $n := |\Omega| \geqslant 5$, and let s be an integer with $1 \leqslant s \leqslant \frac{n}{2}$. Suppose that, $K \leqslant G$ has index $|G : K| < \binom{n}{s}$. Then one of the following holds:*

(i) *For some $\Delta \subset \Omega$ with $|\Delta| < s$ we have $G_{(\Delta)} \leqslant K \leqslant G_{\{\Delta\}}$;*

(ii) *$n = 2m$ is even, K is imprimitive with two blocks of size m, and $|G : K| = \frac{1}{2}\binom{n}{m}$; or*

(iii) *one of six exceptional cases hold where:*

  (a) *K is imprimitive on Ω and $(n, s, |G : K|) = (6, 3, 15)$;*

  (b) *K is primitive on Ω and $(n, s, |G : K|, K) = (5, 2, 6, 5 : 2)$, $(6, 2, 6, PSL_2(5))$, $(7, 2, 15, PSL_3(2))$, $(8, 2, 15, AGL_3(2))$, or $(9, 4, 120, P\Gamma L_2(8))$.*

Remark 11. (1) From part (i) of Lemma 10 we know that K contains the alternating group $G_{(\Delta)} = Alt(\Omega \setminus \Delta)$ of degree $n - s + 1$.

(2) A result similars to Lemma 10 holds for the finite symmetric groups $Sym(\Omega)$ which can be found in [7, Theorem 5.2B].

We will also need some elementary inequalities.

Lemma 12. *Let s and t be two positive integers.*

(i) *If $t > s \geqslant 7$, then $\binom{s+t}{s} > t^4 > s^2t^2$.*

(ii) *If* $s \geqslant 6$ *and* $t \geqslant 2$*, then* $2^{(s-1)(t-1)} > s^4\binom{t}{2}^2$ *implies* $2^{s(t-1)} > (s+1)^4\binom{t}{2}^2$.

(iii) *If* $t \geqslant 6$ *and* $s \geqslant 2$*, then* $2^{(s-1)(t-1)} > s^4\binom{t}{2}^2$ *implies* $2^{(s-1)t} > s^4\binom{t+1}{2}^2$.

(iv) *If* $t \geqslant 4$ *and* $s \geqslant 3$*, then* $\binom{s+t}{s} > s^2t^2$ *implies* $\binom{s+t+1}{s} > s^2(t+1)^2$.

Proof. (i) It is necessary to prove that $\binom{s+t}{s} > t^4$ holds. Since $t > s \geqslant 7$ then $7 \leqslant s \leqslant [\frac{s+t}{2}]$, it follows that $\binom{s+t}{s} \geqslant \binom{t+7}{7} > t^4$.

(ii) Suppose that $s \geqslant 6, t \geqslant 2$ and $2^{(s-1)(t-1)} > s^4\binom{t}{2}^2$. Then

$$2^{s(t-1)} = 2^{(s-1)(t-1)}2^{t-1} > s^4\binom{t}{2}^2 2^{t-1} = (s+1)^4\binom{t}{2}^2\Big(1-\frac{1}{s+1}\Big)^4 2^{t-1}.$$

Combing this with the fact $(1-\frac{1}{s+1})^4 2^{t-1} \geqslant 2 \times (\frac{6}{7})^4 > 1$ gives (ii).

(iii) Suppose that $t \geqslant 6, s \geqslant 2$ and $2^{(s-1)(t-1)} > s^4\binom{t}{2}^2$. Then

$$2^{(s-1)t} = 2^{(s-1)(t-1)}2^{s-1} > s^4\binom{t}{2}^2 2^{s-1} = s^4\binom{t+1}{2}^2\Big(1-\frac{2}{t+1}\Big)^2 2^{s-1}.$$

Combing this with the fact $(1-\frac{2}{t+1})^2 2^{s-1} \geqslant 2 \times (\frac{5}{7})^2 > 1$ gives (iii).

(iv) Suppose that $\binom{s+t}{s} > s^2t^2$. Then

$$\binom{s+t+1}{s} = \binom{s+t}{s}\frac{(s+t+1)}{(t+1)} > s^2t^2\frac{(s+t+1)}{(t+1)} = s^2(t+1)^2\frac{(s+t+1)t^2}{(t+1)^3}.$$

This, together with the inequality $(s+t+1)t^2 > (t+1)^3$, yields (iv). □

3 Proof of Theorem 1

Throughout this paper, we assume that the following hypothesis holds.

HYPOTHESIS: Let $\mathcal{D}$ be a non-symmetric 2-(v,k,λ) design with $(r,\lambda) = 1$, $G \leqslant Aut(\mathcal{D})$ be a flag-transitive automorphism group G with $Soc(G) = A_n$. Let x be a point of P and $H = G_x$.

By Lemma 3, G acts primitively on P. So that H is a maximal subgroup of G by [14, Theorem 8.2] and $v = |G : H|$. Furthermore, by the flag-transitivity of G, we have that b divides G, r divides $|H|$, and $r^2 > v$ by Lemma 6(i).

Suppose first that $n = 6$ and $G \cong M_{10}$, $PGL_2(9)$ or $P\Gamma L_2(9)$. Each of these groups has exactly three maximal subgroups with index greater than 2, and their indices are precisely 45, 36 and 10. By using the computer algebra system GAP [8], for $v = 45$, 36 or 10, we will compute the parameters (v, b, r, k, λ) that satisfy the following conditions:

$$r \mid v-1; \tag{1}$$

$$2 < k < r; \tag{2}$$

$$b = \frac{vr}{k}; \tag{3}$$

$$\lambda = \frac{bk(k-1)}{v(v-1)}; \tag{4}$$

$$(r, \lambda) = 1; \tag{5}$$

$$r \mid |H|. \tag{6}$$

We obtain three possible parameters (v, b, r, k, λ) as follows:

$$(10, 30, 9, 3, 2); \quad (10, 18, 9, 5, 4); \quad (10, 15, 9, 6, 5).$$

Now we consider the existence of flag-transitive non-symmetric designs with above possible parameters. For the parameters $(10, 15, 9, 6, 5)$, we will consider the existence of its complement design which has parameters $(10, 15, 6, 4, 2)$.

Suppose that there exists a 2-$(10, k, \lambda)$ design $\mathcal{D}$ with flag-transitive automorphism group G, where the block size k is $3, 5$ or 4, and $\lambda = 2, 4$ or 2 respectively. Here $v = 10$. Let $P = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, the group $G \cong M_{10}$, $PGL_2(9)$ or $P\Gamma L_2(9)$ as the primitive permutation group of degree 10 acting on P, has the following generators respectively ([2, p.828]):

$$\begin{aligned}
M_{10} &\cong \langle (1, 6, 10, 9, 3, 8, 4, 5)(2, 7), (1, 7, 2, 6, 5, 9, 4, 10)(3, 8) \rangle, \\
PGL_2(9) &\cong \langle (1, 2, 3, 4, 5, 6, 7, 8, 9, 10), (1, 2, 5, 8, 9, 7, 4, 10, 6, 3) \rangle, \\
P\Gamma L_2(9) &\cong \langle (1, 2, 3, 4, 5, 6, 7, 8, 9, 10), (1, 8, 6, 2, 9, 5, 3, 10)(4, 7) \rangle.
\end{aligned}$$

There are totally $\binom{v}{k}$ k-element subsets of P. For any k-element subset $B \subseteq P$, we calculate the length of the G-orbit B^G where $G = M_{10}$, $PGL_2(9)$ or $P\Gamma L_2(9)$ respectively. By using GAP [8], we found that $|B^G| > b$ for any k-element subset B. So G cannot act block-transitively on $\mathcal{D}$, a contradiction.

Now we consider $G = A_n$ or S_n with $n \geqslant 5$. The point stabilizer $H = G_x$ acts both on P and the set $\Omega_n := \{1, 2, \cdots, n\}$. Then by Lemma 7 one of the following holds:

- H is primitive in its action on Ω_n;
- H is transitive and imprimitive in its action on Ω_n;
- H is intransitive in its action on Ω_n.

The proof of Theorem 1 consists of three subsections.

3.1 H acts primitively on Ω_n.

Proposition 13. *Let $\mathcal{D}$ and G satisfy* HYPOTHESIS. *Let the point stabilizer act primitively on Ω_n. Then $\mathcal{D}$ is a* 2-$(6, 3, 2)$ *design or the projective space* $PG(3, 2)$.

Proof. Suppose first that r is even, since $r \mid v-1$ then v is odd. Thus by Lemma 9, $v = 15$ and $G = A_7$, and then $r = 14$. The possible parameters (v, b, r, k, λ) such that $2 < k < r$ and $(r, \lambda) = 1$ are $(15, 35, 14, 6, 5)$, $(15, 21, 14, 10, 9)$.

If there is a design $\mathcal{D}$ with parameters $(15, 21, 14, 10, 9)$, then the complement design $\overline{\mathcal{D}}$ with parameters $(15, 21, 7, 5, 2)$ also exists. However, by [3, Theorem 5.2], we know that 2-$(15, 5, 2)$ design does not exist. So the parameters $(15, 21, 14, 10, 9)$ cannot occur.

Now assume that $(v, b, r, k, \lambda) = (15, 35, 14, 6, 5)$. Let $P = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15\}$. Suppose that there exists a 2-$(15, 6, 5)$ design $\mathcal{D}$ with flag-transitive automorphism group A_7. By [2, p.829], we know

$$A_7 \cong \langle (1, 4, 7, 10, 13)(2, 5, 8, 11, 14)(3, 6, 9, 12, 15), (2, 6, 3, 8, 12, 4, 9)(5, 7, 13, 11, 10, 14, 15) \rangle.$$

There are totally $5005 = \binom{15}{6}$ 6-element subsets of P. For any 6-element subset $B \subseteq P$, using GAP [8], we calculate the length of the A_7-orbit B^{A_7}. It follows that $|B^{A_7}| > 35$ for any 6-element subset B. So A_7 cannot act block-transitively on $\mathcal{D}$, a contradiction.

Then r is odd. Let p be an odd prime divisor of r, then $(p, v) = 1$ according to Lemma 5(ii). Thus H contains a Sylow p-subgroup R of G. Let $g \in G$ be a p-cycle, then there is a conjugate of g belongs to H. This implies that H acting on Ω_n contains an even permutation with exactly one cycle of length p and $n-p$ fixed points. By a result of Jordan [14, Theorem 13.9], we have $n - p \leqslant 2$. Therefore $n - 2 \leqslant p \leqslant n$, $p^2 \nmid |G|$, and so $p^2 \nmid r$. It follows that r is either a prime, namely $n-2, n-1, n$, or the product of two twin primes, namely $r = (n-2)n$. Moreover, since the primitivity of H acting on Ω_n and $H \ngeqslant A_n$ implies that $v \geqslant \frac{[\frac{n+1}{2}]!}{2}$ by [14, Theorem 14.2], combining this with $r^2 > v$ gives

$$r^2 > \frac{[\frac{n+1}{2}]!}{2}.$$

Therefore, $(n, r) = (5, 5), (5, 15), (6, 5), (7, 5), (7, 7), (7, 35), (8, 7)$ or (13,143). From Lemmas 5, 6, the facts $v \geqslant \frac{[\frac{n+1}{2}]!}{2}$ and $[b, v] \mid |G|$, where the condition $[b, v] \mid |G|$ is a consequence of $v \mid |G|$ and $b \mid |G|$, we obtain 3 possible parameters (v, b, r, k, λ) which listed in the following:

$$(6, 10, 5, 3, 2), (15, 21, 7, 5, 2), (15, 35, 7, 3, 1).$$

Case (1): $(v, b, r, k, \lambda) = (6, 10, 5, 3, 2)$. By [2, p.27, p.36], we know that there is up to isomorphism a unique 2-$(6, 3, 2)$ design $\mathcal{D} = (P, \mathcal{B})$ where

$$\begin{aligned} P = \ & \{1, 2, 3, 4, 5, 6\}; \\ \mathcal{B} = \ & \{\{1, 2, 3\}, \{1, 2, 5\}, \{1, 3, 4\}, \{1, 4, 6\}, \{1, 5, 6\}, \\ & \{2, 3, 6\}, \{2, 4, 5\}, \{2, 4, 6\}, \{3, 4, 5\}, \{3, 5, 6\}\}. \end{aligned}$$

Here $G = A_5, S_5, A_6$ or S_6, the stabilizer $G_x = D_{10}, AGL_1(5), A_5$ or S_5 respectively. Assume first that $G = S_5$, A_6 or S_6. As the primitive permutation group of degree 6, S_5, A_6 or S_6 is 3-, 4- or 6-transitive respectively, so G is 3-transitive. If we choose $B = \{1, 2, 3\}$, then $|B^G| = 20 > b = 10$, a contradiction.

Hence $G = A_5$. Without loss of generality, assume that $A_5 = \langle(24)(56), (123)(456)\rangle$. Let $B = \{1, 2, 3\}$ be a block. Then $B^{A_5} = \mathcal{B}$. Now $G_1 = \langle(24)(56), (35)(46)\rangle \cong D_{10}$, and

$$B^{G_1} = \{\{1, 2, 3\}, \{1, 2, 5\}, \{1, 3, 4\}, \{1, 4, 6\}, \{1, 5, 6\}\},$$

which is the G_1-orbit on $\mathcal{B}$ containing B. So that G_1 is transitive on 5 blocks through 1, note that G is also transitive on P, and hence $\mathcal{D}$ is flag-transitive.

Case (2): $(v, b, r, k, \lambda) = (15, 21, 7, 5, 2)$. This can be ruled out by [3, Theorem 5.2].

Case (3): $(v, b, r, k, \lambda) = (15, 35, 7, 3, 1)$. Here $\lambda = 1$, by [5] or Lemma 4, we know that $\mathcal{D}$ is the projective space $PG(3, 2)$ and $G = A_7$ or $A_8 \cong PSL_4(2)$. For completeness, the structure of the design and the proof of flag-transitivity are given below.

Let $P = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15\}$, the group $G \cong A_7$ or A_8, as the primitive group of degree 15 acting on P, has the following generators respectively ([2, p.829]):

$$\begin{aligned} A_7 &\cong \langle(1, 4, 7, 10, 13)(2, 5, 8, 11, 14)(3, 6, 9, 12, 15), (2, 6, 3, 8, 12, 4, 9)(5, 7, 13, 11, 10, 14, 15)\rangle, \\ A_8 &\cong \langle(1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15), (1, 5)(6, 13)(7, 8)(10, 12)\rangle. \end{aligned}$$

There are totally $455 = \binom{15}{3}$ 3-element subsets of P. For any 3-element subset $B \subseteq P$, using GAP [8], we calculate the length of the G-orbit B^G. It follows that up to isomorphism there exists a unique 2-$(15, 3, 1)$ design $\mathcal{D} = (P, \mathcal{B})$ where

$$\begin{aligned} \mathcal{B} &= \{\{1, 2, 13\}, \{1, 4, 5\}, \{1, 6, 11\}, \{4, 7, 8\}, \{1, 7, 9\}, \{4, 9, 14\}, \{1, 3, 10\}, \\ &\{7, 10, 11\}, \{9, 12, 13\}, \{4, 10, 12\}, \{2, 7, 12\}, \{2, 9, 15\}, \{4, 6, 13\}, \{1, 8, 14\}, \\ &\{10, 13, 14\}, \{1, 12, 15\}, \{2, 4, 11\}, \{7, 13, 15\}, \{5, 10, 15\}, \{3, 5, 12\}, \{2, 5, 6\}, \\ &\{3, 9, 11\}, \{11, 14, 15\}, \{3, 4, 15\}, \{5, 7, 14\}, \{6, 9, 10\}, \{5, 11, 13\}, \{3, 8, 13\}, \\ &\{6, 8, 15\}, \{5, 8, 9\}, \{3, 6, 7\}, \{6, 12, 14\}, \{2, 8, 10\}, \{2, 3, 14\}, \{8, 11, 12\}\}. \end{aligned}$$

Let $B = \{1, 2, 13\}$ be a block. Then it is easily known that $B^G = \mathcal{B}$. Now

$$\begin{aligned} G_1 = &\langle(2, 6, 3, 8, 12, 4, 9)(5, 7, 13, 11, 10, 14, 15), (3, 8)(4, 11)(5, 6)(7, 9)(10, 14)(12, 15)\rangle \\ &\cong PSL_3(2) \end{aligned}$$

or

$$G_1 = \langle(2, 5, 8, 3, 15, 11, 7)(4, 14, 10, 12, 6, 9, 13), (3, 14)(7, 12)(8, 10)(9, 15)\rangle \cong AGL_3(2)$$

with $G = A_7$ or A_8 respectively. Then B^{G_1} containing 7 blocks: $\{1, 2, 13\}$, $\{1, 6, 11\}$, $\{1, 3, 10\}$, $\{1, 4, 5\}$, $\{1, 8, 14\}$, $\{1, 7, 9\}$, $\{1, 12, 15\}$. So that G_1 is transitive on r blocks through 1, note that G is also transitive on P, and hence G is flag-transitive and $\mathcal{D} = PG(3, 2)$. □

3.2 H acts transitively and imprimitively on Ω_n.

Proposition 14. *Let $\mathcal{D}$ and G satisfy* HYPOTHESIS. *Let the point stabilizer acts transitively and imprimitively on Ω_n, then $\mathcal{D}$ is a 2-$(10, 6, 5)$ design with $Soc(G) = A_6$.*

Proof. Suppose on the contrary that $\Sigma := \{\triangle_0, \triangle_1, \ldots, \triangle_{t-1}\}$ is a nontrivial partition of Ω_n preserved by H, where $|\triangle_i| = s, 0 \leqslant i \leqslant t-1, s, t \geqslant 2$ and $st = n$. Then

$$\begin{aligned} v &= \frac{\binom{ts}{s}\binom{(t-1)s}{s}\cdots\binom{3s}{s}\binom{2s}{s}}{t!} \\ &= \binom{ts-1}{s-1}\binom{(t-1)s-1}{s-1}\cdots\binom{3s-1}{s-1}\binom{2s-1}{s-1}. \end{aligned} \tag{7}$$

Moreover, the set O_j of j-cyclic partitions with respect to X (a partition of Ω_n into t classes each of size s) is an union of orbits of H on P for $j = 2, \ldots, t$ (see [5, 15] for definitions and details).

(1) Suppose first that $s = 2$, then $t \geqslant 3$, $v = (2t-1)(2t-3)\cdots 5\cdot 3$, and

$$d_j = |O_j| = \frac{1}{2}\binom{t}{j}\binom{s}{1}^j = 2^{j-1}\binom{t}{j}.$$

We claim that $t < 7$. If $t \geqslant 7$, then it is easy to know that $v = (2t-1)(2t-3)\cdots 5\cdot 3 > t^2(t-1)^2$, and as r divides $d_2 = t(t-1)$ it follows that $t(t-1) \geqslant r$, hence $v > t^2(t-1)^2 \geqslant r^2$ which is a contradiction. Thus $t < 7$. For $t = 3, 4, 5$ or 6, we calculate $d = \gcd(d_2, d_3)$ which listed in Table 1 below.

Table 1: Possible d when $s = 2$

t	n	v	d_2	d_3	d
3	6	15	6	4	2
4	8	105	12	16	4
5	10	945	20	40	20
6	12	10395	30	80	10

In each case $r \leqslant d$ which contradicts to the fact $r^2 > v$.

(2) Suppose second that $s \geqslant 3$, then O_j is an orbit of H on P, and $d_j = |O_j| = \binom{t}{j}\binom{s}{1}^j = s^j\binom{t}{j}$. In particular, $d_2 = \binom{t}{2}\binom{s}{1}^2 = s^2\binom{t}{2}$ and $r \mid d_2$. Moreover, from $\binom{is-1}{s-1} = \frac{is-1}{s-1}\cdot\frac{is-2}{s-2}\cdots\frac{is-(s-1)}{1} > i^{s-1}$, for $i = 2, 3, \ldots, t$, we have $v > 2^{(s-1)(t-1)}$. Then

$$2^{(s-1)(t-1)} < v < r^2 \leqslant s^4\binom{t}{2}^2,$$

and so

$$2^{(s-1)(t-1)} < s^4\binom{t}{2}^2. \tag{8}$$

We will calculate all pairs (s,t) satisfying the inequality (8). Since $2^{(s-1)(t-1)} = 2^{25} > s^4\binom{t}{2}^2 = 2^4\cdot 3^6\cdot 5^2$, i.e. the pair $(s,t) = (6,6)$ does not satisfy the inequality (8) but satisfies the conditions of Lemma 12 (ii) and (iii). Thus, we must have $s < 6$ or $t < 6$. It is not hard to get 32 pairs (s,t) satisfying the inequality (8) as follows:

$$\begin{array}{llllllllll} (3,2), & (3,3), & (3,4), & (3,5), & (3,6), & (3,7), & (3,8), & (3,9), & (4,2), & (4,3), \\ (4,4), & (4,5), & (4,6), & (5,2), & (5,3), & (5,4), & (6,2), & (6,3), & (6,4), & (7,2), \\ (7,3), & (8,2), & (8,3), & (9,2), & (10,2), & (11,2), & (12,2), & (13,2), & (14,2), & (15,2), \\ (16,2), & (17,2). \end{array}$$

For each (s,t), we calculate the parameters (v,b,r,k,λ) satisfying Eq.(7), Lemmas 5, 6 and $r \mid d_2$. Then we obtain five possible parameters (v,b,r,k,λ) corresponding to (s,t) are the following:

(2.1) $(s,t) = (3,2)$: $(10,30,9,3,2)$, $(10,18,9,5,4)$, $(10,15,9,6,5)$;

(2.2) $(s,t) = (5,2)$: $(126,525,25,6,1)$, $(126,150,25,21,4)$.

Case (2.1): $(s,t) = (3,2)$. Then $n = 6, v = 10$ and $G \cong A_6$ or S_6. Let $P = \{1,2,3,4,5,6,7,8,9,10\}$, the group $G \cong A_6$ or S_6, as the primitive permutation group of degree 10 acting on P, has the following generators respectively ([2, p.828]):

$$\begin{aligned} A_6 &\cong \langle (1,10,4,7,5)(2,8,6,9,3), (1,3,4,5,7)(2,10,9,8,6) \rangle, \\ S_6 &\cong \langle (1,8,6)(2,3,7,9,10,5), (1,8,9,3,5,6)(2,7,4) \rangle. \end{aligned}$$

Assume first $(v,b,r,k,\lambda) = (10,30,9,3,2)$ or $(10,18,9,5,4)$, and there exists a 2-$(10,k,\lambda)$ design $\mathcal{D}$ with flag-transitive automorphism group G, where $k = 3$ or 5, and $\lambda = 2$ or 4, respectively. There are totally $\binom{v}{k}$ k-element subsets of P. For any k-element subset $B \subseteq P$, we calculate the length of the G-orbit B^G where $G = A_6$ or S_6 respectively. By using GAP [8], we found that $|B^G| > b$ for any k-element subset B. So G cannot act block-transitively on $\mathcal{D}$, a contradiction.

Therefore, $(v,b,r,k,\lambda) = (10,15,9,6,5)$. Here $G \cong A_6$ or S_6 acts transitively on P. There are totally $210 = \binom{10}{6}$ 6-element subsets of P. For any 6-element subset $B \subseteq P$, using GAP [8], we calculate the length of the G-orbit B^G. It follows that up to isomorphism there exists a unique 2-$(10,6,5)$ design $\mathcal{D} = (P,\mathcal{B})$ where

$$\begin{aligned} \mathcal{B} = \; & \{\{1,2,3,4,5,8\}, \{1,2,6,7,8,10\}, \{3,4,5,6,7,10\}, \{4,5,6,8,9,10\}, \{1,2,3,6,9,10\}, \\ & \{1,2,4,5,7,9\}, \{1,3,4,6,7,9\}, \{2,5,6,7,8,9\}, \{2,3,4,8,9,10\}, \{1,3,5,7,8,10\}, \\ & \{2,3,5,7,9,10\}, \{1,3,5,6,8,9\}, \{2,3,4,6,7,8\}, \{1,2,4,5,6,10\}, \{1,4,7,8,9,10\}\}. \end{aligned}$$

Let $B = \{1,2,3,4,5,8\}$ be a block. Then it is easily known that $B^G = \mathcal{B}$. Now

$$G_1 = \langle (2,9,10,3)(4,7,8,5), (2,5,8,10)(3,7,6,4) \rangle \cong 3^2 : 4$$

or

$$\langle (2,10,8,3,9,4)(5,6,7), (3,9,7,8)(4,10,5,6) \rangle \cong 3^2 : D_8$$

with $G = A_6$ or S_6 respectively. Then B^{G_1} containing 9 blocks:

$$\begin{gathered} \{1,2,3,4,5,8\}, \{1,2,4,5,7,9\}, \{1,3,5,7,8,10\}, \\ \{1,4,7,8,9,10\}, \{1,3,5,6,8,9\}, \{1,2,6,7,8,10\}, \\ \{1,2,3,6,9,10\}, \{1,2,4,5,6,10\}, \{1,3,4,6,7,9\}. \end{gathered}$$

So that G_1 is transitive on the blocks through 1, note that G is also transitive on P, and hence $\mathcal{D}$ is flag-transitive.

Case (2.2): $(s,t) = (5,2)$. Then $n = 10$, $v = 126$ and $G \cong A_{10}$ or S_{10}.

For the parameters $(126, 525, 25, 6, 1)$ we have $\lambda = 1$, it can be ruled out by Lemma 4.

Suppose that $(v,b,r,k,\lambda) = (126, 150, 25, 21, 4)$. Since G is flag-transitive, then it is block-transitive and point-transitive. So that G must has subgroups with index 126 and 150. By using MAGMA [1] we know that G has 126 subgroups with index 126. Let $B \in \mathcal{B}$, so that $|G : G_B| = b = 150$ and $|G_B| = 12096$ or 24192 with $G \cong A_{10}$ or S_{10} respectively. Clearly, $|G : G_B| < \binom{10}{4}$. Then by Lemma 10 and [7, Theorem 5.2B], one of 3 cases holds. If Case (i) holds, there exists some $\Delta \subseteq \Omega_{10} = \{1, 2, \ldots, 10\}$ with $|\Delta| < 4$. We have $A_7 \leqslant G_B$ by Remark 11. However, $|A_7| = 7!/2$ does not divide $|G_B|$, a contradiction. Case (ii) can be ruled out by $|G : G_B| = 150 \neq \frac{1}{2}\binom{10}{5}$ and Case (iii) can be ruled out by $n = 10$. Hence, G has no subgroups with index 150, a contradiction. So that $G \ncong A_{10}$ or S_{10}. □

3.3 H acts intransitively on Ω_n.

Proposition 15. *Let $\mathcal{D}$ and G satisfy* HYPOTHESIS. *Then the point stabilizer cannot be intransitive on Ω_n.*

Proof. Suppose on the contrary that H acts intransitively on Ω_n. We have $H = (Sym(S) \times Sym(\Omega \setminus S)) \cap G$, and without loss of generality assume that $|S| = s < n/2$ by Lemma 7(i). By the flag-transitivity of G, H is transitive on the blocks through x, and so H fixes exactly one point in P. Since H stabilizes only one s-subset of Ω_n, we can identify the point x with S. As the orbit of S under G consists of all the s-subsets of Ω_n, we can identify P with the set of s-subsets of Ω_n. So that $v = \binom{n}{s}$, G has rank $s+1$ and the subdegrees are:

$$n_0 = 1, n_{i+1} = \binom{s}{i}\binom{n-s}{s-i}, i = 0, 1, \ldots, s-1. \tag{9}$$

First, we claim that $s \leqslant 6$. Since $r \mid n_i$ for any subdegree n_i of G by Lemma 6(ii) and $n_s = s(n-s)$ is a subdegree of G by (9), then $r \mid s(n-s)$. Combining this with $r^2 > v$, we have $s^2(n-s)^2 > \binom{n}{s}$. Since the condition $s < \frac{n}{2}$ equals to $s < t := n - s$, we have

$$s^2t^2 > \binom{s+t}{s}. \tag{10}$$

Combining it with Lemma 12 (i), we get $s \leqslant 6$.

Case (1): If $s = 1$, then $v = n \geqslant 5$ and the subdegrees are $1, n-1$. The group G is $(v-2)$-transitive on P. Since $2 < k \leqslant v-2$, G acts k-transitively on P. Then $b = |\mathcal{B}| = |B^G| = \binom{n}{k}$ for every block $B \in \mathcal{B}$. From the equality $bk = vr$ we can obtain $\binom{n}{k}k = nr$. On the one hand, by Lemma 5(ii) we have $r \leqslant n-1$, it follows that $\binom{n}{k}k \leqslant n(n-1)$. On the other hand, by $2 < k \leqslant n-2$, we have $n-i \geqslant k-i+2 > k-i+1$

for $i = 2, \ldots, k-1$. So that

$$\binom{n}{k}k = n(n-1) \cdot \frac{n-2}{k-1} \cdot \frac{n-3}{k-2} \cdots \frac{n-k+1}{2} > n(n-1),$$

a contradiction.

Case (2): If $s = 2$, then the subdegrees are 1, $\binom{n-2}{2}$, $2(n-2)$, and G is a primitive rank 3 group acting on P. By Lemma 6(ii), $r \mid \left(\binom{n-2}{2}, 2(n-2) \right) = \frac{n-2}{2}$, $n-2$, or $2(n-2)$ with $n \equiv 2 \pmod 4$, $n \equiv 1 \pmod 4$, or $n \equiv 3 \pmod 4$ respectively.

Assume first that $r \mid \frac{n-2}{2}$. Then $\frac{n(n-1)}{2} = v < r^2 \leqslant \left(\frac{n-2}{2}\right)^2$, which is impossible.

Assume that $r \mid (n-2)$. From Lemma 6(i), $\lambda v = \frac{\lambda n(n-1)}{2} < r^2 \leqslant (n-2)^2$ which forces $\lambda = 1$. It follows from Lemma 4 that $\mathcal{D} = PG(3,2)$ and $G \cong A_7$ or A_8. This contracts the assumption that $v = \binom{n}{2} = 21$ or 28.

Hence $r \mid 2(n-2)$. From above analysis we know that r is even. By Lemma 6(i), $\lambda v = \frac{\lambda n(n-1)}{2} < r^2 \leqslant 4(n-2)^2$ which implies $1 \leqslant \lambda \leqslant 7$. Recall that $(r, \lambda) = 1$, so that λ is odd. Since Lemma 4 implies that $\lambda \neq 1$, we assume that $\lambda = 3$, 5 or 7 in the following.

Let $r = \frac{2(n-2)}{u}$ for some integer u. From Lemma 6(i), we have $\frac{4(n-2)^2}{u^2} > \frac{\lambda n(n-1)}{2}$. It follows that $8 > \frac{8(n-2)^2}{n(n-1)} > \lambda u^2$ which forces $u = 1$. Therefore, $r = 2(n-2)$. By Lemma 5, we get $k = \frac{\lambda(n+1)}{4} + 1$, and $b = \frac{vr}{k} = \frac{4n(n-1)(n-2)}{\lambda(n+1)+4}$, where $\lambda = 3$, 5 or 7.

If $\lambda = 3$, then $(3n+7) \mid 4n(n-1)(n-2)$ since $b \in \mathbb{N}$. And since $(3, 3n+7) = 1$, we have $(3n+7) \mid 36n(n-1)(n-2)$. Recall that $n \equiv 3 \pmod 4$, and from

$$3^2 b = \frac{36n(n-1)(n-2)}{3n+7} = 12n^2 - 64n + 173 - \frac{1211-n}{3n+7} \in \mathbb{N}$$

we have $(3n+7) \mid (1211-n)$, so $n = 7$, 11, 15, 91, 119 or 171.

Similarly, if $\lambda = 5$, then $5^2 b = \frac{100n(n-1)(n-2)}{5n+9} = 20n^2 - 96n + 213 - \frac{n+1917}{5n+9} \in \mathbb{N}$. Now the facts $(5n+9) \mid (n+1917)$ and $n \equiv 3 \pmod 4$ imply $n = 15$, 99, 135 or 211.

If $\lambda = 7$, then $7^2 b = \frac{196n(n-1)(n-2)}{7n+11} = 28n^2 - 128n + 257 - \frac{2827-n}{7n+11} \in \mathbb{N}$. Now the facts $(7n+11) \mid (2827-n)$ and $n \equiv 3 \pmod 4$ imply $n = 7$, 11, 27, 55, 127 or 187.

For each value of n, we compute the possible parameters (n, v, b, r, k) satisfying Lemma 5 and the condition $b > v$ which listed in the following.

$$\begin{array}{lll}
\lambda = 3: & (7, 21, 30, 10, 7), & (11, 55, 99, 18, 10), \\
& (15, 105, 210, 26, 13), & (91, 4095, 10413, 178, 70), \\
& (119, 7021, 18054, 234, 91), & (171, 14535, 37791, 338, 130). \\
\lambda = 5: & (15, 105, 130, 26, 21), & (99, 4851, 7469, 194, 126), \\
& (135, 9045, 14070, 266, 171), & (211, 22155, 34815, 418, 266). \\
\lambda = 7: & (55, 1485, 1590, 106, 99), & (127, 8001, 8890, 250, 225), \\
& (187, 17391, 19499, 370, 330). &
\end{array}$$

Assume first that $(n, v, b, r, k) = (7, 21, 30, 10, 7)$ and there exists a 2-(21, 7, 3) design $\mathcal{D}$ with flag-transitive automorphism group $G \cong A_7$ or S_7. Let $P = \{1, 2, 3, \ldots, 21\}$, the group $G \cong A_7$ or S_7, as the primitive permutation group of degree 21 acting on P, has the following generators respectively ([2, p.830]):

$$\begin{aligned} A_7 &\cong \langle (1,2,3,4,5,6,7)(8,9,10,11,12,13,14)(15,16,17,18,19,20,21), \\ &\quad (1,4)(2,11)(3,9)(5,15)(7,20)(8,13)(10,21)(12,14) \rangle, \\ S_7 &\cong \langle (1,2,3,4,5,6,7)(8,9,10,11,12,13,14)(15,16,17,18,19,20,21), \\ &\quad (1,21,4,10)(2,9,11,3)(5,12,15,14)(7,13,20,8)(17,18) \rangle. \end{aligned}$$

There are totally $\binom{21}{7}$ 7-element subsets of P. For any 7-element subset $B \subseteq P$, we calculate the length of the G-orbit B^G where $G = A_7$ or S_7 respectively. By using GAP [8], we found that $|B^G| > 30$ for any 7-element subset B. So G cannot act block-transitively on $\mathcal{D}$, a contradiction.

For all other possible parameters (n, v, b, r, k) listed above, $G \cong A_n$ or S_n. On the one hand, by the block-transitivity, we have $b = |G : G_B|$ where $B \in \mathcal{B}$. On the other hand, since $|G : G_B| = b < \binom{n}{3}$ for each case, by Lemma 10 and [7, Theorem 5.2B], it is easily known that G has no subgroups of index b, a contradiction.

Case (3): Suppose that $3 \leqslant s \leqslant 6$. Now for each value of s, using the inequality (10) and Lemma 12(iv), we know that t (and hence n) is bounded. For example, let $s = 3$, since $\binom{3+48}{3} > 3^2 \cdot 48^2$, we must have $4 \leqslant t \leqslant 47$ by Lemma 12(iv), and so $7 \leqslant n \leqslant 50$. The bound of n listed in Table 2 below. Here the last column denotes the arithmetical conditions which we used to ruled out each line.

Table 2: Bound of n when $3 \leqslant s \leqslant 6$

s	t	n	Reference
3	$4 \leqslant t \leqslant 47$	$7 \leqslant n \leqslant 50$	(1)-(4), Lemma 6
4	$5 \leqslant t \leqslant 14$	$9 \leqslant n \leqslant 18$	Lemma 6
5	$6, 7, 8, 9$	$11, 12, 13, 14$	Lemma 6
6	7	13	Lemma 6

Note that $v = \binom{n}{s}$, and $n_1 = \binom{n-s}{s}$, $n_s = s(n-s)$ are two subdegrees of G acting on P. Therefore, the 6-tuple (v, b, r, k, λ, n) satisfies the following arithmetical conditions: (1)-(4), $(r, \lambda) = 1$, $r^2 > v$ (Lemma 6(i)), and

$$r \mid d, \text{ where } d = \gcd(n_1, n_s). \tag{11}$$

If $s = 3$, by using GAP [8], it outputs five 6-tuples satisfying above arithmetical conditions:

$$\begin{aligned} &(364, 1001, 33, 12, 1, 14), \quad (1540, 3135, 57, 28, 1, 22), \quad (4960, 7440, 87, 58, 1, 32), \\ &(19600, 19740, 141, 140, 1, 50), \quad (1540, 1596, 57, 55, 2, 22). \end{aligned}$$

The four parameters with $\lambda = 1$ can be ruled out by Lemma 4. For the parameters (1540, 1596, 57, 55, 2, 22), we have $\lambda = 2$, $G \cong A_{22}$ or S_{22}. By the block-transitivity,

$b = |G : G_B| = 1596$ where $B \in \mathcal{B}$. However, since $|G : G_B| = 1596 < \binom{22}{4}$, by Lemma 10 and [7, Theorem 5.2B], it is easily known that A_{22} and S_{22} has no subgroups of index 1596. So the parameters $(1540, 1596, 57, 55, 2, 22)$ cannot occur.

If $s = 4, 5$ or 6, by using GAP [8], there are no parameters (v, k, n) satisfying the conditions. For example, if $s = 6$, then $n = 13$, $v = 1716$, $d = 7$. It follows that $r \leqslant 7$ by (10), then $r^2 < v$ which is impossible. If $s = 5$, then $n = 11, 12, 13$ or 14, $v = 462, 792, 1287$ or 2002 and $d = 6, 7, 8$ or 9, respectively. It is easy to check that $r^2 < v$ for every case. This is the final contradiction. □

Propositions 13-15 finish the proof of Theorem 1.

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