

FLAT SINGULAR INTEGRALS IN PRODUCT DOMAINS

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Abstract

In this paper, we study singular integrals on product domains with kernels in $L(\log L)^2(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$ supported by surfaces of revolutions. We prove that our operators are bounded on L^p under certain convexity assumption on the surfaces. Also, in this paper we prove that the convexity assumption is not necessary for the L^2 boundedness to hold. Moreover, additional related results are presented. Our condition on the kernel is known to be optimal.

1 Introduction

Let $\Gamma_{k,d}(y) = \Gamma_{k,d}(y_1, y_2, \dots, y_k)$ be a smooth k parameter surface in \mathbf{R}^d with $d \geq k+1$. For a pair $(\Gamma_{n,d}, \Gamma_{m,l})$ of smooth n and m parameter surfaces $\Gamma_{n,d}$ and $\Gamma_{m,l}$ in \mathbf{R}^d and \mathbf{R}^l respectively, define the associated singular integral operator $T_{(\Gamma_{n,d}, \Gamma_{m,l})}$ (initially for C_0^∞ functions on $\mathbf{R}^{n+1} \times \mathbf{R}^{m+1}$) by

$$T_{(\Gamma_{n,d}, \Gamma_{m,l})} f(x, y) = \text{p.v.} \int_{\mathbf{R}^n \times \mathbf{R}^m} f(x - \Gamma_{n,d}(u), y - \Gamma_{m,l}(v)) |u|^{-n} |v|^{-m} \Omega(u, v) \, dudv, \quad (1.1)$$

where

$$\Omega \in L^1(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}), \Omega(tx, sy) = \Omega(x, y) \text{ for any } t, s > 0; \quad (1.2)$$

$$\int_{\mathbf{S}^{n-1}} \Omega(u, \cdot) \, d\sigma_n(u) = \int_{\mathbf{S}^{m-1}} \Omega(\cdot, v) \, d\sigma_m(v) = 0. \quad (1.3)$$

Here, \mathbf{S}^{n-1} and \mathbf{S}^{m-1} are the unit spheres in \mathbf{R}^n and \mathbf{R}^m respectively.

The L^p boundedness of singular integral operators on product domains have been studied by many authors ([2], [3], [5], [7], [9], [10], [11], among

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others). When $d = n + 1$, $l = m + 1$, $\Gamma_{n,d}(u) = (u, 0)$, and $\Gamma_{m,l}(v) = (v, 0)$, we shall simply let $T_{(I_n, I_m)} = T_{(\Gamma_{n,d}, \Gamma_{m,l})}$. In [10], R. Fefferman and E. M. Stein proved that $T_{(I_n, I_m)}$ is bounded on $L^p(\mathbf{R}^{n+m})$ for $1 < p < \infty$ if Ω satisfies certain Lipschitz conditions. Subsequently, in [5] Duoandikoetxea established the L^p ($1 < p < \infty$) boundedness of $T_{(I_n, I_m)}$ under the weaker condition $\Omega \in L^q(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$ with $q > 1$. In a recent paper [3], it was shown that the operator $T_{(I_n, I_m)}$ is bounded on L^p ($1 < p < \infty$) provided that Ω satisfies the additional assumption that $\Omega \in L(\log L)^2(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$, i.e.,

$$\int_{\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}} |\Omega(u, v)| (\log 2 + |\Omega(u, v)|)^2 d\sigma_n(u) d\sigma_m(v) < \infty. \quad (1.4)$$

Also, it was shown in [3] that the condition $\Omega \in L(\log L)^2(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$ is nearly optimal in the sense that the exponent 2 in $L(\log L)^2$ can not be replaced by any smaller numbers.

Recently, there have been a considerable amount of research concerning the L^p boundedness of the operators $T_{(\Gamma_{n,d}, \Gamma_{m,l})}$ and their analog in the one parameter setting for various functions Ω and surfaces $\Gamma_{n,d}$ and $\Gamma_{m,l}$ ([1], [4], [12]).

The main focus of this paper is seeking L^p estimates for $T_{(\Gamma_{n,d}, \Gamma_{m,l})}$ provided that each one of the surfaces $\Gamma_{n,d}$ and $\Gamma_{m,l}$ is a hypersurface obtained by rotating a one-dimensional curve around one of the coordinate axes. We also allow each one of our surfaces to have infinite order of contacts with its tangent plane at the origin. More specifically, we assume that $d = n + 1$, $l = m + 1$, $\Gamma_{n,d}(u) = (u, \phi(|u|))$, and $\Gamma_{m,l}(v) = (v, \varphi(|v|))$, where ϕ and φ are real valued functions defined on \mathbf{R}^+ with $\phi(0) = \varphi(0) = 0$. For simplicity, throughout the rest of this paper, we shall let $\Gamma_{\phi,n}(u) = \Gamma_{n,n+1}(u) = (u, \phi(|u|))$ and $\Gamma_{\varphi,m}(v) = \Gamma_{m,m+1}(v) = (v, \varphi(|v|))$. It is worth pointing out here that the functions ϕ and φ can be very flat at the origin in the sense that $\frac{d^k \phi}{dt^k}(0) = \frac{d^k \varphi}{dt^k}(0) = 0$ for all $k \geq 1$. It is natural to ask whether the operator $T_{(\Gamma_{\phi,n}, \Gamma_{\varphi,m})}$ is bounded under the optimal size condition $\Omega \in L(\log L)^2(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$. More precisely we ask the following question:

Question. *Suppose that ϕ and φ are C^2 , convex increasing and that $\Omega \in L(\log L)^2(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$. Is the operator $T_{(\Gamma_{\phi,n}, \Gamma_{\varphi,m})}$ bounded on $L^p(\mathbf{R}^{n+1} \times \mathbf{R}^{m+1})$ for some $p \in (1, \infty)$?*

In this paper, we shall answer this question in the affirmative. For the

L^2 boundedness we shall show that the operator $T_{(\Gamma_{\phi,n}, \Gamma_{\varphi,m})}$ is bounded without convexity or any smoothness conditions on ϕ and φ . In fact, we have the following:

Theorem A. *Suppose that ϕ and φ are real valued functions defined on \mathbf{R}^+ with $\phi(0) = \varphi(0) = 0$. If $\Omega \in L(\log L)^2(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$ and satisfies (1.2)-(1.3), then $T_{(\Gamma_{\phi,n}, \Gamma_{\varphi,m})}$ is bounded on $L^2(\mathbf{R}^{n+1} \times \mathbf{R}^{m+1})$.*

For the L^p boundedness for any $p \in (1, \infty)$, we shall first establish the following result which shows the dependence of the L^p norms on the size of Ω :

Theorem B. *Suppose that $\beta > 1$, ϕ and φ are C^2 , convex increasing. Suppose also that $\Omega \in L^2(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$ and satisfies (1.2)-(1.3). If $\|\Omega\|_{L^2} \leq 2^{4\beta}$ and $\|\Omega\|_{L^1} \leq 1$, then*

$$\left\| T_{(\Gamma_{\phi}, \Gamma_{\varphi})} f \right\|_p \leq \beta^2 C \|f\|_p$$

for $p \in (1, \infty)$. The constant C is independent of the parameter β .

As a consequence of Theorem B, we will obtain the following result:

Theorem C. *Suppose that ϕ and φ are C^2 , convex increasing. If $\Omega \in L(\log L)^2(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$ and satisfies (1.2)-(1.3), then $T_{(\Gamma_{\phi}, \Gamma_{\varphi})}$ is bounded on $L^p(\mathbf{R}^{n+1} \times \mathbf{R}^{m+1})$ for $p \in (1, \infty)$.*

It should be pointed out here that the result of Theorem C still holds for many functions ϕ and φ other than the convex increasing ones. Moreover, a similar result as that in Theorem C can be obtained if we allow the kernel to have an additional roughness in the radial direction. A detailed discussion of these results and some others will be presented in Section 4.

Throughout this paper the letter C will stand for a constant that may vary at each occurrence, but it is independent of the essential variables.

2 Preliminary estimates

In this section, we shall establish certain necessary estimates that we shall need to prove our results.

2.1 Certain oscillatory estimates

Suppose that $\beta > 1$ and $\Omega \in L^1(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$. For $j, k \in \mathbf{Z}$, $\xi \in \mathbf{R}^{n+1}$, $\eta \in \mathbf{R}^{m+1}$, and real valued functions ϕ and φ defined on \mathbf{R}^+ with $\phi(0) = \varphi(0) = 0$, let

$$I_\beta(\xi, \eta, k, j) = \int_{\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}} e^{-i\{\xi \cdot \Gamma_{\phi, n}(2^{\beta k} r u') - \eta \cdot \Gamma_{\varphi, m}(2^{\beta j} s v')\}} \Omega(u', v') d\sigma_n(u') d\sigma_m(v') \quad (2.1)$$

$$J_\beta^n(\xi, k, j) = \int_{\mathbf{S}^{n-1}} e^{-i\xi \cdot \Gamma_{\phi, n}(2^{\beta k} r u')} \Omega(u', v') d\sigma_n(u'), \quad (2.2)$$

$$J_\beta^m(\eta, k, j) = \int_{\mathbf{S}^{m-1}} e^{-i\eta \cdot \Gamma_{\varphi, m}(2^{\beta j} s v')} \Omega(u', v') d\sigma_m(v'). \quad (2.3)$$

Also, let π_n^{n+1} and π_m^{m+1} be the usual projections from \mathbf{R}^{n+1} to \mathbf{R}^n and from \mathbf{R}^{m+1} to \mathbf{R}^m respectively. Thus we have the following:

Proposition 2.1. Suppose that $\Omega \in L^2(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$ that satisfies (1.2)-(1.3). Then

$$\int_{\mathbf{S}^{m-1}} \int_1^{2^\beta} |J_\beta^n(\xi, k, j)| r^{-1} dr d\sigma_n(u') \leq \beta \|\Omega\|_{L^2} \left| 2^{\beta k} \pi_n^{n+1}(\xi) \right|^{-\frac{1}{8}}; \quad (2.4)$$

$$\int_{\mathbf{S}^{n-1}} \int_1^{2^\beta} |J_\beta^m(\eta, k, j)| s^{-1} ds d\sigma_m(v') \leq \beta \|\Omega\|_{L^2} \left| 2^{\beta j} \pi_m^{m+1}(\eta) \right|^{-\frac{1}{8}}; \quad (2.5)$$

$$\int_1^{2^\beta} \int_1^{2^\beta} |I_\beta(\xi, \eta, k, j)| r^{-1} s^{-1} dr ds \leq \beta^2 \|\Omega\|_{L^2} \left| 2^{\beta k} \pi_n^{n+1}(\xi) \right|^{-\frac{1}{8}} \left| 2^{\beta j} \pi_m^{m+1}(\eta) \right|^{-\frac{1}{8}}. \quad (2.6)$$

Proof. We start by proving (2.6). By Cauchy Schwartz inequality, we have

$$\int_1^{2^\beta} \int_1^{2^\beta} |I_\beta(\xi, \eta, k, j)| r^{-1} s^{-1} dr ds \leq \beta \left(\int_1^{2^\beta} \int_1^{2^\beta} |I_\beta(\xi, \eta, k, j)|^2 r^{-1} s^{-1} dr ds \right)^{\frac{1}{2}}. \quad (2.7)$$

Now, it is easy to see that the square of the integral in the right hand side of (2.7) is dominated by

$$\|\Omega\|_{L^2}^2 \int_{(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})^2} A_\beta(k, \xi, u', z') B_\beta(j, \eta, v', w') d\sigma_n(u') d\sigma_m(v') d\sigma_n(z') d\sigma_m(w')^{\frac{1}{2}}, \quad (2.8)$$

where

$$A_\beta(k, \xi, u', z') = \left| \int_1^{2^\beta} e^{-i2^{\beta k} r \pi_n^{n+1}(\xi) \cdot (u' - z')} r^{-1} dr \right|^2 \quad (2.9)$$

$$B_\beta(j, \eta, v', w') = \left| \int_1^{2^\beta} e^{-i2^{\beta j} s \pi_m^{m+1}(\eta) \cdot (v' - w')} s^{-1} ds \right|^2. \quad (2.10)$$

By integration by parts, we immediately obtain

$$A_\beta(k, \xi, u', z') \leq \left| 2^{\beta k} \pi_n^{n+1}(\xi) \cdot (u' - z') \right|^{-2} \quad (2.11)$$

$$B_\beta(j, \eta, v', w') \leq \left| 2^{\beta j} \pi_m^{m+1}(\eta) \cdot (v' - w') \right|^{-2}. \quad (2.12)$$

By (2.11) and the trivial estimate $A_\beta(k, \xi, u', z') \leq \beta$, we get

$$A_\beta(k, \xi, u', z') \leq \beta^{\frac{7}{8}} \left| 2^{\beta k} \pi_n^{n+1}(\xi) \cdot (u' - z') \right|^{-\frac{1}{4}}. \quad (2.13)$$

Similarly, we have

$$B_\beta(j, \eta, v', w') \leq \beta^{\frac{7}{8}} \left| 2^{\beta j} \pi_m^{m+1}(\eta) \cdot (v' - w') \right|^{-\frac{1}{4}}. \quad (2.14)$$

Hence, by combining (2.7)-(2.8) and (2.13)-(2.14), we obtain (2.6). Finally, (2.4) and (2.5) can be obtained by a similar argument. This completes the proof.

Now, let $\{m_{j,k} : j, k \in \mathbf{Z}\}$ be a sequence of Boreal measures defined on $\mathbf{R}^n \times \mathbf{R}^m$ by

$$\int_{\mathbf{R}^n \times \mathbf{R}^m} f dm_{j,k} = \int_{A_k} \int_{A_j} f(\Gamma_{\phi,n}(u), \Gamma_{\varphi,m}(v)) |u|^{-n} |v|^{-m} \Omega(u, v) dudv, \quad (2.15)$$

where $A_k = [2^k, 2^{k+1})$ and $A_j = [2^j, 2^{j+1})$. For these measures, we have the following:

Proposition 2.2. *Suppose that $\Omega \in L^2(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$ that satisfies (1.2)-(1.3). Then for $\beta > 1$, $\xi \in \mathbf{R}^{n+1}$, and $\eta \in \mathbf{R}^{m+1}$, we have*

$$|\hat{m}_{\beta j, \beta k}(\xi, \eta)| \leq \beta^2 \|\Omega\|_{L^1} \quad (2.16)$$

$$|\hat{m}_{\beta j, \beta k}(\xi, \eta)| \leq (\|\Omega\|_{L^2})^{\frac{1}{\beta}} \beta^2 \left| 2^{\beta k} \pi_n^{n+1}(\xi) \right|^{\pm \frac{1}{8\beta}} \left| 2^{\beta j} \pi_m^{m+1}(\eta) \right|^{\pm \frac{1}{8\beta}} \quad (2.17)$$

Here, we used the notation $t^{\pm\alpha} = \min\{t^\alpha, t^{-\alpha}\}$.

Proof. The estimate (2.16) is clear. Therefore, we only need to prove (2.17). By the cancelation property (1.3), it is easy to see that

$$|\hat{m}_{\beta j, \beta k}(\xi, \eta)| \leq \|\Omega\|_{L^2} \beta^2 \left| 2^{\beta k} \pi_n^{n+1}(\xi) \right| \left| 2^{\beta j} \pi_m^{m+1}(\eta) \right|. \quad (2.18)$$

Using polar coordinates and the cancelation property (1.3), we have

$$|\hat{m}_{\beta j, \beta k}(\xi, \eta)| \leq \beta \min\{1, \left| 2^{\beta j} \pi_m^{m+1}(\eta) \right|\} \int_{\mathbf{S}^{m-1}} \int_1^{2^\beta} \left| J_\beta^n(\xi, k, j) \right| r^{-1} dr d\sigma_n(u'), \quad (2.19)$$

$$|\hat{m}_{\beta j, \beta k}(\xi, \eta)| \leq \beta \min\{1, \left| 2^{\beta k} \pi_n^{n+1}(\xi) \right|\} \int_{\mathbf{S}^{n-1}} \int_1^{2^\beta} \left| J_\beta^m(\eta, k, j) \right| s^{-1} ds d\sigma_m(v'). \quad (2.20)$$

Also, it is easy to see that

$$|\hat{m}_{\beta j, \beta k}(\xi, \eta)| \leq \int_1^{2^\beta} \int_1^{2^\beta} \left| I_\beta(\xi, \eta, k, j) \right| r^{-1} s^{-1} dr ds. \quad (2.21)$$

Hence (2.17) follows by (2.18)-(2.21), Proposition 2.1, and an interpolation.

By a careful review of the proof of Proposition 2.1, we notice the following:

Remark 2.3. The estimates in Proposition 2.2 hold for any two real valued functions ϕ and φ defined on \mathbf{R}^+ with $\phi(0) = \varphi(0) = 0$.

2.2 General tools

In this section, we recall some important facts that we need to prove our results. We start by recalling the following version of Theorem 15 in ([2]).

Theorem 2.4 ([2]). *Let $\{\sigma_{k,j} : k, j \in \mathbf{Z}\}$ be a sequence of Borel measures on $\mathbf{R}^N \times \mathbf{R}^M$. Let $L : \mathbf{R}^N \rightarrow \mathbf{R}^{d_1}$ and $Q : \mathbf{R}^M \rightarrow \mathbf{R}^{d_2}$ be linear transformations. Suppose that for some $\alpha_1, \alpha_2, C > 0$, and $B > 1$, the following hold for $k, j \in \mathbf{Z}, (\xi, \eta) \in \mathbf{R}^N \times \mathbf{R}^M$:*

- (i) $|\hat{\sigma}_{k,j}(\xi, \eta)| \leq CB^2 (2^{Bk} |L(\xi)|)^{\pm \frac{\alpha_1}{B}} (2^{Bj} |Q(\eta)|)^{\pm \frac{\alpha_2}{B}};$
- (ii) $\left\| \left(\sum_{k,j \in \mathbf{Z}} |\sigma_{k,j} * g_{k,j}|^2 \right)^{\frac{1}{2}} \right\|_{p_0} \leq CB^2 \left\| \left(\sum_{k,j \in \mathbf{Z}} |g_{k,j}|^2 \right)^{\frac{1}{2}} \right\|_{p_0}$ holds for arbitrary functions $\{g_{k,j}\}$ on $\mathbf{R}^N \times \mathbf{R}^M$ and some $p_0 > 2$.

Then

$$\left\| \sum_{k,j \in \mathbf{Z}} \sigma_{k,j} * f \right\|_{L^p(\mathbf{R}^N \times \mathbf{R}^M)} \leq C_p B^2 \|f\|_{L^p(\mathbf{R}^N \times \mathbf{R}^M)}, \quad (2.22)$$

for all $p'_0 < p < p_0$ and for all f in $L^p(\mathbf{R}^N \times \mathbf{R}^M)$. The constant C_p is independent of B and the linear transformations L and Q .

We shall also need the following result from ([4]):

Lemma 2.5 ([4]). Suppose $\beta > 1$. Let ψ be a C^2 , convex and increasing function satisfying $\psi(0) = 0$. Suppose also that $\Psi \in L^2(\mathbf{S}^{d-1})$ with $\|\Psi\|_{L^1} \leq 1$ and $\|\Psi\|_{L^2} \leq 2^\beta$. Let

$$M_{\beta, \Psi, \psi}^{(d)} f(x, x_{d+1}) = \sup_{j \in \mathbf{Z}} \left| \int_{2^{\beta j} \leq |y| < 2^{\beta(j+1)}} f(x-y, x_{d+1} - \psi(|y|)) |y|^{-d} \Psi(y) dy \right|. \quad (2.23)$$

Then for $1 < p \leq \infty$ there exists a positive constant C_p independent of β such that

$$\left\| M_{\beta, \Psi, \psi}^{(d)} f \right\|_p \leq \beta C_p \|f\|_p \quad (2.24)$$

for every $f \in L^p(\mathbf{R}^{d+1})$.

Now we are in a position to establish the boundedness of the necessary maximal functions:

2.3 Maximal functions

Suppose that $\beta > 1$. Let $M_{\beta, \phi, \varphi}^*$ be the maximal function defined by

$$M_{\beta, \phi, \varphi}^*(f) = \sup_{j,k \in \mathbf{Z}} |m_{\beta j, \beta k}| * f. \quad (2.25)$$

Then we have the following:

Lemma 2.6. Suppose that $\|\Omega\|_{L^2} \leq 2^{4\beta}$ and $\|\Omega\|_{L^1} \leq 1$. Suppose also that ϕ and φ are C^2 , convex increasing. Then for $1 < p < \infty$, there exists a constant C_p independent of β such that $\left\| M_{\beta, \phi, \varphi}^*(f) \right\|_p \leq \beta^2 C_p \|f\|_p$.

Proof. Our proof is based on the same ideas developed in the proof of Proposition 5.5 in ([1]). But, this time, we keep track of the parameter β at each step. For the readers convenience, we shall give a sketch of the proof.

For $j, k \in \mathbf{Z}$, let $\lambda_{k,j,\beta}^{(2,2)}$, $\lambda_{k,j,\beta}^{(1,2)}$, $\lambda_{k,j,\beta}^{(2,1)}$ and $\lambda_{k,j,\beta}^{(1,1)}$ be the measures defined by

$$\begin{aligned}\hat{\lambda}_{k,j,\beta}^{(2,2)}(\xi, \eta) &= \hat{m}_{\beta j, \beta k}(\xi, \eta), \hat{\lambda}_{k,j,\beta}^{(1,2)}(\xi, \eta) = \hat{m}_{\beta j, \beta k}(0, \xi_{n+1}, \eta), \\ \hat{\lambda}_{k,j,\beta}^{(2,1)}(\xi, \eta) &= \hat{m}_{\beta j, \beta k}(\xi, 0, \eta_{m+1}), \hat{\lambda}_{k,j,\beta}^{(1,1)}(\xi, \eta) = \hat{m}_{\beta j, \beta k}(0, \xi_{n+1}, 0, \eta_{m+1}).\end{aligned}$$

Without loss of generality, we may assume that Ω is nonnegative. By similar argument as in Proposition 2.2 and the assumption that $\|\Omega\|_{L^2} \leq 2^{4\beta}$ and $\|\Omega\|_{L^1} \leq 1$, we obtain

$$\left| \hat{\lambda}_{k,j,\beta}^{(2,2)}(\xi, \eta) \right| \leq C\beta^2 \left| 2^{\beta k} \pi_n^{n+1}(\xi) \right|^{\frac{1}{8\beta}} \left| 2^{\beta j} \pi_m^{m+1}(\eta) \right|^{\frac{1}{8\beta}} \quad (2.26)$$

$$\left| \hat{\lambda}_{k,j,\beta}^{(2,2)}(\xi, \eta) - \hat{\lambda}_{k,j,\beta}^{(1,2)}(\xi, \eta) \right| \leq C\beta^2 \left| 2^{\beta k} \pi_n^{n+1}(\xi) \right|^{-\frac{1}{8\beta}} \left| 2^{\beta j} \pi_m^{m+1}(\eta) \right|^{\frac{1}{8\beta}} \quad (2.27)$$

$$\left| \hat{\lambda}_{k,j,\beta}^{(2,2)}(\xi, \eta) - \hat{\lambda}_{k,j,\beta}^{(2,1)}(\xi, \eta) \right| \leq C\beta^2 \left| 2^{\beta k} \pi_n^{n+1}(\xi) \right|^{\frac{1}{8\beta}} \left| 2^{\beta j} \pi_m^{m+1}(\eta) \right|^{-\frac{1}{8\beta}} \quad (2.28)$$

$$\begin{aligned}& \left| \hat{\lambda}_{k,j,\beta}^{(2,2)}(\xi, \eta) - \hat{\lambda}_{k,j,\beta}^{(1,2)}(\xi, \eta) - \hat{\lambda}_{k,j,\beta}^{(2,1)}(\xi, \eta) + \hat{\lambda}_{k,j,\beta}^{(1,1)}(\xi, \eta) \right| \\ & \leq C\beta^2 \left| 2^{\beta k} \pi_n^{n+1}(\xi) \right|^{-\frac{1}{8\beta}} \left| 2^{\beta j} \pi_m^{m+1}(\eta) \right|^{-\frac{1}{8\beta}}.\end{aligned} \quad (2.29)$$

For $l = 1, 2$, $s = 1, 2$, let $M_\beta^{(l,s)}(f) = \sup_{k,j \in \mathbf{Z}} \left| \lambda_{k,j,\beta}^{(l,s)} * f \right|$. Then it is clear that $M_{\beta,\phi,\varphi}^* = M_\beta^{(2,2)}$. Now we claim that

$$\left\| M_\beta^{(l,s)}(f) \right\|_p \leq \beta^2 C_p \|f\|_p \quad (2.30)$$

for all $1 < p < \infty$ and $(l, s) \in \{(1, 2), (2, 1), (1, 1)\}$. To see (2.30), first, let $l = 1, s = 2$, and $\Psi(v) = \int_{\mathbf{S}^{n-1}} |\Omega(u, v)| d\sigma_n(u')$. Then Ψ is a homogenous function of degree 0 on \mathbf{S}^{m-1} with $\|\Psi\|_{L^2} \leq 2^{4\beta}$ and $\|\Psi\|_{L^1} \leq 1$. Let M_β be the maximal function defined by (2.23) with $d = m$. Therefore, by Lemma 2.5, the boundedness of the Hardy-Littlewood maximal function $H_{\mathbf{R}}$ on \mathbf{R} (see [13]), and the observation that $M_\beta^{(1,2)}(f)(x, y) \leq C\beta((\mathbf{I}_{\mathbf{R}^n} \otimes H_{\mathbf{R}}) \otimes M_{\beta,\Psi,\psi}^{(m)})(f)(x, y)$, it is easy to see that $M_\beta^{(1,2)}$ satisfies (2.30). Here $\mathbf{I}_{\mathbf{R}^n}$ is the identity operator on \mathbf{R}^n . Similarly, we can show that $M_\beta^{(l,s)}$, $(l, s) \in \{(2, 1), (1, 1)\}$ satisfy (2.30). Hence, our result follows by following exactly the same bootstrapping argument employed in the proof of Proposition 5.5 in ([1]).

3 Proof of main results

Proof of Theorem B. Suppose that $\beta > 1$, ϕ and φ are C^2 , convex increasing. Suppose also that $\|\Omega\|_{L^2} \leq 2^{4\beta}$ and $\|\Omega\|_{L^1} \leq 1$. Let $\{m_{j,k} : j, k \in \mathbf{Z}\}$ be the sequence of measures defined by (2.15). Then

$$T_{(\Gamma_\phi, \Gamma_\varphi)} f = \sum_{k,j \in \mathbf{Z}} m_{\beta j, \beta k} * f.$$

Therefore, by the fact that $\|\Omega\|_{L^2} \leq 2^{4\beta}$ and $\|\Omega\|_{L^1} \leq 1$, Proposition 2.2, Lemma 2.6, the proof of the lemma in page 544 of ([6]), and Theorem 2.4, the result follows. This completes the proof.

Proof of Theorem C. Assume that $\Omega \in L(\log^+ L)^2(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$. For $w \in \mathbf{N}$, let \mathbf{E}_w be the set of points $(x', y') \in \mathbf{S}^{n-1} \times \mathbf{S}^{m-1}$ that satisfy $2^w \leq |\Omega(x', y')| < 2^{w+1}$. Also, we let \mathbf{E}_0 be the set of those points $(x', y') \in \mathbf{S}^{n-1} \times \mathbf{S}^{m-1}$ that satisfy $|\Omega(x', y')| < 2$. For $w \in \mathbf{N} \cup \{0\}$, set $b_w = \Omega \chi_{\mathbf{E}_w}$ and $\lambda_w = \|b_w\|_1$, where $\chi_{\mathbf{E}_w}$ is the characteristic function of the set \mathbf{E}_w . Set $\mathbf{D} = \{w \in \mathbf{N} : \lambda_w \geq 2^{-3w}\}$ and define the sequence of functions $\{\Omega_w : w \in \mathbf{D} \cup \{0\}\}$ by

$$\begin{aligned} \Omega_0(x, y) &= b_0(x, y) + \sum_{w \notin \mathbf{D}} b_w(x, y) - \int_{\mathbf{S}^{n-1}} b_0(u, y) d\sigma(u) - \int_{\mathbf{S}^{m-1}} b_0(x, v) d\sigma(v) \\ &\quad - \sum_{w \notin \mathbf{D}} \left[\int_{\mathbf{S}^{n-1}} b_w(u, y) d\sigma(u) + \int_{\mathbf{S}^{m-1}} b_w(x, v) d\sigma(v) \right] + \\ &\quad \int \int_{\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}} b_0(u, v) d\sigma(u) d\sigma(v) + \sum_{w \notin \mathbf{D}} \int \int_{\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}} b_w(u, v) d\sigma(u) d\sigma(v) \end{aligned}$$

$$\begin{aligned} \Omega_w(x, y) &= (\lambda_w)^{-1} \left\{ b_w(x, y) - \int_{\mathbf{S}^{n-1}} b_w(u, y) d\sigma(u) \right. \\ &\quad \left. - \int_{\mathbf{S}^{m-1}} b_w(x, v) d\sigma(v) + \int \int_{\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}} b_w(u, v) d\sigma(u) d\sigma(v) \right\}. \end{aligned}$$

Now, for $w \in \mathbf{D}$, set $\theta_w = \lambda_w$ and set $\theta_0 = 1$. Thus, it is easy to see that the following holds:

$$\int_{\mathbf{S}^{n-1}} \Omega_w(u, \cdot) d\sigma_n(u) = \int_{\mathbf{S}^{m-1}} \Omega_w(\cdot, v) d\sigma_m(v) = 0, \quad (3.1)$$

$$\|\Omega_w\|_1 \leq C, \quad \|\Omega_w\|_2 \leq C2^{4(w+1)}, \quad (3.2)$$

$$\Omega(x, y) = \sum_{w \in \mathbf{D} \cup \{0\}} \theta_w \Omega_w(x, y), \quad (3.3)$$

$$\sum_{w \in \mathbf{D} \cup \{0\}} (w+1)^2 \theta_w \leq C \|\Omega\|_{L(\log L)^2(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})}. \quad (3.4)$$

For $w \in \mathbf{D} \cup \{0\}$, let $T_{(\Gamma_\phi, \Gamma_\varphi)}^w$ be the operator defined by (1.1) with Ω replaced by Ω_w . Then the operator $T_{(\Gamma_\phi, \Gamma_\varphi)}$ is decomposed as follows:

$$T_{(\Gamma_\phi, \Gamma_\varphi)} f(x, y) = \sum_{w \in \mathbf{D} \cup \{0\}} \theta_w T_{(\Gamma_\phi, \Gamma_\varphi)}^w f(x, y). \quad (3.5)$$

Now by Theorem B, we have

$$\left\| T_{(\Gamma_\phi, \Gamma_\varphi)}^w f \right\|_p \leq (w+1)^2 C \|f\|_p \quad (3.6)$$

for all $p \in (1, \infty)$; which when combined with (3.5) imply that

$$\left\| T_{(\Gamma_\phi, \Gamma_\varphi)} f \right\|_p \leq C \left\{ \sum_{w \in \mathbf{D} \cup \{0\}} \theta_w (w+1)^2 \right\} \|f\|_p \quad (3.7)$$

for all $p \in (1, \infty)$. Hence, by (3.4) and (3.7), the result follows.

Proof of Theorem A. A proof of Theorem A can be obtained using Plancherel's formula, the decomposition (3.1)-(3.5), the estimates (2.16)-(2.17), and Remark 2.3. We omit the details.

4 Additional results

In this section, we shall present some results that can be obtained by minor modifications of the argument employed in the previous sections.

For $1 < \gamma < \infty$, we let $\Delta_\gamma(\mathbf{R}^+ \times \mathbf{R}^+)$ be the set of all measurable functions $h : \mathbf{R}^+ \times \mathbf{R}^+ \rightarrow \mathbf{R}$ that satisfy

$$\|h\|_{\Delta_\gamma} = \sup_{R_1, R_2 > 0} \left[(R_1 R_2)^{-1} \int_0^{R_1} \int_0^{R_2} |h(t, s)|^\gamma dt ds \right]^{\frac{1}{\gamma}} < \infty.$$

Also, we let $\Delta_\infty(\mathbf{R}^+ \times \mathbf{R}^+) = L^\infty(\mathbf{R}^+ \times \mathbf{R}^+)$. For more information about the space Δ_γ in the one parameter setting, we refer the reader to consult ([6], [8]).

It is clear that $L^\infty(\mathbf{R}^+ \times \mathbf{R}^+) \subset \Delta_\gamma(\mathbf{R}^+ \times \mathbf{R}^+) \subset \Delta_\beta(\mathbf{R}^+ \times \mathbf{R}^+)$ for any $\beta < \gamma$ and the inclusions are proper.

For $h \in \Delta_\gamma(\mathbf{R}^+ \times \mathbf{R}^+)$, $\gamma > 1$, we let $T_{(\Gamma_{\phi, n}, \Gamma_{\varphi, m})}^h$ be the operator given by (1.1) with Ω is replaced by Ωh , $d = n+1$, $l = m+1$, $\Gamma_{n, d}(u) = (u, \phi(|u|))$, and

$\Gamma_{m,l}(v) = (v, \varphi(|v|))$. Then we have the following generalization of Theorem C:

Theorem 4.1. *Suppose that ϕ and φ are C^2 , convex increasing. If $\Omega \in L(\log L)^2(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$ and satisfies (1.2)-(1.3), then $T_{(\Gamma_\phi, \Gamma_\varphi)}^h$ is bounded on $L^p(\mathbf{R}^{n+1} \times \mathbf{R}^{m+1})$ for all p satisfying $|1/p - 1/2| < \min\{1/2, 1/\gamma'\}$.*

It is clear that when $\gamma \geq 2$, then $T_{(\Gamma_{\phi,n}, \Gamma_{\varphi,m})}^h$ in Theorem 4.1 is bounded on $L^p(\mathbf{R}^{n+1} \times \mathbf{R}^{m+1})$ for all $1 < p < \infty$. Hence, if $h = 1$, then Theorem 4.1 reduces to Theorem C.

Proof. We only need to show that all requirements needed to repeat the same proof of Theorem B for p satisfying $|1/p - 1/2| < \min\{1/2, 1/\gamma'\}$. In order to do so, let $\{m_{j,k,h} : j, k \in \mathbf{Z}\}$ and $M_{\beta, \phi, \varphi, h}^*$ be the sequence of measures and the maximal function given in (2.15) and (2.25) respectively with Ω is replaced by Ωh . Then, by Hölder's inequality and the observation that $\|m_{j,k,h}\| \leq \beta^2 \|h\|_{\Delta_\gamma} \|\Omega\|_{L^1}$, we get that Proposition 2.2 still holds for the measures $\{m_{j,k,h} : j, k \in \mathbf{Z}\}$. On the other hand, by Lemma 2.6 and Hölder's inequality, we have that the maximal function $M_{\beta, \phi, \varphi, h}^*$ is bounded on L^p for $\gamma' < p < \infty$. This completes the proof.

Now, the argument in this paper implies the following result:

Theorem 4.2. *Suppose that there exists nonzero real numbers d_1 and d_2 , and positive constants $\{C_{i,l} : 1 \leq i \leq 4, 1 \leq l \leq 2\}$ such that the functions ϕ and φ satisfy*

$$|\phi_l(t)| \leq C_{1,l} t^{d_l}, |\phi_l''(t)| \leq C_{2,l} t^{d_l-2}, C_{3,l} t^{d_l-1} \leq |\phi_l'(t)| \leq C_{4,l} t^{d_l-1} \quad (4.1)$$

for $t \in (0, \infty)$ where $\phi_1 = \phi$ and $\phi_2 = \varphi$. If $\Omega \in L(\log L)^2(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$ and satisfies (1.2)-(1.3), then $T_{(\Gamma_\phi, \Gamma_\varphi)}^h$ is bounded on $L^p(\mathbf{R}^{n+1} \times \mathbf{R}^{m+1})$ for all p satisfying $|1/p - 1/2| < \min\{1/2, 1/\gamma'\}$.

To prove Theorem 4.2, we need the following analog of Lemma 2.5 which can be proved by using the arguments on ([4], p. 168-169, [6], p. 559]):

Lemma 4.3. *Let β and Ψ be as in Lemma 2.5. If ψ satisfies (4.1) with ϕ_l is replaced by ψ . Then the estimate (2.24) holds for $1 < p \leq \infty$.*

Proof (of Theorem 4.2). By Proposition 2.2, Remark 2.3, and the argument in the proof of Theorem 4.1, we only need to show that the result of Lemma 2.6 still holds if the convexity assumptions on the functions ϕ

and φ are replaced by the conditions (4.1). To see this, we repeat exactly the same argument as that in the proof of Lemma 2.6, but this time we use Lemma 4.3 instead of Lemma 2.5. This completes the proof.

By arguing inductively and using the estimates in this paper, we can easily prove the following result:

Theorem 4.4. *Suppose that ϕ and φ are generalized polynomials, i.e., $\phi(t) = \mu_1 t^{d_1} + \dots + \mu_l t^{d_l}$ and $\varphi(t) = \tilde{\mu}_1 t^{\tilde{d}_1} + \dots + \tilde{\mu}_s t^{\tilde{d}_s}$ for some $l, s \in \mathbf{N}$, distinct real numbers d_1, \dots, d_l , distinct real numbers $\tilde{d}_1, \dots, \tilde{d}_s$, and real numbers $\mu_1, \dots, \mu_l, \tilde{\mu}_1, \dots, \tilde{\mu}_s$. If $\Omega \in L(\log L)^2(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$ and satisfies (1.2)-(1.3), then $T_{(\Gamma_\phi, \Gamma_\varphi)}^h$ is bounded on $L^p(\mathbf{R}^{n+1} \times \mathbf{R}^{m+1})$ for all p satisfying $|1/p - 1/2| < \min\{1/2, 1/\gamma'\}$. Moreover, the L^p bounds are independent of the coefficients $\mu_1, \dots, \mu_l, \tilde{\mu}_1, \dots, \tilde{\mu}_s$.*

We end this section by presenting the analogy of our results above for the related truncated maximal functions. The truncated maximal function corresponding to the operator $T_{(\Gamma_{\phi,n}, \Gamma_{\varphi,m})}$ is the operator $(T_{(\Gamma_{\phi,n}, \Gamma_{\varphi,m})})^*$ given by

$$\begin{aligned} & (T_{(\Gamma_{\phi,n}, \Gamma_{\varphi,m})})^* f(x, y) = \\ & = \sup_{\epsilon > 0, \delta > 0} \iint_{|u| > \epsilon, |v| > \delta} f(x - \Gamma_{n,d}(u), y - \Gamma_{m,l}(v)) |u|^{-n} |v|^{-m} \Omega(u, v) \, dudv \end{aligned}$$

where $d = n + 1$, $l = m + 1$, $\Gamma_{n,d}(u) = (u, \phi(|u|))$, and $\Gamma_{m,l}(v) = (v, \varphi(|v|))$.

By the estimates obtained in this paper and the techniques developed in ([1], [2], [4]), one can easily prove the following result:

Theorem 4.5. *Suppose that $\Omega \in L(\log L)^2(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$ and satisfies (1.2)-(1.3). If the functions ϕ and φ are C^2 convex increasing, or satisfy (4.1), or generalized polynomials, then the corresponding operator $(T_{(\Gamma_{\phi,n}, \Gamma_{\varphi,m})})^*$ is bounded on $L^p(\mathbf{R}^{n+1} \times \mathbf{R}^{m+1})$ for all $p \in (1, \infty)$. Moreover, if ϕ and φ are generalized polynomials, then the L^p bounds are independent of the coefficients.*

References

- [1] H. Al-Qassem, *L^p estimates for singular integrals with kernels belonging to certain Block spaces*, Ph. D Thesis, University of Pittsburgh, 1999.

- [2] H. Al-Qassem and Y. Pan, *L^p boundedness for singular integrals with rough kernels on product domains*, Hokkaido Mathematical Journal 31 (2002), 555-613.
- [3] A. Al-Salman, H. Al-Qassem, and Y. Pan, *Singular Integrals on Product Domains*, submitted.
- [4] A. Al-Salman and Y. Pan, *Singular integrals with rough kernels in $Llog^+L(\mathbf{S}^{n-1})$* , J. London Math. Soc. (2) 66 (2002) 153-174.
- [5] J. Duoandikoetxea, *Multiple singular integrals and maximal functions along hypersurfaces*, Ann. Ins. Fourier (Grenoble) 36 (1986) 185-206.
- [6] J. Duoandikoetxea and J. L. Rubio de Francia, *Maximal functions and singular integral operators via Fourier transform estimates*, Invent. Math. 84 (1986), 541-561.
- [7] D. Fan, K. Guo, and Y. Pan, *Singular integrals with rough kernels on product spaces*, Hokkaido Mathematical Journal, 28 (1999), 435-460.
- [8] D. Fan and Y. Pan, *Singular integral operators with rough kernels supported by sub-varieties*, Amer J. Math. 119 (1997), 799-839.
- [9] R. Fefferman, *Singular integrals on product domains*, Bull. Amer. Math. Soc. 4 (1981), 195-201.
- [10] R. Fefferman and E. M. Stein, *Singular integrals on product spaces*, Adv. in Math., 45 (1982), 117-143.
- [11] Y. Jiang and S. Lu, *A class of singular integral operators with rough kernels on product domains*. Hokkaido Mathematical Journal, Vol. 24 (1995), 1-7.
- [12] W. Kim, S. Wainger, J. Wright, and S. Ziesler, *Singular integrals and maximal functions associated to surfaces of revolution*, Bull. London Math. Soc. 28 (1996), 291-296.
- [13] E. M. Stein, *Harmonic analysis real-variable methods, orthogonality and oscillatory integrals*, Princeton University Press, Princeton, NJ, 1993.

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