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FLAT SURFACES IN HYPERBOLIC SPACE AS NORMAL SURFACES TO A CONGRUENCE OF GEODESICS

PEDRO ROITMAN

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Abstract. We first present an alternative derivation of a local Weierstrass representation for flat surfaces in the real hyperbolic three-space, $\boldsymbol{H}^3$, using as a starting point an old result due to Luigi Bianchi. We then prove the following: let $M \subset \boldsymbol{H}^3$ be a flat compact connected smooth surface with $\partial M \neq \emptyset$, transversal to a foliation of $\boldsymbol{H}^3$ by horospheres. If, along ∂M, M makes a constant angle with the leaves of the foliation, then M is part of an equidistant surface to a geodesic orthogonal to the foliation. We also consider the caustic surface associated with a family of parallel flat surfaces and prove that the caustic of such a family is also a flat surface (possibly with singularities). Finally, a rigidity result for flat surfaces with singularities and a geometrical application of Schwarz's reflection principle are shown.

Introduction. In the early 1970s [8] and Volkov and Vladimirova [11] proved independently that in the real hyperbolic three-space $\boldsymbol{H}^3$, i.e. the simply connected real space-form of dimension three with constant curvature -1, the only complete examples of flat surfaces (flat surfaces for short) are horospheres and the equidistant surfaces to geodesics. This lack of complete examples may have contributed to almost three decades without new works on them.

A Weierstrass representation for flat surfaces was derived by Gálvez et al [4]. They have also shown that there is a correspondence between flat surfaces and holomorphic Legendrian curves in $PSL(2, \boldsymbol{C})$. This is interesting, since it somehow connects hyperbolic geometry with complex contact geometry. More recently, Kokubu et al. have shown an alternative way to obtain a Weierstrass representation and have explored some connections between hyperbolic and contact geometry [5, 6]. Both works are inspired by the theory for constant mean curvature one (CMC-1) surfaces in $\boldsymbol{H}^3$, and by the fact shown in [4] that by choosing an appropriate conformal structure for a flat surface, namely that induced by its second fundamental form, one can characterize flat surfaces as those having a holomorphic Gauss map.

We first show how a local version of a Weierstrass representation (Theorem 2.2) is obtained using an old result of Luigi Bianchi, which roughly says the following: the correspondence between the two hyperbolic Gauss maps of a surface $M \subset \boldsymbol{H}^3$ is weakly conformal if and only if M is flat (except for totally umbilical surfaces). We note that integration of the holomorphic data is required to determine a flat surface.

Since there are essentially two possible examples of complete flat surfaces, the Weierstrass representation will furnish, in general, surfaces having singularities. Therefore, if one

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seeks global results, it is natural to consider smooth surfaces with boundary and also surfaces with singularities. For smooth surfaces with boundary we have the following.

THEOREM 2.6. *Let M be a flat compact connected smooth surface with boundary, $\partial M \neq \emptyset$, transversal to a foliation of $\boldsymbol{H}^3$ by horospheres. If, along ∂M, M makes a constant angle with the leaves of the foliation, then M is part of an equidistant surface to a geodesic orthogonal to the foliation.*

It is a well-known fact that the parallel surfaces at distance t, $\{M_t\}$, to a flat surface M are also flat surfaces; see, for instance, [10]. As we shall see, it is natural to study such a family of parallel flat surfaces $\{M_t\}$ instead of a particular member of the family. As mentioned, flat surfaces constructed via the Weierstrass representation will have singularities (except for horospheres and equidistants to a geodesic). These are the singularities treated in this paper.

Let M be a flat surface constructed via the Weierstrass representation. We will consider curves $\gamma \subset M$, that are smooth space curves in $\boldsymbol{H}^3$ and are singular curves of M. This means that M is not an immersion along γ. Such curves will be called *smooth singular curves.*

Now consider the family of parallel surfaces $\{M_t\}$ to a flat surface M, and the set of singularities of all $\{M_t\}$. We call this set the *caustic M_{C} associated with the family $\{M_t\}$*. Note that smooth singular curves γ_t of M_t are curves also on M_{C}. The caustic M_{C} might be a degenerate object such as a curve, but in general it is a surface with singularities. The reader is directed to [1] for a classification of such singularities. In the classical literature caustics are sometimes referred to as surfaces of centers, focal surfaces or evolutes.

For a family of flat surfaces $\{M_t\}$ we have the following relation between $\{M_t\}$ and M_{C}.

THEOREM 3.3. *Let $\{M_t\}$ be a family of parallel flat surfaces and M_{C} the caustic associated with this family. Then M_{C} is flat at smooth points. Furthermore, for some t, let γ_t be a smooth singular curve common to M_t and M_{C}. If M_{C} is smooth in a neighborhood of γ_t, then γ_t is a geodesic of M_{C}.*

We note that one may find a very interesting alternative approach to flat surfaces with singularities in [6].

The consideration of caustics motivates a better understanding of the principal radii of curvature of a flat surface. In Theorem 3.6 we prove that, for umbilic-free flat surfaces, the radius of curvature associated with the principal curvature that is greater than one is a harmonic function, for a chosen conformal structure.

Two applications of Theorem 3.3 are given. First, we will derive an alternative Weierstrass representation for flat surfaces that requires no integration of the holomorphic data (Theorem 3.5). This follows easily from Theorems 2.2 and 3.3, and from the fact that to obtain a caustic from a surface we only have to take first derivatives.

Since flat surfaces are constructed via holomorphic data, it is reasonable to expect some rigidity results. Our second application (Theorem 3.8) confirms this expectation and roughly states that if two flat surfaces have a common singular curve then they are the same surface.

Finally, we give a geometrical application of Schwarz's reflection principle for flat surfaces (Theorem 4.1). Roughly stated, it says that if we have a planar geodesic on a flat surface M, then the plane containing this geodesic is a symmetry plane of M.

This paper is organized as follows. In Section 1 we present the models of $\boldsymbol{H}^3$ to be used and review the adapted moving frames for surfaces in $\boldsymbol{H}^3$. We also present the hyperbolic Gauss maps associated with a surface in $\boldsymbol{H}^3$ and refine Bianchi's characterization of flat surfaces in terms of the hyperbolic Gauss maps. In Section 2 we regard flat surfaces as normal surfaces to a congruence of geodesics and derive a Weierstrass representation for flat surfaces (Theorem 2.2). Some examples are shown and Theorem 2.6 is proved. In Section 3 we discuss caustics of flat surfaces and give proofs of Theorems 3.3, 3.5, 3.6 and 3.8. Section 4 contains the above-mentioned geometrical application of Schwarz's reflection principle (Theorem 4.1).

The author thanks Ricardo Uribe for our discussion regarding caustics.

1. Preliminaries. 1.1. Structure equations in hyperbolic three-space. In some of our calculations we shall use moving frames, and the notation used here coincides with that in [3]. Let e_0, e_1, e_2 and e_3 be a positive oriented frame in $\boldsymbol{L}^4$ (Lorentzian space) satisfying

$$\langle e_\alpha, e_\beta\rangle = \begin{cases} -1 & \text{if } \alpha,\ \beta = 0\,, \\ 0 & \text{if } \alpha \neq \beta\,, \\ 1 & \text{if } \alpha = \beta = 1, 2 \text{ or } 3\,, \end{cases} \tag{1}$$

where $\langle\cdot\,,\cdot\rangle$ denotes the standard Lorentzian inner product. Let ω^i be the dual forms of e_i and ω^α_β the connection forms defined by

$$de_\alpha = e_\beta \omega^\beta_\alpha\,.$$

Using the index range $1 \le i, j, k \le 3$ and writing ω^i for ω^i_0, we have

$$\begin{aligned} de_0 &= e_i\omega^i\,, \\ de_i &= e_0\omega^i + e_j\omega^j_i\,, \\ \omega^i_j &= -\omega^j_i\,. \end{aligned} \tag{2}$$

Differentiation of (2) yields

$$\begin{aligned} d\omega^i &= -\omega^i_j \wedge \omega^j\,, \\ d\omega^i_j &= -\omega^i_k \wedge \omega^k_j - \omega^i \wedge \omega^j\,. \end{aligned} \tag{3}$$

Now, let M denote a connected smooth oriented surface immersed in $\boldsymbol{H}^3$. We shall regard e_0 as its position vector, e_1 and e_2 spanning the tangent plane of M, and e_3 as a normal field to M and in the tangent space of $\boldsymbol{H}^3$. Restricting our forms to M, we obtain $\omega^3 = 0$. From $d\omega^3 = 0$ together with Cartan's lemma we have

$$\begin{pmatrix}\omega^3_1\\ \omega^3_2\end{pmatrix} = \begin{pmatrix}h_{11} & h_{12}\\ h_{21} & h_{22}\end{pmatrix}\begin{pmatrix}\omega^1\\ \omega^2\end{pmatrix}, \tag{4}$$

where the smooth functions h_{ij} are the coefficients of the second fundamental form. Recall that the *Gaussian* and *mean curvatures* are given, respectively, by

$$K = -1 + h_{11}h_{22} - h_{12}^2\,, \tag{5}$$

$$H = (h_{11} + h_{22})/2\,. \tag{6}$$

For more details we refer the reader to [3].

1.2. Upper half-space model. To simplify our local calculations, as well as to visualize some examples, we use the standard upper half-space model, and identify the ideal plane $\partial_\infty \boldsymbol{H}^3 \setminus \{\infty\}$ with the complex plane $\boldsymbol{C}$ with coordinates $z = x + iy$. The third coordinate will be denoted by w, so a point in the upper half-space will be written as (x, y, w), $w > 0$.

1.3. Hyperbolic Gauss maps and Bianchi's characterization of flat surfaces. Let $p \in M$, where M is a smooth oriented surface immersed in $\boldsymbol{H}^3$. Consider an oriented geodesic of $\boldsymbol{H}^3$ passing through p and orthogonal to M at p. This geodesic has two limit points in $\partial_\infty \boldsymbol{H}^3$, the ideal boundary of $\boldsymbol{H}^3$. If we fix an orientation for the geodesics orthogonal to M, we have two maps defined on M. Namely, associate with $p \in M$ the initial limit point and final limit point of the oriented geodesic orthogonal to M.

If we fix an orientation of M, this determines an orientation for the geodesics orthogonal to M. For a fixed orientation we shall denote by $G^+ : M \to \partial_\infty \boldsymbol{H}^3$ (respectively $G^- : M \to \partial_\infty \boldsymbol{H}^3$) the map that associates with $p \in M$ the final (respectively initial) limit point of the geodesic (oriented by the chosen orientation of M) through p and orthogonal to M. These are called the *hyperbolic Gauss maps of M*.

Now, let M be a flat surface. From (5) we see that the extrinsic curvature of M is one and therefore the mean curvature of M never vanishes. From now on we shall orient M by its mean curvature vector so that we have $H \geq 1$.

In terms of moving frames we write $G^+ = [e_0 + e_3]$ and $G^- = [e_0 - e_3]$, where the brackets indicate the equivalence class under the equivalence relation of being in a straight line in the positive null cone of $\boldsymbol{L}^4$. It is standard to identify $\partial_\infty \boldsymbol{H}^3$ with the quotient of the positive null cone under this equivalence relation. The reader may consult [3] for more details.

LEMMA 1.1. *Let M be a flat surface, oriented by its mean curvature vector. Then $G^- : M \to \partial_\infty \boldsymbol{H}^3$ is an immersion, while $G^+ : M \to \partial_\infty \boldsymbol{H}^3$ is an immersion except at umbilical points.*

PROOF. We have

$$dG^\pm = de_0 \pm de_3 = (\omega^1 \mp \omega_1^3)e_1 + (\omega^2 \mp \omega_2^3)e_2\,. \tag{7}$$

Using (3), we may write

$$\begin{aligned}\langle dG^\pm, dG^\pm\rangle = ((1 \mp h_{11})^2 + h_{12}^2)(\omega^1)^2 \mp h_{12}(2 \mp (h_{11} + h_{22}))\omega^1\omega^2 \\ + ((1 \mp h_{22})^2 + h_{12}^2)(\omega^2)^2.\end{aligned}$$

After some manipulation, using (6) and (7), the discriminants $D^{\pm}$ of the corresponding quadratic forms are

$$D^{\pm} = 4(H \mp 1)^2 . \tag{8}$$

For the chosen orientation $H \geq 1$ and, for $p \in M$, $H(p) = 1$ if and only if p is an umbilical point. □

We now state an old result due to Luigi Bianchi that characterizes flat surfaces in terms of a conformal map. Up to notation and terminology, this result is precisely the one proved many years ago in Bianchi's *Lezioni* [2].

PROPOSITION 1.2 (Bianchi [2]). *Let M be a smooth connected flat surface oriented by its mean curvature. Then the local correspondence $G^- \to G^+$ is weakly conformal. Conversely, if M is a connected surface such that there is an orientation that makes the correspondence $G^- \to G^+$ weakly conformal, then either M is totally umbilical or M is a flat surface.*

PROOF. Let us suppose that there is an orientation of M such that locally the correspondence $G^- \to G^+$ is well defined, in other words, that G^- is an immersion. The correspondence $G^- \to G^+$ will be weakly conformal if and only if we have

$$\langle dG^-, dG^- \rangle = \lambda^2 \langle dG^+, dG^+ \rangle , \tag{9}$$

where λ is a smooth real-valued function locally defined on M. From (7) and (9) we may write this as

$$((1 - h_{11})^2 + h_{12}^2) = \lambda^2((1 + h_{11})^2 + h_{12}^2) , \tag{10}$$

$$((1 - h_{22})^2 + h_{12}^2) = \lambda^2((1 + h_{22})^2 + h_{12}^2) \tag{11}$$

and

$$h_{12}(-2 + (h_{11} + h_{22})) = \lambda^2 h_{12}(2 + (h_{11} + h_{22})) . \tag{12}$$

If $\lambda(p) = 0$ for $p \in M$, we must have

$$((1 - h_{11})^2 + h_{12}^2) = ((1 - h_{22})^2 + h_{12}^2) = 0 ,$$

which implies that $h_{11}(p) = h_{22}(p) = 1$ and $h_{12}(p) = 0$, so that $K(p) = 0$ and we have a flat umbilical point.

If $\lambda \neq 0$, we may use (10), (11) and (12) to conclude that $G^- \to G^+$ is weakly conformal if and only if

$$2(h_{22} - h_{11})(1 - (h_{11}h_{22} - h_{12}^2)) + h_{12}(h_{11}^2 - h_{22}) = 0$$

and

$$h_{11}h_{22} - h_{12}^2 = 1 ,$$

in the case $h_{12} \neq 0$. In other words, if for $p \in M$ we have $h_{12}(p) \neq 0$, then $K(p) = 0$. Also, if $h_{12}(p) = 0$, then

$$(h_{22} - h_{11})(1 - h_{11}h_{22}) = 0$$

at p. This means that either $h_{11}(p) = h_{22}(p)$, so that p is umbilical, or $h_{11}(p)h_{22}(p) = 1$, so that $K(p) = 0$.

From Lemma 1.1, if M is flat, it can be oriented by its mean curvature vector and the local correspondence $G^- \to G^+$ is well defined. From the discussion above this correspondence is weakly conformal.

Now suppose that M is not flat and that there is an orientation of M such that the local correspondence $G^- \to G^+$ is weakly conformal. Consider the open set $U = \{p \in M \mid K(p) \neq 0\}$. Let V be a connected component of U. Then V is part of a totally umbilical surface and for such a surface K is constant. Hence, by the continuity of K, $\bar{V} \subset U$ and therefore U is also closed. Since M is connected, $U = M$ and M is totally umbilical. □

2. Flat surfaces as normal surfaces to a congruence of geodesics. 2.1. A Weierstrass representation from Bianchi's characterization. Bianchi's result (Proposition 1.2) suggests a method for constructing surfaces from a pair of meromorphic maps G^- and G^+ defined on a Riemann surface F. We regard them as holomorphic maps $F \to S^2$, and look at the unit sphere S^2 as the ideal boundary $\partial_\infty \boldsymbol{H}^3$. For each $p \in F$, if we have $G^-(p) \neq G^+(p)$, then the pair of points $G^-(p)$ and $G^+(p)$ define a unique geodesic of $\boldsymbol{H}^3$, i.e., the one having these points as limit points. Now note that, if we can find a smooth surface M, normal at each of its points to a geodesic defined by G^- and G^+, and we suppose that locally the correspondence $G^- \to G^+$ is well defined, then, by Bianchi's proposition, M will be a flat surface.

One may compare this approach with that in [4] and [5], where flat surfaces are viewed as projections of holomorphic Legendrian curves in $SL(2, \boldsymbol{C})$, for a canonical complex contact structure. These are easily seen to be equivalent (see Remark 2.3). For this reason we will content ourselves with a local version of the Weierstrass representation, since a global formulation is already available in [4] or [5].

Our reason to present this derivation of a Weierstrass representation is twofold. In the first place we believe that an elementary geometrical proof, stemming from the ideas of a brilliant geometer such as Bianchi, is always welcome. Furthermore, using this classical approach, we are led to consider a harmonic function, admitting a simple geometrical interpretation. This harmonic function will be used in the proof of Theorem 2.6.

REMARK 2.1. It turns out that, in general, given the meromorphic maps G^- and G^+, there exists no smooth flat surface normal to the corresponding geodesics. In this context it is then natural to consider flat surfaces with singularities.

To simplify our local calculations, and to visualize some examples, we will now use the upper half-space model. One then has the following.

THEOREM 2.2. *Let $\Omega \subset \boldsymbol{C}$ be simply connected, $\tau \in R$ and $f : \Omega \to \boldsymbol{C}$ be holomorphic such that $f(z) \neq z$. Then the map*

$$X(z) = c(z) + R(z)(\cos\theta(z)v_1(z) + \sin\theta(z)v_3)\,, \tag{13}$$

where

$$\begin{aligned} c(z) &= \frac{f(z)+z}{2}\,, \quad R(z) = \frac{|f(z)-z|}{2}\,, \quad v_1(z) = \frac{f(z)-z}{|f(z)-z|}\,,\\ \cos\theta(z) &= \frac{1-(\exp 2\,\mathrm{Re}(\int (f(z)-z)^{-1}\,dz-\tau))|f(z)-z|^2}{1+(\exp 2\,\mathrm{Re}(\int (f(z)-z)^{-1}\,dz-\tau))|f(z)-z|^2}\,,\\ \sin\theta(z) &= \frac{2(\exp \mathrm{Re}(\int (f(z)-z)^{-1}\,dz-\tau))|f(z)-z|}{1+(\exp 2\,\mathrm{Re}(\int (f(z)-z)^{-1}\,dz-\tau))|f(z)-z|^2}\,, \end{aligned} \tag{14}$$

and v_3 is a constant vector orthogonal to the z-plane ($\partial_\infty \boldsymbol{H}^3 \setminus \{\infty\}$), yields a parametrized flat surface in the upper half-space model on a subset $\Lambda \subset \Omega$ for which $X(z)$ is an immersion.

If Ω is not simply connected, (13) *defines a parametrized flat surface in the upper half-space model if the integral*

$$\int_\gamma \frac{dz}{f(z)-z} \in i\boldsymbol{R}\,,$$

where γ is any closed loop in Λ.

PROOF. Using the upper half-space model, identify $\boldsymbol{C}$ with $\partial_\infty \boldsymbol{H}^3 \setminus \{\infty\}$, and consider the two-parameter family of geodesics defined by f, that is, those having z and $f(z)$ as limit points. We may parametrize this family of geodesics as

$$X(z,\theta) = c(z) + R(z)(\cos\theta\, v_1(z) + \sin\theta\, v_3)\,, \tag{15}$$

where $c(z) = (f(z)+z)/2$ is the center of the Euclidean circle orthogonal to $\partial_\infty \boldsymbol{H}^3 \setminus \{\infty\}$; $R(z) = |f(z)-z|/2$ is the Euclidean radius of the same circle. Also, $v_1(z) = (f(z)-z)/|f(z)-z|$ and v_3 are unit vectors, in the standard Euclidean metric, defining the plane of the circle. The vector v_3 is constant and orthogonal to the plane $\partial_\infty \boldsymbol{H}^3 \setminus \{\infty\}$; see Figure 1.

According to Bianchi's characterization of flat surfaces (Proposition 1.2), if we can determine a surface M orthogonal to the family of geodesics, it will be flat at smooth points. In terms of our parametrization, one has to determine a function $\theta(z)$ so that the parametrized

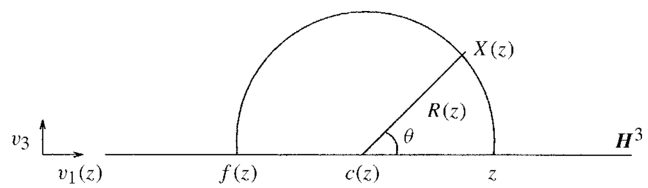

FIGURE 1. Parametrized congruence of geodesics.

surface

$$X(z) = X(z, \theta(z))$$

is orthogonal to the family of geodesics. A necessary condition is

$$\langle dX, -\sin\theta v_1(z) + \cos\theta v_3\rangle_e = 0\,, \tag{16}$$

where $\langle\cdot\,,\cdot\rangle_e$ is the standard Euclidean inner product. From (15), we have

$$dX = dc + dR(\cos\theta v_1 + \sin\theta v_3) + R(d\theta(-\sin\theta v_1 + \cos\theta v_3) + \cos\theta\, dv_1)\,.$$

We may rewrite (16) as

$$R\,d\theta = \sin\theta\langle dc, v_1\rangle_e\,. \tag{17}$$

So, in our coordinates $z = (x, y)$, (17) is written as a differential system for θ:

$$\begin{aligned} \theta_x &= \frac{\sin\theta}{R}\langle c_x, v_1\rangle_e\,,\\ \theta_y &= \frac{\sin\theta}{R}\langle c_y, v_1\rangle_e\,. \end{aligned} \tag{18}$$

Using the expressions for R and c in (14) and writing $f(z) = u(z) + iv(z)$, we obtain

$$\begin{aligned} c_x &= \left(\frac{u_x+1}{2}, \frac{v_x}{2}\right),\\ c_y &= \left(\frac{u_y}{2}, \frac{v_y+1}{2}\right),\\ \frac{v_1}{R} &= 2\frac{(u-x, v-y)}{(u-x)^2+(v-y)^2}\,. \end{aligned}$$

If we define $\alpha = u - x$ and $\beta = v - y$, the system (18) becomes

$$\begin{aligned} \frac{\theta_x}{\sin\theta} &= \frac{\alpha(\alpha_x+2)+\beta(\beta_x)}{\alpha^2+\beta^2} = \frac{1}{2}\ln(\alpha^2+\beta^2)_x + \frac{2\alpha}{\alpha^2+\beta^2}\,,\\ \frac{\theta_y}{\sin\theta} &= \frac{\alpha(\alpha_y)+\beta(\beta_y+2)}{\alpha^2+\beta^2} = \frac{1}{2}\ln(\alpha^2+\beta^2)_y + \frac{2\beta}{\alpha^2+\beta^2}\,. \end{aligned} \tag{19}$$

The general solution $\theta(z)$ of (19) is then easily seen to satisfy

$$\ln\left(\frac{1-\cos\theta(z)}{\sin\theta(z)}\right) + \tau = \ln|f(z)-z| + \mathrm{Re}\left(\int\frac{1}{f(z)-z}\,dz\right), \tag{20}$$

where τ is a real constant, assuming that $\mathrm{Re}(\int(f(z)-z)^{-1}\,dz)$ is well defined.

Solving (20) for $\cos\theta(z)$, we have

$$\cos\theta(z) = \frac{1-(\exp 2\,\mathrm{Re}(\int(f(z)-z)^{-1}\,dz-\tau))|f(z)-z|^2}{1+(\exp 2\,\mathrm{Re}(\int(f(z)-z)^{-1}\,dz-\tau))|f(z)-z|^2}\,,$$

and therefore

$$\sin\theta(z) = \frac{2(\exp\mathrm{Re}(\int(f(z)-z)^{-1}\,dz-\tau))|f(z)-z|}{1+(\exp 2\,\mathrm{Re}(\int(f(z)-z)^{-1}\,dz-\tau))|f(z)-z|^2}\,. \qquad \square$$

REMARK 2.3. If one considers the Weierstrass representation in [5, Theorem 3.3], rewrites it in terms of the upper half-space model (instead of the Hermitian model used there) and sets $G = z$ and $G_* = f(z)$, then one obtains our Theorem 2.2. I thank Masaaki Umehara for this observation.

REMARK 2.4. If we start with f meromorphic instead of holomorphic, we also get a surface for which, at the poles of f, the Euclidean tangent plane, in the chosen upper half-space model, becomes horizontal. It is also clear that points such that $f(z) = z$ correspond to ends of the surface, and points such that $f'(z) = 0$ correspond to umbilical points.

REMARK 2.5. The Weierstrass representation for flat surfaces derived in [4] was inspired by the one known for CMC-1 surfaces in $\boldsymbol{H}^3$. It is worth mentioning that Bianchi considered a Weierstrass representation for CMC-1 surfaces, although he did not write it explicitly; see [2] or [7].

2.2. Examples. (1) *The complete examples*. Families of horospheres are easily seen to correspond to $f(z) = c \in \boldsymbol{C}$. A family of equidistant surfaces to a geodesic corresponds to $f(z) = -z$.

(2) *A periodic example*. Let $f(z) = z + 2$. The surface corresponding to such an f has a horocycle, a non-compact curve, as a singular curve. Furthermore, this surface is invariant under a parabolic translation and curiously the profile curve, in the upper half-space model, is a tractrix. In Figure 2 we show the profile curve and some of the geodesics defined by f.

More elaborate examples are constructed in [6].

2.3. The constant-angle theorem. Let M be a flat surface transversal to a foliation of $\boldsymbol{H}^3$ by horospheres. Without loss of generality, we may describe the above foliation as horizontal planes in the upper half-space model. If M is transversal to the foliation, then for each $p \in M$ there is a unique $q \in \boldsymbol{H}^3$ belonging to the geodesic orthogonal to M at p and to a geodesic orthogonal to the foliation; see Figure 3.

Let $d : M \to \boldsymbol{R}$ be the function defined by $d(p) = \mathrm{dist}_{\boldsymbol{H}^3}(p, q)$. We now prove the following.

THEOREM 2.6. *Let M be a flat compact connected smooth surface with boundary, $\partial M \neq \emptyset$, transversal to a foliation of $\boldsymbol{H}^3$ by horospheres. If, along ∂M, M makes a constant*

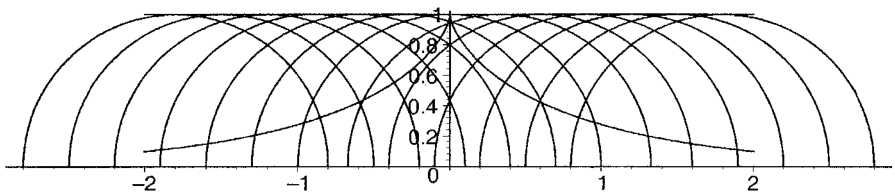

FIGURE 2. An example with a horocycle as a singularity.

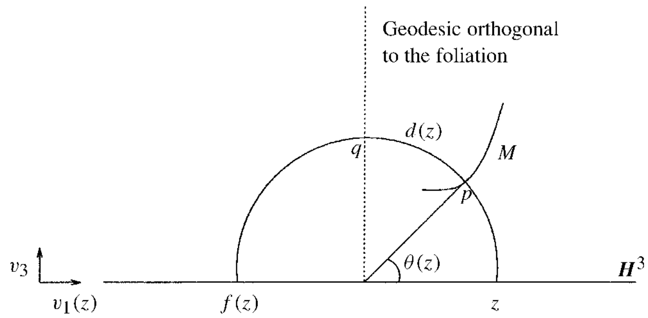

FIGURE 3. The function d.

angle with the leaves of the foliation, then M is part of an equidistant surface to a geodesic orthogonal to the foliation.

PROOF. Consider the conformal structure on M induced by the immersion G^-, namely, the pullback of the conformal structure on $\partial_\infty \boldsymbol{H}^3$ identified with S^2. Since M is transversal to the foliation, we may use the local representation of Theorem 2.2. Then the function d defined above is a harmonic function. Indeed, in terms of coordinates in the upper half-space model d is given by

$$d = \ln\left(\frac{1 - \cos\theta}{\sin\theta}\right),$$

which is seen to be harmonic by (20). Note that the angle θ is the angle of the surface with the foliation. Thus, if θ is constant on ∂M, then d is also constant on ∂M. However, since d is harmonic on M (for the conformal structure just considered), d is constant on M, so θ is constant on M.

Finally, since θ is constant, the system (18) allows one to conclude that $c(z)$ is also constant. Thus M is part of an equidistant surface to a geodesic orthogonal to the foliation. □

REMARK 2.7. Note that the transversality hypothesis is necessary, otherwise a piece of a horosphere, shown in Figure 4, would be a counter-example.

3. Caustics of flat surfaces. As a motivation, consider the family of parallel surfaces to the surface in Figure 2. Note that each of them has a horocycle as a singularity, and as we consider all such horocycles, corresponding to all parallel surfaces, we get a horosphere. So, the surface generated by the singularities (horocycles in this example) is also a flat surface. This is not a mere coincidence, as we now show.

Let M be a flat surface without umbilics, and let $\kappa_1 > 1$ and $\kappa_2 < 1$ denote the principal curvatures of M. Consider a local adapted frame as in Section 2. We shall specialize further and admit that e_1 and e_2 are principal directions. In this way we have $\omega_1^3 = \kappa_1\omega^1$ and $\omega_2^3 =$

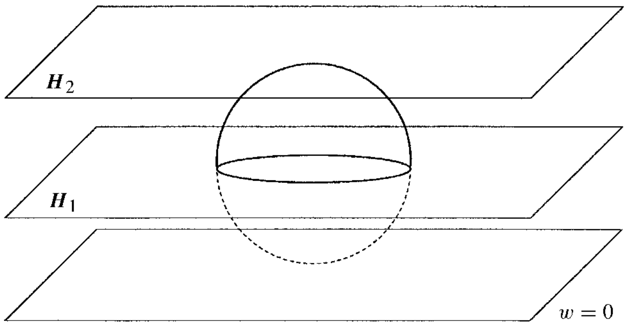

FIGURE 4. A piece of a horosphere not transversal to the foliation.

$\kappa_2\omega^2$. Let ρ_1 and ρ_2 be the radii of curvature associated with the directions e_1 and e_2, which are related to κ_i, $i = 1, 2$ by $\kappa_1 = \coth\rho_1$, $\kappa_2 = \tanh\rho_1$.

We define the map

$$\varepsilon_0 = \cosh\rho_1 e_0 + \sinh\rho_1 e_3 \tag{21}$$

and call it the *caustic M_C associated with M*. It is a simple matter to check that parallel surfaces have the same caustics, so that we may speak of the caustic M_C associated with a family of parallel surfaces $\{M_t\}$.

REMARK 3.1. There are at least three different ways to look at a caustic of a surface: as the loci of the centers of curvature of the surface, as the envelope of geodesics, or a system of rays, and as the loci of the singularities of the parallel surfaces to the given one [1]. Caustics are a classical subject in differential geometry, which are also called evolutes, focal surfaces or surfaces of centers.

REMARK 3.2. In the above considerations about caustics of flat surfaces we have omitted the umbilical points. As can be checked without difficulty, umbilical points correspond to the critical points of f, as appearing in Theorem 2.2, and are isolated. Such points correspond to ends of the caustic surface. Loosely speaking, we may say that the focal point associated with an umbilical point 'is at infinity'. Thus, there is a link between the umbilical set of the surface and the topology of the caustic surface.

As mentioned in the Introduction, the flat surfaces constructed via the Weierstrass representation, in general, have singularities. These are the singularities we shall be concerned with.

Let M be a flat surface constructed via the Weierstass representation. We will consider curves $\gamma \subset M$, that are smooth space curves in $\boldsymbol{H}^3$ and are singular curves of M (i.e., M is not an immersion along γ). We call such curves *smooth singular curves*.

We will also consider the limiting position of the planes tangent to M as one approaches a smooth singular curve γ. Although M is not an immersion along γ, there is a limiting position for the tangent planes. Note that in principle we could have isolated points (cusps) for which there is no limiting position for the tangent plane, so we consider γ to be a smooth space curve to exclude this possibility. So, for $p \in M$ and $p \in \gamma$, we will write T_pM, meaning the limiting position of the tangent planes to M as one approaches p.

THEOREM 3.3. *Let $\{M_t\}$ be a family of parallel flat surfaces, possibly with singularities, and let M_{C} be the caustic associated with this family. Then M_{C} is flat at smooth points. Furthermore, for some t, let γ_t be a smooth singular curve common to M_t and M_{C}. If M_{C} is smooth in a neighborhood of γ_t, then γ_t is a geodesic of M_{C}.*

PROOF. We shall calculate the Gaussian curvature $\tilde{K}$ of the caustic M_{C} using moving frames. First of all, we shall find an adapted orthonormal frame to M_{C}. If we differentiate (21), we get

$$d\varepsilon_0 = d\rho_1(\sinh\rho_1 e_0 + \cosh\rho_1 e_3) + \sinh\rho_1(\kappa_1 - \kappa_2)\omega^2 e_2 .$$

So we may take, as an orthonormal basis for the tangent planes of M_{C}, the following vector fields:

$$\varepsilon_1 = \sinh\rho_1 e_0 + \cosh\rho_1 e_3$$

and

$$\varepsilon_2 = e_2 .$$

If we define $\varepsilon_3 = e_1$, then the vector fields ε_i, $0 \le i \le 3$, constitute an orthonormal frame adapted to M_{C}. We denote by $\tilde{\omega}^i$ and $\tilde{\omega}_i^j$ the corresponding dual and connection forms. By definition

$$\tilde{\omega}_1^2 = \langle d\varepsilon_1, \varepsilon_2\rangle ;$$

thus,

$$\begin{aligned}\tilde{\omega}_1^2 &= \left\langle d\rho_1\varepsilon_0 - \frac{\omega^1}{\sinh\rho_1} e_1 + (\sinh\rho_1 - \cosh\rho_1\kappa_2)\omega^2\varepsilon_2, \varepsilon_2\right\rangle \\ &= (\sinh\rho_1 - \cosh\rho_1\kappa_2)\omega^2 = (\sinh\rho_1 - \cosh\rho_1\tanh\rho_1)\omega^2 = 0 ,\end{aligned}$$

where we have used $\kappa_1\kappa_2 = 1$ and $\kappa_1 = \coth\rho_1$. The above equation shows that if M_{C} is a smooth surface, then the integral curves of ε_1 and ε_2 are geodesics of M_{C} and that $\tilde{K} = 0$, since

$$d\tilde{\omega}_1^2 = -\tilde{K}\tilde{\omega}^1 \wedge \tilde{\omega}^2 .$$

Now consider a smooth curve γ_t common to a flat surface M_t and its caustic M_{C}. The tangent vector γ_t' to γ_t belongs to the tangent planes of M_t and M_{C}, so it must be parallel to $e_2 = \varepsilon_2$. By what we have discussed above, i.e. $\tilde{\omega}_1^2 = 0$, it follows that γ_t, being an integral curve of ε_2, must be a geodesic of M_{C}. □

REMARK 3.4. The integral curves of the vector field ε_1 are also geodesics of M_{C}. However, this is true in general, it has nothing to do with M_t being flat.

From Theorems 2.2 and 3.3 we obtain a Weierstrass representation for flat surfaces where no integration is required.

THEOREM 3.5. *Let $\Omega \subset \boldsymbol{C}$ be an open set and $f : \Omega \to \boldsymbol{C}$ be holomorphic. Then the map*

$$X_{\mathrm{C}}(z) = c(z) + R(z)(\cos\eta(z)v_1(z) + \sin\eta(z)v_3)\,,$$

where

$$c(z) = \frac{f(z)+z}{2}\,, \quad R(z) = \frac{|f(z)-z|}{2}\,, \quad v_1(z) = \frac{f(z)-z}{|f(z)-z|}\,, \tag{22}$$

$$\cos\eta(z) = \frac{1-|f'(z)|^{1/2}}{1+|f'(z)|^{1/2}}\,, \tag{23}$$

$$\sin\eta(z) = \frac{2|f'(z)|^{1/4}}{1+|f'(z)|^{1/2}}\,, \tag{24}$$

and v_3 is a constant vector orthogonal to the z-plane ($\partial_\infty H^3 \setminus \{\infty\}$), is a parametrized flat surface in the upper half-space model, on a subset $\Lambda \subset \Omega$ for which $X_{\mathrm{C}}(z)$ is an immersion.

PROOF. We shall prove that $X_{\mathrm{C}}(z)$ is just the caustic associated with the map $X(z)$ defined in Theorem 2.2. By Theorem 3.3, X_{C} is flat for points of Ω where it defines an immersion.

Indeed, it is a standard fact that the envelope for a map as in (15) is the set of points satisfying

$$\Delta = \det(X_x, X_y, X_\theta) = 0\,,$$

where for instance $X_x = \partial X/\partial x$. If we use (14) and (15), differentiate and carry out some obvious manipulations, we can see that $\Delta = 0$ if and only if

$$\det\begin{bmatrix} \alpha_x(1+\cos\theta)+2 & \beta_x(1+\cos\theta) & (\alpha\alpha_x+\beta\beta_x)\sin\theta \\ -\beta_x(1+\cos\theta) & \alpha_x(1+\cos\theta)+2 & (-\alpha\beta_x-\beta\alpha_x)\sin\theta \\ -\alpha\sin\theta & -\beta\sin\theta & \cos\theta(\alpha^2+\beta^2) \end{bmatrix} = 0\,,$$

where $\alpha(z) = \mathrm{Re}(f(z)-z)$ and $\beta(z) = \mathrm{Im}(f(z)-z)$. After some manipulation, and using $\cos^2\theta = 1-\sin^2\theta$, the above equation is equivalent to the following quadratic polynomial for $\cos\theta$:

$$(|f'(z)|-1)\cos^2\theta + 2(|f'(z)|+1)\cos\theta + |f'(z)| - 1 = 0\,.$$

We now solve the above equation for $\cos\theta$. If we denote the root having absolute value smaller than or equal to one by $\cos\eta(z)$, we have

$$\cos\eta(z) = \frac{1-|f'(z)|^{1/2}}{1+|f'(z)|^{1/2}}\,,$$

and we have $\sin\eta(z) = \sqrt{1-\cos^2\eta(z)}$ given by

$$\sin\eta(z) = \frac{2|f'(z)|^{1/4}}{1+|f'(z)|^{1/2}}\,. \qquad \square$$

The consideration of the caustic associated with a flat surface motivates the study of the function ρ_1, the radius of curvature associated with the principal curvature $\kappa_1 > 1$. We have the following.

THEOREM 3.6. *Let M be a flat surface in $\boldsymbol{H}^3$ without umbilical points and let ρ_1 be the radius of curvature associated with the principal curvature $\kappa_1 > 1$. Then, with respect to the conformal structure on M induced by $G^- : M \to S^2$, $\rho_1 : M \to \boldsymbol{R}$ is a harmonic function.*

PROOF. Without loss of generality we may assume that, for $p \in M$, M is locally transversal to a foliation by horospheres that can be taken to be horizontal planes in the upper half-space. By elementary geometry (see Figure 5), we have

$$\rho_1(p) = d(p) - h(p),$$

where $d(p)$ is the function defined in Subsection 2.3 and $h(p)$ is the corresponding distance from the point of M_C (the caustic of M) to a geodesic orthogonal to the foliation. We know already that d is harmonic (see the proof of Theorem 2.6); it suffices to show that h is also harmonic. In local coordinates, in the z-plane, this function is given by

$$h(z) = \ln \frac{1 - \cos \eta(z)}{\sin \eta(z)},$$

where $\sin \eta(z)$ and $\cos \eta(z)$ are given by (23) and (24). Thus,

$$h(z) = \ln \frac{1 - (1 - |f'(z)|^{1/2})/(1 + |f'(z)|^{1/2})}{2|f'(z)|^{1/4}/(1 + |f'(z)|^{1/2})} = \frac{1}{4} \ln |f'(z)|.$$

Since we assume that M is umbilic free, $f'(z)$ is not zero, and therefore h is harmonic. $\square$

COROLLARY 3.7. *Let M be a smooth compact flat surface with boundary in $\boldsymbol{H}^3$, $\partial M \neq \emptyset$, without umbilical points and let ρ_1 be the radius of curvature associated with the principal curvature $\kappa_1 > 1$. Then the following hold.*

(i) *If ρ_1 is constant along ∂M, M is part of an equidistant surface to a geodesic.*

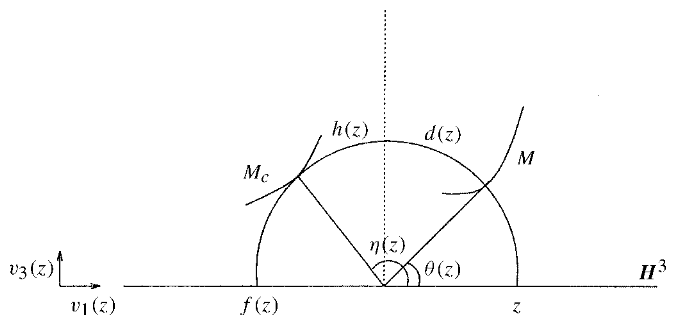

FIGURE 5. Radius of curvature in terms of $h(z)$ and $d(z)$.

(ii) *If ρ_1 has an interior maximum (or minimum), then M is part of an equidistant surface to a geodesic.*

PROOF. By the maximum principle for harmonic functions, ρ_1 must be constant. It is then a standard fact [9] that in this case the surface must be an equidistant surface to a geodesic. □

Our last result in this section shows that, in a sense to be made precise below, a smooth singular curve of a flat surface determines the surface. We will say that M_{E} is a *flat extension* of a flat surface M if M_{E} is flat and $M \subset M_{\mathrm{E}}$.

THEOREM 3.8. *Let $M \subset \boldsymbol{H}^3$ and $\hat{M} \subset \boldsymbol{H}^3$ be flat surfaces and γ a smooth singular curve of M and $\hat{M}$. Then there exists M_{E}, a flat extension of M and $\hat{M}$. In particular, M and $\hat{M}$ coincide in a neighborhood of γ in M_{E}.*

PROOF. First of all, we show that, without loss of generality, γ can be considered as a curve with non-vanishing curvature (as a space curve in $\boldsymbol{H}^3$) and such that M_{C} is smooth along a neighborhood of γ.

In fact, by Theorem 3.3, γ is a geodesic of the caustic M_{C} of M. Since $\gamma \subset M_{\mathrm{C}}$, and M_{C} is flat by Theorem 3.3, γ cannot be a geodesic of $\boldsymbol{H}^3$. The normal curvature of M_{C} in the direction of γ would be zero, a contradiction.

We shall now show that M_{C} cannot be singular along an open set of γ. Indeed, using the frames and notation introduced in this section, and writing

$$d\rho_1 = \rho_{1,1}\omega^1 + \rho_{1,2}\omega^2 , \tag{25}$$

M_{C} will be singular for points where $\rho_{1,1} = 0$, for

$$\tilde{\omega}^1 \wedge \tilde{\omega}^2 = \frac{\rho_{1,1}}{\cosh \rho_1}\omega^1 \wedge \omega^2 .$$

This means that $\nabla \rho_1$, if it is not zero, is tangent to e_2 for points of M corresponding to singular points of M_{C}. On the other hand, $\rho_1 = 0$ along a singular curve such as γ, and γ is an integral curve of the vector field $\varepsilon_2 = e_2$, since it is common to both M and M_{C}. However, this implies that $\nabla \rho_1$ is orthogonal to e_2 along γ. In conclusion, if M_{C} were singular along an open set of γ, we would have $\nabla \rho_1 = 0$ on this open set. Theorem 3.6 shows that the points for which $\nabla \rho_1 = 0$ are isolated, so we get a contradiction.

Thus, possibly restricting ourselves to a piece of γ, we may assume that κ_γ, the curvature vector of γ, as a space curve in $\boldsymbol{H}^3$, does not vanish and that M_C is smooth along γ.

Let $p \in M$, and $p \in \gamma$. Then the vector $\kappa_\gamma(p)$ is orthogonal to $T_p M_{\mathrm{C}}$ (γ is a geodesic on M_{C}). By looking at the relations between adapted frames to M and M_{C} (in the proof of Theorem 3.3), one sees that $\kappa_\gamma(p)$ is orthogonal to $T_p M$. The same reasoning applies to $\hat{M}$ and $\hat{M}_{\mathrm{C}}$.

From the above, we may conclude that M and $\hat{M}$ have the same tangent planes (in the generalized sense explained in this section) along γ. However, this means that the local holomorphic correspondences (as in Theorem 2.2), say $f(z)$ for M and $\hat{f}(z)$ for $\hat{M}$, coincide along a curve in the z-plane. Being holomorphic, the functions $f(z)$ and $\hat{f}(z)$ coincide in a neighborhood of this curve. So M and $\hat{M}$ coincide in a neighborhood of γ.

By analytic continuation, we may construct the extension M_{E}. □

4. Schwarz's reflection principle for flat surfaces. A well-known technique for understanding and constructing minimal surfaces in $\boldsymbol{R}^3$ with symmetries can also be used for flat surfaces in $\boldsymbol{H}^3$. We have the following.

THEOREM 4.1. *Let M be a flat surface and $\gamma \subset P$ be a planar geodesic of M, where P is a hyperbolic plane containing γ. Then M can be extended to a flat surface $\hat{M}$ so that P is a plane of symmetry of $\hat{M}$.*

PROOF. Without loss of generality, we may describe the plane P as a vertical plane in the upper half-space model in such a way that the ideal boundary of P, with a point missing, coincides with the real axis.

Since γ is a geodesic of M, the normal vector to M along γ lies, in this model, in the plane P. Thus, the local correspondence $z \mapsto f(z)$ is real along the real axis, and by Schwarz's reflection principle we have $f(\bar{z}) = \overline{f(z)}$ in a neighborhood V of the real axis. So, the congruence of geodesics associated with $f(z)$ is locally symmetric with respect to the plane P.

Let M^* and $M^\diamond$ be the parts of M on the two closed half-spaces defined by P, corresponding to the points of V. Then the reflection with respect to P of M^*, denoted by $R_P(M^*)$, must be a surface parallel to $M^\diamond$. But M^* and $M^\diamond$ coincide along P, so that $R_P(M^*) = M^\diamond$. By analytic continuation, M can be extended so that it is symmetric with respect to P. □

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Departamento de Matematica
Universidade de Brasilia
Brasilia D.F.
Brasil

E-mail address: roitman@mat.unb.br