Flexible Color Lists in Alon and Tarsi's Theorem, and Time Scheduling with Unreliable Participants

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Submitted: Jul 21, 2008; Accepted: Dec 11, 2009; Published: Jan 5, 2010
Mathematics Subject Classifications: 91A43, 05C15, 05C20, 05C45

Abstract

We present a purely combinatorial proof of Alon and Tarsi's Theorem about list colorings and orientations of graphs. More precisely, we describe a winning strategy for Mrs. Correct in the corresponding coloring game of Mr. Paint and Mrs. Correct. This strategy produces correct vertex colorings, even if the colors are taken from lists that are not completely fixed before the coloration process starts. The resulting strengthening of Alon and Tarsi's Theorem leads also to strengthening of its numerous repercussions. For example we study upper bounds for list chromatic numbers of bipartite graphs and list chromatic indices of complete graphs. As real life application, we examine a chess tournament time scheduling problem with unreliable participants.

Introduction

Alon and Tarsi's Theorem [AlTa] from 1992, about list colorings and orientations of graphs, has many applications in the theory of graph colorings. We will resume and extend most of them in this article. However, Alon and Tarsi's Theorem not only has many applications, it also opened a door to a new very successful algebraic method. This, so called *Polynomial Method*, was explicitly worked out in Alon's paper [Al2], where Alon suggested the name *Combinatorial Nullstellensatz* for the main algebraic tool behind it. We strengthened this Nullstellensatz in [Scha2] with a quantitative formula, and presented some easy-to-apply corollaries and new applications. Our formula led in particular to a quantitative version of Alon and Tarsi's Theorem [Scha2, Corollary 5.5].

Apart from this very successful study of the algebraic method behind Alon and Tarsi's Theorem, combinatorialists always search for purely combinatorial proofs, since this usually helps to understand the situation in more detail. Indeed, Alon and Tarsi asked in their original paper [AlTa] for such a proof. The first main purpose of this article is to present one. Our proof actually gives some insight into the connection between orientations and colorings, but also leads to a new strengthening. Even more, the work on this proof led us to a new coloring game which provides an adequate game-theoretic approach to list coloring problems and time scheduling problems with flexible lists of available time slots. See [Al], [Tu] and [KTV] in order to get an overview of list colorings. We have already presented this *game of Mr. Paint and Mrs. Correct* in [Scha3]. In this article we have demonstrated that, even though the resulting notion of *ℓ-paintability* (Definition 1.2) is stronger than *ℓ-list colorability* (*ℓ-choosability*), many deep theorems about list colorability remain true in the context of paintability. In the present article we continue by giving a combinatorial proof of a paintability strengthening of Alon and Tarsi's Theorem. Afterwards, we show that most applications of Alon and Tarsi's Theorem can be strengthened as well.

In Section 1, we present a reformulated version of the game of Mr. Paint and Mrs. Correct, and define ℓ-paintability as a strengthening of ℓ-list colorability.

In Section 2, we use this to give a purely combinatorial proof of a strengthening of Alon and Tarsi's Theorem (Theorem 2.1).

Section 3 is concerned with classical applications of Alon and Tarsi's Theorem. We use our strengthening to provide paintability versions of Alon and Tarsi's bound of the list chromatic number of bipartite and planar bipartite graphs (Theorem 3.3 and the Corollaries 3.4 and 3.6). We even could refine their techniques, and improved their upper bounds, in particular with respect to the maximal degrees of the vertices inside the two parties (as we call the partition parts) of the graph. Theorem 3.8 is another improvement in this direction.

Furthermore, we present strengthened versions of Fleischner and Stiebitz' Theorem 3.9 about certain 4-regular Hamiltonian graphs, Häggkvist and Janssen's bound (Theorem 3.10) for the list chromatic index of the complete graph K_n, and Ellingham and Goddyn's confirmation of the list coloring conjecture for planar r-regular edge r-colorable multigraphs (Theorem 3.12). Example 3.11 describes a time scheduling problem that demonstrates the advantage of the new painting concept against the list coloring approach with fixed list of available time slots.

We also mention, that in [HKS] we worked out a strengthening of Brooks' Theorem, based on our improved Alon-Tarsi-Theorem. Our result is even stronger than the version by Borodin, Erdős, Rubin and Taylor. Its proof uses the existence of an induced even cycle with at most one chord, and almost acyclic orientation.

1 Mr. Paint and Mrs. Correct

In this short section we lay the game-theoretic foundation for the proof of Alon and Tarsi's Theorem. We introduced the *game of Mr. Paint and Mrs. Correct* in [Scha3]. It is a game with complete information, played on a fixed given graph $G = (V, E)$. Here we use the following equivalent reformulation of the original game (which was first defined in [Scha3, Game 1.6 & Definition 1.8]) : $G = (V, E)$

Game 1.1 (Paint-Correct-Game)**.** *In this reformulation Mr. Paint has just one marker. Mrs. Correct has a finite stack* S_v *of erasers for each vertex* v *in* $G_1 := G$ *. They are lying on the corresponding vertices, ready for use.*

The reformulated game of Mr. Paint and Mrs. Correct works as follows:

$1P$*: Mr. Paint starts, choosing a nonempty set of vertices* $V_{1P} \subseteq V(G_1)$ *and marking them with his marker.*

$1C$*: Mrs. Correct chooses an independent subset* $V_{1C} \subseteq V_{1P}$ *of marked vertices in* G_1 *, i.e.,* $uv \notin E(G_1)$ *for all* $u, v \in V_{1C}$ *. She cuts off the vertices in* V_{1C} *, so that the graph* $G_2 := G_1 \setminus V_{1C}$ *remains. The still marked vertices* $v \in V_{1P} \setminus V_{1C}$ *of* G_2 *have to be cleared. For each such* $v \in V_{1P} \setminus V_{1C}$ *Mrs. Correct has to use (and use up) one eraser from the corresponding stack* S_v *. She loses if she runs out of erasers and cannot do that, i.e., if already* $S_v = \emptyset$ *for a still marked vertex* $v \in V_{1P} \setminus V_{1C}$ *.*

$2P$*: Mr. Paint again chooses a nonempty set of vertices* $V_{2P} \subseteq V(G_2)$ *and marks them with his marker.*

$2C$*: Mrs. Correct again cuts off an independent set* $V_{2C} \subseteq V_{2P}$ *, so that a graph* $G_3 := G_2 \setminus V_{2C}$ *remains. She also uses (and uses up) some erasers to clear the remaining marked vertices* $v \in V_{2P} \setminus V_{2C}$ *.*

⋮ ⋮

End: The game ends when one player cannot move anymore, and hence loses.

Mrs. Correct cannot move if she does not have enough erasers left to clear the vertices she was not able to cut off.

Mr. Paint loses if there are no more vertices left.

We may imagine that after each round the newly cut off vertices are colored with a so far unused color. In this way a win for Mrs. Correct results in a proper coloring of the underlying graph G . Whether this is possible or not possible depends on the sizes of the stacks of erasers S_v at the vertices v of G . We define:

Definition 1.2 (Paintability)**.** Let $\ell = (\ell_v)_{v\in V}$ be defined by $\ell_v := |S_v| + 1$. If there is a winning strategy for Mrs. Correct, then we say that G is *ℓ-paintable*. ℓ, ℓ_v

We write n-"something" instead of $(n\mathbf{1})$-"something", where $\mathbf{1} = (1)_{v\in V}$ and $n \in \mathbb{N}$. $\mathbf{1}$

It is not hard to see that ℓ-paintability is stronger (and in fact strictly stronger) then ℓ-list colorability. The ℓ-paintability may be viewed as a dynamic version of list colorability, where the color lists L_v of sice ℓ_v at the vertices v are not completely fixed before the coloration process starts (see [Scha3] for details). We note down:

$$G \text{ is } \ell\text{-paintable.} \quad \Longrightarrow \quad G \text{ is } \ell\text{-list colorable.} \tag{1}$$

Many people ask if it really makes sense for Mr. Paint to choose in his i^{th} move a proper subset $V_{iP} \subset V(G_i)$ instead of taking the whole set $V(G_i)$. Well, the point is that Mrs. Correct may have a big or somehow advantageous independent set V_{iC} in $V(G_i)$, and that Mr. Paint has to prevent her from cutting off this set by not marking some vertices in it. The not marked and not cut off vertices may become the decisive battlefield of the future. Sometimes patience succeeds. A partial attack $V_{iP} \subset V(G_i)$ may cost less erasers, but can save vantage ground, ground that should be attacked only if the surrounding vertices vertices already have lost more erasers. One example where Paint's winning strategy is like this is $K_{3,3}$ with one eraser at each vertex.

2 Alon and Tarsi's Theorem

In this section we discus a surprising connection between colorings and orientations of graphs. Let $\vec{G} = (V, E, \to)$ be an oriented graph, i.e., a graph $G = (V, E)$ together with an *orientation* $\to\colon E \ni e \longmapsto e^{\to} \in e$. Suppose that we have a cartesian product $\vec{G}, e^{\to}$ $\to$ L

$$L := \prod_{v\in V} L_v \tag{2}$$

of *lists* L_v of sizes ℓ

$$\ell_v := |L_v| > d^+(v) \ , \tag{3}$$

where $d^+(v)$ is the *outdegree* of v in $\vec{G}$. We view the elements $\lambda \in L$ as vertex labellings, $\lambda\colon v \mapsto \lambda_v \in L_v$, and ask: Is there a *proper coloring* $\lambda \in L$ of G ? $d^+(v)$

One could conjecture that there is one, since each list L_v (to each fixed vertex $v \in V$) contains so many colors that – if all "successors" u of v ($v{\to}u$ in $\vec{G}$) are already colored – there is at least one color in L_v that differs from the colors of the successors of v . If we now use this "evasion color" to color the vertex v , and do the same for all other vertices of V , then we obtain a proper coloring of G , since in each edge uv one end "takes care" of the other end (either $v{\to}u$ or $u{\to}v$). $v{\to}u$

However, this train of thought runs on nonexisting rails. We cannot just assume that for each vertex v "all successors u of v are already colored". An example which shows the validity of the desired conclusion is the directed cycle of length 3, which is not colorable with 2 colors. Nevertheless, our consideration contains some plausibility, and one could ask for an additional condition that makes it work. Alon and Tarsi found such a condition in [AlTa]. They proved that ℓ-list colorings exist, if the sets of *even* and *odd Eulerian (spanning) subgraphs* EE and EO of $\vec{G}$ do not have the same size, i.e., EE, EO

$$|EE| \neq |EO| \ ; \tag{4}$$

where a directed graph is *even/odd Eulerian* if it has even/odd many edges, and if the indegree of each single vertex $v \in V$ equals its outdegree. In their paper they work with the set $D_\alpha = D_\alpha(G) = D_\alpha(\vec{G})$ of all orientations φ with *outdegree sequence* $d_\varphi^+ = (d_\varphi^+(v))_{v\in V}$ equal to $\alpha \in \mathbb{Z}^V$. They split this set into the sets $DE_\alpha = DE_\alpha(\vec{G})$ and $DO_\alpha = DO_\alpha(\vec{G})$, of *even* resp. *odd* orientations $\varphi \in D_\alpha$, i.e., those which differ from the fixed given *reference orientation* $\rightarrow$ ($e^\varphi \neq e^\rightarrow$) on even resp. odd many edges $e \in E$. At the end they used the fact that, with $d^+ := d_\rightarrow^+ = (d^+(v))_{v\in V}$, D_α, d_φ^+ DE_α, DO_α d^+

$$|DE_{d^+}| = |EE| \qquad \text{and} \qquad |DO_{d^+}| = |EO| \ . \tag{5}$$

This is not hard to see (see also [Scha1, Lemma 2.6]). In this paper we state our theorems using DE_α and DO_α instead of EO and EE. Of course,

$$DE_\alpha = DO_\alpha = \emptyset \tag{6}$$

if there are no $\varphi \in D(G)$ with $d_\varphi^+ = \alpha$, i.e., no *realizations* of α. This is for example the case if $\alpha_v < 0$ for one $v \in V$, or if

$$\sum_{v\in V} \alpha_v \neq |E| \ , \tag{7}$$

since

$$\sum_{v\in V} d_\varphi^+ = |E| \quad \text{for all orientations } \varphi \in D(G)\,. \tag{8}$$

Alon and Tarsi's work preceded the Combinatorial Nullstellensatz [Al2], which has many applications. In [Scha2] we proved a quantitative strengthening of this Nullstellensatz, which also led to a (weighted) qualitative version of the Alon-Tarsi Theorem. The difference $|DE_\alpha| - |DO_\alpha|$ (which can also be written as *permanent* of an incidence matrix, as in [Scha2, Corrolary 5.5]) equals a weighted sum over certain colorings. Here, we present a paintability strengthening of the result of Alon and Tarsi. Our proof can be generalized to polynomials, as described in [Scha4], and leads to a paintability version of the Combinatorial Nullstellensatz. This version of the Nullstellensatz is more general than the following strengthening of Alon and Tarsi's Theorem. However, Alon and Tarsi have already asked in the original paper [AlTa] for a combinatorial proof of their result. Therefore, we work here in the purely combinatorial frame of orientations of graphs, in order to shed some light on the surprising connection between colorings and orientations of graphs. We have:

Theorem 2.1. *Let $\vec{G}$ be a directed graph and $\alpha \in \mathbb{N}^V$, then*

$$\boxed{|DE_\alpha(\vec{G})| \neq |DO_\alpha(\vec{G})| \implies \vec{G} \text{ is } (\alpha+1)\text{-paintable.}}$$

The proof of this theorem contains an explicit winning strategy. It is a proof by induction, and uses the notations in Game 1.1. We will examine the orientation sets $DE_{\alpha+\mathbb{N}^U}$

$$D_S := \biguplus_{\alpha' \in S} D_{\alpha'} \quad , \qquad DE_S := \biguplus_{\alpha' \in S} DE_{\alpha'} \quad \text{and} \quad DO_S := \biguplus_{\alpha' \in S} DO_{\alpha'} \tag{9}$$

where $\uplus$ stands for disjoint union. Always S will be a set of the form $\uplus$

$\alpha + \mathbb{N}^U$

$$\alpha + \mathbb{N}^U := \{\, \alpha' \geqslant \alpha \mid \alpha'_v = \alpha_v \text{ for all } v \notin U \,\} \tag{10}$$

with $\alpha \in \mathbb{Z}^V$ and $U \subseteq V$ ($\alpha' \geqslant \alpha$ means $\alpha'_v \geqslant \alpha_v$ for all $v \in V$). Note that α does not necessarily has to be a degree sequences, it plays a more general role here. $\geqslant$

One single induction step in the aspirated proof will be partitioned into four parts. In the first part we have to modify the induction hypothesis a little bit. The second part describes the winning strategy of Mrs. Correct; it is mainly contained in the following lemma. In the third part we have to understand why this strategy singles out an independent set. This is also contained in the following lemma (in its very last sentence). The final step is contained in the second lemma below, and will show that the induction hypothesis remains true when we cut off the independent set. Figure 1 illustrates our first lemma, in which we use the standard basis vectors $\mathbb{1}_u = (\delta_{u,v})_{v \in V} \in \{0,1\}^V$ to the indices $u \in V$ with just one nonzero entry at $v = u$: $\mathbb{1}_u$

Lemma 2.2. *Let $\vec{G} = (V, E, \to)$ be a directed graph, $\alpha \in \mathbb{N}^V$, $V_P \subseteq V$ nonempty and $u \in V_P$, then:*

(i) $(\alpha - \mathbb{1}_u) + \mathbb{N}^{V_P} \;=\; \alpha + \mathbb{N}^{V_P} \;\uplus\; (\alpha - \mathbb{1}_u) + \mathbb{N}^{V_P \setminus u}$.

(ii) $DE_{(\alpha-\mathbb{1}_u)+\mathbb{N}^{V_P}} \;=\; DE_{\alpha+\mathbb{N}^{V_P}} \uplus DE_{(\alpha-\mathbb{1}_u)+\mathbb{N}^{V_P\setminus u}}$ *and*
$DO_{(\alpha-\mathbb{1}_u)+\mathbb{N}^{V_P}} \;=\; DO_{\alpha+\mathbb{N}^{V_P}} \uplus DO_{(\alpha-\mathbb{1}_u)+\mathbb{N}^{V_P\setminus u}}$.

(iii) $|DE_{\alpha+\mathbb{N}^{V_P}}| \neq |DO_{\alpha+\mathbb{N}^{V_P}}|$ *implies that*
$|DE_{(\alpha-\mathbb{1}_u)+\mathbb{N}^{V_P}}| \neq |DO_{(\alpha-\mathbb{1}_u)+\mathbb{N}^{V_P}}|$ *or*
$|DE_{(\alpha-\mathbb{1}_u)+\mathbb{N}^{V_P\setminus u}}| \neq |DO_{(\alpha-\mathbb{1}_u)+\mathbb{N}^{V_P\setminus u}}|$.

(iv) $|DE_{\alpha+\mathbb{N}^{V_P}}| \neq |DO_{\alpha+\mathbb{N}^{V_P}}|$ *implies that there is a $V_C \subseteq V_P$ and an $0 \leqslant \alpha' \leqslant \alpha$ s.t.*
$|DE_{\alpha'+\mathbb{N}^{V_C}}| \neq |DO_{\alpha'+\mathbb{N}^{V_C}}|$, $\alpha'|_{V_C} \equiv 0$ *and* $\alpha'_v < \alpha_v$ *for all* $v \in V_P \setminus V_C$.
Each such set V_C is independent in $\vec{G}$.

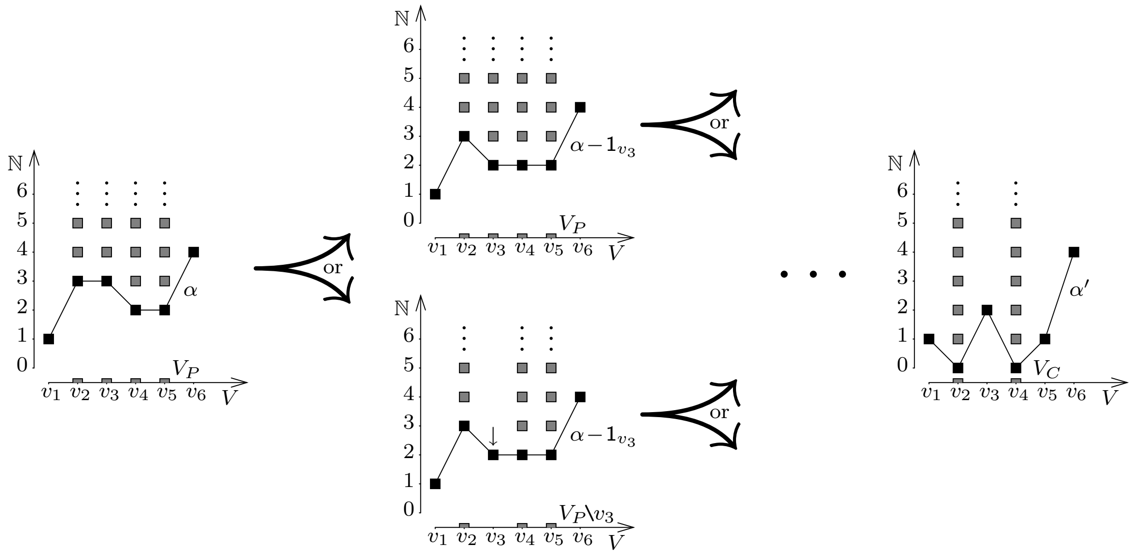

Figure 1: $v \longmapsto \alpha_v$ *and* $\alpha + \mathbb{N}^{V_P}$ *in Lemma 2.2.*

Proof. The elements σ of the set $(\alpha - \mathbf{1}_u) + \mathbb{N}^{V_P}$ on the left side of Equation (i) fulfill $\sigma_u \geqslant \alpha_u - 1$. On the right side we simply distinguish between those with $\sigma_u > \alpha_u - 1$ and those with $\sigma_u = \alpha_u - 1$.

In order to obtain part (ii), we just have to take the preimages of the sets in (i) under the mapping $\varphi \longmapsto d^+_\varphi$, which we viewed either as a mapping defined on the set DE of all even orientations, or as a mapping defined on the set DO of all odd orientations.

DE
DO

Now, we consider the cardinalities of the sets in part (ii) and obtain

$$|DE_{(\alpha-\mathbf{1}_u)+\mathbb{N}^{V_P}}| \quad = \quad |DE_{\alpha+\mathbb{N}^{V_P}}| \;+\; |DE_{(\alpha-\mathbf{1}_u)+\mathbb{N}^{V_P\backslash u}}| \quad \text{and} \tag{11}$$
$$|DO_{(\alpha-\mathbf{1}_u)+\mathbb{N}^{V_P}}| \quad = \quad |DO_{\alpha+\mathbb{N}^{V_P}}| \;+\; |DO_{(\alpha-\mathbf{1}_u)+\mathbb{N}^{V_P\backslash u}}| \quad . \tag{12}$$

If we extend this system of linear equations with

$$|DE_{(\alpha-\mathbf{1}_u)+\mathbb{N}^{V_P}}| \quad = \quad |DO_{(\alpha-\mathbf{1}_u)+\mathbb{N}^{V_P}}| \quad \text{and} \tag{13}$$
$$|DE_{(\alpha-\mathbf{1}_u)+\mathbb{N}^{V_P\backslash u}}| \quad = \quad |DO_{(\alpha-\mathbf{1}_u)+\mathbb{N}^{V_P\backslash u}}| \quad , \tag{14}$$

it follows that

$$|DE_{\alpha+\mathbb{N}^{V_P}}| \quad = \quad |DO_{\alpha+\mathbb{N}^{V_P}}| \quad . \tag{15}$$

Part (iii) is the contraposition to this conclusion.

In order to prove part (iv), we may use part (iii), as illustrated in Figure 1, to produce sequences

$$\alpha =: \alpha^0 \gneqq \alpha^1 \gneqq \cdots \gneqq \alpha^t \geqslant 0 \qquad \text{and} \qquad V_P =: V_C^0 \supseteq V_C^1 \supseteq \cdots \supseteq V_C^t \tag{16}$$

with the property

$$|DE_{\alpha^i+\mathbb{N}^{V_C^i}}| \;\neq\; |DO_{\alpha^i+\mathbb{N}^{V_C^i}}| \qquad \text{for } i=0,1,\ldots,t\,. \tag{17}$$

Note that

$$\alpha^t|_{V_C^t} \;\equiv\; 0 \tag{18}$$

if and only if the sequence of componentwise nonnegative α^i in (16) can no longer be extended through application of part (*iii*); hence, in this case part (*iv*) holds, if we set

$$\alpha' := \alpha^t \qquad \text{and} \qquad V_C := V_C^t\;\;. \tag{19}$$

It remains to show that the existence of an edge uv with both ends in V_C would lead to a contradiction: Suppose there is one. Then turning this edge uv around gives rise to a fixpoint free involution

$$\Theta_{uv}:\;\; D_{\mathbb{N}^V}(G) \;\xrightarrow{\;\cong\;}\; D_{\mathbb{N}^V}(G)\;\;. \tag{20}$$

This involution can be restricted to an involution

$$D_{\alpha'+\mathbb{N}^{V_C}} \;\xrightarrow{\;\cong\;}\; D_{\alpha'+\mathbb{N}^{V_C}}\;\;, \tag{21}$$

since – if we apply Θ_{uv} to an orientation $\varphi\in D_{\alpha'+\mathbb{N}^{V_C}}$ – the two changing outdegrees $d_\varphi^+(u)$ and $d_\varphi^+(v)$ are irrelevant for its membership to $D_{\alpha'+\mathbb{N}^{V_C}}$. That is because

$$\alpha'_u \;=\; 0 \quad \text{and} \quad \alpha'_v \;=\; 0\;\;, \tag{22}$$

by Equation (18), and because if $\sigma := d_\varphi^+$ belongs to $\alpha'+\mathbb{N}^{V_C}$ then each $\sigma'\geqslant 0$, which differs from σ only on vertices $w\in V_C$ with $\alpha'_w = 0$, belongs to $\alpha'+\mathbb{N}^{V_C}$ as well. Altogether, as Θ_{uv} maps even orientations to odd orientations and *vice versa*, we see that

$$|DE_{\alpha'+\mathbb{N}^{V_C}}| \;=\; |DO_{\alpha'+\mathbb{N}^{V_C}}|\;\;, \tag{23}$$

a contradiction. □

Now we come to our second lemma which allows us to cut off independent sets $V_C\subseteq V$. For our main theorem we will need only the case $V_P = V_C$:

Lemma 2.3. *Let* $\vec{G}=(V,E,\to)$ *be a directed graph,* $\alpha\in\mathbb{N}^V$, $V_P\subseteq V$, $uv\in E$, $u\to v$, $E'\subseteq E$ *and let* $V_C\subseteq V$ *be an independent set in* $\vec{G}$, *then:*

(i) $|DE_{\alpha+\mathbb{N}^{V_P}}(\vec{G})| \;=\; |DE_{(\alpha-1_u)+\mathbb{N}^{V_P}}(\vec{G}\backslash uv)| \;+\; |DO_{(\alpha-1_v)+\mathbb{N}^{V_P}}(\vec{G}\backslash uv)|$ *and*

$|DO_{\alpha+\mathbb{N}^{V_P}}(\vec{G})| \;=\; |DO_{(\alpha-1_u)+\mathbb{N}^{V_P}}(\vec{G}\backslash uv)| \;+\; |DE_{(\alpha-1_v)+\mathbb{N}^{V_P}}(\vec{G}\backslash uv)|\;\;.$

(ii) $|DE_{\alpha+\mathbb{N}^{V_P}}(\vec{G})| \;\neq\; |DO_{\alpha+\mathbb{N}^{V_P}}(\vec{G})|$ *implies that*

$|DE_{(\alpha-1_u)+\mathbb{N}^{V_P}}(\vec{G}\backslash uv)| \;\neq\; |DO_{(\alpha-1_u)+\mathbb{N}^{V_P}}(\vec{G}\backslash uv)|$ *or*

$|DE_{(\alpha-1_v)+\mathbb{N}^{V_P}}(\vec{G}\backslash uv)| \;\neq\; |DO_{(\alpha-1_v)+\mathbb{N}^{V_P}}(\vec{G}\backslash uv)|\;\;.$

(iii) $|DE_{\alpha+\mathbb{N}^{V_P}}(\vec{G})| \neq |DO_{\alpha+\mathbb{N}^{V_P}}(\vec{G})|$ *implies that there is an* $0 \leqslant \alpha' \leqslant \alpha$ *such that* $|DE_{\alpha'+\mathbb{N}^{V_P}}(\vec{G} \setminus E')| \neq |DO_{\alpha'+\mathbb{N}^{V_P}}(\vec{G} \setminus E')|$.

(iv) $|DE_{\alpha+\mathbb{N}^{V_P}}(\vec{G})| \neq |DO_{\alpha+\mathbb{N}^{V_P}}(\vec{G})|$ *implies that there is an* $0 \leqslant \alpha'' \leqslant \alpha|_{V\setminus V_C}$ *s.t.* $|DE_{\alpha''+\mathbb{N}^{V_P\setminus V_C}}(\vec{G} \setminus V_C)| \neq |DO_{\alpha''+\mathbb{N}^{V_P\setminus V_C}}(\vec{G} \setminus V_C)|$.

Proof. When we restrict an orientation φ of G to $E\backslash uv$, we obtain an orientation of the smaller graph $G\backslash uv$. This restricted orientation $\varphi|_{E\backslash uv}$ has the same parity (either even or odd) as φ if $u \overset{\varphi}{\to} v$, and the opposite parity in the other case. Conversely, each orientation φ' of the smaller graph $G\backslash uv$ extends to one orientation of G with the same parity as φ', and to one orientation with the opposite orientation as φ'. The restriction of the orientations leads to bijections

$$DE_{\alpha+\mathbb{N}^{V_P}}(\vec{G}) \xrightarrow{\cong} DE_{(\alpha-\mathbf{1}_u)+\mathbb{N}^{V_P}}(\vec{G}\backslash uv) \uplus DO_{(\alpha-\mathbf{1}_v)+\mathbb{N}^{V_P}}(\vec{G}\backslash uv) \quad \text{and} \tag{24}$$

$$DO_{\alpha+\mathbb{N}^{V_P}}(\vec{G}) \xrightarrow{\cong} DO_{(\alpha-\mathbf{1}_u)+\mathbb{N}^{V_P}}(\vec{G}\backslash uv) \uplus DE_{(\alpha-\mathbf{1}_v)+\mathbb{N}^{V_P}}(\vec{G}\backslash uv) \quad , \tag{25}$$

and part (i) follows.

As in the proof of Lemma 2.2(iii), we deduce part (ii) from part (i). Likewise, iteration of part (ii) yields part (iii), we just have to use that in inequalities of the form

$$|DE_{\alpha+\mathbb{N}^{V_P}}(\vec{G})| \neq |DO_{\alpha+\mathbb{N}^{V_P}}(\vec{G})| \tag{26}$$

negative values of α may be replaced by zeros, as

$$DE_\alpha(\vec{G}) = \emptyset = DO_\alpha(\vec{G}) \quad \text{for } \alpha \ngeq 0 . \tag{27}$$

In order to prove part (iv), at first, we remove the set $E(U,W)$

$$E' := E(V_C, V\setminus V_C) \tag{28}$$

of all edges between V_C and $V\setminus V_C$. Let $0 \leqslant \alpha' \leqslant \alpha$ be as in part (iii). As V_C is independent, the vertices of V_C are isolated in $\vec{G} \setminus E'$, so that, for all orientations $\varphi\colon E \setminus E' \to V$ and all $v \in V_C$,

$$d^+_\varphi(v) = 0 \tag{29}$$

and

$$d^+_\varphi(v) \in \alpha'_v + \mathbb{N} \iff 0 = \alpha'_v \iff d^+_\varphi(v) = \alpha'_v \quad . \tag{30}$$

It follows that

$$D_{\alpha'+\mathbb{N}^{V_P}}(\vec{G} \setminus E') = D_{\alpha'+\mathbb{N}^{V_P\setminus V_C}}(\vec{G} \setminus E') \quad , \tag{31}$$

and if we set

$$\alpha'' := \alpha'|_{V\setminus V_C} \quad , \tag{32}$$

this extends to

$$D_{\alpha'+\mathbb{N}^{V_P}}(\vec{G}\setminus E') \;=\; D_{\alpha'+\mathbb{N}^{V_P\setminus V_C}}(\vec{G}\setminus E') \;=\; D_{\alpha''+\mathbb{N}^{V_P\setminus V_C}}(\vec{G}\setminus V_C)\ , \tag{33}$$

where we have used that

$$E(\vec{G}\setminus E') \;=\; E(\vec{G}\setminus V_C)\ . \tag{34}$$

Moreover, these equalities also hold when we replace D with DE or DO, so that the inequality in part (iv) follows from those in part (iii). □

With this we are prepared to describe the winning strategy required in the main proof:

Proof of Theorem 2.1. We present a winning strategy for Mrs. Correct, described in the terms of Game 1.1. We suppose that, when the game has reached the i^{th} round, Mrs. Correct has (at least) α_v^i erasers left at each vertex v of $\vec{G}_i$, and that she has managed to ensure

$$|DE_{\alpha^i}(\vec{G}_i)| \;\neq\; |DO_{\alpha^i}(\vec{G}_i)|\ , \tag{35}$$

where $\alpha^i = (\alpha_v^i)_{v\in V(\vec{G}_i)} \in \mathbb{N}^{V(\vec{G}_i)}$. (For $i=1$, $\vec{G}_1 := \vec{G}$ and $\alpha^1 := \alpha$ this holds.)

Now Mr. Paint makes his i^{th} move:

iP: Mr. Paint chooses a nonempty subset $V_{iP} \subseteq V(\vec{G}_i)$, and marks the vertices in V_{iP} with his marker. If already $V(\vec{G}_i) = \emptyset$, then the game ends here, Mr. Paint is defeated and Mrs. Correct wins.

Now, after Mr. Paint's preselection, Mrs. Correct makes her i^{th} move in the following way, which is always possible, so that the game does not stop when it is her turn and she indeed does not lose:

iC: Mrs. Correct knows from the induction hypothesis (35) that

$$D_{\alpha^i}(\vec{G}_i) \;\neq\; \emptyset\ , \tag{36}$$

and, using double counting, she concludes that

$$\sum_{v\in V(\vec{G}_i)} \alpha_v^i \;=\; |E(\vec{G}_i)| \tag{37}$$

With the same reasoning she then sees, that

$$D_{\alpha^i}(\vec{G}_i) \;=\; D_{\alpha^i+\mathbb{N}^{V_{iP}}}(\vec{G}_i) \tag{38}$$

so that the induction hypothesis (35) can be rewritten as

$$|DE_{\alpha^i+\mathbb{N}^{V_{iP}}}(\vec{G}_i)| \;\neq\; |DO_{\alpha^i+\mathbb{N}^{V_{iP}}}(\vec{G}_i)|\ . \tag{39}$$

Now, she applies the algorithm used in the proof of Lemma 2.2 (iv) to $\vec{G}_i$, α^i and V_{iP} in place of $\vec{G}$, α and V_P, and obtains an independent set $V_{iC} := V_C$ and a tuple $\alpha'^i := \alpha'$.

Mrs. Correct knows from Lemma 2.2 (iv) that V_{iC} is independent, and she cuts it off. She further knows that for all still marked vertices $v \in V_{iP} \setminus V_{iC}$:

$$\alpha_v^i \;>\; \alpha_v'^i \geqslant 0 \quad , \tag{40}$$

so that there are enough erasers to clear the remaining markings. Moreover, at least $\alpha_v'^i$ erasers remain at each vertex v of $\vec{G}_i$, and this will be enough to establish the induction hypothesis for

$$\vec{G}_{i+1} := \vec{G}_i \setminus V_C \quad : \tag{41}$$

As Mrs. Correct knows from Lemma 2.2 (iv),

$$|DE_{\alpha'^i + \mathbb{N}^{V_{iC}}}(\vec{G}_i)| \;\neq\; |DO_{\alpha'^i + \mathbb{N}^{V_{iC}}}(\vec{G}_i)| \quad . \tag{42}$$

Therefore, she can apply the algorithm behind Lemma 2.3 (iv) using the input $(\vec{G}, V_P, V_C, \alpha) := (\vec{G}_i, V_{iC}, V_{iC}, \alpha'^i)$. She obtains a tuple $\alpha^{i+1} := \alpha'' \in \mathbb{N}^{V(\vec{G}_{i+1})}$ such that

$$|DE_{\alpha^{i+1}}(\vec{G}_{i+1})| \;\neq\; |DO_{\alpha^{i+1}}(\vec{G}_{i+1})| \quad . \tag{43}$$

This is exactly the induction hypothesis required for the next round, and since

$$\alpha_v^{i+1} \;\leqslant\; \alpha_v'^i \quad \text{for all } v \in V(\vec{G}_{i+1}) \,, \tag{44}$$

the values α_v^{i+1} in this hypothesis are actually covered by the numbers of erasers in the remaining stacks S_v .

The graph $\vec{G}_{i+1}$ and the reduced stacks S_v of size (at least) α_v^{i+1} will be passed to the next round. After some finite time $t \in \mathbb{N}$, the graph $\vec{G}_t$ will be empty, Mr. Paint cannot move any more, and Mrs. Correct's strategy succeeds. □

3 Applications of Alon and Tarsi's Theorem

There are several "classical" applications of Alon and Tarsi's Theorem. The proofs in these applications lead to paintability statements, if we use our Theorem 2.1 instead of the original version from Alon and Tarsi. In the first examples below, we also could refine the originally used techniques, and obtain slightly better upper bounds and new corollaries. These examples are based on the following definition, where, as in this whole section, we always assume that our graph G has edges, $E(G) \neq \emptyset$:

$L(G)$, $\check{L}(G)$

Definition 3.1.

$$L(G) \;:=\; \max_{H \leqslant G} \tfrac{|E(H)|}{|V(H)|} \quad \text{and} \quad \check{L}(G) \;:=\; \max_{H \leqslant G} \tfrac{|E(H)|-1}{|V(H)|} \;\leqslant\; (1 - \tfrac{1}{|E(G)|})\, L(G) \quad .$$

This definition means that, if G is oriented, so that

$$|E(H)| \;=\; \sum_{v\in V(H)} d_H^+(v)\ , \tag{45}$$

then $L(G)$ is simply the maximum value of the average outdegrees of the subgraphs $H \neq \emptyset$ of G. Since average outdegrees are bounded by maximal outdegrees, this means that we never will find an orientation $\varphi\colon E \longrightarrow V$ of G with *maximal outdegree* $\Delta^+(\varphi)$ strictly smaller than $L(G)$, i.e., $\Delta^+(\varphi)$

$$\Delta^+(\varphi) \;\geqslant\; L(G) \tag{46}$$

for all orientations $\varphi\colon E \longrightarrow V$ of G. However, the rounded up number $\lceil L(G)\rceil \geqslant L(G)$ is exactly the lowest possible maximal outdegree, as, e.g., shown in [AlTa, Lemma 3.1]. We extend this a little bit using the notation $\lfloor x\rfloor \leqslant x$ for rounded down numbers $x \in \mathbb{R}$: $\lceil \ldots \rceil$ $\lfloor \ldots \rfloor$

Lemma 3.2. *Any graph* $G = (V, E)$ *has an orientation* $\varphi\colon E \longrightarrow V$ *with*

$$\Delta^+(\varphi) \;=\; \lceil L(G)\rceil \;=\; \big\lfloor \check{L}(G) + 1\big\rfloor\ .$$

Proof. Subdividing each edge $e \in E$ with a new vertex $\bar{e}$ yields a bipartite graph B with vertex set

$$V(B) \;=\; V \uplus \bar{E}\ . \tag{47}$$

Replacing the original vertices $v \in V \subseteq V(B)$ with

$$m := \big\lfloor \check{L}(G) + 1\big\rfloor \;>\; \check{L}(G) \tag{48}$$

copies $(v, 1)$, $(v, 2)$, …, (v, m) of v we obtain a bipartite graph B^m, where the inserted vertices $\bar{e} \in \bar{E}$ have degree $2m$. Now, it is sufficient to find a matching of $\bar{E}$ in B^m. Such a matching $\bar{e} \longmapsto (v_{\bar{e}}, i_{\bar{e}})$ would induce an orientation $\bar{\varphi}\colon E \longrightarrow V$ of G via

$$e \;\longmapsto\; \bar{e} \;\longmapsto\; (v_{\bar{e}}, i_{\bar{e}}) \;\longmapsto\; v_{\bar{e}} =: \bar{\varphi}(e)\ . \tag{49}$$

Then the indegrees of $\bar{\varphi}$ wold not exceed m, so that the opposite orientation φ would fulfill

$$\lceil L(G)\rceil \overset{(46)}{\leqslant} \Delta^+(\varphi) \;\leqslant\; m \;=\; \big\lfloor \check{L}(G) + 1\big\rfloor \;\leqslant\; \lceil L(G)\rceil\ , \tag{50}$$

and the lemma would follow.

However, the existence of such a matching follows from Hall's Theorem. We only have to show that each nonempty vertex set $\bar{F} \subseteq \bar{E}$ has more than $|\bar{F}| - 1$ neighbors in B^m. To this end, let $F \subseteq E$ be the set of edges in G corresponding to $\bar{F} \subseteq \bar{E}$. Let $G[F] \leqslant G$ be its induced subgraph, and let $\bigcup F = V(G[F]) \subseteq V$ be the set of all end-vertices of edges in F. Then, indeed, the number of neighbors of $\bar{F}$ in B^m is

$$m\,|{\textstyle\bigcup} F| \;>\; \check{L}(G)\,|{\textstyle\bigcup} F| \;\geqslant\; \frac{|E(G[F])| - 1}{|V(G[F])|}\,|{\textstyle\bigcup} F| \;=\; |\bar{F}| - 1\ . \tag{51}$$

□

Note that it can be advantageous to use the second expression $\lfloor \check{L}(G)+1 \rfloor$, instead of $\lceil L(G) \rceil$, when one wants to utilize an upper bound for $L(G)$. We will see this below. At first, we combine our results in the following theorem, similar to [AlTa, Theorem 3.4]:

Theorem 3.3. *Bipartite graphs G are k-paintable for*

$$k := \lceil L(G)+1 \rceil = \lfloor \check{L}(G)+2 \rfloor \quad .$$

Proof. Bipartite directed graphs $\vec{G}$ do not contain odd Eulerian subgraphs, so that

$$|DO_{d^+}(\vec{G})| \overset{(5)}{=} |EO(\vec{G})| = 0 < |\{\emptyset\}| \leqslant |EE(\vec{G})| \overset{(5)}{=} |DE_{d^+}(\vec{G})| \quad , \tag{52}$$

and Theorem 2.1 applies. We just have to choose the orientation $\rightarrow$ of $\vec{G}$ in accordance with Lemma 3.2. □

As in [AlTa, Corollary 3.4] we obtain as corollary:

Corollary 3.4. *Bipartite planar graphs are* 3*-paintable.*

Proof. Every planar graph G is contained in a triangulation with $3|V|-6$ edges, and we have to remove at least one third of the edges (at least one edge from each triangular face) to obtain the original bipartite graph G. Hence, G contains at most $2|V|-4$ edges, and it follows that $L(G)<2$ (since each subgraph $H \leqslant G$ is bipartite and planar as well). □

As it can be difficult to calculate $L(G)$ or $\check{L}(G) \leqslant (1-\frac{1}{|E(G)|})L(G)$ in Theorem 3.3, we provide the following upper bounds:

Lemma 3.5. *Let $G=(V_1 \uplus V_2, E)$ be a bipartite graph with parties V_1 and V_2, and let $\Delta_i(G) := \max_{v \in V_i} d(v)$ be the maximal degree inside V_i ($i=1,2$), then* $\Delta_i(G)$

$$L(G) \leqslant \frac{1}{1/\Delta_1(G) + 1/\Delta_2(G)} \leqslant \tfrac{1}{2}\Delta(G) \quad .$$

Proof. Since $E \neq \emptyset$, we may alow in the definition of $L(G)$, and in the minima below, only subgraphs H with $E(H) \neq \emptyset$, and can conclude:

$$\begin{aligned} \frac{1}{L(G)} &= \min_{H \leqslant G} \frac{|V(H)|}{|E(H)|} \geqslant \min_{H \leqslant G} \frac{|V(H) \cap V_1|}{|E(H)|} + \min_{H \leqslant G} \frac{|V(H) \cap V_2|}{|E(H)|} \\ &= \frac{1}{\Delta_1(G)} + \frac{1}{\Delta_2(G)} \geqslant \frac{2}{\Delta(G)} \quad . \end{aligned} \tag{53}$$

□

With the "partite" maximal degrees $\Delta_1(G)$, $\Delta_2(G)$ from Lemma 3.5, we obtain the following corollary to Theorem 3.3. Note that the upper bound in this corollary about bipartite graphs is significantly better than those in Brooks' Theorem (namely $\Delta(G)$), even if we replace $\Delta_1(G)$ and $\Delta_2(G)$ with $\Delta(G)$:

Corollary 3.6. *Bipartite graphs G are $\left\lfloor \dfrac{1-1/|E|}{1/\Delta_1(G)\,+\,1/\Delta_2(G)} + 2 \right\rfloor$-paintable.*

If we apply this corollary to $K_{2,3}\backslash e$ ($K_{2,3}$ minus one edge), it tells us that this graph is 2-paintable, which would not follow if we would have based our corollary only on the expression $\lceil L(G)+1 \rceil$ in Theorem 3.3. The small improvement $\check{L}(G) \leqslant (1-\frac{1}{|E(G)|})L(G)$ makes a difference, even though $\lceil L(G)+1 \rceil = \lfloor \check{L}(G)+2 \rfloor$.

We want to go a little bit more into detail, and examine the possible orientations in the bipartite case again. With the "partite" maximal degrees $\Delta_1(G)$, $\Delta_2(G)$ from Lemma 3.5 we obtain, in analogy to Lemma 3.2:

Lemma 3.7. *Let $G = (V_1 \uplus V_2, E)$ be a bipartite graph with the two parties V_1 and V_2. For numbers $L_1, L_2 \in \mathbb{N}$ with $\frac{L_1}{\Delta_1(G)} + \frac{L_2}{\Delta_2(G)} > 1-\frac{1}{|E|}$ holds:*

$$\text{There exists an orientation } \varphi\colon E \longrightarrow V_1 \uplus V_2 \text{ with } d^+_\varphi(v) \leqslant \begin{cases} L_1 & \text{for all } v \in V_1\,, \\ L_2 & \text{for all } v \in V_2\,. \end{cases}$$

Proof. The proof works exactly as those of Lemma 3.2. We just have to construct a graph B^{L_1,L_2} with L_i copies of the vertices in V_i ($i = 1,2$), instead of the graph B^m. Hall's theorem is applicable in the modified proof, as each subset $F \subseteq E$ of edges in G "meets" at least $|F|/\Delta_i(G)$ vertices in V_i, and this means that each subset $\bar{F} \subseteq \bar{E}$ of new vertices has at least

$$\frac{L_1|\bar{F}|}{\Delta_1(G)} + \frac{L_2|\bar{F}|}{\Delta_2(G)} \; > \; (1-\frac{1}{|E|})|\bar{F}| \; \geqslant \; |\bar{F}|-1 \tag{54}$$

neighbors in B^{L_1,L_2}. □

It follows, exactly as in the more special case Theorem 3.3:

Theorem 3.8. *Let $G = (V_1 \uplus V_2, E)$ be a bipartite graph with the parties V_1 and V_2. For numbers $L_1, L_2 \in \mathbb{N}$ with $\frac{L_1}{\Delta_1(G)} + \frac{L_2}{\Delta_2(G)} > 1-\frac{1}{|E|}$ holds:*

$$G \text{ is } \ell\text{-paintable for } \ell = (\ell_v)_{v\in V_1 \uplus V_2} \text{ defined by } \ell_v := \begin{cases} L_1+1 & \text{if } v \in V_1\,, \\ L_2+1 & \text{if } v \in V_2\,. \end{cases}$$

If we apply this theorem to $L_1 = L_2 := \lfloor \frac{1-1/|E|}{1/\Delta_1(G)+1/\Delta_2(G)} + 1 \rfloor > \frac{1-1/|E|}{1/\Delta_1(G)+1/\Delta_2(G)}$, it leads to the same upper bound as in Corollary 3.6.

Fleischner and Stiebitz examined in [FlSt] 4-regular Hamiltonian graphs, and solved a coloring problem of Erdős (see also [Tu]). They made the following observation about Eulerian subgraphs, which implies 3-paintability by Theorem 2.1 and (5):

Theorem 3.9. *If a directed graph* $\vec{G}$ *is the edge-disjoint union of a directed Hamiltonian cycle and some mutually vertex-disjoint, cyclically oriented triangles, then*

$$|EE(\vec{G})| - |EO(\vec{G})| \equiv 2 \pmod 4 \quad ,$$

and, consequently, $\vec{G}$ *is* 3*-paintable.*

Häggkvist and Janssen found in [HäJa, Theorem 3.1] a bound for the list chromatic index of the complete graph K_n. This bound is best possible for odd n. Using Theorem 2.1 instead of Alon and Tarsi's classical version (which they use at the end of the proof of [HäJa, Preposition 2.4]) we get:

Theorem 3.10. *The complete graph* K_n *is edge* n*-paintable.*

Sketch of the proof. The line graph LK_n of K_n consists of n cliques $Q_v = K_{n-1}$, one for each vertex $v \in V(K_n)$. Häggkvist and Janssen extend each Q_v to a K_n by adding a new vertex $\bar{v}$ to it. Afterwards, they define a tuple $\alpha \leqslant n-1$ such that the extended line graph $\overline{LK_n}$ has exactly one orientation φ with outdegree sequence $d_\varphi^+ = \alpha$ and with no directed cycle inside one of the cliques Q_v. Therefore, $|D_\alpha(G)|$ is odd, the Alon-Tarsi Condition in Theorem 2.1 applies, and the $(\alpha+\mathbf{1})$-paintability of $\overline{LK_n}$ follows. Hence, LK_n is n-paintable and K_n is edge n-paintable. Häggkvist and Janssen just use a different notation and say $\alpha_{\bar{v}}$ is *blocked out* in Q_v. In this way they come back to the examination of orientations of the original line graph LK_n. □

Based on this theorem we give an example that demonstrates the advantage of the new painting concept against the list coloring approach:

Example 3.11 (Chess Tournaments)**.** Five chess players are organizing a round robin chess tournament. Each player shall play against each other exactly one time. In each round at most $\lfloor \frac{5}{2} \rfloor = 2$ games can be played, as no player should play more then one game at a time. Therefore, the $\binom{5}{2} = 10$ chess parties have to be played in at least 5 rounds, say on the five working days of a week. Indeed, König's Theorem tells us that the *chromatic index* of K_5 is 5 so that 5 rounds are enough. Our five friends only have to figure out how to do it in detail.

Now, Häggkvist and Janssen's Theorem tells us that the *list chromatic index* of K_5 is 5 as well. What does this mean for our chess tournament? For example, it means that the tournament can be scheduled in seven rounds/days, in such a way, that each player is allowed to be unavailable on one day. Each player has to announce the day of his absence in advance. If player A is not available on Sunday and player B does not have time on Wednesday, then still five days are left for their game to take place. In other words, we have a list of at least five available time slots at each edge of K_5, and Häggkvist and Janssen's Theorem guarantees that appropriate appointments can be made.

Now, our strengthening of Häggkvist and Janssen's Theorem tells us that even the *paintability index* of K_5 is 5. In other words, Mrs. Correct has a winning strategy in the corresponding edge coloring game if there are $5-1=4$ erasers at each edge of K_5 (only 3 erasers per edge would not be enough). What dose this mean? Again, it means that our tournament can be organized on seven days, in such a way, that each player is allowed to miss one day. The new thing is that the players do not have to announce their absence in advance. On each particular day of the tournament, it is the job of Mr. Paint to look around and see who shows up. He then suggests that each present player should play against all those other present players against whom he has not played so far. Since each player should play only one party at a time, his suggestion might be quite inapplicable, and our smart Mr. Correct has to correct this proposal by selecting a matching inside the suggested subgraph. Our Theorem 3.10 guarantees that she can do this in such a way that after seven days all 10 games are played. This is because Mr. Paint will suggest each edge of K_5 at least five times, if it is rejected again and again, and Mrs. Correct can reject it at most four times.

This example can be generalized to tournaments of arbitrarily many players. However, it could be that some days/rounds can be saved. For even n König's Theorem states that the chromatic index of K_n is $n-1$, so that, e.g., another sixth player can join the five-day tournament above, the $\binom{6}{2}=15$ possible parties can be played on 5 days as well. In other words, in the even case with always available players one day can be saved.

According to the List Coloring Conjecture, the list chromatic index of any graph equals its chromatic index. Therefore, $n-1$ should also be the list chromatic index of K_n if n is even, so that we should be able to save one day in the corresponding tournament with disclosed days of absences. This would be best possible since the so-called *total chromatic number* of K_{2m} is $\Delta(K_{2m})+2=2m+1$, see [AlWi]. However, there seems to be no published proof of the List Coloring Conjecture in the case of even complete graphs.

Beyond this open problems, one could conjecture that the list coloring index of a graph equals it paintability index, which would extend the List Coloring Conjecture and would apply to the third case in our example above. In this case $n+1$ rounds would be enough, provided that n is even. If $n \geqslant 2$ is odd, our bound of $n+2$ rounds is best possible. That is because the n players may jointly attend the first $n-1$ rounds. Afterwards, there are two players, say A and B, who have not played against each other. Now, if A does not show up in the n^{th} round, and B does not show up in the $(n+1)^{\text{th}}$ round, their game has to be staged in the remaining $(n+2)^{\text{th}}$ round.

A further generalization concerns the allowed number of absences. It seams that $2k$ additional rounds are needed if k absences are allowed. Note also that our proof of Theorem 3.10 and its repercussions is based on Häggkvist and Janssen's proof and the proof of Theorem 2.1. This leads to a quite tricky strategy for Mrs. Correct, an algorithm with exponential running time. Brute force calculations might be as good.

We conclude this section with another special case of the List Coloring Conjecture. Ellingham and Goddyn's confirmed the List Coloring Conjecture for planar r-regular edge r-colorable multigraphs G (see [ElGo] or the end of Section 5 in [Scha2]), and this can be generalized to paintability. In their original proof, they show that the difference

$$|DE_{r-1}(\vec{LG})| - |DO_{r-1}(\vec{LG})| \quad , \tag{55}$$

where $\vec{LG}$ is the arbitrarily oriented line graph of G, equals the number of edge r-colorings of G (up to a constant factor). Thus, the existence of an edge r-coloring implies the assumptions of Theorem 2.1, and hence the r-paintability of $\vec{LG}$:

Theorem 3.12. *Planar r-regular edge r-colorable multigraphs are edge r-paintable.*

For arbitrary graphs, the trick behind this theorem does not work. This is because the corresponding difference of even and odd orientations usually equals just a weighted sum over certain colorings [Scha2, Corollary 5.5(i)], so that the contributions of the different colorings may cancel each other.

Acknowledgement:
We are grateful to Melanie Win Myint, Stefan Felsner, Anton Dochtermann, Caroline Faria, Stephen Binns and Ayub Khan for helpful comments. Furthermore, the author gratefully acknowledges the support provided by the King Fahd University of Petroleum and Minerals, the Technical University of Berlin and the Eberhard Karls University of Tübingen during this research.

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