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Flexible Multibody Simulation and Choice of Shape Functions

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Abstract. The approach most widely used for the modelling of flexible bodies in multibody systems has been called the floating frame of reference formulation. In this methodology the flexible body motion is subdivided into a reference motion and deformation. The displacement field due to deformation is approximated by the Ritz method as a product of known shape functions and unknown coordinates depending on time only. The shape functions may be obtained using finite-element-models of flexible bodies in multibody systems, resulting in a detailed system representation and a high number of system equations. The number of system equations of such a nodal approach can be reduced considerably using a modal representation of deformation. This modal approach, however, leads to the fundamental problem of selecting the shape functions.

The floating frame of reference formulation is reviewed here using a generic flexible body model, from which the various body models used in multibody simulations may be derived by formulation of specific constraint equations. Special attention is given in this investigation to the following subjects:

- The separation of flexible body motion into a reference motion and deformation requires the definition of a body reference frame, which in turn affects the choice of shape functions. Some alternatives will be outlined together with their advantages and disadvantages.
- Assuming the body deformation to be small, the system equations can be linearized. This may require considering geometric stiffening terms. The problem of how to compute these terms has been solved in literature on the instability of structures under critical loads. For finite element models the geometric stiffening terms are obtained from the tangential stiffness matrix.
- The generality of the flexible body model allows the definition of an object oriented data base to describe the system bodies. Such a data base includes a general interface between multibody- and finite-element-codes.
- By combining eigenfunctions and static deformation modes to represent body deformation one obtains a set of so-called quasi-comparison functions. When selected properly these functions can be shown to improve the representation of stresses significantly.

Keywords: Multibody simulation, flexible body modelling, interaction of multibody- and finite-element-codes, shape functions and quasi-comparison functions, nodal and modal coordinates.

1. Introduction

A general multibody system (MBS), as considered here, is shown in Figure 1. The elements of the model are bodies, force elements, joints and a global reference frame. The bodies of the system may be rigid or flexible, and they are the only system components, which are assumed to have inertia. On the surface of the bodies there are parts, called nodes, at which the joints and force elements are attached. The force elements are used to model applied forces and

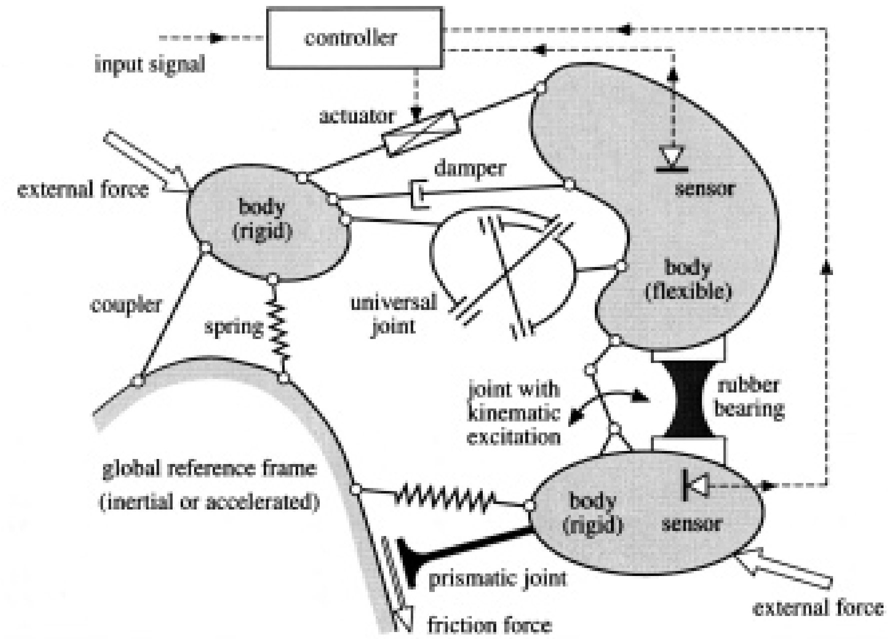

Figure 1. General multibody system model and its elements.

torques. As shown in Figure 1, they may represent external forces, e.g. due to gravity, or interaction between the bodies, resulting from dampers, springs, actuators or contact. All of these forces and torques are functions of the position and velocity of the system bodies, the constraint forces (e.g. in case of dry friction) and the state of a control system in case of the actuators. The joints are any devices that constrain the relative motion of the nodes on the bodies, and they result in unknown constraint forces and torques. Joint deformations as a result of the interaction between the system bodies are not considered. The global reference frame is used to model a known global system motion in inertial space.

Multibody formalisms are computer oriented procedures to generate the equations of motion for systems of the general form shown in Figure 1, based on data, which describe the system elements and system topology, i.e. the way the nodes on the system bodies are interconnected by force elements and joints. Two groups of formalisms may be distinguished. They result in basically different types of equations of motion. The first group yields the Lagrangian equations of type 1, which contain the unknown generalized constraint forces in terms of Lagrangian multipliers. These differential equations are accompanied by a set of algebraic constraint equations. The resulting representation of the system motion is sometimes called the descriptor form of the equations of motion. It is simple to generate, but it requires the numerical solution of differential-algebraic equations [1]. By contrast, the second group of formalisms provides the state space representation of motion, i.e. a minimal set of first order (kinematical and dynamical) differential equations, in which the constraint forces have been eliminated. Numerical methods for solving these equations are often considered to be more mature with respect to computational efficiency. The starting point for the development of both types of formalisms are the equations describing the motion of a representative system body i, acted upon by the applied external and internal forces and torques due to the force elements and the unknown internal joint forces and torques between the system bodies.

Methods of modelling flexible bodies in a multibody system have been reviewed in [2]. Here the floating frame of reference formulation will be used. In this methodology the motion of a flexible body is subdivided into a reference motion and a deformation. The former is the motion of the body in its reference configuration. It may be described as the motion of a body reference frame, i.e. by six variables depending on time t only. Deformation is the motion of the points of the body with respect to its reference frame, and it is given by variables depending on material coordinates and time. Introducing a Ritz approximation, one obtains a representation of the body deformation by variables, which depend on time t only. Thus the motion of an arbitrary body i is described by position- and velocity-variables of the form

$$\mathbf{z}^i_I(t) = [z^i_{Ij}(t)] \quad \text{and} \quad \mathbf{z}^i_{II}(t) = [z^i_{IIj}(t)], \quad j = 1, 2, \ldots, n^i_z. \tag{1}$$

In case of rigid bodies $n^i_z = 6$, whereas $n^i_z > 6$ for flexible bodies. It is important to realize that reference motion and deformation are defined only after specifying location and orientation of the body reference frame [3–5].

The representation of the motion of an arbitrary body in the system, as proposed above, has the following two consequences:

1. The system equations can be linearized, assuming the deformations to be small, whereas the reference motion may be large and fast. The assumption is true in many applications [5, chap. 1]. The simplifications due to linearization can be exploited, to increase the computational efficiency, but as emphasized in [6], so-called geometric stiffening effects need to be considered. The problem has been detailed for beams in [7]. In case of high accelerations of the reference motion, large inertia forces act upon the system bodies. The deformation of bodies remains small only, if their stiffness against the large inertial effects is high. In such cases geometric stiffness terms appear in the linearized equations, describing body deformation in the directions, along which the stiffness coefficients are low [8–11]. Besides the inertia forces, large external and interaction forces applied in directions of high body stiffness require the consideration of geometric stiffening.
2. The Ritz method [12; 13, vol. I, p. 150] requires the selection of shape functions. Their choice is tied to the definition of the body reference frame [4, 5]. Different sets of shape functions and the corresponding definitions of body reference frames result in different magnitudes of the two displacements (reference motion and deformation), into which the body motion has been subdivided. For linearization one would like to select the body frame in such a way that the variables describing deformation are as small as possible. It has been demonstrated in [14] that a so-called Buckens-, Gylden-, Tisserand- or mean-axis-frame (see, e.g., [15–18]) guarantees such an optimal choice. This does not imply of course that such a frame is the best alternative in all applications.

Based on the equations of motion of a representative body i in terms of the variables (1), the following topics will be discussed in the paper:

1. definition of data describing a flexible body in a multibody system;
2. computation of geometric stiffening terms for finite-element models of the body;
3. representation of body deformation using various sets of shape functions.

The last point requires special attention. Convergence of the Ritz method towards the solution of the partial differential equation describing body deformation is assured, when the shape functions form a complete set of functions and when they satisfy the *geometrical* boundary conditions [11, p. 40]. Such functions are called admissible functions [19]. The expansion theorem [13, vol. I, p. 311] or [20, p. 111] states the conditions, under which eigenfunctions or eigenmodes form a complete set of admissible functions. Using eigenfunctions, which violate the dynamical boundary conditions, the convergence of the Ritz method is often poor, especially when evaluating the internal forces of a flexible body. Convergence can be improved significantly by introducing an expanded class of admissible functions, called quasi-comparison functions [19, 21–24]. These are admissible functions allowing to satisfy the dynamical boundary conditions. But not each function has to satisfy the dynamical boundary conditions individually. Rather a linear combination of quasi-comparison functions has to be capable of satisfying them to any desired degree of accuracy.

Eigenfunctions will be used as shape functions here. Poor convergence, when representing deformation and in particular the resulting internal forces by these functions only, is demonstrated by simple examples. They suggest that the inability of the eigenfunctions to satisfy the dynamical boundary conditions is the prime reason for unsatisfactory results. Expanding the system of eigenfunctions by static deformation modes to obtain a set of quasi-comparison functions, convergence is improved vastly. The results are true for eigenfunctions corresponding to any boundary conditions, but two points need to be considered, when using the representation of flexible body motion described above to develop multibody formalisms:

1. Different eigenfunctions correspond to different definitions of body reference frames. A multibody formalism, in which any set of eigenfunctions is to be used, has to allow for any definition of the body reference frame. This is true in particular for the so-called $O(n)$-formalisms. For them a specific choice of the body reference frame results in computational advantages [25, p. 82; 26]. When sticking to such a specific choice, the eigenfunctions need to be transformed accordingly, to satisfy the geometric boundary conditions before using them to represent body deformation [4].
2. For linearization the deformations have to be as small as possible. This condition is satisfied by a mean-axis-frame, which requires to use eigenfunctions of free structures. As compared to other body reference frames, belonging to eigenfunctions of supported structures, the mean-axis-frame enlarges the range of applicability of the linearized equations of motion, often used in the floating frame of reference formulation.

These statements will be explained now in more detail. They are verified by examples in a companion paper on the representation of stress in the flexible system bodies.

The notation used here is as described in [5]. Vectors considered as invariants are represented by underlined letters from any alphabet, and matrices are denoted by boldface letters. The ith row and the jth column of a matrix $\mathbf{A} = [A_{ij}]$ are $\mathbf{A}_{i*}$ and $\mathbf{A}_{*j}$, respectively. Tilded symbols, such as $\tilde{\mathbf{v}}$, denote the skew symmetric matrices, assigned to the coordinates $\mathbf{v}$ of a vector $\underline{v}$, as required to represent its cross product $\underline{v} \times \underline{w}$ with vector $\underline{w}$,having coordinates $\mathbf{w}$, by the matrix product $\tilde{\mathbf{v}}\mathbf{w}$. Vector arrays, a convenient device to specify the resolution of vectors in a basis, are denoted by underlined boldface letters. Greek subscripts always range from 1 to 3. A Cartesian coordinate system is denoted by $\{O, \underline{\mathbf{e}}\}$, where O is its origin and where $\underline{\mathbf{e}} = [\underline{e}_\alpha]$ denotes its basis given by a set of three orthonormal base vectors $\underline{e}_\alpha$. The symbol ${}^i\mathrm{d}\underline{v}/\mathrm{d}t$ represents the relative derivative of a vector $\underline{v}$ with respect to a coordinate system $\{O^i, \underline{\mathbf{e}}^i\}$.

2. General Model of a Representative Body

Models of flexible bodies used in multibody system simulation are finite element models and models of beams, plates or shells, i.e. continuum models with internal constraints. The latter

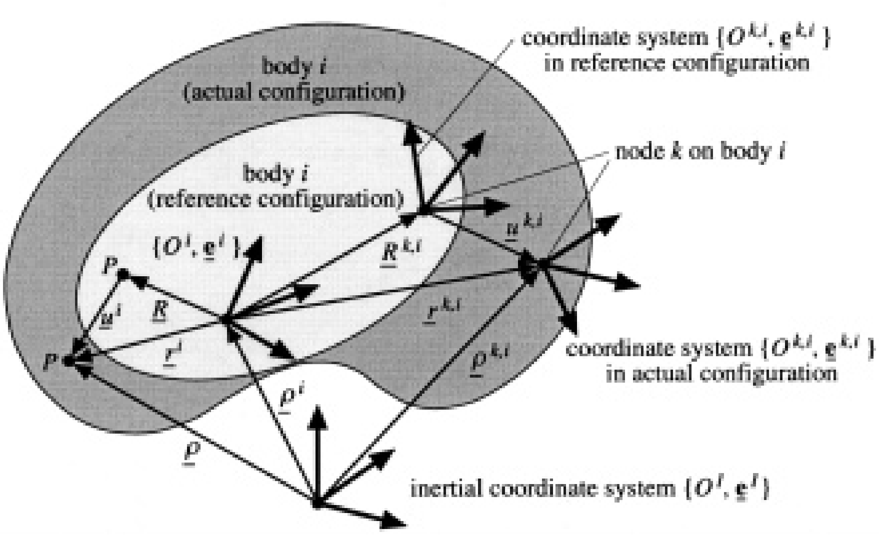

Figure 2. General model of a representative body i.

term, most probably, is due to Volterra [27, 28]. All of these models are summarised in a general body model, shown in Figure 2.

In the most general case the body i may be modelled as suggested in continuum mechanics [29, 30]. In its reference configuration, which is shown in light grey in Figure 2, the location of an arbitrary point P of body i with respect to a body reference frame $\{O^i, \underline{\mathbf{e}}^i\}$ is given by the material coordinates

$$\underline{R} = \underline{\mathbf{e}}^{iT}\,\mathbf{R}, \quad \mathbf{R} = [R_\alpha]. \tag{2}$$

Any of the body models, as used in multibody system simulation, is obtained from the general continuum model by constraining the motion of its points P. Examples of such constraint equations may be found in [15]. In many cases the formulation of the constraint equations requires introducing coordinate systems $\{P, \underline{\mathbf{e}}\}$ at the points of the body characterized by the material coordinates $\mathbf{R}$ (see, e.g., [31]). The orientation of the basis $\underline{\mathbf{e}} = \underline{\mathbf{e}}(\mathbf{R})$ with respect to $\underline{\mathbf{e}}^i$ is described by a rotation matrix $\mathbf{\Gamma}^i = \mathbf{\Gamma}^i(\mathbf{R})$:

$$\underline{\mathbf{e}}(\mathbf{R}) = \mathbf{\Gamma}^i(\mathbf{R})\underline{\mathbf{e}}^i, \quad \mathbf{\Gamma}^i = [\Gamma^i_{\alpha\beta}]. \tag{3}$$

In case of an Euler–Bernoulli beam the coordinate systems $\{P, \underline{\mathbf{e}}\}$ are fixed in the rigid cross sections of the beam, and in case of finite element models such coordinate systems often exist at the nodes of the model.

The existence of coordinate systems $\{P, \underline{\mathbf{e}}\}$ can be mandatory at the attachment points of joints and force elements, to obtain a meaningful multibody system model: obviously, a rigid surface element is required on a flexible body to attach a joint, which remains undeformed. More precisely, such a surface element and the associated coordinate system $\{P, \underline{\mathbf{e}}\}$ is required, if a joint constrains rotation or if a force element transmits torque. When using flexible body models without appropriate coordinate systems $\{P, \underline{\mathbf{e}}\}$, their existence at the attachment points of joints and force elements has to be assured by boundary conditions. They state that the strains are zero on parts of the surface of the body, at which the elements are fixed.

Thus, coordinate systems $\{P, \underline{\mathbf{e}}\}$ exist at least at the attachment points of*) joints, i.e. at the corresponding nodes of the body model. With the symbol k, i denoting the node k on

*) force elements and

body i, a coordinate system at node k, i is $\{O^{k,i}, \underline{\mathbf{e}}^{k,i}\}$. The location of its origin is given in the reference configuration by the coordinates $\mathbf{R} = \mathbf{R}^{k,i}$. The orientation of its basis $\underline{\mathbf{e}}^{k,i} = \underline{\mathbf{e}}(\mathbf{R}^{k,i})$ with respect to the body reference frame is described in the reference configuration in view of Equation (3) as

$$\underline{\mathbf{e}}^{k,i} = \mathbf{\Gamma}^{k,i}\ \underline{\mathbf{e}}^{i}, \quad \mathbf{\Gamma}^{k,i} = \mathbf{\Gamma}^{i}(\mathbf{R}^{k,i}). \tag{4}$$

In many applications the matrices $\mathbf{\Gamma}^{k,i}$ are given by the unit matrix $\mathbf{E}$.

3. Kinematics of a Representative Body

Orientation and angular velocity of the body reference frame with respect to an inertial frame, i.e. the absolute motion of $\{O^i, \underline{\mathbf{e}}^i\}$, is represented in the following way:

$$\underline{\mathbf{e}}^{i}(t) = \mathbf{A}^{i}(t)\ \underline{\mathbf{e}}^{I}, \quad \mathbf{A}^{i}(t) = [A^{i}_{\alpha\beta}(t)], \quad \text{where} \quad \mathbf{A}^{i}(t) = \mathbf{A}^{i}(\boldsymbol{\alpha}^{i}(t), \quad \boldsymbol{\alpha}^{i}(t) = [\alpha^{i}_{\alpha}(t)], \tag{5}$$

$$\underline{\omega}^{i}(t) = \underline{\mathbf{e}}^{i^T}(t)\ \boldsymbol{\omega}^{i}(t) = [\omega^{i}_{\alpha}(t)], \quad \text{where} \quad \tilde{\boldsymbol{\omega}}^{i} = \mathbf{A}^{i}\ \dot{\mathbf{A}}^{i^T}. \tag{6}$$

With the angular velocity given by Equation (6) and with reference to Figure 2, one obtains the location and velocity of O^i with respect to O^I as

$$\underline{\rho}^{i}(t) = \underline{\mathbf{e}}^{i^T}(t)\ \boldsymbol{\rho}^{i}(t), \quad \boldsymbol{\rho}^{i}(t) = [\rho^{i}_{\alpha}(t)], \tag{7}$$

$$\frac{{}^{I}\mathrm{d}\rho^{i}}{\mathrm{d}t} = \underline{\dot{\rho}}^{i}(t) = \underline{v}^{i}(t) = \underline{\mathbf{e}}^{i^T}(t)\ \mathbf{v}^{i}(t), \quad \mathbf{v}^{i} = [v^{i}_{\alpha}], \quad \text{where} \quad \mathbf{v}^{i} = \dot{\boldsymbol{\rho}}^{i} + \tilde{\boldsymbol{\omega}}^{i}\ \boldsymbol{\rho}^{i}. \tag{8}$$

In view of Figure 2, the motion of the representative points P of body i with respect to the body reference frame is given by the displacement field

$$\underline{u}^{i}(\mathbf{R}, t) = \underline{\mathbf{e}}^{i^T}(t)\ \mathbf{u}^{i}(\mathbf{R}, t), \quad \mathbf{u}^{i} = [u^{i}_{\alpha}]. \tag{9}$$

The absolute location and velocity of P is given by (see Figure 2)

$$\underline{\rho}(\mathbf{R}, t) = \underline{\mathbf{e}}^{i^T}(t)\ {}^{I}\boldsymbol{\rho}(\mathbf{R}, t), \quad \boldsymbol{\rho} = [\rho_{\alpha}], \quad {}^{I}\boldsymbol{\rho} = [{}^{I}\rho_{\alpha}], \tag{10}$$

$$\underline{v}(\mathbf{R}, t) = \underline{\dot{\rho}}(\mathbf{R}, t) = \underline{\mathbf{e}}^{i^T}(t)\ \mathbf{v}(\mathbf{R}, t), \quad \mathbf{v} = [v_{\alpha}]. \tag{11}$$

Using the preceding equations and Figure 2, one concludes

$${}^{I}\boldsymbol{\rho}(\mathbf{R}, t) = \mathbf{A}^{i^T}(t)\ \boldsymbol{\rho}(\mathbf{R}, t), \tag{12}$$

$$\boldsymbol{\rho}(\mathbf{R}, t) = \boldsymbol{\rho}^{i}(t) + \mathbf{R} + \mathbf{u}^{i}(\mathbf{R}, t) \tag{13}$$

$$\mathbf{v}(\mathbf{R}, t) = \mathbf{v}^{i}(t) + \dot{\mathbf{u}}^{i}(\mathbf{R}, t) + \tilde{\boldsymbol{\omega}}^{i}(t)(\mathbf{R} + \mathbf{u}^{i}(\mathbf{R}, t)). \tag{14}$$

In these equations $\boldsymbol{\rho}^{i}(t)$ and $\mathbf{A}^{i}(t)$ and their time derivatives represent the reference motion, whereas $\mathbf{u}^{i}(\mathbf{R}, t)$ and its derivatives describe the deformation of body i, when modelling it as a general continuum. For flexible bodies, in which the motion of P is constrained, one has to consider the motion of a basis $\underline{\mathbf{e}}$ at P as well. In the actual configuration the orientation of $\underline{\mathbf{e}}(\mathbf{R}, t)$ with respect to $\underline{\mathbf{e}}^i$ is described as

$$\underline{\mathbf{e}}(\mathbf{R}, t) = \mathbf{\Theta}^i(\mathbf{R}, t)\, \mathbf{\Gamma}^i(\mathbf{R})\, \underline{\mathbf{e}}^i, \quad \mathbf{\Theta}^i = [\Theta^i_{\alpha\beta}]. \tag{15}$$

For small deformations one obtains in linear approximation

$$\mathbf{\Theta}^i(\mathbf{R}, t) = \mathbf{E} - \tilde{\boldsymbol{\vartheta}}^i(\mathbf{R}, t), \quad \boldsymbol{\vartheta}^i = [\vartheta^i_\alpha]. \tag{16}$$

The absolute orientation and angular velocity of $\underline{\mathbf{e}}(\mathbf{R}, t)$ is described by

$$\underline{\mathbf{e}}(\mathbf{R}, t) = \mathbf{A}(\mathbf{R}, t)\, \underline{\mathbf{e}}^I, \quad \mathbf{A} = [A_{\alpha\beta}], \tag{17}$$

$$\underline{\omega}(\mathbf{R}, t) = \underline{\mathbf{e}}^{i^T}(t)\, \boldsymbol{\omega}\, (\mathbf{R}, t), \quad \boldsymbol{\omega} = [\omega_\alpha]. \tag{18}$$

From these equations one concludes with (15), (5) and (6) when the deformations are assumed to be small that

$$\mathbf{A}(\mathbf{R}, t) = (\mathbf{E} - \tilde{\boldsymbol{\vartheta}}^i(\mathbf{R}, t))\mathbf{\Gamma}^i(\mathbf{R})\mathbf{A}^i(t), \tag{19}$$

$$\boldsymbol{\omega}(\mathbf{R}, t) = \boldsymbol{\omega}^i(t) + \dot{\boldsymbol{\vartheta}}^i(\mathbf{R}, t). \tag{20}$$

Here $\mathbf{A}^i(t)$ and $\boldsymbol{\omega}^i(t)$ describe the rotational reference motion and $\boldsymbol{\vartheta}^i(\mathbf{R}, t)$ represents the effects due to body deformation.

To summarize, small deformations of a representative body i are described by the functions $\mathbf{u}^i(\mathbf{R}, t)$ and $\boldsymbol{\vartheta}^i(\mathbf{R}, t)$. Introducing a Ritz approximation of the form

$$\left.\begin{array}{ll} \mathbf{u}^i(\mathbf{R}, t) = \mathbf{\Phi}^i(\mathbf{R})\, \mathbf{q}^i(t), & \mathbf{\Phi}^i = [\Phi^i_{\alpha k}] \\ \boldsymbol{\vartheta}^i(\mathbf{R}, t) = \mathbf{\Psi}^i(\mathbf{R})\, \mathbf{q}^i(t), & \mathbf{\Psi}^i = [\Psi^i_{\alpha k}] \end{array}\right\} \quad \mathbf{q}^i(t) = [q^i_k(t)], \quad k = 1, 2, \ldots, n^i_q, \tag{21}$$

the variables representing position and velocity of a body i, as introduced by Equation (1), can be written as

$$\mathbf{z}^i_I(t) = [z^i_{Ij}(t)] = \begin{bmatrix} \boldsymbol{\rho}^i(t) \\ \boldsymbol{\alpha}^i(t) \\ \mathbf{q}^i(t) \end{bmatrix}, \quad \mathbf{z}^i_{II}(t) = [z^i_{IIj}(t)] = \begin{bmatrix} \mathbf{v}^i(t) \\ \boldsymbol{\omega}^i(t) \\ \dot{\mathbf{q}}^i(t) \end{bmatrix},$$

$$j = 1, 2, \ldots, n^i_z, n^i_z = 6 + n^i_q. \tag{22}$$

These variables satisfy kinematical equations of motion of the form

$$\dot{\mathbf{z}}^i_I = \mathbf{Z}^i(\mathbf{z}^i_I)\, \mathbf{z}^i_{II} \quad \text{with} \quad \mathbf{Z}^i = \begin{bmatrix} \mathbf{E} & \tilde{\boldsymbol{\rho}}^i & \mathbf{0} \\ \mathbf{0} & \mathbf{Z}^i_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{E} \end{bmatrix}, \tag{23}$$

where $\mathbf{Z}^i_r$ is a 3×3-matrix, relating the angular velocity $\boldsymbol{\omega}^i$ to the time derivatives of the angles $\boldsymbol{\alpha}^i$, introduced in Equation (5) to parameterize matrix $\mathbf{A}^i$.

The motion of the coordinate systems $\{O^{k,i}, \underline{\mathbf{e}}^{k,i}\}$ at the nodes k, i, as required for computation of forces and torques at the nodes*) and for evaluation and comparison of results, can be obtained from the variables (22). With reference to Figure 2, the absolute location of $O^{k,i}$ is given by $\underline{\rho}^{k,i}(t) = \underline{\rho}(\mathbf{R}^{k,i}, t)$, i.e. because of Equation (10) by $\boldsymbol{\rho}^{k,i}(t) = \boldsymbol{\rho}(\mathbf{R}^{k,i}, t)$. Similarly,

*) , for the formulation of the joint constraint equations

in view of Equation (5), the absolute orientation of $\underline{\mathbf{e}}^{k,i}$ is given by $\boldsymbol{\alpha}^{k,i}(t) = \boldsymbol{\alpha}(\mathbf{R}^{k,i}, t)$. The relative motion of $\{O^{k,i}, \underline{\mathbf{e}}^{k,i}\}$ with respect to the body reference frame is described by the variables $\mathbf{u}^{k,i}(t) = \mathbf{u}(\mathbf{R}^{k,i}, t)$ and $\boldsymbol{\vartheta}^{k,i}(t) = \boldsymbol{\vartheta}(\mathbf{R}^{k,i}, t)$ (see Equation (21)) and $\mathbf{r}(\mathbf{R}^{k,i}, t) = \mathbf{r}^{k,i}(t) = \mathbf{R}^{k,i} + \mathbf{u}^{k,i}(t)$.

The description of absolute location and orientation of $\{P, \underline{\mathbf{e}}\}$ requires six variables, whereas it is given by the right-hand sides of Equations (12, 13) and (19) by the 12 variables $\boldsymbol{\rho}^i(t), \boldsymbol{\alpha}^i(t), \mathbf{u}^i(\mathbf{R}, t)$ and $\boldsymbol{\vartheta}^i(\mathbf{R}, t)$. This clearly demonstrates that six constraint equations are still required to uniquely and completely define the variables of Equation (22). These equations are obtained by defining location and orientation of the body reference frame $\{O^i, \underline{\mathbf{e}}^i\}$. This definition can be given using kinematical and dynamical relations (see Section 5).

4. Dynamics of a Representative Body

The model of a multibody system, as shown in Figure 1, may be obtained from the general model of continuum mechanics by formulating two types of constraint equations. They result from

1. definition of the models of bodies (i.e. finite element models, beams, etc.) to be used for a specific analysis;
2. constraints due to the joints between the nodes on the bodies.

The explicit form of the type 1 constraints relates the displacement field of all the points of the multibody system to the variables of Equation (22). The constraints of type 2 result from the joints only and their explicit form represents the redundant variables of Equation (22) in terms of an independent set of system state variables. Formulating both types of constraint equations and applying one of the principles of dynamics, one can generate the descriptor form or the state space form of the system equations [5].

An intermediate result of such a derivation of the equations of motion for a system of n bodies is the virtual power expression in terms of the velocities $\mathbf{z}^i_{II}$ defined in Equation (22)

$$\delta P = \sum_{i=1}^{n} \boldsymbol{\delta}\, \mathbf{z}^{iT}_{II}(\mathbf{M}^i \dot{\mathbf{z}}^i_{II} - \mathbf{h}^i_a - \mathbf{h}^i_c). \tag{24}$$

The matrices $\boldsymbol{\delta}\, \mathbf{z}^i_{II}$ are the virtual velocities, belonging to the generalized velocities $\mathbf{z}^i_{II}$, and $\mathbf{M}^i, \mathbf{h}^i_a$ and $\mathbf{h}^i_c$ are the corresponding generalized masses, applied forces and constraint forces, respectively. Jourdain's principle states that the virtual power of the constraint forces, resulting from both types of constraints, is zero. In Equation (24), the virtual power expressions resulting from type 1 constraints do not appear, but the virtual power due to type 2 constraints has been kept explicit. Because of the latter constraints, the elements of $\mathbf{z}^i_{II}$ and $\boldsymbol{\delta}\, \mathbf{z}^i_{II}$ are independent only for a multibody system without joints. In such a case $\mathbf{h}^i_c \equiv \mathbf{0}$, and Jourdain's principle yields in view of Equation (24) the system equations of motion

$$\mathbf{M}^i\, \dot{\mathbf{z}}^i_{II} = \mathbf{h}^i_a, \quad i = 1, 2, \ldots, n. \tag{25}$$

The result suggests a physical interpretation of the expression enclosed by parentheses in Equation (24). The equations

$$\mathbf{M}^i\, \dot{\mathbf{z}}^i_{II} = \mathbf{h}^i_a + \mathbf{h}^i_c, \quad i = 1, 2, \ldots, n \tag{26}$$

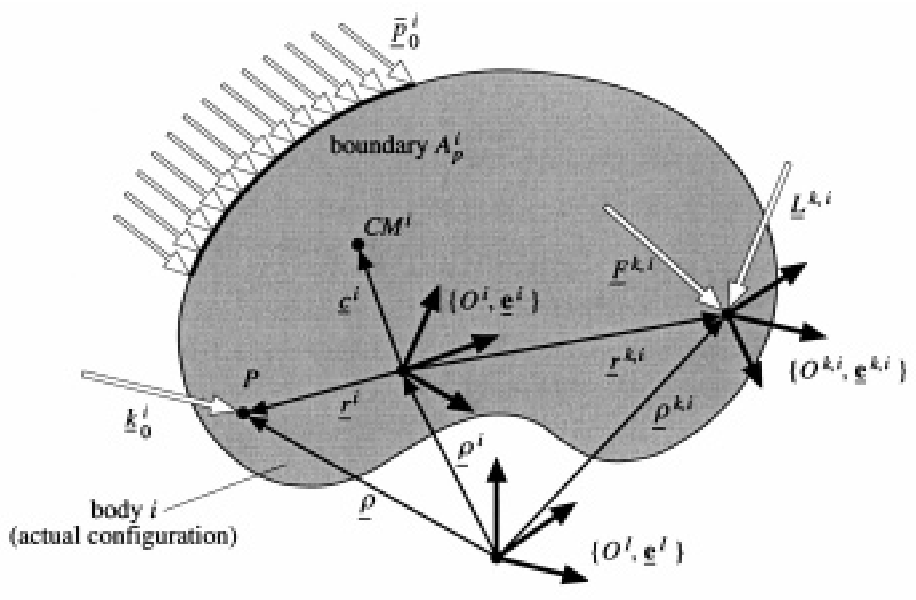

Figure 3. Forces and moments upon a typical body i.

are the equations of motion of a system body i, as shown in Figure 3. It is obtained from the entire multibody system by cutting all force elements and joints. The generalized applied forces in Equation (26) may be separated into

$$\mathbf{h}_a^i = \mathbf{h}_\omega^i + \mathbf{h}_g^i + \mathbf{h}_e^i + \mathbf{h}_p^i + \mathbf{h}_f^i. \tag{27}$$

In this expression $\mathbf{h}_\omega^i$ are generalized inertia forces due to the angular velocity $\boldsymbol{\omega}^i$ of the reference motion. They are distributed over the body volume V_0^i. This is also true for the external forces, which are applied at point P, as shown in Figure 3 by the body force $\underline{k}_0^i$ per unit volume. An important example are gravitational forces, which yield the generalized forces $\mathbf{h}_g^i$ in Equation (27). Matrix $\mathbf{h}_e^i$ represents the generalized internal forces, resulting from elastic body deformation, and $\mathbf{h}_p^i$ results from external surface forces given by the external stresses $\underline{\overline{p}}_0^i$, applied at the boundary A_p^i of body i. The generalized forces $\mathbf{h}_f^i$ result from force elements attached at nodes k, i. Rigid surface elements have been assumed to exist at the nodes. Any forces distributed over such a rigid surface element at node k, i may be replaced by a resultant force $\underline{F}^{k,i}$ and a moment $\underline{L}^{k,i}$ at $O^{k,i}$. Both, $\underline{F}^{k,i}$ and $\underline{L}^{k,i}$, are known functions of the relative motion of the two nodes interconnected by the force element (and possibly of additional quantities, as detailed in the description of the multibody model shown in Figure 1). Forces $\underline{F}^{k,i}$ and moments $\underline{L}^{k,i}$ result as well from joints fixed at nodes k, i, but these are unknown. They yield the unknown generalized constraint forces $\mathbf{h}_c^i$ in Equation (26).

The generalized mass matrix $\mathbf{M}^i$ in Equation (26) can be partitioned according to the partitioning of $\mathbf{z}_I^i$ and $\mathbf{z}_{II}^i$ in Equation (22) as follows

$$\mathbf{M}^i = \begin{bmatrix} \mathbf{M}_{tt}^i & & \text{sym.} \\ \mathbf{M}_{rt}^i & \mathbf{M}_{rr}^i & \\ \mathbf{M}_{et}^i & \mathbf{M}_{er}^i & \mathbf{M}_{ee}^i \end{bmatrix} = \begin{bmatrix} m^i\,\mathbf{E} & & \text{sym.} \\ m^i\,\tilde{\mathbf{c}}^i & \mathbf{I}^i & \\ \mathbf{C}_t^i & \mathbf{C}_r^i & \mathbf{M}_e^i \end{bmatrix}. \tag{28}$$

Indices t, r and e refer to translation and rotation of the reference motion and to elastic deformation, respectively. The symbol m^i in Equation (28) denotes the mass of body i and $\mathbf{I}^i$ is the inertia matrix of the body (in its deformed configuration) with respect to the origin O^i of its reference frame. It is the resolution of the inertia tensor $\underline{\underline{I}}^i$ in basis $\underline{\mathbf{e}}^i$. Matrix $\mathbf{c}^i$ contains the coordinates of vector $\underline{c}^i$ in $\underline{\mathbf{e}}^i$. As shown in Figure 3, it locates the center of mass CM^i of the body (in its deformed configuration) with respect to O^i. Submatrix $\mathbf{M}_e^i$ contains the generalized masses corresponding to the velocities $\dot{\mathbf{q}}^i$, and $\mathbf{C}_t^i$ and $\mathbf{C}_r^i$ are matrices representing the coupling the xxx between reference motion and deformation. They have a simple interpretation: The separation of the motion of body i into reference motion and deformation results in a corresponding separation of the linear and angular momentum vectors of the body. Resolving linear and angular momentum due to deformation in basis $\underline{\mathbf{e}}^i$ and denoting the coordinates by ${}_O\mathbf{J}^i$ and ${}_O\mathbf{H}^i$, respectively, one finds that the matrices $\mathbf{C}_t^i$ and $\mathbf{C}_r^i$ represent these momentum vectors in the form

$$ {}_O\mathbf{J}^i = \mathbf{C}_t^i\,\dot{\mathbf{q}}^i, \quad {}_O\mathbf{H}^i = \mathbf{C}_r^i\,\dot{\mathbf{q}}^i \quad \text{where} \quad \mathbf{C}_r^i = \mathbf{C}_r^i(\mathbf{q}^i). \tag{29} $$

All of the generalized forces and masses in Equation (26) are found to be algebraic expressions, containing the variables (22) and integrals over the shape functions $\mathbf{\Phi}^i(\mathbf{R})$ and $\mathbf{\Psi}^i(\mathbf{R})$ [15].

5. Definition of Body Reference Frames

A unique representation of the motion of body i by the variables of Equation (22) requires the definition of the body reference frame $\{O^i, \underline{\mathbf{e}}^i\}$. A widely used option is to identify it with the frame $\{P, \underline{\mathbf{e}}\}$ at one of the nodes of body i, e.g. the node characterized by the material coordinates $\mathbf{R} = \mathbf{0}$

$$ \left\{O^i, \underline{\mathbf{e}}^i\right\} = \left\{P, \underline{\mathbf{e}}\right\}\Big|_{\mathbf{R}=\mathbf{0}}. \tag{30} $$

As the reference motion represents the motion of $\{P, \underline{\mathbf{e}}\}$ at $\mathbf{R} = \mathbf{0}$, one concludes that

$$ \mathbf{u}^i(\mathbf{0}, t) = \mathbf{0} \quad \text{and} \quad \boldsymbol{\vartheta}^i(\mathbf{0}, t) = \mathbf{0} \tag{31} $$

holds for the variables of Equation (21) representing deformation. These are geometrical boundary conditions, which need to be satisfied by the shape functions, i.e.

$$ \mathbf{\Phi}^i(\mathbf{0}) = \mathbf{0} \quad \text{and} \quad \mathbf{\Psi}^i(\mathbf{0}) = \mathbf{0}. \tag{32} $$

This clearly demonstrates that the definition of the body reference frame and the choice of shape functions are interrelated. When using eigenfunctions of a flexible body to represent deformation as proposed in Equation (21) and when the body reference frame is considered to satisfy Equation (30), the eigenfunctions have to satisfy Equation (32), i.e. belong to a structure clamped at $\mathbf{R} = \mathbf{0}$. A definition of the body reference frame as suggested in Equation (30) is often used in so-called $O(n)$-formalisms: the associated generation of the system equations of motion becomes particularly efficient, when the body reference frame coincides with the frame $\{P, \underline{\mathbf{e}}\}$ at the attachment point of the inboard joint on the body [25]. When being forced to use a set of eigenfunctions violating Equation (32) (e.g. because they are the only ones, which are available for a given finite element model and because their recomputation for the desired boundary conditions is expensive) one has two choices:

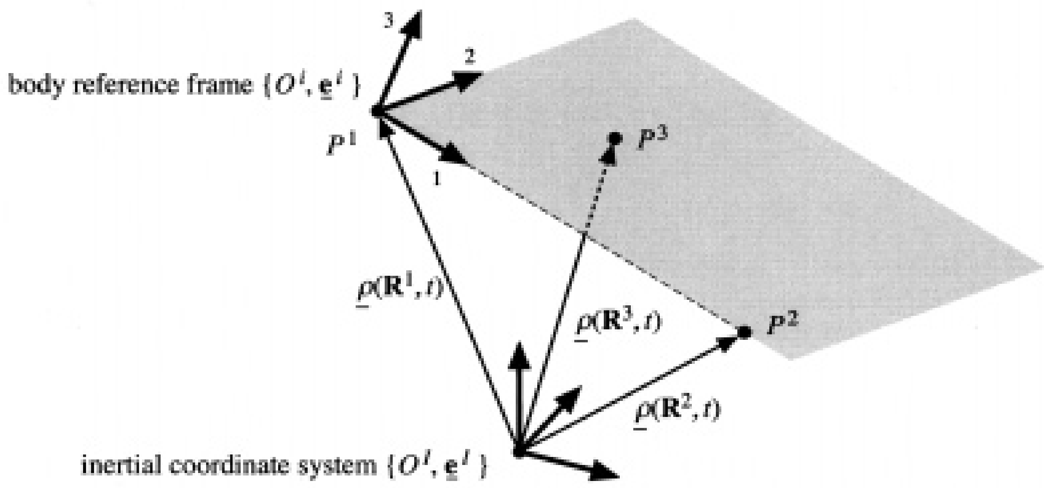

Figure 4. Definition of the body reference frame based on three points of the body.

1. the eigenfunctions are transformed to satisfy Equation (32) as proposed in [4];
2. the definition of the body reference frame is modified as required by the functions, which are available.

An example of other eigenfunctions are those belonging to simply supported structures. In such a case $\{O^i, \underline{\mathbf{e}}^i\}$ is given by three points P^1, P^2 and P^3 of the body, which are characterized by their material coordinates $\mathbf{R}^1$, $\mathbf{R}^2$ and $\mathbf{R}^3$. The origin O^i is defined as

$$O^i = P\big|_{\mathbf{R}=\mathbf{R}^1}. \tag{33}$$

The basis $\underline{\mathbf{e}}^i$ is obtained as suggested by Figure 4: vector $\underline{e}_1^i$ points to point P^2, $\underline{e}_3^i$ is defined to be perpendicular to the plane spanned by the three points P^1, P^2 and P^3 and $\underline{e}_2^i = \underline{e}_3^i \times \underline{e}_1^i$. Such a definition results in the six boundary conditions

$$\mathbf{u}^i(\mathbf{R}^1, t) = \mathbf{0}, \quad u_2^i(\mathbf{R}^2, t) = 0, \quad u_3^i(\mathbf{R}^2, t) = 0, \quad u_3^i(\mathbf{R}^3, t) = 0, \tag{34}$$

which implies for the shape functions

$$\mathbf{\Phi}^i(\mathbf{R}^1) = \mathbf{0}, \quad \mathbf{\Phi}_{2*}^i(\mathbf{R}^2) = \mathbf{0}, \quad \mathbf{\Phi}_{3*}^i(\mathbf{R}^2) = \mathbf{0}, \quad \mathbf{\Phi}_{3*}^i(\mathbf{R}^3) = \mathbf{0}. \tag{35}$$

In a similar way other body frames may be defined by tying the frame to specific points or lines on the body. A complete definition of $\{O^i, \underline{\mathbf{e}}^i\}$ requires six equations, which implies that six boundary conditions are needed. In particular, they have to exclude any rigid body motion of the flexible body.

Instead of the kinematical relations used heretofore, the body reference frame may be defined as well, using dynamical concepts. A particularly favourable choice is to select its location and orientation in such a way that linear and angular momentum due to body deformation are zero, i.e. in view of Equation (29) that

$$\mathbf{C}_t^i\,\dot{\mathbf{q}}^i = \mathbf{0} \quad \text{and} \quad \mathbf{C}_r^i\,\dot{\mathbf{q}}^i = \mathbf{0} \quad \text{where} \quad \mathbf{C}_r^i = \mathbf{C}_r^i(\mathbf{q}^i). \tag{36}$$

A body frame satisfying these constraint equations is called Tisserand-, Gylden- or mean-axis-frame, and in particular Buckens-frame, when using the linearized form of Equations (36),

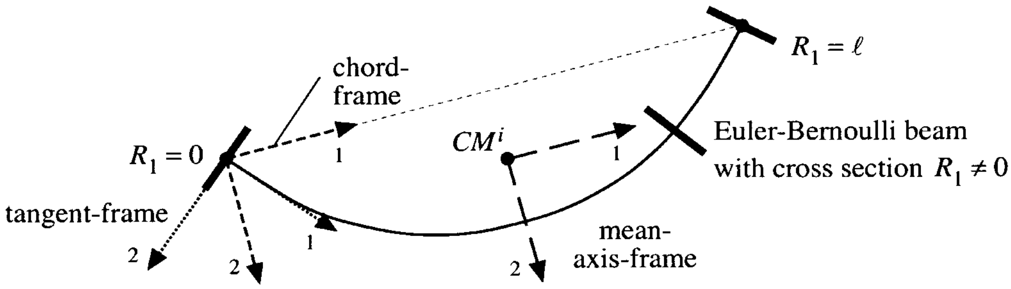

Figure 5. Deformation of an Euler–Bernoulli beam with respect to a tangent-, chord- and mean-axis-frame.

assuming the deformations described by $\mathbf{q}^i$ and $\dot{\mathbf{q}}^i$ to be small [15–18]. Its origin is located at the center of mass CM^i of the deformed body, and its basis $\underline{\mathbf{e}}^i$ rotates as to guarantee that ${}_O\mathbf{H}^i \equiv \mathbf{0}$. Such a frame has three remarkable properties:

1. Because of Equation (36), which implies $\mathbf{c}^i = \mathbf{0}$, the mass matrix in Equation (28) has block diagonal form.
2. The separation of the body motion into reference motion and deformation implies a corresponding separation of the kinetic energy. As demonstrated in [14], the choice of a mean-axis-frame results in a minimum value of the kinetic energy due to body deformation, i.e.

$$\int\limits_{V_0^i} \dot{\mathbf{u}}^{i^T}(\mathbf{R}, t)\dot{\mathbf{u}}^i(\mathbf{R}, t)\, \mathrm{d}V = \min . \tag{37}$$

3. The constraint relations as defined by Equation (36) are nonlinear, because matrix $\mathbf{C}_r^i$ is a function of $\mathbf{q}^i$. When linearizing, assuming the deformations to be small, Equation (37) is no longer satisfied, but

$$\int\limits_{V_0^i} \mathbf{u}^{i^T}(\mathbf{R}, t)\mathbf{u}^i(\mathbf{R}, t)\, \mathrm{d}V = \min . \tag{38}$$

   A Buckens-frame yields the smallest deformations, which are possible, a choice of $\{O^i, \underline{\mathbf{e}}^i\}$, which is most favourable for linearization.

For beams, the frames defined by Equations (31) and (34) are often called tangent- and chord-frame, respectively. Figure 5 visualizes the deformations to be smallest, when measured with respect to a mean-axis-frame. Also, deformations measured in a chord-frame are much smaller than those measured with respect to a tangent-frame.

There is a basic difference of the conditions given by Equation (36) as compared to Equations (31) and (34). The latter are boundary conditions, applying to specific points $\mathbf{R}$ of the body only, whereas the former takes into account the motion of all the body points. The conditions need to be satisfied, when solving the system equations. It is shown in [14] that this can be achieved in a most simple way: using the eigenfunctions of unsupported structures for the shape functions in Equation (21) and deleting the rigid body modes, the constraint equations (36) are automatically satisfied.

6. Linearization and Geometric Stiffening

In many applications the dynamical equations of motion (26) can be linearized, assuming the deformations as described by Equation (21) to be small. This may require considering geometric stiffness terms [7]. A detailed analysis for arbitrary structures demonstrates that geometric stiffening has to be considered, if the following conditions are satisfied [5]:

1. Body i is soft in directions associated with coordinates described by a subset $\mathbf{q}_w^i$ of the variables $\mathbf{q}^i$, but it is stiff in the directions associated with deformations described by the remainder $\mathbf{q}_s^i$ of these variables.[1]
2. Forces resulting in deformations described by $\mathbf{q}_w^i$ are small, which implies that these deformations themselves and the corresponding stresses are small. Forces resulting in large deformations described by $\mathbf{q}_w^i$ do not exist for the models to be considered.
3. Forces yielding deformations $\mathbf{q}_s^i$ are large. Despite the magnitude of the forces, the deformations described by $\mathbf{q}_s^i$ are considered to have the same order of magnitude as those described by $\mathbf{q}_w^i$. This requires the corresponding stiffness of the body to be large. As a consequence, the large forces result in large stresses $\boldsymbol{\sigma}_s^i$.

A frequently studied example satisfying these conditions is a slender beam under large axial loading. It is soft against bending and stiff against stretching (condition 1), the forces resulting in bending are assumed to be small when compared with the longitudinal forces (condition 2) and because of high longitudinal stiffness the large longitudinal forces result in small longitudinal deformations (condition 3). Other examples of beams, which require the modelling of geometric stiffening, have been discussed in [7]. The various cases are found easily from two sets of equations, the potential energy expression in terms of so-called deformation quantities and the transformations of these variables into displacement quantities, which yield a simple representation of kinetic energy.

If the three conditions are satisfied, the geometric stiffness terms appear in the linearized equations (26) and they can be computed for any flexible body model from the equilibrium conditions between the large forces acting on the body and the stresses $\boldsymbol{\sigma}_s^i$. Small forces need not be considered in this context. If the large forces do not depend on the accelerations $\dot{\mathbf{z}}_{II}^i$ (examples are Coriolis- and centrifugal forces and forces $\underline{F}^{k,i}$ and moments $\underline{L}^{k,i}$ at the nodes) the geometric stiffness terms are obtained from ordinary, i.e. non-differential equations. As the high frequency motions due to high stiffness of the body need not be considered in such cases, the efficiency of the numerical solution of the multibody system equations is increased by an order of magnitude [32]. On the contrary, if the large forces are inertia forces, depending on the accelerations $\dot{\mathbf{z}}_{II}^i$, high frequency terms appear in the linearized equations of motion, slowing down their numerical integration.

Geometric stiffness terms are found for beam models as described in [7]. An analysis of various approximations is available in [33]. The computation of the terms for finite element models is detailed in [34, 35]. In [5] it is shown of how to obtain the terms from the total tangential stiffness matrix $\mathbf{K}_T^i$ of a finite element model of body i. With reference to [36, p. 287]

$$\mathbf{K}_T^i = \mathbf{K}^i + \mathbf{K}_\sigma^i + \mathbf{K}_\nu^i. \tag{39}$$

Here, $\mathbf{K}^i$ is the small displacement stiffness matrix, $\mathbf{K}_\sigma^i$ is the initial stress matrix, and $\mathbf{K}_\nu^i$ is the large displacement matrix. Considering small displacements of bodies without initial stress

[1] Index w of $\mathbf{q}_w^i$ refers to the German word ‘weich’ for ‘soft’.

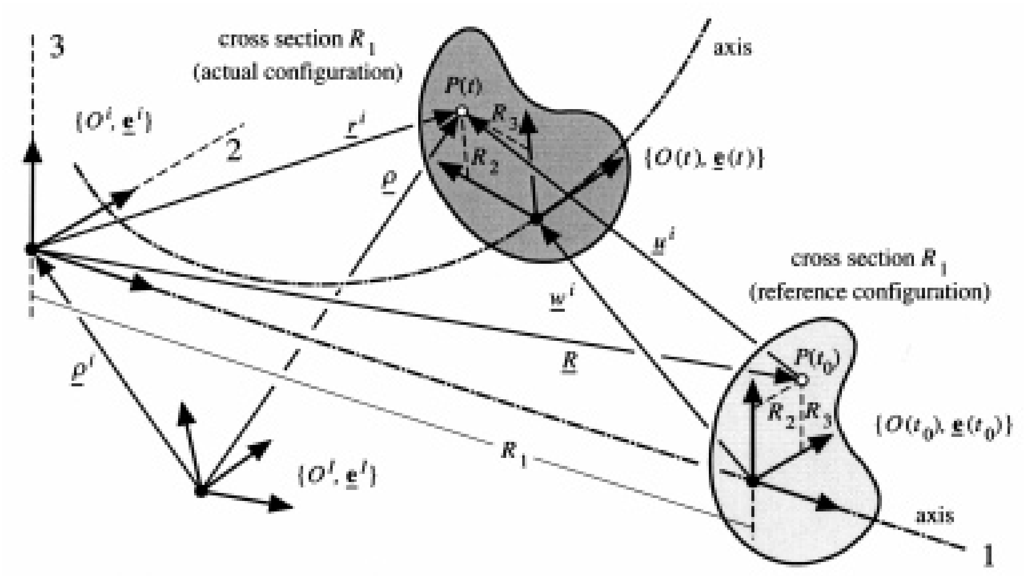

Figure 6. Model of a straight Euler–Bernoulli beam and notation to represent its motion.

and the corresponding linearized equations of motion, it can be shown that $\mathbf{K}^i_\nu = \mathbf{0}$ and that $\mathbf{K}^i_\sigma$ is identical with the geometric stiffness matrix $\mathbf{K}^i_{\text{geo}}$, i.e. $\mathbf{K}^i_\sigma = \mathbf{K}^i_{\text{geo}}$, when $\mathbf{K}^i_\sigma$ is computed using the stresses $\boldsymbol{\sigma}^i_s$, due to large forces yielding the deformations $\mathbf{q}^i_s$. As a consequence one obtains the geometric stiffness matrix in such cases as

$$\mathbf{K}^i_{\text{geo}} = \mathbf{K}^i_T - \mathbf{K}^i, \tag{40}$$

i.e. from the tangential stiffness matrix $\mathbf{K}^i_T$ and the small displacement stiffness matrix $\mathbf{K}^i$.

7. Multibody System Data

As mentioned in Section 4, one may develop any type of multibody formalism by introducing type 2 constraints into the equations of motion (26) and applying the principles of dynamics. The resulting procedure for generation of the equations of motion provides the definition of data to describe a multibody system. They may be subdivided into data describing bodies, joints, force elements, the motion of the global reference frame and the system topology. All of these have been well discussed in the literature on dynamics of systems of rigid bodies. In [37], an object oriented data model has been proposed for such systems. It has been augmented by a set of standard input data describing flexible bodies in multibody systems [38]. The definition of these data is based on Equation (26), which represents the motion of a general flexible body i as shown in Figure 3. The specific equations for models of beams, plates, shells or finite element structures are found by formulating the constraint equations (21), relating deformations, as described by $\mathbf{u}^i(\mathbf{R}, t)$ and $\boldsymbol{\vartheta}^i(\mathbf{R}, t)$, to the generalized coordinates $\mathbf{q}^i(t)$. Such equations are given for beams and finite element models here – computational details are found in [5].

As an example consider small deformations of a straight Euler–Bernoulli beam. With reference to Figure 6, the displacement field of its points is given by

$$\mathbf{u}^i(\mathbf{R}, t) = \mathbf{w}^i(R_1, t) - \tilde{\mathbf{R}}^C\, \boldsymbol{\vartheta}^i(R_1, t), \quad \mathbf{R}^C = [0 \;\; R_2 \;\; R_3]^T. \tag{41}$$

Here, $\mathbf{w}^i(R_1, t)$ represents the displacements $\underline{w}^i$ of the points on the beam's axis in the basis $\underline{\mathbf{e}}^i$ and angles

$$\boldsymbol{\vartheta}^i(R_1, t) = [\vartheta_1^i(R_1, t) \quad -w_3'^i(R_1, t) \quad w_2'^i(R_1, t)]^T \tag{42}$$

describe the orientation of the coordinate system $\{O(t), \underline{\mathbf{e}}(t)\}$, fixed in a cross section of the beam in its actual configuration, with respect to $\{O(t_0), \underline{\mathbf{e}}(t_0)\}$ in the reference configuration. Angle $\vartheta_1^i(R_1, t)$ represents torsional motion. Introducing the Ritz approximation

$$\mathbf{w}^i(R_1, t) = \mathbf{W}^i(R_1)\, \mathbf{q}^i(t), \quad \vartheta_1^i(R_1, t) = \boldsymbol{\Psi}_{1*}^i(R_1)\, \mathbf{q}^i(t), \tag{43}$$

one obtains the matrices $\boldsymbol{\Phi}^i(\mathbf{R})$ and $\boldsymbol{\Psi}^i(\mathbf{R})$ appearing in Equation (21) as

$$\boldsymbol{\Psi}^i(\mathbf{R}) = [\boldsymbol{\Psi}_{1*}^i(R_1) \quad -\mathbf{W}_{3*}'^i(R_1) \quad \mathbf{W}_{2*}'^i(R_1)]^T, \quad \boldsymbol{\Phi}^i(\mathbf{R}) = \mathbf{W}^i(R_1) - \tilde{\mathbf{R}}^C\, \boldsymbol{\Psi}^i(\mathbf{R}). \tag{44}$$

In the classical Ritz method the shape functions in Equation (43) have to be 'admissible functions' [19]. They satisfy the geometrical boundary conditions only. Convergence, and in particular the representation of internal forces due to body deformation, are improved when using 'comparison functions'. These are requested to satisfy the dynamical boundary conditions as well, but such functions are hard to produce. Fortunately, the improvements are obtained also when selecting 'quasi-comparison functions' [21–24]. These are admissible functions with the additional property that a small, limited number of linear combinations of them are able to satisfy all of the boundary conditions including the dynamical ones. Often quasi-comparison functions are found by combining eigenfunctions and static deformation modes. When selecting shape functions based on these results, the conditions resulting from the choice of the body reference frame as discussed in Section 5 need to be considered.

To consider finite element structures in multibody system models, two routes have been pursued, which result in the so-called *nodal* and *modal* approaches. Both of them will be discussed now. In terms of the nodal coordinates $\mathbf{z}_F^i(t)$, describing the motion of an unconstrained structure with n_E^i elements, the displacement field of a body i is represented as

$$\mathbf{u}^i(\mathbf{R}, t) = \sum_{e=1}^{n_E^i} \boldsymbol{\Gamma}^{e^T}\, \mathbf{N}^e(\mathbf{x})\, \mathbf{T}^e\, \mathbf{z}_F^i(t) \quad \text{where} \quad \mathbf{x} = \boldsymbol{\Gamma}^e(\mathbf{R} - \mathbf{R}^e). \tag{45}$$

With reference to Figure 7, the coordinates $\mathbf{R}^e$ of $\underline{R}^e$ in $\underline{\mathbf{e}}^i$ specify the location of the origin of an element reference frame $\{O^e, \underline{\mathbf{e}}^e\}$ with respect to the body reference frame $\{O^i, \underline{\mathbf{e}}^i\}$, and matrix $\boldsymbol{\Gamma}^e$ describes the orientation of $\underline{\mathbf{e}}^e$ by the relation $\underline{\mathbf{e}}^e = \boldsymbol{\Gamma}^i\, \underline{\mathbf{e}}^i$ (see also Equation (3)). Matrices $\mathbf{N}^e(\mathbf{x})$ contain the element shape functions with $\mathbf{N}^e(\mathbf{x}) \equiv \mathbf{0}$ for points $\mathbf{x}$ not in the element e. Matrix $\mathbf{T}^e$ is an orthogonal transformation matrix [39, p. 482], which relates the nodal coordinates $\mathbf{z}^e(t)$ of an element e to the global coordinates $\mathbf{z}_F^i(t)$ by $\mathbf{z}^e(t) = \mathbf{T}^e\, \mathbf{z}_F^i(t)$.

In case of structures modelled by beam- and plate-elements, the coordinates $\mathbf{z}_F^i(t)$ represent translations $\mathbf{u}^{k,i}$ and rotations $\boldsymbol{\vartheta}^{k,i}$ of coordinate systems $\{O^{k,i}, \underline{\mathbf{e}}^{k,i}\}$ at the nodes k, i of the finite element model of body i, i.e.

$$\mathbf{z}_F^i = \left[z_{Fj}^i\right] = \begin{bmatrix} \vdots \\ \begin{bmatrix} \mathbf{u}^{k,i} \\ \boldsymbol{\vartheta}^{k,i} \end{bmatrix} \\ \vdots \end{bmatrix}, \quad j = 1, 2, \ldots n_F^i. \tag{46}$$

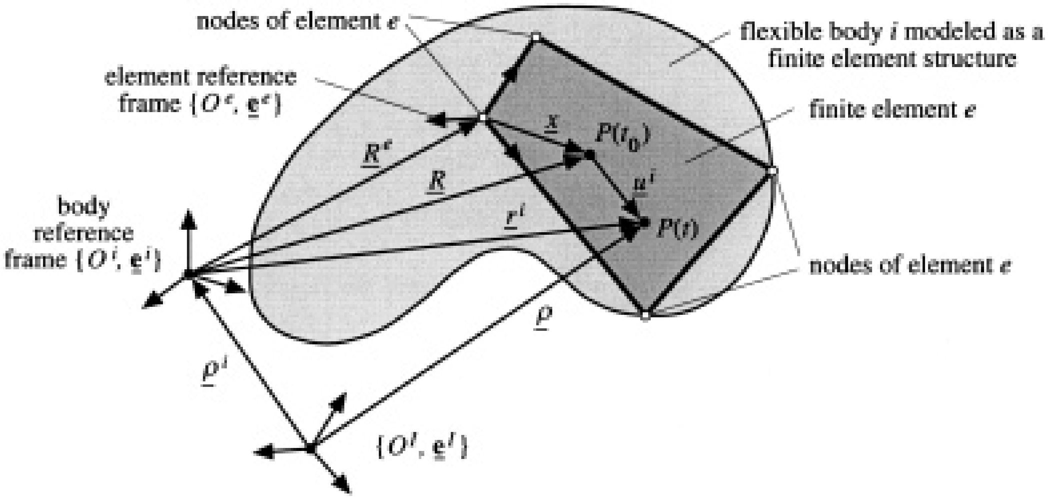

Figure 7. Finite element model of body i and notation to represent its motion.

When using other elements, matrix $\mathbf{z}_F^i$ contains a subset of these quantities (see, e.g., description of elements in [40]). The existence of translational and rotational displacements $\mathbf{u}^{k,i}(t) = \mathbf{u}^i(\mathbf{R}^{k,i}, t)$ and $\boldsymbol{\vartheta}^{k,i}(t) = \boldsymbol{\vartheta}^{k,i}(\mathbf{R}^{k,i}, t)$ is guaranteed in such cases at least at the attachment points of force elements and joints by the boundary conditions mentioned in Section 2.

With reference to Section 5, boundary conditions need to be satisfied as well, when defining the body reference frame by kinematical relations. They describe, in which way the structure is supported, to satisfy the six conditions specifying location and orientation of the body reference frame. In terms of the nodal coordinates (46) these conditions can be written in the form

$$\mathbf{z}_F^i(t) = \bar{\bar{\mathbf{T}}}^i \, \bar{\bar{\mathbf{z}}}_F^i(t) \quad \text{where} \quad \bar{\bar{\mathbf{z}}}_F^i = \left[\bar{\bar{z}}_{Fj}^i\right], \quad j = 1, 2, \ldots \bar{\bar{n}}_F^i, \quad \bar{\bar{n}}_F^i = n_F^i - 6, \tag{47}$$

with a matrix $\bar{\bar{\mathbf{T}}}^i$, whose elements are either zero ore one. Thus, variables $\bar{\bar{\mathbf{z}}}_F^i(t)$ are a subset of the variables $\mathbf{z}_F^i(t)$.

In the *nodal approach* one identifies the nodal variables $\bar{\bar{\mathbf{z}}}_F^i(t)$ with the generalized coordinates $\mathbf{q}^i(t)$ describing deformation, i.e.

$$\mathbf{q}^i(t) \equiv \bar{\bar{\mathbf{z}}}_F^i(t). \tag{48}$$

The entire finite element model is considered in a multibody system simulation in this methodology, resulting in a high number of system variables as given by Equation (22). This number is reduced in the *modal approach*. The eigenmodes of the finite element structure, described by variables $\bar{\bar{\mathbf{z}}}_F^i(t)$, are given by the eigenvectors $\mathbf{z}_{F\nu}^i$, $\nu = 1, 2, \ldots \bar{\bar{n}}_F^i$. A set of static deformations, due to suitably selected loads, is $\mathbf{z}_{F\mu}^i$, $\mu = 1, 2, \ldots n_{\text{stat}}^i$. In these terms the variables $\mathbf{q}^i(t)$ are defined by the relation

$$\bar{\bar{\mathbf{z}}}_F^i(t) = \boldsymbol{\Phi}_F^i \, \mathbf{q}^i(t) \quad \text{where} \quad \boldsymbol{\Phi}_F^i = \left[\bar{\bar{\mathbf{z}}}_{F\nu}^i, \bar{\bar{\mathbf{z}}}_{Fs\mu}^i\right], \;\; \nu = 1, 2, \ldots n_{\text{eig}}^i, \;\; \mu = 1, 2, \ldots n_{\text{stat}}^i. \tag{49}$$

Usually one considers just a small subset of $n_{\text{eig}}^i \ll \bar{\bar{n}}_F^i$ eigenvectors, thereby reducing the number of generalized coordinates significantly. Such variables are called modal coordinates.

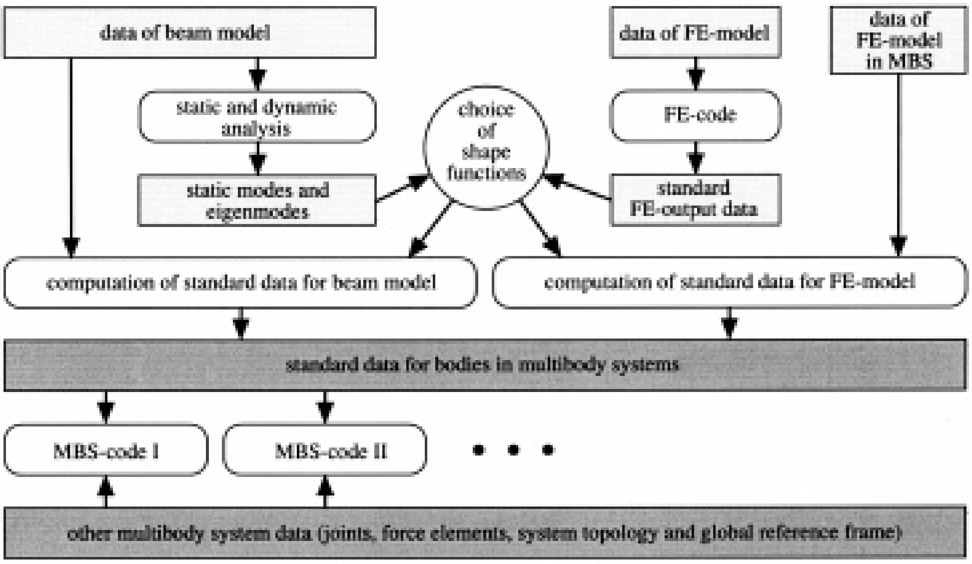

Figure 8. Multibody system data and pre-processors for computation of body data as required for system simulation with any multibody system-code.

In [5] the computation of the generalized masses and forces appearing in the linearized equations (26) of a body i is described in detail, based on the general model shown in Figures 2 and 3. The evaluation of the corresponding equations requires the knowledge of the body data. Most of them are found to be integrals over the body volume V_0^i, whose integrands are given by simple algebraic expressions including the shape functions $\mathbf{\Phi}^i$ and $\mathbf{\Psi}^i$ from Equation (21). Collecting all of these data, the set of *standard data for bodies in multibody systems*, shown in Figure 8, has been defined [5, 38].

Considering the specific forms of $\mathbf{\Phi}^i$ and $\mathbf{\Psi}^i$, as suggested by Equation (44) or by Equations (45) to (49), one can generate the specific data required to model flexible bodies as beams or finite element structures. These data may be computed by pre-processors, as shown in Figure 8. Examples of such pre-processors are described in [41, 42]. In particular, such pre-processors allow to use general finite element codes, to compute the data required for an analysis based on multibody simulation. As exemplified in Figure 8, the generality of the definition of the body data basis allows to combine any finite element code with any multibody program.

The generality just mentioned has its price. To compute the body data, one has to use the output of finite element codes, as available in any code. Usually the output files contain the system matrices of the finite element model, the loads resulting in static deformations $\bar{\bar{\mathbf{z}}}^i_{Fs\mu}$ and the eigenvectors $\bar{\bar{\mathbf{z}}}^i_{F\nu}$ due to a linear system analysis, together with the matrices $\bar{\bar{\mathbf{T}}}^i$ describing the way the structure is supported. To maintain generality, the data required for the description of bodies in a multibody system need to be computed from these data, solely. For some data, only approximations can be found under these restrictions, as described in [5, 43]. To be specific, the data are found by comparing the kinetic energy expressions of body i, when represented in terms of the variables $\dot{\mathbf{z}}^i_F$ and $\mathbf{z}^i_{II}$ (see Equations (46) and (22)). The nodal coordinates $\mathbf{z}^i_F$ of a finite element model must be able to represent a rigid body motion. Therefore, variables $\mathbf{v}^i$ and $\boldsymbol{\omega}^i$, appearing in Equation (22), can be represented in terms of $\dot{\mathbf{z}}^i_F$. Introducing these relations into the kinetic energy expression, as given by $\dot{\mathbf{z}}^i_F$, and comparing with the expression in terms of $\mathbf{z}^i_{II}$, one obtains the relations representing the submatrices of $\mathbf{M}^i$, given in Equation (28), in terms of the generalized mass matrix of the finite element structure. These are the equations (in some cases just approximations) to compute the body data for multibody simulation from the information available in the output files of finite element codes.

8. Conclusions

A floating frame of reference formulation, based on a generic flexible body model, allows a definition of general data describing flexible bodies in multibody systems. Such a data base provides an interface between any multibody- and finite-element-codes. It facilitates the usage of finite element models in multibody system simulation.

The floating frame of reference formulation is based on a separation of the flexible body motion into a reference motion and deformation. The definition of the two motions requires the specification of a body reference frame, which in turn is tied to the choice of shape functions in the Ritz method used to discretize deformation. Two approaches may be pursued. In the nodal approach the shape functions are given by the interpolation functions used in a finite element model of the flexible body. The method allows a detailed representation of deformation and internal forces but it also results in a high system order and corresponding computational costs. The modal approach may reduce the number of system variables and costs considerably, but it raises the problem of how to select the shape functions. In the classical formulation of the Ritz method so-called admissible functions satisfying the geometrical boundary conditions are used. A popular choice are eigenfunctions of a linear, often explicitly solvable problem, in which the flexible body moves in a similar way as when embedded in the multibody system. Recently, quasi-comparison functions have been proposed as an alternative. They improve the convergence of the Ritz method and the representation of internal forces considerably. Suitable quasi-comparison functions may be obtained by combining eigenfunctions and static deformation modes.

The separation of the motion of a flexible body into reference motion and deformation is introduced primarily to linearize the equations of motion assuming the deformation to be small. The magnitude of deformation depends on the choice of the body reference frame. A so-called Buckens frame yields a minimal deformation, but this does not imply that such a frame is the best choice in all application problems: Shape functions belonging to the conditions of other frames may be better suited to represent body deformation in specific situations.

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References

1. Eich-Soellner, E. and Führer C., *Numerical Methods in Multibody Dynamics*, Teubner, Stuttgart, 1998.
2. Shabana, A. A., ‘Flexible multibody dynamics: review of past and recent developments’, *Multibody System Dynamics* **1**, 1997, 189–222.

3. Koppens, W. P., Sauren, A. A., Veldpaus, F. E., and van Campen, D. H., 'The dynamics of a deformable body experiencing large displacements', *Journal of Applied Mechanics* **55**, 1988, 676–680.
4. Shabana, A. A., 'Resonance conditions and deformable body coordinate systems', *Journal of Sound and Vibration* **192**(1), 1996, 389–398.
5. Schwertassek, R. and Wallrapp, O., *Dynamik flexibler Mehrkörpersysteme*, Vieweg, Braunschweig, 1999.
6. Kane, T. R., Ryan, R. R., and Banerjee, A. K., 'Dynamics of a cantilever beam attached to a moving base', *Journal of Guidance, Control, and Dynamics* **10**(2), 1987, 139–151.
7. Schwertassek, R., 'Flexible bodies in multibody systems', in *Computational Methods in Mechanical Systems*, J. Angeles and E. Zakhariev (eds.), Springer-Verlag, Berlin, 1997, pp. 329–363.
8. Prandtl, L., 'Kipp-Erscheinungen – Ein Fall von instabilem elastischem Gleichgewicht', Dissertation, Ludwigs-Maximilian Universität zu München, Sektion II der philosophischen Fakultät, 1899.
9. Weidenhammer, F., 'Gekoppelte Biegeschwingungen von Laufschaufeln im Fliehkraftfeld', *Ingenieur-Archiv* **39**, 1970, 281–292.
10. Truckenbrodt, A., *Bewegungsverhalten und Regelung hybrider Mehrkörpersysteme mit Anwendung auf Industrieroboter*, Fortschritt-Bericht VDI, Reihe 8: Meß-, Steuerungs- und Regelungstechnik, Nr. 33, VDI-Verlag, Düsseldorf, 1980.
11. Bremer, H. and Pfeiffer, F., *Elastische Mehrkörpersysteme*, Teubner, Stuttgart, 1992.
12. Ritz, W., 'Theorie der Transversalschwingungen einer quadratischen Platte mit freien Rändern', *Annalen der Physik* **IV**, 1909, 737–786.
13. Courant, R. and Hilbert, D., *Methoden der mathematischen Physik*, Springer-Verlag, Berlin, 1968.
14. Veubeke, B. F., 'The dynamics of flexible bodies', *International Journal of Engineering Sciences* **14**, 1976, 895–913.
15. Gylden, H., 'Recherches sur la rotation de la Terre', *Actes de la Société Royale des Sciences d'Upsal,* April 5, 1871, 1–21.
16. Tisserand, F., *Traite de Mechanique Celeste*, Gauthier-Villars, Paris, 1891.
17. Canavin, J. R. and Likins, P. W., 'Floating reference frames for flexible spacecraft', in *AIAA 15th Aerospace Sciences Meeting*, Los Angeles, CA, 1977, Paper No. 77-66.
18. Agrawal, O. P. and Shabana, A. A., 'Application of deformable-body mean axis to flexible multibody system dynamics', *Computer Methods in Applied Mechanics and Engineering* **56**, 1986, 217–245.
19. Meirovitch, L. and Kwak, M. K., 'Convergence of the classical Rayleigh–Ritz method and the finite element method', *AIAA Journal* **28**(8), 1990, 1509–1516.
20. Tychonoff, A. N. and Samarski, A. A., *Differentialgleichungen der mathematischen Physik*, Deutscher Verlag der Wissenschaften, Berlin, 1959.
21. Meirovitch, L. and Kwak, M. K., 'On the modeling of flexible multi-body systems by the Rayleigh–Ritz method', Paper presented at the AIAA Dynamics Specialists Conference, Long Beach, CA, April 1990.
22. Meirovitch, L. and Hagedorn, P., 'A new approach to the modelling of distributed non-self-adjoined systems', *Journal of Sound and Vibration* **178**(2), 1994, 227–241.
23. Hagedorn, P., 'Quasi-comparison functions in the dynamics of elastic multibody systems', in *Dynamics and Control of Large Structures, Proceedings of the 8th VPI&SU Symposium*, Blacksburg, VA, May 6–8, L. Meirovitch (ed.), 1991, pp. 97–108.
24. Hagedorn, P., 'The Rayleigh–Ritz method with quasi-comparison functions in nonself-adjoint problems', *Journal of Vibration and Acoustics* **115**, 1993, 280–284.
25. Schwertassek, R. and Rulka, W., 'Aspects of efficient and reliable multibody system simulation', in *Real-Time Integration Methods for Mechanical System Simulation*, E. J. Haug and R. C. Deyo (eds.), Springer-Verlag, Berlin, 1991, pp. 55–96.
26. Roberson, R. E. and Schwertassek, R., *Dynamics of Multibody Systems*, Springer-Verlag, Berlin, 1988.
27. Volterra, E., 'The equations of motion for curved elastic bars deduced by the use of the 'method of internal constraints', *Ingenieur-Archiv* **XXIII**, 1955, 402–409.
28. Volterra, E., 'The equations of motion for curved and twisted elastic bars deduced by the use of the 'method of internal constraints', *Ingenieur-Archiv* **XXIV**, 1956, 392–400.
29. Fung, Y. C., *Foundations of Solid Mechanics*, Prentice-Hall, Englewood Cliffs, NJ, 1965.
30. Washizu, K., *Variational Methods in Elasticity and Plasticity*, 3rd edition, Pergamon Press, Oxford, 1982.
31. Pascal, M., 'Dynamics analysis of a system of hinge-connected flexible bodies', *Celestial Mechanics* **41**, 1988, 253–274.

32. Sachau, D., ‘Berücksichtigung von Körperverformung und Fügestellen in der Mehrkörpersimulation mit Anwendung auf aktive Raumfahrtstrukturen’, Dissertation, Universität Stuttgart, 1996.
33. Padilla, C. E. and von Flotow, A. H., ‘Nonlinear strain-displacement relations and flexible multibody dynamics’, *AIAA Journal of Guidance, Control and Dynamics* **15**, 1992, 128–136.
34. Banerjee, A. K. and Dickens, J. M., ‘Dynamics of an arbitrary flexible body in large rotation and translation’, *AIAA Journal of Guidance, Control and Dynamics* **13**(2), 1990, 221–227.
35. Banerjee, A. K., ‘Block-diagonal equations for multibody elastodynamics with geometric stiffness and constraints’, *AIAA Journal of Guidance, Control and Dynamics* **16**(6), 1993, 1092–1100.
36. Zienkiewicz, O. C. and Taylor, R. L., *The Finite Element Method: Solid and Fluid Mechanics Dynamics and Non-linearity*, Vol. 2, McGraw-Hill, London, 1991.
37. Otter, M., Hocke, M., and Leister, G., ‘An object oriented data model for multibody systems’, in *Advanced Multibody System Dynamics – Simulation and Software Tools,* W. Schiehlen (ed.), Kluwer, Dordrecht, 1993, pp. 19–48.
38. Wallrapp, O., ‘Standard input data of flexible members for multibody system codes’, in *Advanced Multibody System Dynamics – Simulation and Software Tools,* W. Schiehlen (ed.), Kluwer, Dordrecht, 1993, pp. 445–450.
39. Shames, I. H. and Dym, C. L., *Energy and Finite Element Methods in Structural Mechanics*, Hemisphere, New York, 1985.
40. Swanson-Analysis-System, *ANSYS User’s Manual for Rev. 5.0*, Vol. I to IV, Swanson Analysis System, Houston, PA, 1992.
41. Wallrapp, O., ‘Beam – A pre-processor for mode shape analysis of straight beam structures and generation of the SID file for MBS codes, User’s manual’, Report Deutsche Forschungsanstalt für Luft- und Raumfahrt (DLR), Institut für Robotik und Systemdynamik, Oberpfaffenhofen, 1994.
42. Wallrapp, O., Eichberger, A., and Gerl, J., ‘FEMBS – An interface between FEM codes and MBS codes, User manual for ANSYS, NASTRAN, and ABAQUS’, Report INTEC GmbH, Wessling, 1997.
43. Mangler, D., ‘Die Berechnung geometrisch nichtlinearer Probleme der Dynamik unter Nutzung linearer FEM-Programme’, Dissertation, Otto-von-Guericke-Universität, Magdeburg, 1994.