Flexible Representation of Quantum Images and Its Computational Complexity Analysis

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Abstract: Flexible Representation of Quantum Images (FRQI) is proposed to provide a representation of images that enables efficient preparation and image processing operators on quantum computers. FRQI captures colors and their corresponding positions in an image into a quantum state. The preparation with polynomial simple operations for FRQI turning a quantum computer from the initialized state to the FRQI state is shown by the proposed Polynomial Preparation theorem. The Quantum Image Compression (QIC) algorithm reduces simple operations used in the preparation process based on the minimization of Boolean equations are from same color groups of positions. Quantum image processing operators on FRQI based unitary transforms dealing with only colors, colors at some positions and the combination by the quantum Fourier transform of both colors and positions are addressed. The experiments for the storage and retrieval of images using FRQI are implemented in Matlab. The compression ratio of QIC among same color groups ranges from 6.67% to 31.62% on the Lena image.

1. Introduction

With quantum computers [5], the research on image processing and analysis has faced difficulties. Firstly, preparation processes turning quantum computers from their initial states to the image state has to be efficient in term of using a polynomial number of simple operations. Secondly, all of image processing operators on a quantum computer are invertible transforms applying on the image state. A representation for images on a quantum computer that utilizing both the efficient preparation process and the invertible image processing operators is needed to overcome the difficulties.

On a quantum computation model, research on image object began with quantum image representations [1], [2] and quantum transforms related to signal processing such as quantum Fourier transform [5], quantum discrete cosine transform [3], quantum Wavelet transform [5].

The Flexible Representation of Quantum Image (FRQI) which captures information about colors and their corresponding positions in image into a quantum state is proposed. Then the following computational and image processing aspects on FRQI are studied:

- The complexity of the preparation process for FRQI,
- The method to reduce number of simple operations that are used in the image state preparation step,
- Three types of invertible processing operators on FRQI.

2. Flexible Representation of Quantum Images and Its Polynomial Preparation

The information for representing an image in natural way is colors and their corresponding positions of these colors. In conventional computers, this way is used for storing images in pixel representation of images. Inspired by the fact, the quantum flexible representation of image (FRQI) is proposed. This proposal integrates information of an image into a quantum state having its formula in (1)

$$|I\rangle = \frac{1}{2^n}\sum_{i=0}^{2^{2n}-1}\left(\cos\theta_i|0\rangle + \sin\theta_i|1\rangle\right)\otimes|i\rangle, \quad \theta_i \in \left[0,\frac{\pi}{2}\right], i = 1,2,\ldots,2^{2n}-1, \tag{1}$$

capturing colors and corresponding positions of those colors, where $\otimes$ is the tensor product notation, $|0\rangle,|1\rangle$ are 2-D computational basis quantum states, $|i\rangle, i = 0,1,\ldots,2^{2n}-1$ are 2^{2n} -D computational basis quantum states and θ_i are angles encoding colors.

There are two parts in the quantum representation of an image; $\left(\cos\theta_i|0\rangle + \sin\theta_i|1\rangle\right)$ and $|i\rangle$ which encode the colors and their corresponding positions in the image, respectively. An example of 2×2 image is shown in Fig. 1.

The quantum state in FRQI form is a normalized state, i.e. $\left\| |I\rangle \right\| = 1$ as given by

$$\left\| |I\rangle \right\| = \frac{1}{2^n}\sqrt{\sum_{i=0}^{2^{2n}-1}\left(\cos^2\theta_i + \sin^2\theta_i\right)} = \frac{1}{2^n}\sqrt{2^{2n}} = 1. \quad (2)$$

This property is very useful for further unitary operations on the image state.

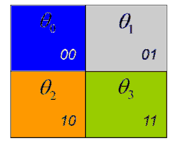
Lemma 2.1. *Given a set* $\{\theta_i\}, i = 0,1,\ldots,2^{2n}-1 (n \in N)$ *of angles encoding colors at the position* i^{th}*, there is a unitary transform P that turns quantum computers from the initialized state* $|0\rangle^{\otimes 2n+1}$ *to the quantum image state,* $|I\rangle = \frac{1}{2^n}\sum_{i=0}^{2^{2n}-1}\left(\cos\theta_i|0\rangle + \sin\theta_i|1\rangle\right)\otimes|i\rangle$*, by using Hadamard transforms and controlled-rotation transforms.*

Corollary 2.2. *The unitary transform P in lemma 2.1 for a given* $\{\theta_i\}, i = 0,1,\ldots,2^{2n}-1 (n \in N)$ *can be implemented by simple quantum gates, specifically by 2n Hadamard gates and* 2^{2n} *generalized-* $C^{2n}\left(R_y(2\theta_i)\right)$ *gates, where* $R_y(2\theta_i)$ *are the rotations about* $\hat{y}$ *axis by the angles* $2\theta_i, i = 1,2,\ldots,2^{2n}-1$.

Theorem 2.3 (Polynomial Preparation Theorem) *Given a set* $\{\theta_i\}, i = 0,1,\ldots,2^{2n}-1 (n \in N)$ *of angles encoding colors at the* i^{th} *position, there is an efficiently implemented quantum transformation P turning a quantum system from initialized state* $|0\rangle^{\otimes 2n+1}$ *to the quantum image state using polynomial number of simple gates.*

3. Quantum Image Compression based on Minimization of Boolean Expression

Classical image compression techniques such as JPEG [9] reduce the amount of computational resources, used to restore or reconstruct images. Similarly, quantum image compression is the procedure that reduces quantum resources used to prepare or reconstruct quantum images. The main resource in quantum computation is the quantity of simple quantum gates or simple operations used in the computation. Therefore, the process of decreasing the quantity of simple quantum gates in preparation and reconstruction of quantum images is the Quantum Image Compression (QIC). The preparation and reconstruction of quantum images are the same, however, in sense of computation.



$$|I\rangle = \frac{1}{2}\left[\left(\cos\theta_0|0\rangle + \sin\theta_0|1\rangle\right)\otimes|00\rangle + \left(\cos\theta_1|0\rangle + \sin\theta_1|1\rangle\right)\otimes|01\rangle + \right.$$
$$\left. + \left(\cos\theta_2|0\rangle + \sin\theta_2|1\rangle\right)\otimes|10\rangle + \left(\cos\theta_3|0\rangle + \sin\theta_3|1\rangle\right)\otimes|11\rangle\right]$$

Fig. 1 A simple image and its quantum state.

There are several reasons why image compression must be considered in the FRQI. Firstly, studies in classical image processing point out that there is redundancy in the image that can be reduced for compression in quantum image as well. Secondly, as shown in section 2, preparing a quantum image needs a large number of simple gates. For example a 2^{16} position image needs 2^{32} simple gates for preparation. Thirdly, in physical experiments, only a limited number of simple gates can be prepared. For all of these reasons, the reduction of simple gates is necessary for both theoretical and practical aspects of the FRQI.

The Quantum Image Compression (QIC) is proposed in order to reduce controlled-rotation gates in same color groups based on the minimization of their Boolean expression as shown in Fig. 2. The algorithm starts with the information about positions in a same color group and ends with the minimized form of the Boolean expression.

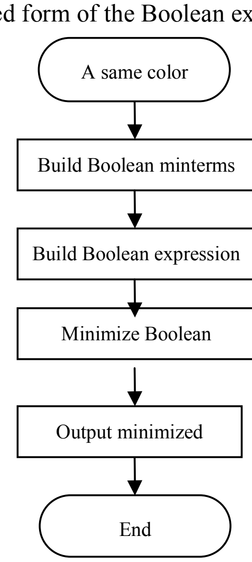


Fig. 2 Quantum Image Compression Algorithm

The result of QIC algorithm is the minimized Boolean expressions that are used to construct a quantum circuit. The number of product terms in a minimized Boolean expression indicates the number of conditioned-rotation gates that is needed to use for the corresponding group of same color positions. The laterals in a product term in each minimized expression point out the condition part of the conditioned-rotation gate. The Boolean variables with complement laterals use extra pair of NOT gates for each complement.

4. Image Processing Operators on Quantum Images based on Unitary Transforms

FRQI absorb information on colors and their related spiral positions into a quantum state as the fundamental object for further quantum transforms. The quantum image processing operators are unitary operators with image processing purposes. These operators are divided into 3 categories based on 3 types of unitary transforms, G_1, G_2 and G_3, applied to FRQI dealing with only colors, colors at some specific positions and the combination of colors and positions, respectively.

The unitary transforms of the first group, G_1 and the second one, G_2 are in following form

$$G_1 = U \otimes I^{\otimes n}, \tag{2}$$

$$G_2 = U \otimes C + I \otimes \overline{C}, \tag{3}$$

where U are single qubit unitary transforms on colors, C are matrices regarding eligible positions, $\overline{C}$ are matrices regarding ineligible positions, $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is the identity operator and n is the number of qubits encoding positions.

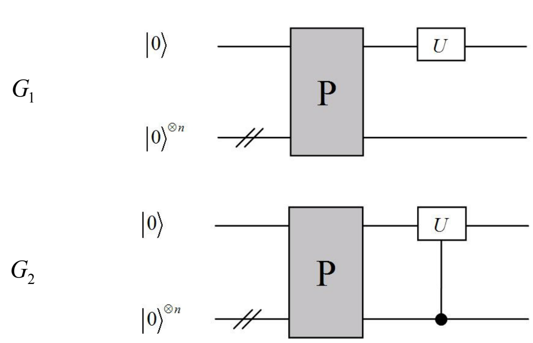


Fig. 3 The circuits of G1 and G2 image processing operations.

The shifting color operator, S is defined as an operator in the group G_1, $S = U \otimes I^{\otimes n}$, by using rotation matrices $U = R_y(2\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, where θ is the shifting parameter. The following calculation produces the result $|I_S\rangle$ of the application of S on $|I\rangle$,

$$\begin{aligned} |I_S\rangle &= S(|I\rangle) \\ &= S\left(\frac{1}{2^n}\sum_{i=0}^{2^{2n}-1}(\cos\theta_i|0\rangle + \sin\theta_i|1\rangle)\otimes|i\rangle\right) \\ &= \left(U \otimes I^{\otimes n}\right)\left(\frac{1}{2^n}\sum_{i=0}^{2^{2n}-1}(\cos\theta_i|0\rangle + \sin\theta_i|1\rangle)\otimes|i\rangle\right) \\ &= \frac{1}{2^n}\sum_{i=0}^{2^{2n}-1}U(\cos\theta_i|0\rangle + \sin\theta_i|1\rangle)\otimes|i\rangle \\ &= \frac{1}{2^n}\sum_{i=0}^{2^{2n}-1}(\cos(\theta_i+\theta)|0\rangle + \sin(\theta_i+\theta)|1\rangle)\otimes|i\rangle. \end{aligned} \tag{15}$$

The quantum image $|I_S\rangle$ has all of its colors coming from the original image $|I\rangle$ by shifting the θ angle.

The changing color of some points in an image depends on the specific positions of the points in the image. Information about the positions is used as conditions encoded in the matrix C of the conditioned-gate G_2 to construct the processing operators. For instance, let us consider an 2×2 image and the changing color at positions $|0\rangle, |2\rangle$. The matrix $C = |0\rangle\langle 0| + |2\rangle\langle 2|$ and $\overline{C} = |1\rangle\langle 1| + |3\rangle\langle 3|$ are used to construct the conditioned-gate G_2,

$$G_2 = U \otimes (|0\rangle\langle 0| + |2\rangle\langle 2|) + I \otimes (|1\rangle\langle 1| + |3\rangle\langle 3|).$$

The action of this particular G_2 on a general 2×2 image in FRQI form, $|I\rangle = \frac{1}{2}\sum_{i=0}^{3}(\cos\theta_i|0\rangle + \sin\theta_i|1\rangle)\otimes|i\rangle$, is given by

$$\begin{aligned} G_2|I\rangle &= \left[U \otimes (|0\rangle\langle 0| + |2\rangle\langle 2|) + I \otimes (|1\rangle\langle 1| + |3\rangle\langle 3|)\right]|I\rangle \\ &= \frac{1}{2}(\cos\theta_0 U|0\rangle + \sin\theta_0 U|1\rangle)\otimes|0\rangle + \\ &\quad + \frac{1}{2}(\cos\theta_2 U|0\rangle + \sin\theta_2 U|1\rangle)\otimes|2\rangle + \\ &\quad + \frac{1}{2}(\cos\theta_1|0\rangle + \sin\theta_1|1\rangle)\otimes|1\rangle + \\ &\quad + \frac{1}{2}(\cos\theta_3|3\rangle + \sin\theta_3 U|1\rangle)\otimes|3\rangle. \end{aligned} \tag{16}$$

The calculation in (16) shows that the action of operator U for changing color affects only on the specific positions $|0\rangle, |2\rangle$.

The number of simple gates used for an operation in G_1 category is one gate while the number of controlled-rotations used for an operator in G_2 category depends linearly on number of positions involving in the operator.

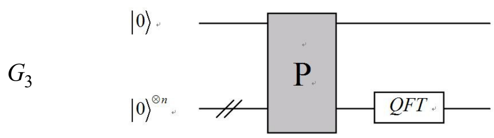


Fig. 4 The circuit of the operator G_3 based on QFT

5. Experiments on Quantum Images

The essential requirements for a representation of classical or quantum images are simplicity and efficiency in storage and retrieval the images. The storage of quantum image is preparation process that is ensured by the proposed EPT in section 2. The measurement of the quantum image state produces a probability distribution that is used for the retrieval of the image, shown in fig. 5.

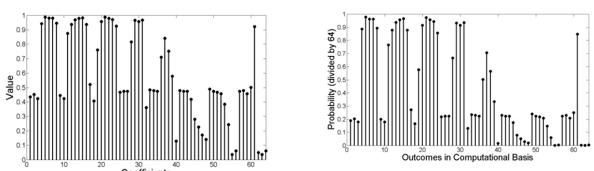


Fig. 5 Coefficients of quantum state and probability distribution

In experiment, compression ratios among groups are estimated based on analysis of minimizations of Boolean expressions derived from the 256×256 gray image of Lena as in the Fig. 6. The minimization step in QIC algorithm is done by Logic Friday free software. The compression ratios range from 6.67% to 31.62% between groups.

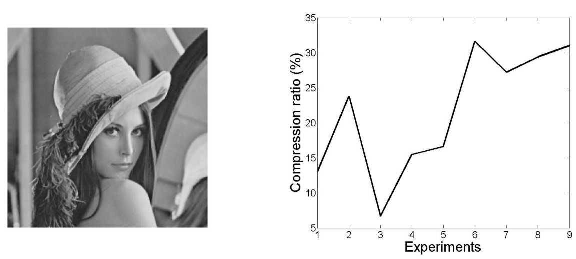


Fig. 6 Compression ratios of same color groups in Lena image

6. Discussion

FRQI captures colors and their corresponding positions in an image into a quantum state. The preparation with polynomial simple operations for FRQI turning a quantum computer from the initialized state to the FRQI state is shown by the proposed Polynomial Preparation theorem. The Quantum Image Compression (QIC) algorithm reduces simple operations used in the preparation process based on the minimization of Boolean equations are from same color groups of positions. Quantum image processing operators on FRQI based unitary transforms dealing with only colors, colors at some positions and the combination by the quantum Fourier transform of both colors and positions are addressed. The experiments for the storage and retrieval of images using FRQI are implemented in Matlab. The compression ratio of QIC among same color groups ranges from 6.67% to 31.62% on the Lena image.

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