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Flexural–torsional behavior of thin-walled composite beams

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Abstract

This paper presents a flexural–torsional analysis of I-shaped laminated composite beams. A general analytical model applicable to thin-walled I-section composite beams subjected to vertical and torsional load is developed. This model is based on the classical lamination theory, and accounts for the coupling of flexural and torsional responses for arbitrary laminate stacking sequence configuration, i.e. unsymmetric as well as symmetric. Governing equations are derived from the principle of the stationary value of total potential energy. Numerical results are obtained for thin-walled composites under vertical and torsional loading, addressing the effects of fiber angle, and laminate stacking sequence.

Keywords: Thin-walled structures; Laminated composites; Flexural–torsional response; Finite element method

1. Introduction

Fiber-reinforced plastics (FRP) have been increasingly used over the past few decades in a variety of structures that require high ratio of stiffness and strength to weight. In the construction industry, recent applications have shown the structural and cost efficiency of FRP structural shapes, such as thin-walled open sections through pultrusion manufacturing process. Thin-walled open section members made of isotropic materials have been studied by many researchers [1,2]. Bauld and

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Tzeng [3] extended Vlasov's thin-walled bar theory [1] to symmetric fiber-reinforced laminates to develop the linear and nonlinear theories for the bending and twisting of thin-walled composite beams. Davalos et al. [4] studied the bending response of various I and box sections experimentally and analytically. Ascione et al. [5] presented the statical behavior of fiber-reinforced polymer thin-walled beams taking into account the effects of shear deformation. Shin et al. [6,7] presented analytical results for bending and torsional response of symmetrically laminated composite open section beams.

In this paper, a general analytical model applicable to the flexural, torsional and flexural–torsional behavior of an I-section composite beams subjected to vertical and torsional load is developed. This model is based on the classical lamination theory, and accounts for the coupling of flexural and torsional responses for arbitrary laminate stacking sequence configuration, i.e. unsymmetric as well as symmetric. Governing equations are derived from the principle of the stationary value of total potential energy. Numerical results are obtained for thin-walled composites under vertical and torsional loading, addressing the effects of fiber angle, and laminate stacking sequence.

2. Kinematics

This paper requires three sets of coordinate systems: an orthogonal Cartesian coordinate system (x, y, z), an orthogonal coordinate system (n, s, z), and a contour coordinate s along the profile of the section with its origin at any point O on the profile section. Three sets of coordinate systems are mutually interrelated and shown in Fig. 1. The n axis as shown in Fig. 1 is normal to the middle surface of a plate element, the s axis is tangent to the middle surface and is directed along the contour line of the cross-section. The basic assumptions regarding the kinematics of thin-walled composites are stated as follows:

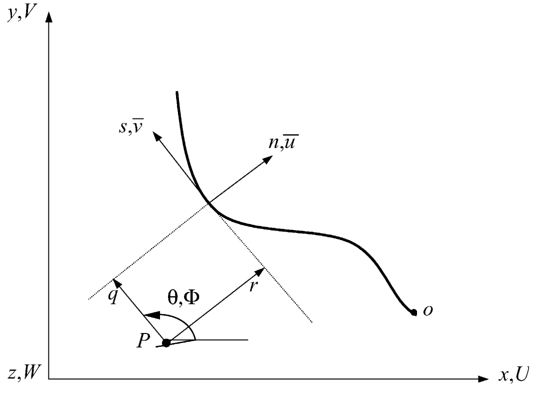

Fig. 1. Definition of coordinates in thin-walled.

1. The contour of the thin wall does not deform in its own plane.
2. The shear strain $\bar{\gamma}_{sz}$ of the middle surface is zero in each element.
3. The Kirchhoff–Love assumption in classical plate theory remains valid for laminated composite thin-walled beams.
4. The time-dependent behavior is neglected.

The midsurface displacement components $\bar{u}$, $\bar{v}$, $\bar{w}$ in the contour coordinate system, respectively, mean perpendicular, lateral and axial displacements. Such plane elements as these $\bar{u}$, $\bar{v}$, $\bar{w}$ can be expressed in terms of beam elements as these U, V, W and Φ.

$$\bar{u}(s,z) = U(z)\sin\theta(s) - V(z)\cos\theta(s) - \Phi(z)q(s) \tag{1a}$$

$$\bar{v}(s,z) = U(z)\cos\theta(s) + V(z)\sin\theta(s) + \Phi(z)r(s) \tag{1b}$$

$$\bar{w}(s,z) = W(z) - U'(z)x(s) - V'(z)y(s) - \Phi'(z)\omega(s) \tag{1c}$$

where Φ means the rotation angle about the pole axis of the pole P; the prime ($'$) is used to indicate differentiation with respect to z; and ω is the so-called sectorial coordinate or warping function given by

$$\omega(s) = \int r(s)\,\mathrm{d}s \tag{2}$$

The displacement components u, v, w representing the deformation of any generic point on the profile section are given with respect to the midsurface displacements $\bar{u}$, $\bar{v}$, $\bar{w}$ by assumption 3.

$$u(s,z,n) = \bar{u}(s,z) \tag{3a}$$

$$v(s,z,n) = \bar{v}(s,z) - n\frac{\partial \bar{u}(s,z)}{\partial s} \tag{3b}$$

$$w(s,z,n) = \bar{w}(s,z) - n\frac{\partial \bar{u}(s,z)}{\partial z} \tag{3c}$$

3. Strains

The strains associated with the small-displacement theory of elasticity are given by

$$\varepsilon_s = \bar{\varepsilon}_s + n\bar{\kappa}_s \tag{4a}$$

$$\varepsilon_z = \bar{\varepsilon}_z + n\bar{\kappa}_z \tag{4b}$$

$$\gamma_{sz} = n\bar{\kappa}_{sz} \tag{4c}$$

where

$$\bar{\varepsilon}_s = \frac{\partial \bar{v}}{\partial s}, \quad \bar{\varepsilon}_z = \frac{\partial \bar{w}}{\partial z} \tag{5a}$$

$$\bar{\kappa}_s = -\frac{\partial^2 \bar{u}}{\partial s^2}, \quad \bar{\kappa}_z = -\frac{\partial^2 \bar{u}}{\partial z^2}, \quad \bar{\kappa}_{sz} = -2\frac{\partial^2 \bar{u}}{\partial s \partial z} \tag{5b}$$

All the other strains are identically zero. In Eq. (5a,b), $\bar{\varepsilon}_s$ and $\bar{\kappa}_s$ are assumed to be zero, and $\bar{\varepsilon}_z$, $\bar{\kappa}_z$ and $\bar{\kappa}_{sz}$ are midsurface axial strain and biaxial curvatures of the shell, respectively. The above shell strains can be converted to beam strain components by substituting Eqs. (1a–c) and (3a–c) into Eq. (5a,b)

$$\bar{\varepsilon}_z = \varepsilon_z^o + x\kappa_y + y\kappa_x + \omega\kappa_\omega \tag{6a}$$

$$\bar{\kappa}_z = \kappa_y \sin\theta - \kappa_x \cos\theta - \kappa_\omega q \tag{6b}$$

$$\bar{\kappa}_{sz} = \kappa_{sz} \tag{6c}$$

where ε_z^o κ_x, κ_y, κ_ω and κ_{sz} are axial strain, biaxial curvatures in the x and y direction, warping curvature with respect to the shear center, and twisting curvature in the beam, respectively defined as

$$\varepsilon_z^o = W' \tag{7a}$$

$$\kappa_x = -V'' \tag{7b}$$

$$\kappa_y = -U'' \tag{7c}$$

$$\kappa_\omega = -\Phi'' \tag{7d}$$

$$\kappa_{sz} = 2\Phi' \tag{7e}$$

The resulting strains can be obtained from Eqs. (4a–c) and (6a–c) as

$$\varepsilon_z = \varepsilon_z^o + (x + n\sin\theta)\kappa_y + (y - n\cos\theta)\kappa_x + (\omega - nq)\kappa_\omega \tag{8a}$$

$$\gamma_{sz} = n\kappa_{sz} \tag{8b}$$

4. Variational formulation

Total potential energy of the system is calculated by sum of strain energy and potential energy,

$$\Pi = u + v \tag{9}$$

where u is the strain energy

$$u = \frac{1}{2}\int_v (\sigma_z \varepsilon_z + \sigma_{zs}\gamma_{zs})\,\mathrm{d}v, \tag{10}$$

The strain energy is calculated by substituting Eq. (6a–c) into Eq. (10)

$$u = \int_v \{\sigma_z[\varepsilon_z^o + (x + n\sin\theta)\kappa_y + (y - n\cos\theta)\kappa_x + (\omega - nq)\kappa_\omega] + \sigma_{sz} n\kappa_{sz}\}\,\mathrm{d}v \tag{11}$$

The variation of strain energy, Eq. (11), can be stated as

$$\delta u = \int_0^l \{N_z\delta\varepsilon_z^o + M_y\delta\kappa_y + M_x\delta\kappa_x + M_\omega\delta\kappa_\omega + M_t\delta\kappa_{sz}\}\,\mathrm{d}z \tag{12}$$

where N_z, M_x, M_y, M_ω and M_t are axial force, bending moments in the x and y

directions, warping moment (bimoment), and tortional moment with respect to the centroid, respectively, defined by integrating over the cross-sectional area A as

$$N_z = \int_A \sigma_z \, \mathrm{d}s\mathrm{d}n \tag{13a}$$

$$M_y = \int_A \sigma_z (x + n\sin\theta) \, \mathrm{d}s\mathrm{d}n \tag{13b}$$

$$M_x = \int_A \sigma_z (y - n\cos\theta) \, \mathrm{d}s\mathrm{d}n \tag{13c}$$

$$M_\omega = \int_A \sigma_z (\omega - nq) \, \mathrm{d}s\mathrm{d}n \tag{13d}$$

$$M_\mathrm{t} = \int_A \sigma_{z\mathrm{s}} n \, \mathrm{d}s\mathrm{d}n \tag{13e}$$

The variation of the work done by external force can be stated as

$$\delta v = -\int_0^l (q\delta V + t\delta\Phi) \, \mathrm{d}z \tag{14a}$$

where q is transverse load and t is applied torque. Using the principle that the variation of the total potential energy is zero, the following weak statement is obtained:

$$0 = \int_0^l \{N_z \delta W' - M_y \delta U'' - M_x \delta V'' - M_\omega \delta\Phi'' + 2M_\mathrm{t}\delta\Phi' + q\delta V + t\delta\Phi\} \, \mathrm{d}z \tag{14b}$$

5. Constitutive equations of plate elements

The constitutive equation in an arbitrary z–s coordinate system is then written

$$\begin{Bmatrix} \sigma_z \\ \sigma_\mathrm{s} \\ \sigma_{\mathrm{s}z} \end{Bmatrix}^k = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}^k \begin{Bmatrix} \varepsilon_z \\ \varepsilon_\mathrm{s} \\ \gamma_{\mathrm{s}z} \end{Bmatrix} \tag{15}$$

where $\bar{Q}_{ij}$ are transformed reduced stifnesses [8] and are made up of material property with respect to each layer. The above constitutive equation can be simplified by using free stress in contour direction ($\sigma_\mathrm{s} = 0$) or free strain in contour direction ($\varepsilon_\mathrm{s} = 0$) assumption as

$$\begin{Bmatrix} \sigma_z \\ \sigma_{\mathrm{s}z} \end{Bmatrix}^k = \begin{bmatrix} \bar{Q}^*_{11} & \bar{Q}^*_{16} \\ \bar{Q}^*_{16} & \bar{Q}^*_{66} \end{bmatrix}^k \begin{Bmatrix} \varepsilon_z \\ \gamma_{\mathrm{s}z} \end{Bmatrix} \tag{16}$$

for free stress assumption,

$$\bar{Q}^*_{11} = \bar{Q}_{11} - \frac{\bar{Q}^2_{12}}{\bar{Q}_{22}} \tag{17a}$$

$$\bar{Q}^*_{16} = \bar{Q}_{16} - \frac{\bar{Q}_{12}\bar{Q}_{26}}{\bar{Q}_{22}} \tag{17b}$$

$$\bar{Q}^*_{66} = \bar{Q}_{66} - \frac{\bar{Q}^2_{26}}{\bar{Q}_{22}} \tag{17c}$$

and for free strain assumption,

$$\bar{Q}^*_{11} = \bar{Q}_{11} \tag{18a}$$

$$\bar{Q}^*_{16} = \bar{Q}_{16} \tag{18b}$$

$$\bar{Q}^*_{66} = \bar{Q}_{66} \tag{18c}$$

In Eq. (13a), N_z, M_x, M_y, M_x, M_ω, and M_{t} can now be expressed with respect to the generalized strains ($\varepsilon^{\mathrm{o}}_z$, κ_y, κ_x, κ_ω, κ_{sz}) by combining Eqs. (13a), (16) and (8). Consequently, the constitutive equations for a thin-walled laminated composite are obtained as

$$\begin{Bmatrix} N_z \\ M_y \\ M_x \\ M_\omega \\ M_{\mathrm{t}} \end{Bmatrix} = \begin{bmatrix} E_{11} & E_{12} & E_{13} & E_{14} & E_{15} \\ & E_{22} & E_{23} & E_{24} & E_{25} \\ & & E_{33} & E_{34} & E_{35} \\ & & & E_{44} & E_{45} \\ \mathrm{sym.} & & & & E_{55} \end{bmatrix} \begin{Bmatrix} \varepsilon^o_z \\ \kappa_y \\ \kappa_x \\ \kappa_\omega \\ \kappa_{\mathrm{sz}} \end{Bmatrix} \tag{19}$$

In Eq. (19), E_{ij} are stiffness of the thin-walled composite. It appears that the laminate stiffnesses E_{ij} depend on the cross-section of the composites, and the explicit expressions for I-section (Fig. 2) are given in [9].

6. Governing equations

The governing equations of the present approach can be derived by integrating the derivatives of the varied quantities by parts and collecting the coefficients of δU, δV, δW and $\delta\Phi$:

$$N'_z = 0 \tag{20a}$$

$$M''_y = 0 \tag{20b}$$

$$M''_x + q = 0 \tag{20c}$$

$$M''_\omega + 2M'_{\mathrm{t}} + t = 0 \tag{20d}$$

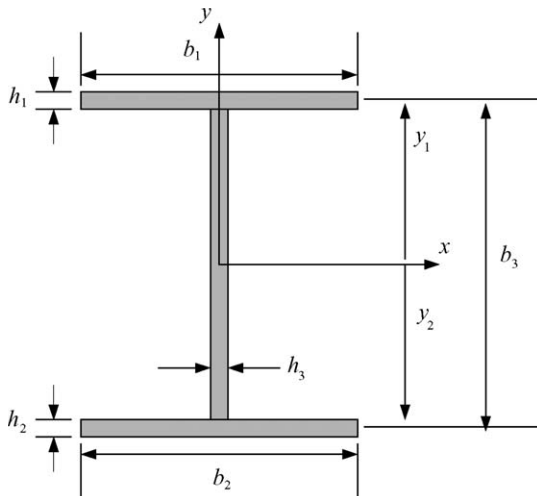

Fig. 2. Geometry of a thin-walled composite open section.

By substituting Eqs. (19) and (7a–e) into Eq. (20a–d), the explicit form of the governing equations yield:

$$E_{11}W'' - E_{12}U''' - E_{13}V''' + 2E_{15}\Phi'' = 0 \tag{21a}$$

$$E_{12}W''' - E_{22}U^{iv} - E_{24}\Phi^{iv} + 2E_{25}\Phi''' = 0 \tag{21b}$$

$$E_{13}W''' - E_{33}V^{iv} - E_{34}\Phi^{iv} + 2E_{35}\Phi''' + q = 0 \tag{21c}$$

$$\begin{aligned} &- E_{24}U^{iv} - E_{34}V^{iv} - E_{44}\Phi^{iv} + 2E_{15}W'' - 2E_{25}U''' - 2E_{35}V''' \\ &\quad + 4E_{35}\Phi'' + t = 0 \end{aligned} \tag{21d}$$

Eqs. (21a–d) are most general form for flexural–torsional behavior of a thin-walled laminated composite with an I-section, and the dependent variables, U, V, W and Φ are fully coupled.

7. Finite element model

The present theory for thin-walled composite beams described in the previous section was implemented via a displacement-based finite element method. The generalized displacements are expressed over each element as a linear combination of the one-dimensional Lagrange interpolation function Ψ_j and Hermite-cubic interpolation function ψ_j associated with node j and the nodal values;

$$W = \sum_{j=1}^{n} w_j\Psi_j, \quad U = \sum_{j=1}^{n} u_j\psi_j, \quad V = \sum_{j=1}^{n} v_j\psi_j, \quad \Phi = \sum_{j=1}^{n} \phi_j\psi_j \tag{22}$$

Substituting these expressions into the weak statement in Eq. (14b), the finite element model of a typical element can be expressed as

$$[K]\{\Delta\} = \{f\} \tag{23}$$

where $[K]$ is the element stiffness matrix, and $\{f\}$ is the element force vector

$$[K] = \begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} \\ & K_{22} & K_{23} & K_{24} \\ & & K_{33} & K_{34} \\ \text{sym.} & & & K_{44} \end{bmatrix} \tag{24}$$

The explicit form of $[K]$ is given in [9] and $\{f\}$ is given by

$$\{f\} = \{\,0 \quad 0 \quad f_3 \quad f_4\,\}^{\mathrm{T}} \tag{25}$$

where

$$f_i^3 = \int_0^l q\psi_i \,\mathrm{d}z \tag{26a}$$

$$f_i^4 = \int_0^l t\psi_i \,\mathrm{d}z \tag{26b}$$

In Eq. (23), $\{\Delta\}$ is the unknown nodal displacements

$$\{\Delta\} = \{\,W \quad U \quad V \quad \Phi\,\}^{\mathrm{T}} \tag{27}$$

8. Numerical results and discussion

For the verification purpose, the results by present approach are compared with the previous results [6,7] and ABAQUS [10]. In ABAQUS analysis, S9R5 shell elements are used. A simply supported I-section beam under uniformly distributed load of 1 kN/m with dimension of ($50 \times 50 \times 2.08$ mm) is considered. The following engineering constants are used.

$$E_1 = 53.78 \text{ GPa}, \quad E_2 = E_3 = 17.93 \text{ GPa} \tag{28a}$$

$$G_1 = 3.45 \text{ GPa}, \quad G_2 = G_3 = 8.96 \text{ GPa} \tag{28b}$$

$$\nu_{23} = 0.34, \quad \nu_{12} = \nu_{13} = 0.25 \tag{28c}$$

The maximum vertical deflections based on two different assumptions ($\sigma_s = 0$ and $\varepsilon_s = 0$) are compared with the previous result [6] and ABAQUS [10] in Table 1 for several stacking sequences. All the results are in good agreement. Especially, the results by $\sigma_s = 0$ assumption show excellent agreement with ABAQUS solution.

The next example shows the angle of twist for various stacking sequences with the same configuration as the previous example except the concentrated torque $T = 100$ N cm is applied at one end of the beam. The maximum angles of twist are given in Table 2. For strong coupling stacking sequence $[\theta]_{16}$, the solution by

Table 1
Deflections of a simply supported I-section beam under uniformly distributed load

Stacking sequence	Ref. [5]	ABAQUS	Present	
			$\varepsilon_s = 0$	$\sigma_s = 0$
$[0]_{16}$	6.103	6.340	6.103	6.233
$[15/-15]_{4s}$	6.611	6.989	6.610	6.899
$[30/-30]_{4s}$	8.282	9.360	8.281	9.290
$[45/-45]_{4s}$	11.343	13.479	11.340	13.421
$[60/-60]_{4s}$	15.124	17.023	15.119	16.962
$[75/-75]_{4s}$	17.641	18.490	17.643	18.411
$[0/90]_{4s}$	9.153	9.400	9.153	9.299
$[0/-45/90/45]_{4s}$	10.130	10.851	10.130	10.777

$\varepsilon_s = 0$ assumption remarkably underestimates the angle of twist while the solution by $\sigma_s = 0$ assumption shows good agreements. From these two examples, it is shown that the solution based on the assumption $\sigma_s = 0$ yield more accurate results.

In order to investigate the flexural–torsional coupling effects, different stacking sequences are given for the top and bottom flanges; and the web is assumed to be unidirectional. Sectional dimension of I-beam is (100×200 mm) in 1 m length made of 16 plies with each of them 0.05 mm in thickness. The material properties considered in this investigation are:

$$E_1/E_2 = 25 \tag{29a}$$

$$G_{12}/E_2 = 0.5 \tag{29b}$$

$$v_{12} = 0.25 \tag{29c}$$

Table 2
Angle of twist of a simply supported I-section beam under concentered torque

Stacking sequence	Ref. [6]	ABAQUS	Present	
			$\varepsilon_s = 0$	$\sigma_s = 0$
$[0]_{16}$	0.2481	0.2490	0.2481	0.2481
$[15/-15]_{4s}$	0.2073	0.2100	0.2073	0.2073
$[30/-30]_{4s}$	0.1563	0.1600	0.1599	0.1563
$[45/-45]_{4s}$	0.1396	0.1430	0.1388	0.1396
$[60/-60]_{4s}$	0.1571	0.1590	0.1559	0.1571
$[75/-75]_{4s}$	0.2080	0.2080	0.2072	0.2080
$[0/90]_{4s}$	0.2481	0.2480	0.2481	0.2481
$[0/-45/90/45]_{4s}$	0.1891	0.1910	0.1879	0.1891
$[45]_{16}$	0.1686	0.1710	0.1389	0.1687
$[30]_{16}$	0.1686	0.1720	0.1561	0.1687
$[60]_{16}$	0.1971	0.1972	0.1561	0.1972

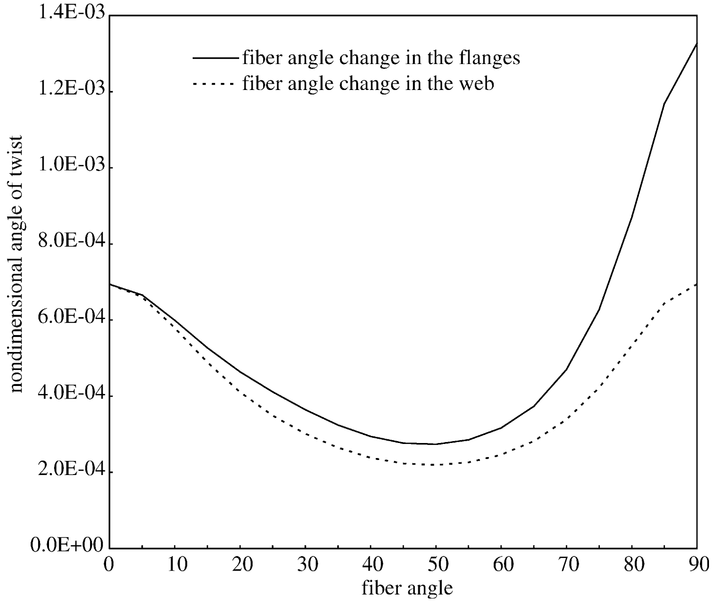

Fig. 3. Variation of the torsional displacements with respect to fiber angle change of a cantilever beam under torque at free end.

For convenience, the following nondimensional values of vertical displacements and angle of twist are used:

$$\bar{\phi} = \frac{\phi p l}{G_{12} b t^2} \tag{30a}$$

$$\bar{v} = \frac{v p l^3}{E_2 b^3 t} \tag{30b}$$

The beam is assumed to be under torque at free end. The fiber angle is varied in two ways; first case, the top and bottom flanges are considered as antisymmetric angle-ply laminates $[\theta/\theta/-\theta/-\theta]$, and the web laminate is assumed to be unidirectional; second case, antisymmetric angle-ply laminates $[\theta/\theta/-\theta/-\theta]$ in the web, and unidirectional fiber orientation in the flanges. Four layers with equal thicknesses are considered. For all the analysis, the assumption $\sigma_s = 0$ is made. Variation of the torsional displacement of free end with respect to fiber angle change in the flanges and web is shown in Fig. 3. It is found that the beam with fiber angle change in the flanges is more sensitive to angle of twist than that of fiber angle change in the web. For both cases, the minimum angle of twist occurs near $\theta = 45^\circ$, that is, because the torsional rigidity E_{55} becomes maximum value at $\theta = 45^\circ$.

The last example presents a cantilever beam under point load instead of torsional load at free edge (Fig. 4.). This case is that both the flanges are antisym-

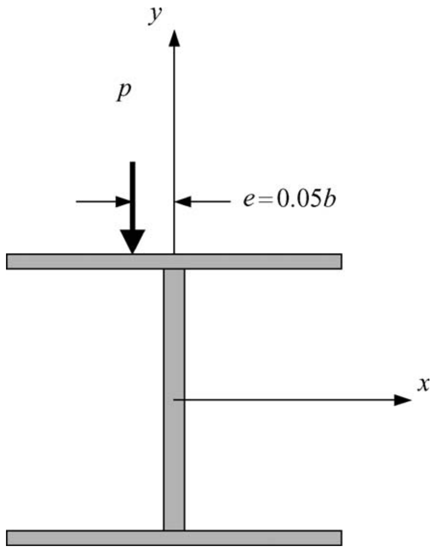

Fig. 4. I-section composite cantilever beam under eccentric load at free end.

metric angle-ply stacking sequence, and the other conditions are the same as the previous example. Stacking sequence of top and bottom flanges are $[\theta/\theta/-\theta/-\theta]$, $[\theta/-\theta/\theta/-\theta]$, respectively, and the web laminate is assumed to be unidirectional

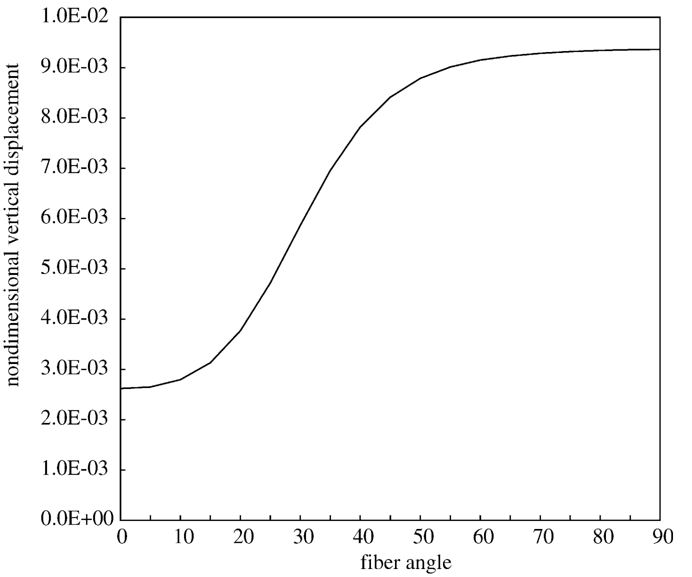

Fig. 5. Variation of the vertical displacements with respect to the fiber angle change of a cantilever beam under eccentric load at free end.

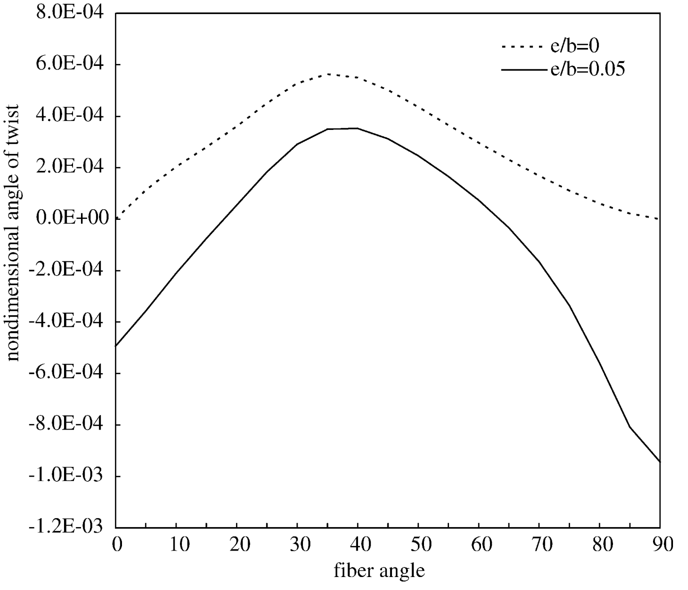

Fig. 6. Variation of the torsional displacement with respect to the fiber angle change of a cantilever beam under eccentric at free end.

and thus exhibit flexural–torsional coupling. The vertical displacements at the free end are shown in Fig. 5 with respect to fiber angle variation. It shows that the load eccentricity does not affect the vertical displacements. On the other hand, the maximum torsional displacements show substantial changes for eccentricity with respect to fiber angle variation (Fig. 6). Even for no eccentricity ($e/b = 0$), the torsional displacement becomes nonzero as fiber angle goes off-axis implying that the coupling stiffnesses E_{15} and E_{35} drive flexural–torsional coupling. Vice versa, for $e/b = 0.05$, the torsional displacement can vanish for specific value of fiber angle (near 18° and 63°), implying that the angle of twist can be suppressed with carefully tailored stacking sequence even for applied torque.

9. Concluding remarks

An analytical model was developed to study the flexural–torsional behavior of a laminated composite beam with an I-section. The model is capable of predicting accurate deflection as well as angle of twist shapes of various configuration including boundary conditions, laminate orientation and ratio of elastic moduli. To formulate the problem, a one-dimensional displacement-based finite element method is employed. The assumption that normal stress in contour direction vanishes ($\sigma_s = 0$) seems more appropriate than the free strain assumption in contour direction. The model presented is found to be appropriate and efficient in analyzing flexural–torsional problem of a thin-walled laminated composite beam.

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