

FLIP THEOREM AND THE EXISTENCE OF MINIMAL MODELS FOR 3-FOLDS

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To Professor M. Nagata on his 60th birthday

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0. INTRODUCTION

One of the main achievements of the 19th century Italian school of algebraic geometry was a complete understanding of the birational geometry of surfaces, including the construction of minimal models. During the past several years, a program has emerged to construct minimal models in higher dimensions using something called extremal rays. This paper completes the final step of that Minimal Model Program (MMP) in dimension 3 (cf. an excellent introduction [Ko2] for nonexperts).

We should mention that, even in dimension 3, there are several things yet to be done in classification theory (e.g. [R3, §4]).

Before formulating our main theorem (0.2.5), let us explain some corollaries and the background.

We will work over the field \mathbb{C} of complex numbers.

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(0.1) **Corollaries to Minimal Model Program for 3-folds.** Here are three important corollaries indicating that the canonical divisor K_X of a 3-fold X has decisive information.

(0.1.1) **Theorem.** *For every nonsingular projective 3-fold X , the graded canonical ring $\bigoplus_{\nu \geq 0} H^0(X, \mathcal{O}(\nu K_X))$ is finitely generated.*

This follows from MMP by virtue of the works of Fujita [Ft], Benveniste [B1] and Kawamata [Ka1], and it is considered to be a necessary step in the construction of moduli of 3-folds.

(0.1.2) **Theorem.** *A nonsingular projective 3-fold X is uniruled if and only if $H^0(X, \mathcal{O}(\nu K_X)) = 0$ for all $\nu > 0$, where an n -fold Z is uniruled if there is a dominating rational map $\mathbb{P}^1 \times Y \dashrightarrow Z$ for some $(n - 1)$ -fold Y .*

This follows from MMP by virtue of the works of Miyaoka [My1, My2, My3], and Miyaoka and Mori [MM]. Indeed some model X' birational to X is shown to have a finer structure (cf. (0.3)).

(0.1.3) **Theorem.** *Every birational morphism $f: X \rightarrow Y$ between nonsingular projective 3-folds is a composition of divisorial contractions and directed flips (cf. (0.2)).*

This gives (in principle) a factorization of birational morphisms of 3-folds with mild singularities (cf. (0.3.11)), and generalizes the first half of Danilov's factorization [Da] of toric birational morphisms of 3-folds to arbitrary birational morphisms.

We should point out that the development of the theory of extremal rays was originally motivated by Hironaka's approach [H] to the factorization problem using "cones" (although the techniques were based on Kleiman's criterion of ampleness [Kl]).

Let us look at the Minimal Model Conjecture more closely.

(0.2) **The Minimal Model Conjecture.** We will review the necessary material only briefly and more detailed treatment is given in [KMM, Mr4, Wi].

(0.2.1) **Definition.** We say that a Weil divisor D on a normal variety X is \mathbb{Q} -Cartier if νD is Cartier for some $\nu > 0$. We note that X with only *canonical* or *terminal* singularities has \mathbb{Q} -Cartier K_X , leaving the technical definition to (1.0). We say that X has only \mathbb{Q} -factorial singularities if every Weil divisor is \mathbb{Q} -Cartier.

Let X be a normal projective variety. Let $\rho(X)$ denote the Picard number ($\in \mathbb{N}$), the rank of the Neron-Severi group $\text{NS}(X)$ of X . We say that X is a *canonical* (resp. *minimal*) *model* if X has only canonical (resp. terminal) singularities and K_X is ample (resp. nef), where a \mathbb{Q} -Cartier divisor H is *nef* iff $(H \cdot C) \geq 0$ for every irreducible curve C .

We say that X has a canonical (resp. minimal) model if there is a canonical (resp. minimal) model birational to X . We remark that such a canonical model is unique while a minimal model is not.

(0.2.2) MMC, in short, claims that every variety X has a minimal model iff X is not uniruled. To make it more explicit, we will review the theory of extremal rays built by Benveniste, Kawamata, Kollár, Mori, Reid, Shokurov (cf. [KMM, Mr4, R3, or Wi]). To simplify the exposition, we treat the core of the theory as a black box; it implies the following.

(0.2.3) **Theorem.** *Let X be a projective variety with only \mathbb{Q} -factorial terminal singularities. If X is not a minimal model, then there is a surjective morphism $f: X \rightarrow Y$ to a normal projective variety Y with connected fibers such that $\rho(X) = \rho(Y) + 1$, $-K_X$ is f -ample and one of the following holds.*

- (a) $\dim X > \dim Y$ (\mathbb{Q} -Fano fibering).
- (b) f is birational and contracts a divisor (divisorial contraction).
- (c) f is birational and contracts no divisors (small contraction).

(0.2.4) Let the notation and the assumptions be as in (0.2.3). If f is a divisorial contraction, it is easy to see that Y has only \mathbb{Q} -factorial terminal singularities. The case of a small contraction is treated by our main theorem if X is a 3-fold.

(0.2.5) **Main Theorem (Flip Theorem).** *Let $f: X \rightarrow Y$ be a birational morphism between normal projective 3-folds such that X has only \mathbb{Q} -factorial terminal singularities, $\rho(X) = \rho(Y) + 1$, $-K_X$ is f -ample and f contracts no divisors (to curves or points). Then there is a birational morphism $f': X' \rightarrow Y$ from a projective 3-fold X' with only \mathbb{Q} -factorial terminal singularities such that f' contracts no divisors and $K_{X'}$ is f' -ample. (Let us emphasize the change in the sign of the canonical divisor.)*

The map $X \dashrightarrow X'$ is isomorphic in codimension 1 (i.e. induces $X - Z \simeq X' - Z'$ for some closed subsets Z and Z' of $\text{codim} \geq 2$), $\rho(X) = \rho(X')$, and f' (or X') is called the *directed flip* (or *flip*) of f . If $\dim X \geq 4$, the existence of the directed flip is unknown.

There are a couple of ways to put (0.2.3)–(0.2.5) together to state the Minimal Model Program. One of the formulations is the following. We note that (c) is conjectural if $\dim X \geq 4$.

(0.2.6) **Minimal Model Program** (Reid [R2]: an observation based on [Mr2] and Francia [Fr]). Let X be a projective 3-fold with only \mathbb{Q} -factorial terminal singularities. If X is not a minimal model, then there is a morphism $f: X \rightarrow Y$ which is a \mathbb{Q} -Fano fibering, a divisorial contraction or a small contraction. We treat three cases.

Case (a). No divisorial or small contractions exist: f is a \mathbb{Q} -Fano fibering.

Case (b). f is a divisorial contraction: Since Y has only \mathbb{Q} -factorial terminal singularities (0.2.4), we may continue the program by replacing X with Y . We note $\rho(Y) = \rho(X) - 1$.

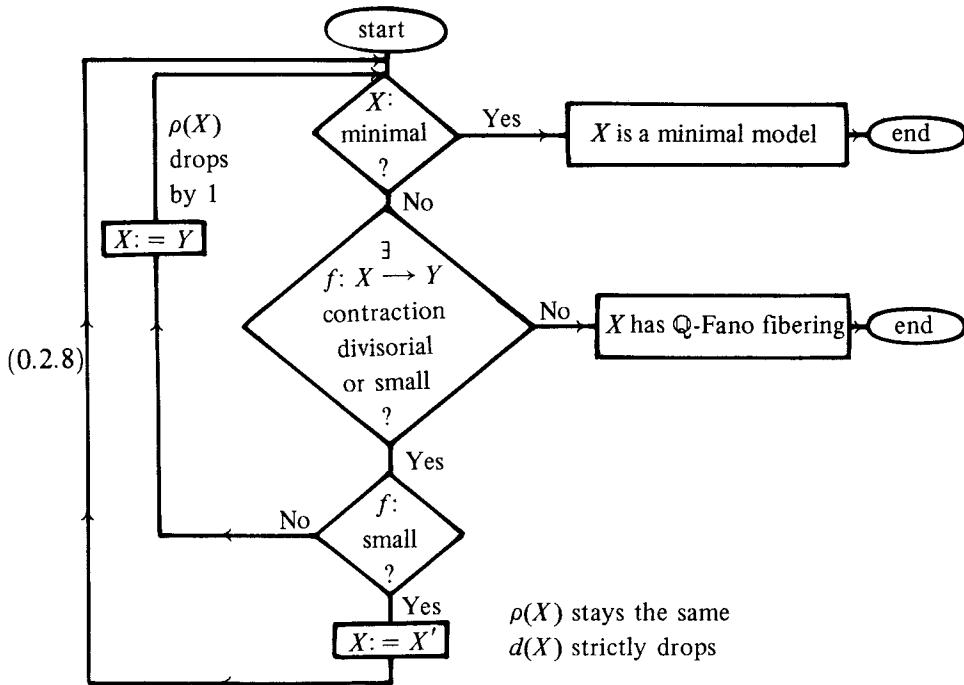
Case (c). f is a small contraction: Since the directed flip X' of $X \rightarrow Y$ has only \mathbb{Q} -factorial terminal singularities (0.2.5), we may continue by replacing X with X' . We note $\rho(X') = \rho(X)$.

Since $\rho(X)$ is a natural number, (b) does not occur infinitely many times, and the following theorem of Shokurov shows that (c) does not repeat infinitely many times either.

(0.2.7) **Theorem** (Shokurov [Sh]). *To each algebraic 3-fold with only terminal singularities is associated a nonnegative integer $d(X)$ called the difficulty, and one has $d(X') < d(X)$ for X and X' as in (0.2.6(c)).*

Though the existence of flip is unknown if $\dim X \geq 4$, the termination of flips (0.2.7) is generalized to 4-folds X by Kawamata, Matsuda, and Matsuki [KMM].

These can be put together into the flow chart for 3-folds.



We will show how the results in (0.1) follow from the Minimal Model Program for 3-folds and state some more corollaries.

(0.3) **Results in (0.1) and further corollaries.** The Minimal Model Program is thus completed for 3-folds and

(0.3.1) **Theorem.** *Every 3-fold has a minimal model or a model with a \mathbb{Q} -Fano fibering.*

We recall that a nonsingular projective variety X is of *general type* if the rational mapping associated to $H^0(X, \mathcal{O}(\nu K_X))$ is birational for some $\nu > 0$, and that a \mathbb{Q} -Cartier divisor D is *semiample* if νD is a Cartier divisor such that $\mathcal{O}(\nu D)$ is generated by global sections for some $\nu > 0$.

We quote several results.

(0.3.2) **Theorem** (Kawamata [Ka1, KMM] and Benveniste [B1]). *Let X be a nonsingular projective variety of general type which has a minimal model (say, Y). Then K_Y is semiample, and hence X has a canonical model and the canonical ring of X is finitely generated.*

This implies (0.1.1), since [Ft] settles (0.1.1) when X is a 3-fold which is not of general type. We will explain (0.1.2).

(0.3.3) **Theorem** (Miyaoka [My1, My2, My3]). *If a nonsingular projective 3-fold X has a minimal model, then $H^0(X, \mathcal{O}(\nu K_X)) \neq 0$ for some $\nu > 0$.*

We note that the generalization of (0.3.3) to the case of $\dim X \geq 4$ is unknown.

Thus if we start from a nonsingular projective 3-fold X such that

$$H^0(X, \mathcal{O}(\nu K_X)) = 0 \quad \text{for all } \nu > 0,$$

then we never get a minimal model and hence we get a model X' with a \mathbb{Q} -Fano fibering. Then we apply the following to get (0.1.2).

(0.3.4) **Theorem** (Miyaoka and Mori [MM]). *If $f: X \rightarrow Y$ is a \mathbb{Q} -Fano fibering, then X is uniruled.*

Most of the above results in the absolute cases have been generalized to the relative case based on Kollár's vanishing theorem [Ko1]. For instance, (0.3.2) is a special case of the following (0.3.5), which is also a special case of Kawamata's relative Base Point Free Theorem.

(0.3.5) **Theorem** (Kawamata [KMM, Theorem 3-3-1]). *Let $f: X \rightarrow S$ be a projective morphism with connected fibers from a variety X with only terminal singularities to a variety S . If a general fiber of f is a variety of general type and K_X is f -nef, then K_X is f -semiample.*

(0.3.6) **Remark.** In the above, a divisor D on X is f -nef if $(D \cdot C) \geq 0$ for all irreducible curves C on X such that $f(C)$ is a point, and it is f -semiample if the natural map $f^* f_* \mathcal{O}(nD) \rightarrow \mathcal{O}(nD)$ is surjective for sufficiently divisible $n > 0$.

Unlike the absolute case, there are sometimes only trivial minimal models in the relative case.

(0.3.7) **Proposition.** *Let $g: Z \rightarrow X$ be a birational projective morphism of normal varieties such that K_Z is \mathbb{Q} -Cartier and X has only \mathbb{Q} -factorial terminal singularities. If g is not an isomorphism, then K_Z is not g -nef.*

(0.3.8) *Remark.* This is proved for smooth projective 3-folds X and Z in [Mr2, (3.40)], whose proof works for (0.3.7) modulo trivial changes.

(0.3.9) The Minimal Model Program works equally well for 3-folds with \mathbb{Q} -factorial terminal singularities which are projective over a fixed irreducible reduced algebraic variety S as mentioned after (0.3.4) because (0.2.5) is proved in a more general form (0.4.1). We state a theorem which is an immediate corollary to the Minimal Model Program for 3-folds in the relative setting.

(0.3.10) **Theorem.** *Let $f: X \rightarrow S$ be a projective morphism from a 3-fold X with only \mathbb{Q} -factorial terminal singularities to a variety S . Then via a composition of divisorial contractions and directed flips, X is S -birational to an S -projective 3-fold $f': X' \rightarrow S$ such that X' has only \mathbb{Q} -factorial terminal singularities and satisfies one of the following.*

- (a) $K_{X'}$ is f' -nef, or
- (b) X' has a surjective S -morphism to an S -projective variety $Z \rightarrow S$ such that $\dim Z < \dim X$, $-K_{X'}$ is relatively ample over Z , and the relative Picard number $\rho(X'/Z)$ of X' over Z is 1.

(0.3.10.1) *Remark.* In the important special case where f is a semistable projective morphism onto a smooth curve, we note (i) “minimal” models (maybe nonalgebraic) over S were constructed by Kulikov [Ku] and Persson and Pinkham [PP] if the general fibers of f are surfaces with $K \sim 0$ (the first breakthrough in this direction) and by Morrison [Mrr1] if the general fibers of f are surfaces with $2K \sim 0$ and $K \not\sim 0$ (they did not use extremal rays), (ii) Tsunoda [Mn, T], Shokurov and Mori independently proved (0.3.10) if the general fibers of f are minimal surfaces, and (iii) Kawamata [Ka2] proved (0.3.10) with no assumptions on the general fibers.

By virtue of (0.3.7), the following holds.

(0.3.11) **Theorem.** *Every birational projective morphism $f: X \rightarrow S$ between 3-folds with only \mathbb{Q} -factorial terminal singularities is a product of divisorial contractions and directed flips.*

We note that (0.1.3) is only a special case of (0.3.11).

By (0.3.5) quoted earlier, we obtain

(0.3.12) **Theorem.** *Let S be a 3-fold. Then there exist projective birational morphisms $f: X \rightarrow S$ and $g: Y \rightarrow S$ such that*

- (i) X has only \mathbb{Q} -factorial terminal singularities and K_X is f -semiample, and
- (ii) Y has only canonical singularities, K_Y is f -ample, and there is an S -morphism $h: X \rightarrow Y$ such that $K_X = h^*K_Y$.

We note that such Y is unique while such X is unique only in codimension 1, and we call X a \mathbb{Q} -factorial terminal modification and Y the canonical modification of S .

These results have two kinds of rather technical generalizations.

(0.3.13) *Remark.* When S is the germ (S, s) of an analytic space at a point s or a compact analytic space, the results corresponding to (0.3.10)–(0.3.12) hold by Nakayama [N] because (0.2.5) is proved in the analytic space case (0.4.1), where an analytic space Z over S is said to have only \mathbb{Q} -factorial singularities if every global Weil divisor on Z is \mathbb{Q} -Cartier.

(0.3.14) *Remark.* By virtue of the definition of the extremal ray, the results in (0.2) extend to G -threefolds in Manin's sense [Ma] (hence to 3-folds over any field k of characteristic 0) as indicated by [Mr2, Chapter 2, §3]. Let us take a finite group G for simplicity. The results in (0.2) hold true if we interpret varieties (resp. subvarieties, morphisms) as varieties with regular G -action (resp. G -stable subvarieties, G -equivariant morphisms). Here, $G\mathbb{Q}$ -factoriality (the modified \mathbb{Q} -factoriality) means that all the G -stable Weil divisors are \mathbb{Q} -Cartier, and $G\rho(X)$ (the modified $\rho(X)$) denotes the rank of the G -invariant part of $\text{NS}(X)$. This way, (0.3.1), (0.3.10), (0.3.11), and (0.3.12) hold true for G -threefolds.

Let us refine (0.3.1) (cf. [I, Ma, and Ko3, (3.6)]), where $pG3ft$ is an abbreviation of “projective G -threefold with only $G\mathbb{Q}$ -factorial terminal singularities.”

(0.3.14.1) **Theorem.** Any $pG3ft$ is G -birational to either

- (i) a $pG3ft$ X with nef K_X (G -minimal model), or
- (ii) a $pG3ft$ X with a surjective G -morphism f to a normal projective G -variety Y with connected fibers such that $G\rho(X) = G\rho(Y) + 1$, $-K_X$ is f -ample and $\dim Y < \dim X$ ($G\mathbb{Q}$ -Fano fibering).

Let us explain what is done in this paper to prove (0.2.5).

(0.4) **What is to be done to prove the Flip Theorem.** Let us modify (0.2.5) to a statement which is formulated in both algebraic and analytic contexts without \mathbb{Q} -factoriality.

(0.4.1) **Flip Theorem.** Let $f: X \rightarrow Y$ be a proper birational morphism of normal algebraic (resp. analytic) 3-folds such that X has only terminal singularities, f contracts no divisors, and $-K_X$ is f -ample. Then there is a proper birational morphism $f': X' \rightarrow Y$ from an algebraic (resp. analytic) 3-fold X' with only terminal singularities such that f' contracts no divisors and $K_{X'}$ is f' -ample.

We note that X' is unique and its existence is equivalent to the finite generation of $\bigoplus_{n \geq 0} f_* \mathcal{O}(nK_X)$ as an \mathcal{O}_X -algebra. The birational map $X \dashrightarrow X'$ is isomorphic in codimension 1 and f' (or X') is called the *directed flip* (or *flip*) of f .

(0.4.1.1) *Remark.* (i) The conclusion of (0.4.1) holds even if X has canonical singularities. This can be proved by the arguments as in [Ka2, Ko3], or by applying the Minimal Model Program over Y , which are both based on the original (0.4.1), and

(ii) the two versions of (0.4.1), the algebraic $(0.4.1)_{\text{alg}}$ and the analytic $(0.4.1)_{\text{an}}$, are equivalent as in [Ka2, Proposition 8.4]. It is based on the rationality of the singularities of Y and Shokurov's termination of directed flips (0.2.7).

(0.4.2) It is easy to see $(0.4.1)_{\text{alg}} \Rightarrow (0.2.5)$, and $(0.4.1)_{\text{an}}$ is to be proved. The advantage is the following: The fundamental set Σ on Y for f^{-1} is discrete and we may set $X' - f^{-1}\Sigma = Y - \Sigma$. Thus the problem is local at each point of Σ and we may assume Y is a germ (Y, y) at a point. For $f: X \rightarrow (Y, y)$ as in (0.4.1), we call the germ $X \supset f^{-1}(y)_{\text{red}}$ of an analytic 3-fold X along $f^{-1}(y)_{\text{red}}$ an *extremal nbd* (as in (1.1)).

By treating irreducible components of $f^{-1}(y)$ one by one [Ka2, Proposition 8.4], one can reduce $(0.4.1)_{\text{an}}$ to

(0.4.3) **Lemma.** $(0.4.1)_{\text{an}}$ holds if we assume further that Y is the germ (Y, y) at a point y and $f^{-1}(y)$ is irreducible.

We should mention that $f^{-1}(y) \simeq \mathbb{P}^1$ above. This follows from the rationality of the singularity of (Y, y) (1.3).

Kawamata's approach [Ka2] to the directed flip is as follows:

(0.4.4) Take a double cover V of X with branch locus $D \in |-2K_X|$. If V has only canonical singularities then so does the double cover W of Y with branch locus $f(D)$ and one can use finite generation of $\bigoplus_{n \geq 0} \mathcal{O}_W(nK_Y)$ [Ka2, Theorem 6.1'] (a simpler proof is given in [Ko3]) to get finite generation of its \mathbb{Z}_2 -invariant part $\bigoplus_{n \geq 0} \mathcal{O}_Y(nK_Y) = \bigoplus_{n \geq 0} f_* \mathcal{O}_X(nK_X)$.

Thus [Ka2, Proposition 8.7] reduces (0.4.3) to the following.

(0.4.5) **Theorem.** Let $X \supset C \simeq \mathbb{P}^1$ be an extremal nbd. Then one of the following on the linear system $|-aK_X|$ ($a = 1$ or 2) holds.

- (i) $|-K_X|$ has a member D with only rational double points, or
- (ii) $|-2K_X|$ has a member D so that the double cover Z of X with branch locus D has only canonical singularities.

Kawamata [Ka2] proved (0.4.5) in the case of 1-parameter semistable family of surfaces to prove the result mentioned in (0.3.10.1(iii)).

It is (0.4.5) that is actually proved in this paper (Theorems (7.3) and (9.10), cf. also Appendix B). The main information we use for the extremal nbd $X \supset C \simeq \mathbb{P}^1$ is

$$(0.4.6) \quad H^1(\mathcal{O}_Z) = H^1(\omega_X \otimes \mathcal{O}_Z) = 0 \quad \forall Z \subset X \text{ with } Z_{\text{red}} = C.$$

These conditions are similar to those in [Mr2, §9] (cf. (1.2.1), §10), and so are some of our arguments.

The following is the smooth case of (0.4.5). We will illustrate our approach by proving (0.4.7) via (0.4.6). We should emphasize that we are interested in the arguments not the results. The results show that the smooth X is the blow-up of a smooth 3-fold (Y, Q) along a smooth curve (Z, Q) and C is the fiber over Q , which means that extremal nbds must have singular points (cf. §10).

(0.4.7) **Proposition.** *Let $X \supset C \simeq \mathbb{P}^1$ be a smooth extremal nbd. Then $\mathcal{O}_C(K_X) \simeq \mathcal{O}_C(-1)$, $I_C/I_C^2 \simeq \mathcal{O}_C \oplus \mathcal{O}_C(1)$, and $|-K_X|$ has a smooth member, where I_C is the ideal sheaf of C in X .*

(0.4.8) *Proof.* We see $(K_X \cdot C) = -1$ from $(K_X \cdot C) < 0$ and $H^1(\mathcal{O}_C(K_X)) = 0$ (0.4.6). By the standard exact sequence

$$0 \rightarrow I_C/I_C^2 \rightarrow \Omega_X^1 \otimes \mathcal{O}_C \rightarrow \mathcal{O}(K_C) \rightarrow 0,$$

one has a natural isomorphism

$$(0.4.8.1) \quad \alpha_1: \wedge^2(I_C/I_C^2) \xrightarrow{\sim} \mathcal{O}(K_X) \otimes \mathcal{O}_C(-K_C),$$

and $\deg I_C/I_C^2 = (K_X \cdot C) - \deg K_C = 1$. By the exact sequence

$$0 \rightarrow I_C/I_C^2 \otimes \mathcal{O}_C(-1) \rightarrow \mathcal{O}_X/I_C^2 \otimes \mathcal{O}_X(K_X) \rightarrow \mathcal{O}_C(-1) \rightarrow 0$$

and (0.4.6), we see $H^1(I_C/I_C^2 \otimes \mathcal{O}_C(-1)) = 0$. Whence $I_C/I_C^2 \otimes \mathcal{O}_C(-1) \simeq \mathcal{O}_C \oplus \mathcal{O}_C(-1)$ since it has degree $1 - 2 = -1$. Hence follows the second assertion. Let $x \in C$. Let (D, x) be a smooth divisor in the germ (X, x) intersecting transversally with (C, x) , then (D, x) extends to a divisor D' of X since X is the germ along the curve C . Then $(D' \cdot C) = 1 = (-K_X \cdot C)$. Hence $D' \in |-K_X|$ since $\text{Pic } X \ni L \mapsto (L \cdot C) \in \mathbb{Z}$ induces $\text{Pic } X \simeq \mathbb{Z}$ by (0.4.6) (cf. (0.1.3)). \square

(0.4.9) The arguments for singular extremal nbds $X \supset C \simeq \mathbb{P}^1$ are more elaborate. We will indicate how the arguments in (0.4.8) are modified to obtain results similar to (0.4.7).

We start with easy definitions.

(0.4.9.1) **Definition.** For a coherent sheaf \mathcal{F} , let $\text{gr}_C^0(\mathcal{F}) = \mathcal{F} \otimes \mathcal{O}_C/T$, where $T (\subset \mathcal{F} \otimes \mathcal{O}_C)$ is the maximal subsheaf of finite length. The sheaf $\text{gr}_C^0(\mathcal{F})$ satisfies $\mathcal{F} \otimes \mathcal{O}_C \simeq \text{gr}_C^0(\mathcal{F}) \oplus T$. For $n > 0$, let $I_C^{(n)}$ be the n th symbolic power of I_C which is determined by

$$\mathcal{O}_X/I_C^{(n)} = (\mathcal{O}_X/I_C^n)/(\text{maximal subsheaf of finite length}).$$

(0.4.10) **Analogues of “ $\omega_X \otimes \mathcal{O}_C \simeq \mathcal{O}_C(-1)$.”** Let us look at $\text{gr}_C^0 \omega$ ($= \text{gr}_C^0(\omega_X)$). Let m be a positive integer such that mK_X is a Cartier divisor.

Since

$$(\omega_X \otimes \mathcal{O}_C)^{\otimes m} \rightarrow \mathcal{O}_C(mK_X)$$

induces an injection

$$(0.4.10.1) \quad \beta_0: (\text{gr}_C^0 \omega)^{\otimes m} \hookrightarrow \mathcal{O}_C(mK_X) \quad (2.2),$$

we see $\deg \text{gr}_C^0 \omega < 0$ by $(K_X \cdot C) < 0$. Thus we see one analogue

$$(0.4.10.2) \quad \text{gr}_C^0 \omega \simeq \mathcal{O}_C(-1) \quad (2.3.2)$$

by $H^1(\omega_X \otimes \mathcal{O}_C) = 0$ (0.4.6), and another

$$(0.4.10.3) \quad (K_X \cdot C) = -1 + \sum_P w_P(0) \quad (2.3.2)$$

by $\beta_0: \mathcal{O}(-m) \hookrightarrow \mathcal{O}_C(mK_X)$, where $w_P(0) = (\text{len}_P \text{Coker } \beta_0)/m \ (\geq 0)$ for $P \in C$. In particular, we have

$$(0.4.11.1) \quad -1 \leq (K_X \cdot C) < 0 \text{ (cf. [B2, Théorème 0])}.$$

(0.4.11) **Analogues of “ $I_C/I_C^2 \simeq \mathcal{O} \oplus \mathcal{O}(1)$.”** Let us look at $\text{gr}_C^0 I$ ($= \text{gr}_C^0(I_C)$). (It is equal to $\text{gr}_C^1 \mathcal{O} = I_C/I_C^{(2)}$ (2.2) in the text.) It is a locally free \mathcal{O}_C -module of rank 2. Instead of α_1 in (0.4.8.1), we have a natural injection

$$(0.4.11.1) \quad \alpha_1: \bigwedge^2 \text{gr}_C^0 I \hookrightarrow \text{gr}_C^0 \omega \otimes \mathcal{O}_C(-K_C) \simeq \mathcal{O}(1) \quad (2.2.1).$$

For $P \in C$, let $i_P(1) = \text{len}_P \text{Coker } \alpha_1$ so that

$$(0.4.11.2) \quad \deg \text{gr}_C^0 I = 1 - \sum_P i_P(1) \quad (2.3.2).$$

A local computation shows

$$(0.4.11.3) \quad \textbf{Lemma (= (2.15)).} \text{ If } P \text{ is a singular point of } X \text{ then } i_P(1) > 0.$$

Our proof for the existence of good members of $|-K_X|$ or $|-2K_X|$ is based on two kinds of classifications of singularities of $X \supset C$, one local and the other global.

Let us see several results without going into technical details.

(0.4.12) **Classification of germs** $(X, P) \supset (C, P)$. Using $w_P(0)$ and $i_P(1)$ and a similar invariant $w_P^*(1)$, one can give a classification of $(X, P) \supset (C, P)$ without too much trouble (it is done in §3 and (4.2), cf. Appendix A for results).

Let P be a singular point on an extremal nbd $X \supset C \simeq \mathbb{P}^1$. We recall that the *index* of P is the smallest positive integer m such that mK_X is Cartier at P . The index m is invariant under deformation and has a topological interpretation:

$$\varprojlim_{U: \text{open nbd of } P} \pi_1(U - P) \simeq \mathbb{Z}_m.$$

We say that P is *ordinary* (4.5) if (X, P) is an ordinary double point or a cyclic quotient terminal singularity. The classification has a simple but important

(0.4.12.1) **Corollary.** *An arbitrary singular point P of $X \supset C$ has an open neighborhood U so that general deformations of the pair $U \supset C \cap U$ have only ordinary points.*

Any deformation of the pair $U \supset C \cap U$ can be extended to a deformation of the pair $X \supset C$ (§1b, §10).

(0.4.12.2) **Deformation Lemma.** *Arbitrary extremal nbds $X \supset C \simeq \mathbb{P}^1$ deform to extremal nbds $X' \supset C'$ with only ordinary points.*

To be rigorous, the lemma is wrong as it is: The deformation X' of X often contains finitely many compact curves other than C' but it is true that the germ $X'^o \supset C'$ of X' along C' is an extremal nbd. This is, however, a matter of technical details.

(0.4.12.3) **Remark.** The strategy behind (0.4.12.2) is that, for certain problems on extremal nbds, it is enough to treat extremal nbds with only ordinary points. Though we cannot apply the strategy directly to the flip theorem, we can still apply it to various auxiliary problems (e.g. (0.4.13.1)).

Here is an easy example to use in our introduction.

(0.4.12.4) **Example.** Let μ_m act on $(X^\sharp, P^\sharp) = (\mathbb{C}^3, 0)$ by

$$\zeta(x, y, z) = (\zeta x, \zeta^a y, \zeta^{-1} z) \quad (\zeta \in \mu_m),$$

where a is an integer prime to m such that $0 < a < m$. Let $C^\sharp \subset X^\sharp$ be the x -axis. Then $(X, P) = (X^\sharp, P^\sharp)/\mu_m \supset (C, P) = (C^\sharp, P^\sharp)/\mu_m$ is an example of an ordinary point of index m .

We note that any ordinary (X, P) of index $m > 1$ is of the above form, but in general C^\sharp is singular at P^\sharp .

The following are easy explicit computations.

- (i) $\mathcal{O}_{C,P} = \mathbb{C}\{x^m\}$,
- (ii) $\text{gr}_C^0 \omega = \mathcal{O}_C(x^{m-a} dx \wedge dy \wedge dz)$,
 $\mathcal{O}_C(mK_X) = \mathcal{O}_C(dx \wedge dy \wedge dz)^m$ near P ,
- (iii) $w_P(0) = (m - a)/m$,
- (iv) $\text{gr}_C^0 I = \mathcal{O}_C(x^{m-a} y) \oplus \mathcal{O}_C(xz)$ near P , and
- (v) $i_P(1) = 1$.

(0.4.13) **Classification of extremal nbds $X \supset C$.** This part is done in §6. Arguments involving $w_P(0)$, $i_P(1)$ and other higher order numerical invariants prove the following coarse classification of extremal nbds $X \supset C$ (cf. Appendix B for the precise results).

(0.4.13.1) **Proposition.** *Let $X \supset C \simeq \mathbb{P}^1$ be an extremal nbd. Then we have*

- (i) X has at most one singular point with index 1 on C ,
- (ii) X has at most two points with index > 1 on C , and
- (iii) if X has three singular points, one of them has index 2.

We present two arguments concerning (0.4.13.1).

(0.4.13.2) Here, only to give the idea about our approach, let us prove that X has at most three singular points on C . The complex $I_C/I_C^2 \rightarrow \mathcal{O}_X/I_C^2 \rightarrow \mathcal{O}_C \rightarrow 0$ induces an exact sequence

$$0 \rightarrow \text{gr}_C^0 I \rightarrow \mathcal{O}_X/I_C^{(2)} \rightarrow \mathcal{O}_C \rightarrow 0.$$

Since $H^0(\mathcal{O}_X/I_C^{(2)}) \rightarrow H^0(\mathcal{O}_C) = \mathbb{C}$ is surjective, one has $H^1(\text{gr}_C^0 I) = 0$ by $H^1(\mathcal{O}_X/I_C^{(2)}) = 0$ (0.4.6). Thus $\text{deg gr}_C^0 I \geq -2$ and it is now immediate to see that X has at most three singular points on C by (0.4.11.2) and (0.4.11.3). \square

(0.4.13.3) We sketch an easy topological proof of (ii) of (0.4.13.1). We will disprove the case of an extremal nbd $X \supset C \simeq \mathbb{P}^1$ with exactly three singular points P, Q, R , which are cyclic quotient singular points of indices $a, b, c > 1$. (The general case can be either reduced to this case by Deformation Lemma (0.4.12.2), or treated in a similar way using a certain nbd U for each P such that $\pi_1(U - P) \simeq \mathbb{Z}_{(\text{index } P)}$.)

Here is a homotopical description of $X - \{P, Q, R\}$: In $C - \{P, Q, R\} = S^2 - \{P, Q, R\}$, let ρ (resp. σ, τ) be small loops around P (resp. Q, R) in S^2 . Embed ρ, σ, τ into lens spaces in certain ways: $\rho \subset S^5/\mu_a, \sigma \subset S^5/\mu_b, \tau \subset S^5/\mu_c$. Then $S^2 - \{P, Q, R\}, S^5/\mu_a, S^5/\mu_b, S^5/\mu_c$ glued together by ρ, σ, τ become homotopically equivalent to $X - \{P, Q, R\}$.

The (class of the) loop ρ need not generate $\pi_1(S^5/\mu_a)$. But the case $\pi_1(S^5/\mu_a) \neq \langle \rho \rangle$ (cf. (0.4.16)) can be reduced to the case $\pi_1(S^5/\mu_a) = \langle \rho \rangle$ in an easy way ((1.13) and (1.16)). Hence we may assume that $\pi_1(S^5/\mu_a) = \langle \rho \rangle, \dots, \pi_1(S^5/\mu_c) = \langle \tau \rangle$. Thus

$$\pi_1(X - \{P, Q, R\}) \simeq \langle \rho, \sigma, \tau \rangle / \{ \rho^a = 1, \sigma^b = 1, \tau^c = 1, \rho\sigma\tau = 1 \}$$

by Van Kampen's Theorem. This group is known to have a finite quotient group in which the image of ρ (resp. σ, τ) is exactly of order a (resp. b, c). The associated covering space of $X - \{P, Q, R\}$ is extended to a finite covering of X which is a smooth extremal nbd. This is impossible by the comment preceding (0.4.7). \square

Based on these, our case-dependent arguments go as follows.

(0.4.14) **Local methods to find members** $\in |-K_X|, |-2K_X|$. This part is done in §7 and generalizes an argument in (0.4.8). For each $P \in C$ of index > 1 (say m) of X , let D_P be a local divisor defined by a local section of $\mathcal{O}(-K_X)$ at P so that $D_P \cap C = \{P\}$. Then $-K_X - \sum_P D_P$ is a Cartier divisor and

$$(0.4.14.1) \quad \sum_P (D_P \cdot C) \equiv (-K_X \cdot C) \pmod{\mathbb{Z}}.$$

If one can choose D_P 's such that $\sum (D_P \cdot C) \leq 1$, then $\sum (D_P \cdot C) = (-K_X \cdot C)$ (0.4.10.4) and this shows that $\sum D_P \in |-K_X|$.

Let us work on Example (0.4.12.4).

(0.4.14.2) **Local description.**

(i) Let P be as in Example (0.4.12.4). Then $\mathcal{O}(-K_X)$ near P is the invariant part of $\mathcal{O}_{X^2}(dx \wedge dy \wedge dz)^{-1}$. Since $(y - x^a)/(dx \wedge dy \wedge dz)$ is μ_m -invariant, it is a local section of $\mathcal{O}(-K_X)$ and defines $D_P = \{y = x^a\}/\mu_m$. Then

$$(D_P \cdot C) = \frac{1}{m}(\{y = x^a\} \cdot (x\text{-axis})) = \frac{a}{m} < 1.$$

(ii) If P in (i) is the only point of $X \supset C$ with index > 1 , then $D_P \in |-K_X|$ and $D_P \simeq (\mathbb{C}^2, 0)/\mu_m$ with $\zeta(x, z) = (\zeta x, \zeta^{-1} z)$. So D_P has a rational double point of type A_{m-1} at P .

(0.4.14.3) One can argue similarly with $\mathcal{O}(-2K_X)$. These methods work when X has at most one point P with index > 2 on C . We should point out that the computation is needed only at P (note: $-2K_X$ is Cartier on $X - P$) and it is similar to (0.4.14.2).

(0.4.15) **A global method to find members** $\in |-K_X|$. This part is done in §9. In view of the classification (0.4.13.1) and the local methods (0.4.14.3), we only treat $X \supset C \simeq \mathbb{P}^1$ with exactly two singular points, which are of indices ≥ 3 . In this case, an isomorphism

$$\text{gr}_C^0 I \simeq \mathcal{O} \oplus \mathcal{O}(-1)$$

is proved and the singularities of $X \supset C$ and the infinitesimal structure of X along C are further studied by the arguments (9.1)–(9.8) which are similar to but more elaborate than (0.4.13.2). Let us concentrate on the ordinary case for simplicity. (The nonordinary case is treated similarly.) Here are the results.

(0.4.15.1) **Further classification.** Let $X \supset C \simeq \mathbb{P}^1$ be an extremal nbd with two ordinary points of indices ≥ 3 . Then

- (i) $X \supset C$ at each ordinary point P is given by Example (0.4.12.4) for suitable a , and furthermore for suitable choices of coordinates in (0.4.12.4) at these points one has
- (ii) the (unique) direct summand $\mathcal{L} = \mathcal{O} \subset \text{gr}_C^0 I \simeq \mathcal{O} \oplus \mathcal{O}(-1)$ restricts to $\mathcal{O}_C(xz) \subset \mathcal{O}_C(x^{m-a}y) \oplus \mathcal{O}_C(xz)$ at each P , and
- (iii) there is a direct summand $\mathcal{M} \simeq \mathcal{O}(-1)$ of $\text{gr}_C^0 I$ restricting to $\mathcal{O}_C(x^{m-a}y)$ at each P .

This might not sound strong enough to be called a classification. However, there is an explicit description of all the extremal nbds $X \supset C \simeq \mathbb{P}^1$ with two points of indices ≥ 3 based on (0.4.15.1), which will be published elsewhere.

We note that an explicit local description of this kind is the core of the ℓ -structure introduced in §8. Out of this, one can pick up a homomorphism $\omega_X \rightarrow \text{gr}_C^0 I$ inducing an isomorphism $\text{gr}_C^0 \omega \simeq \mathcal{M}$. Then we extend it to $\omega_X \rightarrow I_C/I_C^{(n)}$ for all $n > 0$ in (9.9).

Let us illustrate the extension mechanism by arguing backward. From now on till the end of (0.4.15.5), the sheaves \mathcal{L} and \mathcal{M} are chosen and fixed as in (0.4.15.1).

(0.4.15.2) **What is to be expected.** Assume that a member $D (\supset C)$ of $|-K_X|$ is the zero locus of a homomorphism $\omega_X \rightarrow I_C$ extending $\omega_X \rightarrow \mathcal{M} \subset \text{gr}_C^0 I$ above. Then, for suitable (formal) coordinates as in (0.4.15.1), D restricts to $D_P = \{y = 0\}/\mu_m$ near each P (an exercise using (0.4.15.1(iii))). This D_P has only rational double points as in (ii) of (0.4.14.2) and D is smooth outside the ordinary points.

The Weil divisors nC of D ($n > 0$) are defined in X (not D) by ideals $J_n (\subset \mathcal{O}_X)$. For an ideal $J \subset \mathcal{O}_X$, we denote by $F^1 J$ the kernel of the natural map $J \rightarrow \text{gr}_C^0(J)$. We note that $\text{gr}_C^0(J) = J/F^1 J$ and $F^1 J \supset I_C J$.

By the construction, J_2 is the ideal uniquely determined by $I_C \supset J_2 \supset I_C^{(2)}$ and $J_2/I_C^{(2)} = \mathcal{M} \subset \text{gr}_C^0 I$, and the ideals

$$J_2 \supset J_3 \supset \dots \supset J_n \supset \dots$$

enjoy the following properties ($n \geq 2$).

- (a)_n $\text{gr}_C^0(J_n) = J_n/F^1 J_n$ is a locally free \mathcal{O}_C -module of rank 2,
- (b)_n the inclusion $J_n \subset J_2$ induces a natural surjection

$$\lambda_n: \text{gr}_C^0(J_n) \rightarrow \mathcal{M} \quad (\subset \text{gr}_C^0 I)$$

(λ_n corresponds to α_n in (9.9.6)) fitting in the exact sequence

$$E_n: 0 \rightarrow \text{Ker } \lambda_n \rightarrow \text{gr}_C^0(J_n) \xrightarrow{\lambda_n} \mathcal{M} \rightarrow 0,$$

- (c)_{n+1} $J_n \supset J_{n+1} \supset F^1 J_n$, and $J_{n+1}/F^1 J_n$ is an invertible subsheaf \mathcal{M}_n of $\text{gr}_C^0(J_n)$ giving a splitting of E_n .

Remark. We note that $\text{deg Ker } \lambda_n \geq 0$ by (a)_n, (b)_n, (c)_ν ($\nu \leq n$). Indeed $J_n \supset I_C^n$ by (c)_ν's, and the multiplication $\otimes^n(I_C) \rightarrow J_n$ induces a homomorphism $S^n(\text{gr}_C^0 I) \rightarrow \text{gr}_C^0(J_n)$ and an injection $\mathcal{O} \simeq \mathcal{L}^n \hookrightarrow \text{Ker } \lambda_n$ (0.4.15.1). In particular, $\text{deg Ker } \lambda_n > \text{deg } \mathcal{M} = -1$ hence E_n is split.

Here is an explicit description at each ordinary point P .

(0.4.15.3) **Local description.** There is a (formal) coordinate system (x, y, z) as in (0.4.15.1) with the following for all n :

- (d)_n J_n is the invariant part of (y, z^n) at P , and
- (e)_n $\text{gr}_C^0(J_n) = \mathcal{O}_C(x^{m-a}y) \oplus \mathcal{O}_C(x^{n-m[n/m]}z^n)$ at P , and λ_n is the projection to the first factor $\mathcal{M} = \mathcal{O}_C(x^{m-a}y)$.

We note that (e)_n follows from (d)_n easily.

(0.4.15.4) **The extension step.** Now without D , we inductively construct J_n 's from J_2 with the properties (a)_n, ..., (d)_n. If J_ν with the properties (a)_ν, ..., (d)_ν are chosen ($\nu \leq n$), then $\text{Ker } \lambda_n \geq 0$ and E_n is split by the remark in

(0.4.15.2). Let $\mathcal{M}_n \subset \text{gr}_C^0(J_n)$ be an arbitrary splitting submodule for E_n , and J_{n+1} as in (c) $_{n+1}$. Then (a) $_{n+1}$ and (b) $_{n+1}$ are satisfied. Hence to satisfy (d) $_{n+1}$, we will choose a proper \mathcal{M}_n using $\text{deg Ker } \lambda_n \geq 0$ above.

(0.4.15.5) **Local description.** Let us illustrate this using the coordinates in (0.4.15.1). Let $n \geq 2$. Assume that (d) $_n$ holds. ((d) $_2$ follows from (iii) of (0.4.15.1).) Via (e) $_n$, we have

$$\mathcal{M}_n = \mathcal{O}_C(x^{m-a}y + cx^{n-m[n/m]}z^n) \quad \text{near } P$$

for some $c \in \mathbb{C}\{x^m\}$. For (d) $_{n+1}$ to hold, c has to satisfy certain conditions. Let us illustrate it when $n \equiv 0 \pmod{m}$ by treating the cases $c = 0, 1$ to avoid inessential coordinate changes.

Case $c = 0$. One has $\mathcal{M}_n = \mathcal{O}_C(x^{m-a} \cdot y)$ and J_{n+1} is the invariant part of $(y, z^{n+1}) = (y) + (y, z)(y, z^n)$.

Case $c = 1$. One has $\mathcal{M}_n = \mathcal{O}_C(x^{m-a}y + z^n)$ and J_{n+1} is the invariant part of $(y^2, yz, z^{n+1}, x^{m-a}y + z^n) = (x^{m-a}y + z^n) + (y, z)(y, z^n)$.

Thus (d) $_{n+1}$ does not hold in the case $c = 1$ when $n \equiv 0 \pmod{m}$, and the choice of \mathcal{M}_n should not be arbitrary. In general, one can see that (d) $_{n+1}$ is satisfied if $c(0) = 0$. Hence we can choose an \mathcal{M}_n satisfying the two conditions (there are two points) because \mathcal{M}_n depends on two parameters by $\text{deg Ker } \lambda_n > \text{deg } \mathcal{M} = -1$.

Though we use these J_n 's to compute the general sections of $\omega_X^* \otimes I_C/I_C^{(n)}$ for $n \gg 0$ in (9.9.6)–(9.9.13), it is intuitively clearer to say that the good member D of $| -K_X |$ can be recovered (cf. (0.4.15.2)) as the formal Weil divisor $\varprojlim C_n$ of the completion \widehat{X} of X along C , where C_n is the closed subscheme defined by J_n .

(0.4.16) **Comment on imprimitive points.** A point P on an extremal nbd $X \supset C \simeq \mathbb{P}^1$ is *imprimitive* (1.7) if

$$\varprojlim \pi_1(U \cap C - P) \simeq \mathbb{Z} \rightarrow \varprojlim \pi_1(U - P) \simeq \mathbb{Z}/(\text{index } P)$$

(cf. (0.4.12)) is not surjective.

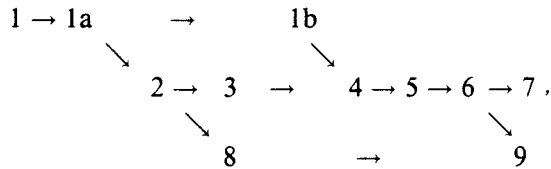
If $X \supset C \simeq \mathbb{P}^1$ has an imprimitive point P , then $X \supset C$ has a finite cover $X' \supset C'$ which is an extremal nbd primitive everywhere along any irreducible component of C' (C' is reducible). Therefore we may work on X' instead of X . Nevertheless we treat imprimitive points as extra cases and classify them. It is not just because we want to classify them. Our proof of the flip theorem needs them. Indeed if we do not flip the imprimitive X directly, we have to flip several curves on X' instead, since C' above is reducible. If the 3-fold picks up an imprimitive point after one flip, then one has to go up to a finite cover of the flipped 3-fold. Therefore, we need to disprove the possibility:

(0.4.16.1) One has to go up to covers for infinitely many times and hence the processes do not stop.

The impossibility of (0.4.16.1) is not obvious to us. We can settle it, but our proof even uses the flip theorem.

(0.4.17) **The descriptions of the sections.** Since §§6, 7, and 9 are described in (0.4.13), (0.4.14), and (0.4.15), let us now describe other sections. §1 gives basic definitions and the corollaries to the vanishing (0.4.6). §1a reviews the necessary material on terminal singularities. The terminal lemma (1a.6) in it is used all over in this paper and is the key in the number matching calculation mentioned above. §1b gives us some tools to deform extremal nbds (cf. (0.4.12.2)), and it says that being an extremal nbd is an open condition. §§2 and 8 prepare basic formulas (cf. (0.4.10) and (0.4.11)) and terminology (e.g. l -structures mentioned in (0.4.15.1)), and §2 gives formulas covering all the cases while §8 gives carefully tuned formulas covering only the delicate case. §§3 and 4 classify the local structure of $X \supset C$ at a singular point x of X into several cases (cf. (0.4.12) and Appendix A), and §5 gives practical formulas to work with in §§6 and 9. §10 states a slight generalization of our main theorem (0.4.5) together with a short explanation and it also makes comments on (1a.7).

We put a diagram indicating logical dependence among sections:



where $X \rightarrow \cdots \rightarrow Y$ means that §Y depends on some results in §X. For instance, it does not mean that the results in §1b are quoted only in §4; they are also quoted in §9.

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I should add gratefully that the referee’s constructive criticism and patience helped me revise the introduction very much.

Conventions. A variety (or an algebraic variety) means an irreducible reduced separated scheme of finite type over \mathbb{C} , the field of complex numbers. A curve in a variety means a 1-dimensional closed subscheme. An analytic space is meant to be Hausdorff and have countable open bases. The word “bimeromorphic” is used interchangeably with “birational” by abuse of language.

“Complex” and “0-sequence” are used interchangeably.

The set of rational numbers is denoted by \mathbb{Q} ; the set of real numbers is denoted by \mathbb{R} ; the set of integers is denoted by \mathbb{Z} ; the set of nonnegative integers $\{z \in \mathbb{Z} \mid z \geq 0\}$ is denoted by \mathbb{Z}_+ ; and the set of positive integers $\{z \in \mathbb{Z} \mid z > 0\}$ is denoted by \mathbb{Z}_{++} and it is equal to \mathbb{N} which denotes the set of natural numbers.

For $r \in \mathbb{R}$, $[r]$ denotes the integer d such that $d \leq r < d + 1$, and $\lceil r \rceil$ the integer u such that $u - 1 < r \leq u$. For $i, j \in \mathbb{Z}$, $\delta_{i,j}$ denotes Kronecker’s δ , i.e.

$$\delta_{i,j} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

For $m \in \mathbb{N}$, we set $\mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z}$ and $\mu_m = \{z \in \mathbb{C} \mid z^m = 1\}$.

List of notation and terminology. This list can be found at the end of the paper.

1. PRELIMINARIES AND BASIC DEFINITIONS

The main purpose of this section is to recall some known results and make basic definitions together with auxiliary results.

(1.0) **Definition (Reid [R1]).** Let (X, P) be a germ of a normal analytic (or algebraic) variety. Let $f: Y \rightarrow (X, P)$ be a *resolution*, i.e. a proper bimeromorphic morphism from a complex manifold Y . We say that the singularity (X, P) is *rational* if $R^i f_* \mathcal{O}_Y = 0$ for all $i > 0$ and all resolutions $f: Y \rightarrow (X, P)$. We say that (X, P) is a *canonical* (resp. *terminal*) singularity if

- (i) there is an integer $r > 0$ such that rK_X is a Cartier divisor (the smallest such r is called the *index* of (X, P)), and
- (ii) let $f: Y \rightarrow (X, P)$ be an arbitrary resolution, and let E_1, \dots, E_n be all the exceptional divisors. Then one has $rK_Y = f^*(rK_X) + \sum_i a_i E_i$ with all $a_i \geq 0$ (resp. $a_i > 0$).

Let us introduce the notion of an *extremal nbd* which we will study throughout this paper.

(1.1) Let X be an analytic 3-fold with at most terminal singularities and $C \subset X$ a reduced connected curve, and we consider the germ of X along C . We say that $X \supset C$ is an *extremal nbd* if there is a proper bimeromorphic morphism $f: X \rightarrow (Y, Q)$ so that $f_* \mathcal{O}_X = \mathcal{O}_Y$, $f^{-1}(Q) = C$ (as sets), $f|_{X-C}$

is an isomorphism, and $-K_X$ is f -ample. We call $f: X \rightarrow (Y, Q)$ the *total contraction*. (In case of reducible C , f contracts all the components of C , which is why we put the adjective “total.”) We note that X is the blow-up of (Y, Q) by a divisorial ideal $f_*\mathcal{O}(-mK_X) \simeq \mathcal{H}om(\omega_Y^{\otimes m}, \mathcal{O}_Y)$ for sufficiently divisible $m \in \mathbb{N}$, because $X \rightarrow Y$ is an isomorphism in codimension 1. Since (Y, Q) is an isolated singularity, it is algebraic by Artin [A2], hence so is $f: X \rightarrow Y$. We note that our notion of total contraction of extremal nbd coincides with Kawamata’s flipping singularity [Ka2, §8]. We quote his result [Ka2, §8] with the Grauert-Riemenschneider vanishing theorem [GR].

(1.2) **Theorem.** $R^i f_* \mathcal{O}_X = R^i f_* \omega_X = 0 \quad (i > 0)$.

(1.2.1) *Remark.* For an arbitrary ideal J such that $\text{Supp } \mathcal{O}_X/J \subset C$, one sees

$$H^1(\mathcal{O}_X/J) = H^1(\omega_X/J\omega_X) = H^1((\omega_X/J\omega_X)/\mathcal{F}_{or}) = 0$$

by (1.2) because fibers of f have dimension ≤ 1 , where \mathcal{F}_{or} denotes the largest submodule of $\omega_X/J\omega_X$ of finite length.

The following two corollaries are in [BS, §8, Ka2, §3]. We give the proofs for the readers’ convenience.

(1.3) **Corollary.** (i) C is a union of \mathbb{P}^1 ’s, whose configuration is a tree.

(ii) If C is a union of r \mathbb{P}^1 ’s ($r \geq 1$), then

$$\text{Pic } X \simeq H^2(C, \mathbb{Z}) \simeq \mathbb{Z}^{\oplus r}.$$

Since (1.3(ii)) is used in this paper only to compare a finite number of divisors at a time, we simply shrink $X \supset C$ if needed and we need not worry about the inductive limit.

Proof. (i) follows from the exact sequence $R^1 f_* \mathcal{O}_X \rightarrow H^1(C, \mathcal{O}_C) \rightarrow 0$. From (1.2) and the exact sequence

$$0 \rightarrow \mathbb{Z}_X \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_X^* \rightarrow 1,$$

we have $\text{Pic } X \simeq R^2 f_* \mathbb{Z}_X \simeq H^2(C, \mathbb{Z}_C) \simeq \mathbb{Z}^{\oplus r}$, since X is a germ along C [Go, Théorème 4.11.1]. \square

(1.4) **Corollary.** If L is an invertible sheaf on X such that $(L \cdot C_i) \geq 0$ for all irreducible components C_i , then $f^* f_* L \rightarrow L$ and $R^1 f_* L = 0$.

Proof. For each i , choose $P_i \in C_i$ so that $P_i \notin C_j$ for all $j \neq i$, and a Cartier divisor H_i of X such that $H_i \cdot C_i = P_i$ and $H_i \cap C_j = \emptyset$ for all $j \neq i$ (note that X is a germ along curve C). Then $L \simeq \mathcal{O}(\sum_i (L \cdot C_i) H_i)$ by (1.3). By choosing different P_i ’s, one obtains a surjection $\mathcal{O}_X^{\oplus 2} \rightarrow L$ and hence $f^* f_* L \rightarrow L$. Since $\dim f^{-1}(Q) = 1$, one has $0 = R^1 f_* \mathcal{O}_X^{\oplus 2} \rightarrow R^1 f_* L$. \square

(1.5) **Corollary.** If C' is a curve contained in C , then the germ of X along C' is an extremal nbd.

Proof. Under the notation of proof of (1.4), the morphism associated to the base-point-free linear system $|\sum_{i \in I} H_i|$ contracts components of C' only, where $I = \{i \in I \mid C_i \not\subset C'\}$. \square

We want to study the subgroup of the divisor class group $\text{Cl } X$ consisting of Weil divisor classes D which are \mathbb{Q} -Cartier (i.e. nD is Cartier for some $n \in \mathbb{N}$). We call this the *semi-Cartier divisor class group* and denote it by $\text{Cl}^{\text{sc}} X$. Then we quote

(1.6) **Proposition** (Reid, Ue [Ka2, Lemma 5.1]). *If (V, P) is a 3-fold terminal singularity of index m , then $\text{Cl}^{\text{sc}}(V, P) \simeq \mathbb{Z}/(m)$ and it is generated by the class $[\omega_V]$ of ω_V .*

We note that (1.6) implies that if P is a terminal singular point of index m of an algebraic (or even formal) 3-fold V then $\mathbb{Z}/(m)[\omega_V]$ is equal to the torsion part of $\text{Cl}^{\mathcal{O}_{V,P}}$.

(1.7) **Corollary-Definition.** *Under the notation and assumptions of (1.6), let (W, P) be a smooth curve $\subset (V, P)$. Then*

(i) *for an arbitrary $\xi \in \text{Cl}^{\text{sc}}(V, P)$, there exists an effective divisor $D (\ni P)$ such that $[D] = \xi$ and $D \cap W = \{P\}$.*

(ii) $\xi \mapsto (mD \cdot C)/m$ *induces a well-defined homomorphism*

$$\text{cl}(W, P): \text{Cl}^{\text{sc}}(V, P) \rightarrow \frac{1}{m}\mathbb{Z}/\mathbb{Z} \subset \mathbb{Q}/\mathbb{Z}.$$

We say that $V \supset W$ is locally primitive at P (and P is a primitive point) if $\text{cl}(W, P)$ is an isomorphism and that an extremal nbd $X \supset C \simeq \mathbb{P}^1$ is locally primitive if it is so at each point of C . A point $P \in W$ is called imprimitive if $V \supset W$ is not primitive at P , and

(iii) *let*

$$\pi: (V^\sharp, P^\sharp) \rightarrow (V, P)$$

be the canonical cover (cf. [R2]) and $W^\sharp = \pi^{-1}(W)_{\text{red}}$. Then W^\sharp has exactly $|\text{Ker cl}(W, P)|$ irreducible components. In particular, $V \supset W$ is locally primitive iff W^\sharp is irreducible. We call $|\text{Ker cl}(W, P)|$ (resp. $m/|\text{Ker cl}(W, P)|$) the splitting degree (resp. subindex) of $V \supset W$ at P (or simply of P).

Proof. Let $I \subset \mathcal{O}_{V,P}$ be a divisorial ideal representing $-\xi$. Let ϕ be an element of $\text{Hom}(I, \mathcal{O}_V)$ which does not vanish along any irreducible components of (W, P) . Then the divisor D defined by ϕI satisfies the conditions in (i). Let D be an effective Weil divisor with class $[\omega_V]$ such that $D \cap W = \{P\}$ as in (i). Let $\phi \in \mathcal{O}_{V,P}$ be such that $(\phi) = mD$ and $n = (mD \cdot W)$. Then $\phi|_W = t^n$ for some coordinate t of (W, P) , and $|\text{Ker cl}(W, P)| = (m, n)$. One sees that $\mathcal{O}_{W^\sharp, P^\sharp} \supset \mathcal{O}_{W,P}[z]/(z^m - \phi|_W) = \mathbb{C}\{t\}[z]/(z^m - t^n)$ is birational because $V^\sharp \rightarrow \{(v, z) \in V \times \mathbb{C} \mid z^m = \phi\}$ is isomorphic above general points of W . Thus W^\sharp has (m, n) irreducible components. \square

(1.8) Let $X \supset C \simeq \mathbb{P}^1$ be an extremal nbd. Let H be an effective Cartier divisor on X such that $(H \cdot C) = 1$ and $H \cap C$ is a smooth point of X . Let $P_1, \dots, P_n \in C$ be all the points of X of indices > 1 (say, m_1, \dots, m_n) and D_1, \dots, D_n effective \mathbb{Q} -Cartier divisors such that $D_i \cap C = \{P_i\}$ and $[D_i]$ is a generator of $\text{Cl}^{\text{sc}}(X, P_i)$ for all i (1.7). Then these imply that the following natural 0-sequence is an exact sequence

$$(1.8.1) \quad 0 \rightarrow \text{Pic } X \rightarrow \text{Cl}^{\text{sc}} X \rightarrow \bigoplus_i \text{Cl}^{\text{sc}}(X, P_i) \rightarrow 0,$$

and one has

(1.9) **Corollary.** $\text{Cl}^{\text{sc}}(X)$ is generated by $[D_1], \dots, [D_n]$ and $[H]$, and $m_i[D_i] - (m_i D_i \cdot C)[H] = 0$ ($1 \leq i \leq n$) are the generating relations, and

(1.10) **Corollary.** Under the notation and assumptions of (1.8), the following are equivalent.

- (i) $(D \cdot C) = 1/m_1 \cdots m_n$ for some $D \in \text{Cl}^{\text{sc}}(X)$.
- (ii) $\text{Cl}^{\text{sc}}(X) \simeq \mathbb{Z}$.
- (iii) $\text{Cl}^{\text{sc}}(X)$ is torsion-free.
- (iv) $X \supset C$ is locally primitive (i.e. $(m_i, (m_i D_i \cdot C)) = 1$ for all i) and $(m_i, m_j) = 1$ for all i and j such that $i \neq j$.

Proof of (1.10). By $[\text{Cl}^{\text{sc}}(X) : \text{Pic } X] = m_1 \cdots m_n$, (i) implies (ii), and (ii) \Rightarrow (iii) is obvious. It remains to prove (iii) \Rightarrow (iv) and (iv) \Rightarrow (i). By (1.9), it is clear that $(m_i, (m_i D_i \cdot C)) = 1$ for all i if $\text{Cl}^{\text{sc}}(X)$ is torsion-free. Thus we may assume $(m_i, (m_i D_i \cdot C)) = 1$ for all i , to prove (iii) \Rightarrow (iv) and (iv) \Rightarrow (i). Let $k_i, n_i \in \mathbb{Z}$ be such that $m_i k_i + (m_i D_i \cdot C) n_i = 1$. Then $\delta_i = n_i [D_i] + k_i [H]$ ($i \in [1, n]$) and $[H]$ form a set of generators of $\text{Cl}^{\text{sc}}(X)$ such that $m_i \delta_i = [H]$, whence we see that the generating relations are $m_a \delta_a = [H]$ ($1 \leq a \leq n$). Then (iii) \Rightarrow (iv) is obvious by $m_i \delta_i = m_j \delta_j$. If m_1, \dots, m_n are pairwise prime, then $n_a = \prod_{i \neq a} m_i$ ($a \in [1, n]$) generate the unit ideal \mathbb{Z} and hence $(\prod m_i) \delta_a = n_a H_a$ ($a \in [1, n]$) generate $\text{Pic } X$, whence (iv) \Rightarrow (i). \square

(1.11) **Proposition-Definition.** Let X be a normal analytic space, D a Weil divisor and $d \in \mathbb{N}$ such that dD is a principal divisor (ϕ) for some meromorphic function ϕ on X . Let $\pi: X' \rightarrow X$ be the normalization of $\mathcal{O}_X[z]/(z^d - \phi)$ on X , which is a finite Galois μ_d -morphism. We call it the μ_d -cover associated to $[D, \phi]$ and denote it by $\pi[D, d, \phi]: X[D, d, \phi] \rightarrow X$ (we may use the divisor class of D instead of D and may omit ϕ in the expression if there is no danger of confusion). One has

- (i) $\pi[D|_U, d, \phi|_U] = \pi[D, d, \phi] \times_X U$ for open sets U of X ,
- (ii) the pull back of D by the finite morphism $\pi[D, d]$ is the principal divisor (z) ,
- (iii) if X is connected and if the exponential map $t \mapsto e^t$ induces a surjection $\exp: H^0(\mathcal{O}_X) \twoheadrightarrow H^0(\mathcal{O}_X^*)$, then the number of connected components of $X[D, d]$ is exactly $d/(\text{order of } [D] \text{ in } \text{Cl } X)$, and

- (iv) *the morphism $\pi[D, d]$ is étale in codim 1. Furthermore, the ramification index of $\pi[D, d]$ at points above $P \in X$ is the order of the class $(D, P) \in \text{Cl}(X, P)$ induced by D .*

Proof. (i) and (ii) are obvious. For (iii), let n be the order of $[D]$ in $\text{Cl}X$ and c be the number of connected components of $X[D, d]$. By the hypothesis, the d/n th root of ϕ is a meromorphic function on X because nD is a principal divisor. Hence $z^d - \phi$ has at least d/n prime factors, and it is clear that $c \geq d/n$. Let Y be a connected component of $X[D, d]$. Then $Y \rightarrow X$ is a Galois $\mu_{d/c}$ -morphism and the norm $N_{Y/X}(z)$ of z for Y/X satisfies $(N_{Y/X}(z)) = (d/c)D$, whence $d/c \geq n$. Thus $c = d/n$ and (iii) is done. (iv) follows from (iii) applied to a small contractible nbd of P . \square

(1.12) **Corollary.** *Let $X \supset C$ be an extremal nbd, $f: X \rightarrow (Y, Q)$ the total contraction, and $D \in \text{Cl}X$ be of order d . Then $X^\sharp = X[D, d] \supset C^\sharp = \pi[D, d]^{-1}(C)_{\text{red}}$ is an extremal nbd and $f^\sharp: X[D, d] \rightarrow (Y^\sharp, Q^\sharp) = Y[f_*D, d]$ is the total contraction. Using the notation of (1.8), we consider $\pi^\sharp = \pi[D, d]: X^\sharp \rightarrow X$ in two special cases when $C \simeq \mathbb{P}^1$ (cf. (1.10)).*

(1.12.1) *Case $d = (m_a, (m_a D_a \cdot C)) > 1$ for some $a \in [1, n]$ and $D = (m_a D_a - (m_a D_a \cdot C)H)/d$. In this case, P_a is an imprimitive point of splitting degree d , subindex $\bar{m} = m_a/d$, π^\sharp is étale over $X - \{P_a\}$, $\pi^{\sharp-1}(P_a)$ is one point P_a^\sharp of index \bar{m} , C^\sharp is a union of d \mathbb{P}^1 's meeting only at P_a^\sharp , and each irreducible component is locally primitive at P_a^\sharp . In this case, we call the extremal nbd $X^\sharp \supset C^\sharp$ the splitting cover of $X \supset C$ associated to P_a , and*

(1.12.2) *Case $X \supset C$ is locally primitive, $d = (m_a, m_b) > 1$ for some distinct $a, b \in [1, n]$, and $D = \alpha D_a + \beta D_b + \gamma H$, where $\alpha, \beta, \gamma \in \mathbb{Z}$ satisfy $(\alpha, m_a) = m_a/d$ and $(\beta, m_b) = m_b/d$ (cf. proof of (1.10), (iii) \Rightarrow (iv)). In this case, π^\sharp is étale over $X - \{P_a, P_b\}$, $\pi^{\sharp-1}(P_a)$ (resp. $\pi^{\sharp-1}(P_b)$) is one point P_a^\sharp (resp. P_b^\sharp) of index m_a/d (resp. m_b/d), and $C^\sharp \simeq \mathbb{P}^1$.*

(1.12.3) *Remark.* In (1.12), it is easy to see that one can choose D and d as in (1.12.1) or (1.12.2) if $\text{Cl}^{\text{sc}}(X)$ has a nonzero torsion (cf. (1.10)).

Proof. We note that $\text{exp}: H^0(\mathcal{O}_X) = H^0(\mathcal{O}_Y) \rightarrow H^0(\mathcal{O}_{X^*}) = H^0(\mathcal{O}_{Y^*})$ is surjective since (Y, Q) is a germ. Since D and f_*D are of order d , X^\sharp and Y^\sharp are connected by (1.11(iii)), whence there is only one point $Q^\sharp \in Y^\sharp$ over Q . By construction, f^\sharp is bimeromorphic and étale over $Y^\sharp - \{Q^\sharp\}$, whence f^\sharp is isomorphic outside $C^\sharp = f^{\sharp-1}(Q^\sharp)$. Since (Y^\sharp, Q^\sharp) is normal, C^\sharp is connected (ZMT). Since π^\sharp is étale in codimension 1, one sees $\pi^{\sharp*}K_X = K_{X^\sharp}$, whence $X^\sharp \supset C^\sharp$ is an extremal nbd and f^\sharp is the total contraction.

In (1.12.1), D is Cartier outside $\{P_a\}$, and of order d at P_a . Thus (1.11(iv)) implies the first part and $\pi_1(C - P_a) = 0$ implies the second part. The

last assertion follows because (X^\sharp, P_a^\sharp) and (X, P_a) have isomorphic canonical covers.

In (1.12.2), the proof is the same except for the assertion $C^\sharp \simeq \mathbb{P}^1$. If C^\sharp is reducible, then each irreducible component meets some component at P_a and some others at P_b , because π^\sharp is Galois. Then C^\sharp contains a cycle of \mathbb{P}^1 's, whence $H^1(\mathcal{O}_{C^\sharp}) \neq 0$, which contradicts (1.3(i)). Thus $C^\sharp \simeq \mathbb{P}^1$ (1.3(i)). \square

(1.13) **Corollary.** *Let $X \supset C \simeq \mathbb{P}^1$ be an extremal nbd. Then $X \supset C$ does not have more than one imprimitive point. If it has an imprimitive point P , then the splitting cover $X^\sharp \supset C^\sharp$ of $X \supset C$ associated to P is locally primitive along an arbitrary irreducible component of C^\sharp .*

Proof. Assume that $X \supset C$ has two imprimitive points P, P' ($P \neq P'$) with splitting degrees d, d' (> 1), respectively. Let D (resp. D') be the divisor as given in (1.12.1) for P (resp. P'). Let $X^\sharp \supset C^\sharp$ (resp. $X^b \supset C^b$) be the extremal nbd obtained from D (resp. D') as in (1.12). Let $\pi^\sharp = \pi[D, d]: X^\sharp \rightarrow X$. Since D' has order d' globally and also at P' , $\pi^{\sharp*}D'$ has order d' on X^\sharp because π^\sharp is étale over $X - \{P\}$ (1.12.1). Hence $X^\dagger = X^\sharp[\pi^{\sharp*}D', d'] = X^\sharp \times_X X^b \supset C^\dagger = C^\sharp \times_X C^b$ is an extremal nbd. By (1.12.1), C^\dagger is a Galois $\mu_d \times \mu_{d'}$ -cover of \mathbb{P}^1 with dd' irreducible components such that each component meets $d - 1$ (resp. $d' - 1$) other components at every point above P (resp. P'). Thus C^\dagger contains a cycle of \mathbb{P}^1 's and $H^1(\mathcal{O}_{C^\dagger}) \neq 0$, which contradicts (1.3(i)). Hence $X \supset C$ has at most 1 imprimitive point, and the last assertion follows from (1.12.1). \square

Let $X \supset C$ be an extremal nbd and $D (\subset C)$ an arbitrary reduced curve. Let $\text{gr}_D^0 \omega$ be $\omega_X \otimes \mathcal{O}_D$ modulo the maximal subsheaf of finite length (which is compatible with the definition in (2.2)).

(1.14) **Proposition.** *Let $X \supset C$ be an extremal nbd. Then*

- (i) $\text{gr}_D^0 \omega \simeq \mathcal{O}_D(-1)$ for each irreducible component $D (\simeq \mathbb{P}^1)$ of C , and
- (ii) if $C(1), \dots, C(r)$ are the irreducible components of C , then

$$\text{gr}_C^0 \omega \simeq \bigoplus_{1 \leq i \leq r} \text{gr}_{C(i)}^0 \omega.$$

Proof. (i) Let $m \in \mathbb{N}$ be such that mK_X is a Cartier divisor. Then $\omega_X^{\otimes m} \rightarrow \mathcal{O}(mK_X) \otimes \mathcal{O}_D$ induces $(\omega_X \otimes \mathcal{O}_D)^{\otimes m} \rightarrow (\text{gr}_D^0 \omega)^{\otimes m} \hookrightarrow \mathcal{O}_D(mK_X)$. Thus $\text{deg gr}_D^0 \omega < 0$ by $(K_X \cdot D) < 0$. On the other hand, one has $H^1(\text{gr}_D^0 \omega) = 0$ by (1.2.1), whence $\text{gr}_D^0 \omega \simeq \mathcal{O}_D(-1)$.

(ii) By construction, we have an injection

$$\text{gr}_C^0 \omega \rightarrow \bigoplus_{1 \leq i \leq r} \text{gr}_{C(i)}^0 \omega$$

whose cokernel F is of finite length. Since $H^1(\text{gr}_C^0 \omega) = 0$ (1.2.1), we have $0 \leq \chi(\text{gr}_C^0 \omega) = -\text{length}(F)$ by (i). Thus $F = 0$. \square

By (1.14(ii)), we have

(1.15) **Corollary.** *Let $X \supset C$ be an extremal nbd such that two irreducible components of C meet at P . Then P has index > 1 .*

(1.16) **Corollary.** *Let $X \supset C \simeq \mathbb{P}^1$ be an extremal nbd and P an imprimitive point. Then P has subindex > 1 .*

Indeed, we immediately see (1.16) by applying (1.15) to the splitting cover of $X \supset C$ associated to P .

1a (APPENDIX 1a). RESULTS ON 3-FOLD TERMINAL SINGULARITIES

In Appendix 1a, we list only some of the necessary results and definitions on 3-fold terminal singularities for convenience of quotation. We refer the reader to [Mr3], or [R4] for more polished treatments.

We start by quoting a general

(1a.1) **Theorem** ([Fl, El2, SB]). *Canonical (hence terminal) singularities are rational in all dimensions.*

Let (X, P) be a 3-fold terminal singularity of index $m \geq 1$, and let $\pi^\sharp: (X^\sharp, P^\sharp) \rightarrow (X, P)$ be the canonical cover, which is a Galois μ_m -cover. Then (X^\sharp, P^\sharp) is a 3-fold terminal singularity of index 1 [R1, (3.1)], and the μ_m -action on $X^\sharp - \{P^\sharp\}$ is free because the singularity of (X, P) is isolated. We will fix the meaning of these symbols.

The essential result is

(1a.2) **Definition-Theorem** (Reid [R1]). *A 3-fold hypersurface singularity (Y, Q) is a cDV point if a general hyperplane section H through Q has at most a rational double point at Q . Depending on the type of H , (Y, Q) is said to be of type cA , cD , or cE . Then*

- (i) *the canonical cover (X^\sharp, P^\sharp) introduced above is an isolated cDV point. Depending on the type of (Y^\sharp, P^\sharp) , (X, P) is said to be of type cA , cD , or cE , and*
- (ii) *a cDV point (Y, Q) is a canonical singularity, and if it is isolated then it is a terminal singularity.*

We note that (X, P) is of cA type iff (X^\sharp, P^\sharp) is smooth (cA_0 type) or the quadratic part of the defining equation of the normal hypersurface singularity (X^\sharp, P^\sharp) has rank ≥ 2 as a quadratic form.

From now on we assume that $m > 1$. Then the μ_m -action on (X^\sharp, P^\sharp) is to be analyzed. We fix a character χ generating $X(\mu_m) = \text{Hom}(\mu_m, \mathbb{C}^*) = \mathbb{Z}_m$. For a μ_m -semi-invariant v , we write as $\text{wt } v \equiv a \pmod{m}$ if $g(v) = \chi(g)^a \cdot v$ for all $g \in \mu_m$. For a sequence of semi-invariants $v = (v_1, \dots, v_r)$, we write as

$\text{wt } v \equiv (a_1, \dots, a_r) \pmod{m}$ if $\text{wt } v_i \equiv a_i \pmod{m}$ for all i . (We note that χ will be chosen to be an ℓ -character (2.5) in the text.)

The combinatorially hardest cA_0 -singularity is analyzed by

(1a.3) **Theorem** (Danilov [Da], Morrison and Stevens [MS]). *If (X^τ, P^τ) is smooth, it is μ_m -isomorphic to $(x\text{-space}, 0)$ such that*

$$\text{wt } x \equiv (a, b, -a) \pmod{m}$$

for some $a, b \in \mathbb{Z}$ prime to m . Conversely $(x\text{-space}, 0)/\mu_m$ for every such action is terminal.

Other cases are combinatorially easier, though one has to take the defining equation into account.

(1a.4) **Theorem** ([Mr3, Theorems 12, 23, 25]). *If (X^τ, P^τ) is singular, it is μ_m -isomorphic to a hypersurface $\phi = 0$ in $(x\text{-space}, 0)$ such that*

- (i) $\text{wt}(x, \phi) \equiv (a, b, -a, 0, 0) \pmod{m}$, or
- (ii) $m = 4$, and $\text{wt}(x, \phi) \equiv (a, b, -a, 2, 2) \pmod{m}$ for some $a, b \in \mathbb{Z}$ prime to m .

We introduce the following notion to give a unified treatment of these three cases in (1a.3) and (1a.4).

(1a.5) **Definition-Corollary.** *Let us consider μ_m -equivariant embeddings*

$$e_x : (X^\tau, P^\tau) \xrightarrow{\sim} \{\phi = 0\} \subset (x\text{-space}, 0) = (\mathbb{A}_x^4, 0)$$

such that $x = (x_1, \dots, x_4)$ and ϕ are μ_m -semi-invariants and $\text{wt } x_i \equiv 0 \pmod{m}$ for some i if (X^τ, P^τ) is smooth. We call such x an ℓ -coordinate system and $\phi \in \mathbb{C}\{x\}$ an ℓ -equation of (X, P) . For an ℓ -coordinate system x , we call the set of points in \mathbb{A}_x^4 at which the μ_m -action is not free the fixed axis and denote it by F_x . Then

- (i) given an ℓ -coordinate system x , F_x is the x_i -axis for some i . Thus the name of fixed axis is justified, and
- (ii) given two ℓ -coordinate systems x and y , there exists a μ_m -isomorphism

$$c_{x,y} : (\mathbb{A}_y^4, 0) \xrightarrow{\sim} (\mathbb{A}_x^4, 0)$$

such that $c_{x,y} \circ e_y = e_x$ and hence $c_{x,y}(F_y) = F_x$. Thus the fixed axis may be denoted simply by F without causing confusion, whence

- (iii) given an ℓ -coordinate system x and an ℓ -equation ϕ , the multiplicity of 0 in the equation $\phi|_F = 0$ makes sense and is independent of choice of x and ϕ . We call it the axial multiplicity of (X, P) .

We note that this axial multiplicity coincides with Morrison’s weight [Mrr2] when the singularity is not “exceptional,” that is if we are not in case (1a.4(ii)).

Proof. Indeed (i) follows from (1a.3) and (1a.4) because there exists exactly one i such that $\text{wt } x_i$ is not prime to m , and (iii) immediately follows from (i) and (ii). Let x and y be two ℓ -coordinate systems. Then modulo permutation of y_i 's, we may assume that $\text{wt } x \equiv \text{wt } y \pmod{m}$ and that $\text{wt } x_4 \equiv \text{wt } y_4 \equiv 0 \pmod{m}$ if (X^\sharp, P^\sharp) is smooth. We treat two cases for (ii).

(1a.5.1) *Case (X^\sharp, P^\sharp) is smooth.* We see by (1a.3) that $\mathcal{O}_{X^\sharp, P^\sharp} = \mathbb{C}\{x'\} = \mathbb{C}\{y_1, y_2, y_3\}$, and there exist semi-invariant functions $f = (f_1, \dots, f_4) \in \mathbb{C}\{x'\}$ such that $f_i(x') + \delta_{i,4}x_4 = y_i$, where $x' = (x_1, x_2, x_3)$, and $\delta_{a,b}$ is Kronecker's delta. Then we set $c_{y,x}(x) = f(x') + (0, 0, 0, x_4)$. Since $\text{wt } f_4 \equiv 0 \pmod{m}$ and $(\text{wt } x_i, m) = 1$ for all $i \in [1, 3]$, we see $f_4 \in (x')^2$ by $m > 1$. Since $\partial(f_1, f_2, f_3)/\partial x'(0) \neq 0$, we see that $c_{y,x}$ is a μ_m -isomorphism.

(1a.5.2) *Case (X^\sharp, P^\sharp) is singular.* Both x and y are minimal sets of generators of the maximal ideal of $\mathcal{O}_{X^\sharp, P^\sharp}$. Thus there exist semi-invariant functions $f = (f_1, \dots, f_4) \in \mathbb{C}\{x\}$ such that $f_i(x) = y_i$ in $\mathcal{O}_{X^\sharp, P^\sharp}$. Then $\det \partial f / \partial x(0) \neq 0$, and we set $c_{y,x}(x) = f(x)$. It is clear that this is a μ_m -isomorphism. \square

By (1a.3) and [Mr3, Theorems 12, 23, 25], we have

(1a.6) **Theorem.** *Let x and ϕ be an ℓ -coordinate system and an ℓ -equation of (X, P) . Then modulo permutation of x_i 's we may assume that*

$$(*) \quad (\text{wt } x_i, m) = 1 \quad (i \in [1, 3]), \quad \text{wt } x_1 x_3 \equiv 0, \quad \text{wt } \phi \equiv \text{wt } x_4(m),$$

and we have two cases: (i) $\text{wt } \phi \equiv 0 \pmod{m}$, or (ii) $m = 4$ and $\text{wt } \phi \equiv 2 \pmod{4}$. Assuming $\phi \in (x)^2$, we have one of the following.

(i) *Case $\text{wt } \phi \equiv 0 \pmod{m}$.*

(i.a) *P is a cA -point, and up to μ_m -change of ℓ -coordinate system and multiplication of ϕ by invariant units, we have*

$$\phi = x_1 x_3 + f(x_2^m, x_4).$$

(i.b) *$m = 3$, P is a cD -point, $\text{wt } x \equiv \pm(1, 1, 2, 0) \pmod{3}$ modulo permutation, and x_4^2 and x_3^3 must appear in ϕ (i.e. $\partial^2 \phi / \partial x_4^2(0) \neq 0$ and $\partial^3 \phi / \partial x_3^3(0) \neq 0$).*

(i.c) *$m = 2$, P is a cA , cD , or cE_6 -point, $\text{wt } x \equiv (1, 1, 1, 0) \pmod{2}$ (cf. [Mr3, Theorems 23, 25] for details).*

(ii) *Case $m = 4$ and $\text{wt } \phi \equiv 2 \pmod{4}$. P is a cA -point, $\text{wt } x \equiv \pm(1, 1, 3, 2) \pmod{4}$ modulo permutation of x_i 's, and then x_3^2 must appear in ϕ (i.e. $\partial^2 \phi / \partial x_3^2(0) \neq 0$).*

We note that (1a.6) is only a necessary condition. Sufficient conditions are found in [Mr3, Remarks 12.1, 23.1, 25.1] and [KSB]. We quote only what we need (cf. (10.7)–(10.9)).

(1a.7) **Theorem.** Let μ_m act diagonally on x -space with $x = (x_1, \dots, x_4)$ and $\phi \in (x)^2$ a μ_m -semi-invariant so that $(\text{wt } x_i, m) = 1$ for $i \in [1, 3]$ and $\text{wt } x_4 \equiv \text{wt } \phi (m)$. Let $(X, P) = (\{\phi = 0\}, 0) / \mu_m$. Then

- (i) ([KSB, (6.7)], cf. also (10.7)–(10.9)) if $\text{wt } x_4 \equiv 0 (m)$ and the quadratic part g of $\phi(x_1, x_2, x_3, 0)$ has rank ≥ 2 as a quadratic form and if (X, P) has an isolated singularity, then (X, P) is terminal and of index m , and
- (ii) ([Mr3, Remark 12.1]) if $m = 4$ and $\text{wt}(x, \phi) \equiv \pm(1, 1, 3, 2, 2) \pmod 4$ and if $\phi = x_1x_2 + x_3^2 + x_4^{2n+1}$ ($n \in \mathbb{N}$), then (X, P) is a terminal singularity of index 4 as in (1a.6(ii)).

Idea of proof. Since (ii) is stated in [Mr3, Remark 12.11], we only comment on how to reduce (i) to a result in [KSB, (6.7)].

First we show, modulo permutation of x_1, x_2 , and x_3 and μ_m -change of $x_1x_2x_3$ -space, $g(x_1, x_2, 0, 0) = x_1x_2$ in two cases.

Case 1. $m = 2$: Since $\text{wt } x_i \equiv 1 (2)$ for $i \in [1, 3]$, we may assume $g = x_1x_2 + \lambda x_3^2$ for some $\lambda \in \mathbb{C}$ modulo a linear change of x_1, x_2 , and x_3 , which is a μ_m -change of coordinates.

Case 2. $m > 2$: Since $(\text{wt } x_i, m) = 1$ for $i \in [1, 3]$, we see no x_i^2 appear in g and that at most two of x_1x_2, x_2x_3 , and x_3x_1 appear in g . Modulo a permutation of x_1, x_2 , and x_3 , we may write $g = \mu x_1x_2 + \nu x_1x_3$ ($\mu, \nu \in \mathbb{C}, \mu \neq 0$). We note that $\text{wt } x_2 \equiv \text{wt } x_3$ if $\nu \neq 0$. Thus $(x_1, x_2, x_3) \mapsto (x_1, \bar{x}_2, x_3)$ with $\bar{x}_2 = \mu x_2 + \nu x_3$ is a μ_m -change of coordinates, and $g(\bar{x}) = x_1\bar{x}_2$.

Then one can show that one can write

$$\phi(x_1, x_2, x_3, x_4) = x_1x_2 + f(x_3^m, x_4)$$

modulo a μ_m -change of coordinates by approximation for some convergent power series f [A1]. Now [KSB, (6.7)] shows that (X, P) is a terminal singularity of index m . \square

1b (APPENDIX 1b). DEFORMATION OF EXTREMAL NBDS

In Appendix 1b, we will study how to construct a small deformation of an extremal nbd and obtain a nearby extremal nbd from it in three steps: local process, local-to-global process, and global process. We should note that only the local description (1b.3) is needed for many arguments in the text but the explicit construction in (1b.8.1) is also needed for some delicate arguments (e.g. in §9). The formal scheme case is treated by [Wa].

(1b.1) Let $X \supset C \simeq \mathbb{P}^1$ be an extremal nbd. Let $P \in C$ be a point of X of index m and $\pi^\sharp: (X^\sharp, P^\sharp) \rightarrow (X, P)$ the canonical cover and $C^\sharp = \pi^{\sharp^{-1}}(C)_{\text{red}}$. Fix an embedding $(X, P) \subset (\mathbb{C}^N, P)$ and a μ_m -equivariant embedding $(X^\sharp, P^\sharp) \subset (\mathbb{C}^4, 0)$.

The following are the three steps to construct a small deformation of extremal nbd $X \supset C \simeq \mathbb{P}^1$ with parameter λ .

(L) Construct a small embedded μ_m -equivariant deformation of the germs

$$(X^\sharp, P^\sharp) \supset (C^\sharp, P^\sharp)$$

in (\mathbb{C}^4, P) which, if $m > 1$, satisfies (1b.3(i)) and one of (1b.3(ii)), (1b.5(ii')) or (1b.5(ii'')).

As the μ_m -quotients, one has a small embedded deformation $X_{P,\lambda} \supset C_{P,\lambda}$ of $(X, P) \supset C$ in $(\mathbb{C}^N, P) = (\mathbb{C}^4, P)/\mu_m$ with only terminal singularities, by (1b.3) and (1b.5) if $m > 1$, and by Elkik's Theorem [El1] if $m = 1$. Then

(LG) extend the deformation $X_{P,\lambda} \supset C_{P,\lambda}$ of the germs $(X, P) \supset (C, P)$ to a deformation $X_\lambda \supset C_\lambda$ of $X \supset C$, which is trivial outside a small nbd of P (1b.8).

If $X_\lambda \supset C_\lambda$ is a small deformation of $X \supset C \simeq \mathbb{P}^1$, then

(G) X_λ is a modification of some Stein variety if $0 < |\lambda| \ll 1$ by Grauert's theorem [Gra]. Thus we have a proper bimeromorphic morphism $f_\lambda: X_\lambda \rightarrow Y_\lambda$ so that C_λ is a part of the exceptional set E_λ , and f_λ is the total contraction of the extremal nbd $X_\lambda \supset E_\lambda$ by (1b.10). Thus by (1.5), the germ $X_\lambda^\circ \supset C_\lambda$ of $X_\lambda \supset C_\lambda$ along C_λ is an extremal nbd if $0 < |\lambda| \ll 1$.

We call such an extremal nbd a *nearby* extremal nbd of $X \supset C$.

(1b.2) Let $(X, 0)$ be a 3-fold terminal singularity of index $m > 1$, and $(X^\sharp, 0)$ the μ_m -canonical cover. Let $(x) = (x_1, \dots, x_4)$ and ϕ be an ℓ -coordinate system and an ℓ -equation of (X, P) , and let $F^\sharp \subset x$ -space be the fixed axis (1a.5).

(1b.3) **Proposition.** *Let ψ be a semi-invariant $\in (x)\mathbb{C}\{x\}$ such that $\text{wt } \psi \equiv \text{wt } \phi \pmod{m}$ and $\{\phi = \psi = 0\}$ is a surface. Let $X_\lambda^\sharp = \{\phi + \lambda\psi = 0\}$, $(X_\lambda, 0) = (X_\lambda^\sharp, 0)/\mu_m$ ($\lambda \in \mathbb{C}$), $S^\sharp = \text{Sing}\{\phi = \psi = 0\}$, and let $S, F_\lambda \subset X_\lambda$ the images of $S^\sharp, F^\sharp \cap X_\lambda^\sharp$, respectively. Assume that*

- (i) $S^\sharp \cap F^\sharp \subset \{0\}$, and
- (ii) $(X_\lambda, 0)$ is terminal if $0 < |\lambda| \ll 1$.

For $\varepsilon > 0$ and $\lambda \in \mathbb{C}$ such that $1 \gg \varepsilon \gg |\lambda|$, let

$$X_{\lambda,\varepsilon}^\sharp = \{(x) \in X_\lambda^\sharp \mid |x_i| < \varepsilon \forall i\} \quad \text{and} \quad X_{\lambda,\varepsilon} = X_{\lambda,\varepsilon}^\sharp / \mu_m.$$

Then $X_{\lambda,\varepsilon}$ has only terminal singularities; $X_{\lambda,\varepsilon} - (S \cup F_\lambda)$ is smooth; $X_{\lambda,\varepsilon} \cap S - F_\lambda$ are Gorenstein points; $X_{\lambda,\varepsilon} \cap F_\lambda - (S \cup \{0\})$ consists of cyclic quotient singular points of the same index m' which is a factor of m ; 0 is a point of index m .

We note that $m' \neq m$ in (1b.3) only if (X, P) is exceptional, i.e. in case (1a.6(ii)), where $m = 4$ and $m' = 2$.

Remark (1b.3.1). Under the notation and assumptions of (1b.3), let $(C^\sharp, 0) \subset (X^\sharp, 0)$ be a μ_m -stable reduced curve such that $(C, 0) = (C^\sharp, 0)/\mu_m$ is smooth

and $(C^\sharp, 0)$ μ_m -equivariantly deforms with $(X^\sharp, 0)$ in \mathbb{C}^4 so that $C_\lambda^\sharp \subset X_\lambda^\sharp$. Then for $\varepsilon \ll 1$ and $P_\lambda \in C_{\lambda, \varepsilon} = (C_\lambda^\sharp \cap X_{\lambda, \varepsilon}^\sharp) / \mu_m$, $X_{\lambda, \varepsilon} \supset C_{\lambda, \varepsilon}$ is locally primitive at P_λ iff $(C_\lambda^\sharp, P_\lambda^\sharp)$ is irreducible, where $P_\lambda^\sharp \in C_\lambda^\sharp$ is a point above P_λ .

Indeed this follows from (1.7(iii)), because $(X_{\lambda, \varepsilon}^\sharp, P_\lambda^\sharp) \rightarrow (X_{\lambda, \varepsilon}, P_\lambda)$ is the canonical cover and $(C_\lambda^\sharp, P_\lambda^\sharp)$ is the preimage of C_λ .

We recall a special case of Bertini's Theorem.

(1b.4) **Lemma (Bertini).** *Let ϕ and ψ be convergent power series in x_1, \dots, x_n , which are prime to each other. Then $\{\phi + \lambda\psi = 0\} \cap B_\varepsilon$ is smooth outside the singular locus of $\{\phi = \psi = 0\} \cap B_\varepsilon$ if $0 < |\lambda| \ll \varepsilon \ll 1$, where $B_\varepsilon = \{(x) \mid |x_i| < \varepsilon (\forall i)\}$.*

Proof. Let $(x, \lambda) = (x(t), \lambda(t))$ ($|t| \ll 1$) be a parametrized curve with parameter t such that $(x(0), \lambda(0)) = 0$ and $x(t)$ is a singular point of $\phi + \lambda(t)\psi = 0$ and $\lambda(t) \neq 0$. It is enough to show that $x(t)$ is a singular point of $\{\phi = \psi = 0\}$ for all t . By the hypothesis, one has

$$(*) \quad \frac{\partial \phi}{\partial x_i}(x(t)) + \lambda(t) \frac{\partial \psi}{\partial x_i}(x(t)) = 0 \quad (i \in [1, n]).$$

From $\phi(x(t)) + \lambda(t)\psi(x(t)) = 0$, one has

$$\sum_i \left(\frac{\partial \phi}{\partial x_i} + \lambda \frac{\partial \psi}{\partial x_i} \right) \frac{dx_i}{dt} + \frac{d\lambda}{dt} \psi = 0,$$

whence $(d\lambda/dt)\psi(x(t)) = 0$ and $\psi(x(t)) = 0$ by $\lambda(t) \neq \text{const.}$ Thus one has $\phi(x(t)) = \psi(x(t)) = 0$, and $(*)$ above implies

$$\text{rank} \left(\frac{\partial(\phi, \psi)}{\partial x} \Big|_{x=x(t)} \right) \leq 1$$

which means that $x(t)$ is a singular point of $\{\phi = \psi = 0\}$ by Jacobian criterion. \square

Proof of (1b.3). We may assume $\text{wt } x_4 \equiv \text{wt } \phi, \text{ wt } x_1 x_2 \equiv 0 \pmod{m}, (\text{wt } x_4, m) \neq 1$, and $\text{wt } x_1, \text{ wt } x_2, \text{ wt } x_3$ are prime to m (modulo a permutation of x_1, \dots, x_4), so that $F^\sharp = x_4$ -axis (1a.6). Since the μ_m -action on $X^\sharp - \{0\}$ is free, $\phi|_{F^\sharp} \neq 0$. By (1b.4), one sees that $(\phi + \lambda\psi)|_{F^\sharp} = 0$ has no multiple roots x_4 such that $0 < |x_4| \ll 1$ if $|\lambda| \ll 1$. Let $\varepsilon \in \mathbb{R}$ be small enough and $B_\varepsilon = \{(x_1, \dots, x_4) \mid |x_1|, \dots, |x_4| < \varepsilon\}$. Then

- (1) $X_{\lambda, \varepsilon}^\sharp = X_\lambda^\sharp \cap B_\varepsilon$ is smooth in $B_\varepsilon - S^\sharp$,
- (2) $X_{\lambda, \varepsilon}^\sharp$ has only isolated rational singularities, and
- (3) $X_{\lambda, \varepsilon}^\sharp$ intersects transversally with F^\sharp in $B_\varepsilon - \{0\}$,

if $|\lambda| \ll \varepsilon \ll 1$, where (1) follows from (1b.4), and (2) follows from [E11]. Let $\pi_\lambda^\sharp: X_{\lambda, \varepsilon}^\sharp \rightarrow X_{\lambda, \varepsilon}$ be the quotient map. Let $Q^\sharp \in X_{\lambda, \varepsilon}^\sharp - \{0\}$ for λ such that $|\lambda| \ll \varepsilon$. If $Q^\sharp \notin F^\sharp$, then π_λ^\sharp is étale at Q^\sharp and $X_{\lambda, \varepsilon}$ is smooth at $\pi_\lambda^\sharp(Q^\sharp)$

by (1) if $Q^\sharp \notin F^\sharp \cup S^\sharp$, and has Gorenstein terminal singularity at $\pi_\lambda^\sharp(Q^\sharp)$ by (2) if $Q^\sharp \in S^\sharp - F^\sharp$. If $Q^\sharp \in F^\sharp - \{0\}$, then $x_1(Q^\sharp) = x_2(Q^\sharp) = x_3(Q^\sharp) = 0$ and x_1, x_2, x_3 form a G_{Q^\sharp} -semi-invariant coordinate system for $(X_{\lambda,\varepsilon}^\sharp, Q^\sharp)$ by (3), where $G_{Q^\sharp} \subset \mu_m$ is the stabilizer at Q^\sharp and $|G_{Q^\sharp}| = (\text{wt } x_4, m)$. Since $\text{wt } x_1 x_2 \equiv 0 \pmod{m}$ and $\text{wt } x_1, \text{wt } x_2, \text{wt } x_3$ are prime to m , $X_{\lambda,\varepsilon}^\sharp$ has at most a terminal cyclic quotient singularity at $\pi_\lambda^\sharp(Q^\sharp)$ by (1a.3). \square

(1b.5) **Corollary.** *In (1b.3) above, (ii) follows from either one of the following:*

- (ii') $(\phi, \psi) \notin (x_1, \dots, x_4)^2$,
- (ii'') $(\phi, \psi) \in (x_1, \dots, x_4)^2$, $\text{wt } x_4 \equiv 0 \pmod{m}$, $(\phi + \lambda\psi)(x_1, x_2, x_3, 0)$ has a quadratic part of rank ≥ 2 in x_1, x_2, x_3 if $0 < |\lambda| \ll 1$.

Proof. By (1a.3), (ii') implies (ii). Let us assume (ii''). By (1a.4(i)), $(\text{wt } x_i, m) = 1$ for $i \in [1, 3]$. Since $(X, 0)$ has an isolated singularity, $(X_\lambda, 0)$ has an isolated singularity if $|\lambda| \ll 1$. Then such $(X_\lambda, 0)$ is terminal by (1a.7(i)). \square

(1b.6) **Proposition.** *Let $\mathcal{X} \supset \mathcal{E}$ be analytic subspaces of germ $(\mathbb{C}^N \times \mathbb{C}_\lambda^1, P \times 0)$ flat over $(\mathbb{C}_\lambda^1, 0)$ such that $C = \mathcal{E} \cap (\mathbb{C}^N \times 0) \subset \mathbb{C}^N$ is a smooth curve. Then there is a biholomorphic automorphism σ of $(\mathbb{C}^N \times \mathbb{C}_\lambda^1, P \times 0)$ over $(\mathbb{C}_\lambda^1, 0)$ such that*

$$\sigma|_{\mathbb{C}^N \times 0} = \text{id} \quad \text{and} \quad \sigma(\mathcal{E}) = C \times \mathbb{C}_\lambda^1.$$

Proof. We choose coordinates (x_1, \dots, x_N) of (\mathbb{C}^N, P) such that $x(P) = 0$ and $C = x_1$ -axis. Then by flatness $\mathcal{E} \subset \mathbb{C}^N \times \mathbb{C}_\lambda^1$ is defined by $x_i - \lambda f_i(x, \lambda) = 0$ ($i \in [2, N]$) for some convergent power series f_i . We set

$$\sigma(x, \lambda) = (x_1, x_2 - \lambda f_2, \dots, x_N - \lambda f_N, \lambda). \quad \square$$

Let $y = (y_1, \dots, y_N)$, and let $(X, P) \subset (ty\text{-space}, 0) \simeq (\mathbb{C}^{N+1}, 0)$ be the germ of a pure r -dimensional analytic subspace such that $X \supset C = t$ -axis and $(\text{Sing } X) \cap C \subset \{P\}$. Then $I_{C,P}/I_{C,P}^2$ is a rank $(r - 1)\mathcal{O}_{C,P}$ -module, where $I_{C,P} \subset \mathcal{O}_{X,P}$ is the defining ideal of C in (X, P) . Let $(\mathcal{X}, P \times 0) \subset (ty\lambda\text{-space}, 0) \simeq (\mathbb{C}^{N+2}, 0)$ be the germ of an analytic subspace defined by ideal $J \subset \mathbb{C}\{t, y, \lambda\}$ such that the projection $f: (\mathcal{X}, P \times 0) \rightarrow (\lambda\text{-line}, 0)$ is flat, $X = f^{-1}(0)$, and $\mathcal{X} \supset \mathcal{E} = t\lambda$ -plane. Then

$$\frac{(y)}{\lambda(y) + (y)^2 + J} \xrightarrow{\sim} \frac{(y) + (\lambda)}{(\lambda) + (y)^2 + J} \simeq I_{C,P}/I_{C,P}^2 \quad (\text{isomorphism theorem}).$$

Here and in (1b.7), ideals (e.g. (y)) mean ideals in $\mathbb{C}\{t, y, \lambda\}$ unless otherwise denoted.

(1b.7) **Proposition.** *Under the above notation and assumptions, let $u = (u_1, \dots, u_{r-1})$ be elements of the ideal (y) such that $I_{C,P}/I_{C,P}^2 + (u)\mathcal{O}_{X,P}$ is of finite length. Then for small enough $\alpha > \beta > 0$, the natural map*

$$\mathcal{X} \left(\begin{matrix} I \times \gamma \times \delta \\ u < \delta \end{matrix} \right) = \left\{ \begin{matrix} (t, y, \lambda) \in \mathcal{X}; \\ |t| \in I, |y| < \gamma \\ |u| < \delta, |\lambda| < \delta \end{matrix} \right\} \rightarrow \mathcal{U} \left(\begin{matrix} I \times \delta \times \delta \\ u \end{matrix} \right) = \left\{ \begin{matrix} (t, u, \lambda); \\ |t| \in I \\ |u|, |\lambda| < \delta \end{matrix} \right\}$$

is biholomorphic for all $\gamma, \delta \in \mathbb{R}$ such that $0 < \delta \ll \gamma \ll \beta$, where $|y| = \text{Max}_i |y_i|$, $|u| = \text{Max}_j |u_j|$, and I is the interval (β, α) .

Proof. By the natural isomorphisms

$$\begin{aligned} \frac{J + \lambda(y) + (y)^2}{\lambda(y) + (y)^2} &\simeq \text{Ker} \left(\frac{y}{\lambda(y) + (y)^2} \rightarrow \frac{(y) + (\lambda)}{(\lambda) + (y)^2 + J} \right) \\ &\simeq \text{Ker} \left(\bigoplus_{1 \leq i \leq N} \mathcal{O}_{C,P} y_i \rightarrow I_{C,P}/I_{C,P}^2 \right), \end{aligned}$$

one sees that $J + \lambda(y) + (y)^2/\lambda(y) + (y)^2$ is a free $\mathcal{O}_{C,P}$ -module of rank $N - r + 1$. Thus it is generated by images of some $f = (f_r, \dots, f_N) \in J$. Let $h_{ij} \in \mathbb{C}\{t, y, \lambda\}$ ($i, j \in [1, N]$) be such that $u_i = \sum_j h_{ij} y_j$ ($i < r$) and $f_i = \sum_j h_{ij} y_j$ ($i \geq r$). Since $(y)/\lambda(y) + (y)^2 + (u) + (f)$ is of finite length, $\det(h_{ij})(t, 0, 0)$ is not identically zero. For arbitrary small enough $\alpha > \beta > 0$, one sees $(\partial(u, f)/\partial y)(t, y, \lambda) \neq 0$ if $\alpha \geq |t| \geq \beta$ and $|y|, |\lambda| \ll \beta$, because

$$\frac{\partial(u, f)}{\partial y}(t, 0, 0) = \det(h_{ij})(t, 0, 0).$$

By the Implicit Function Theorem, y_j 's are expressed as convergent power series $y_j = y_j(u, f)$ in (u) and (f) such that $y_j(0, 0) = 0$ depending holomorphically on t and λ such that $\alpha > |t| > \beta$ and $\delta > |\lambda|$ if $0 < \delta \ll \beta$. In other words, (t, u, f, λ) forms a coordinate system for the open set $\{(t, y, \lambda) \mid \beta < |t| < \alpha, |y| < \gamma, |\lambda| < \delta\}$ of \mathbb{C}^{N+1} if $0 < \delta \ll \gamma \ll \beta$. Imposing extra conditions $f = 0$ and $|u| < \delta$ and making δ smaller, we see (1b.7). \square

(1b.8) **Definition.** Let $X \supset C$ be the germ of a normal analytic space X along a smooth irreducible compact curve C such that $C \not\subset \text{Sing } X$. Let $P \in C$ and $(X, P) \subset (t\gamma\text{-space}, 0) \simeq (\mathbb{C}^{N+1}, 0)$ an embedding of a germ so that (C, P) is the t -axis, where $y = (y_1, \dots, y_N)$. Let $X_\lambda \supset C_\lambda$ be a deformation of pair $X \supset C$ of germs at P . By (1b.6), we may embed $X_\lambda \supset C_\lambda$ as $(\mathbb{C}^{N+1} \times \mathbb{C}_\lambda^1, P \times 0) \supset \mathcal{X} \supset \mathcal{C} = t\lambda\text{-plane}$ which are flat over $(\mathbb{C}_\lambda^1, 0)$.

Let $u = (u_1, \dots, u_{r-1})$ be elements of $(y)\mathbb{C}\{t, y, \lambda\}$ such that

$$(1b.8.1) \quad I_C/I_C^2 + (u)\mathcal{O}_X \text{ is of finite length at } P,$$

where I_C is the sheaf of defining ideals of C in X . Let $u(\cdot, c) = (u_1(\cdot, c), \dots, u_{r-1}(\cdot, c)) \in \mathbb{C}\{t, y\}$ be the restrictions of u to $\lambda = c$.

Then applying (1b.7) to both \mathcal{X} and the trivial deformation $(X \times \mathbb{C}_\lambda^1, P \times 0) \subset (t\gamma\lambda\text{-space}, 0)$, we see that

$$\rho: \mathcal{X} \left(\begin{matrix} (\beta, \alpha) \times \gamma \times \delta \\ u < \delta \end{matrix} \right) \xrightarrow{\sim} (X \times \mathbb{C}_\lambda^1) \left(\begin{matrix} (\beta, \alpha) \times \gamma \times \delta \\ u(\cdot, 0) < \delta \end{matrix} \right)$$

such that

$$\rho^* u_i(\cdot, 0) = u_i \quad (i \in [1, r - 1]), \quad \rho|_{\mathcal{E}} = \text{id}: \mathcal{E} \rightarrow t\lambda\text{-plane}$$

by the identification

$$\mathcal{X} \left(\begin{matrix} (\beta, \alpha) \times \gamma \times \delta \\ u \end{matrix} \right) = \mathcal{X} \left(\begin{matrix} (\beta, \alpha) \times \gamma \times \delta \\ u(\cdot, 0) \end{matrix} \right) \quad \text{if } 1 \gg \alpha > \beta \gg \gamma \gg \delta > 0.$$

Thus if $1 \gg \alpha' > \alpha > \beta \gg \gamma \gg \delta > 0$, one may patch

$$\mathcal{X} \left(\begin{matrix} [0, \alpha] \times \gamma \times \delta \\ u < \delta \end{matrix} \right) \quad \text{and} \quad (X \times \mathbb{C}_\lambda^1) \left(\begin{matrix} (\beta, \alpha') \times \gamma \times \delta \\ u(\cdot, 0) < \delta \end{matrix} \right)$$

via the above isomorphism to get a deformation of $X \left(\begin{matrix} [0, \alpha'] \times \gamma \\ u(\cdot, 0) < \delta \end{matrix} \right)$ which is trivial on $X \left(\begin{matrix} (\beta, \alpha') \times \gamma \\ u(\cdot, 0) < \delta \end{matrix} \right)$, where

$$X \left(\begin{matrix} I \times \gamma \\ v < \delta \end{matrix} \right) = \{(t, y) \in (X, P) \mid |t| \in I, |y| < \gamma, |v| < \delta\}$$

for an interval I . Thus replacing X with a smaller nbd of C , we can extend this deformation to the deformation $X_\lambda \supset C_\lambda = C$ of $X \supset C$. For simplicity of notation, we set (omitting $\alpha, \beta, \gamma, \delta$)

$$\mathcal{X}_{\text{out}}(u) = X \times \{\lambda \mid |\lambda| < \delta\} - (X \times \mathbb{C}_\lambda^1) \left(\begin{matrix} [0, \beta] \times \gamma \times \delta \\ u(\cdot, 0) < \delta \end{matrix} \right) \supset \mathcal{E}_{\text{out}}(u),$$

$$\mathcal{X}_{\text{in}}(u) = \mathcal{X} \left(\begin{matrix} [0, \alpha] \times \gamma \times \delta \\ u < \delta \end{matrix} \right) \supset \mathcal{E}_{\text{in}}(u),$$

$$\mathcal{X}_{\text{mid}}(u) = \mathcal{X}_{\text{out}}(u) \cap \mathcal{X}_{\text{in}}(u) \supset \mathcal{E}_{\text{mid}}(u), \quad \text{and}$$

$$\mathcal{X}_{\text{all}}(u) = \mathcal{X}_{\text{out}}(u) \cup \mathcal{X}_{\text{in}}(u) \supset \mathcal{E}_{\text{all}}(u),$$

where $\mathcal{E}_{\text{out}}(u), \dots, \mathcal{E}_{\text{all}}(u)$ are defined in the obvious way.

We note that $X_\lambda \supset C_\lambda = C$ is obtained by patching

$$X_{\lambda, \text{in}} = X_\lambda \left(\begin{matrix} [0, \alpha] \times \gamma \\ u(\cdot, \lambda) < \delta \end{matrix} \right) \quad \text{and} \quad X_{\lambda, \text{out}} = X - X \left(\begin{matrix} [0, \beta] \times \gamma \\ u(\cdot, 0) < \delta \end{matrix} \right)$$

via

$$\rho_\lambda: X \left(\begin{matrix} (\beta, \alpha) \times \gamma \\ u(\cdot, \lambda) < \delta \end{matrix} \right) \xrightarrow{\sim} X \left(\begin{matrix} (\beta, \alpha) \times \gamma \\ u(\cdot, 0) < \delta \end{matrix} \right)$$

such that $\rho_\lambda^* u_i(\cdot, 0) = u_i(\cdot, \lambda)$ ($i \in [1, r - 1]$), on $X_{\lambda, \text{mid}}$ “ = ” $X_{\lambda, \text{in}} \cap X_{\lambda, \text{out}}$, and $C_{\lambda, \text{in}}, \dots, C_{\lambda, \text{mid}}$ are defined similarly.

We call $\mathcal{X}_{\text{all}}(u) \supset \mathcal{E}_{\text{all}}(u)$ (or $X_\lambda \supset C_\lambda$) the *twisted extension* of $\mathcal{X} \supset \mathcal{E}$ (or $(X_\lambda, P) \supset (C_\lambda, P)$) by u .

(1b.8.2) **Corollary.** *Let $X \supset C$ be the germ of a normal analytic space along a complete smooth irreducible curve such that $C \not\subset \text{Sing } X$. Let $P \in C$. Then every deformation of germs $(X, P) \supset (C, P)$ can be extended to a deformation of $X \supset C$ so that the deformation is trivial outside some small nbd of P .*

(1b.8.3) **Proposition.** *Let the notation and assumptions be as in (1b.8.1). Let I (resp. I_λ) be the sheaf of defining ideals of C (resp. C_λ) in X (resp. X_λ), and let $I^{(2)}$ be the symbolic square of I . Then*

(i) *the closed subscheme of X defined by $I^{(2)}$ deforms in a flat family as subschemes defined by $(I^{(2)})_\lambda$ iff $\chi(I_\lambda/(I_\lambda)^{(2)}) = \chi(I/I^{(2)})$ for each $\lambda > 0$ such that $|\lambda| \ll 1$. If so, then $(I^{(2)})_\lambda = (I_\lambda)^{(2)}$ (hence denoted by $I_\lambda^{(2)}$) and $I_\lambda/I_\lambda^{(2)}$ is a flat family of locally free sheaves on C_λ over $(\mathbb{C}_\lambda^1, 0)$, and*

(ii) *assume that the closed subscheme defined by $I^{(2)}$ deforms and that $I/I^{(2)}$ contains an invertible subsheaf L such that L_P is generated by (the image of) $u_1(\cdot, 0)$. Then L deforms as invertible subsheaf L_λ of $I_\lambda/I_\lambda^{(2)}$ and L_λ is generated at $P_\lambda (= P)$ by $u_1(\cdot, \lambda)$.*

Proof. (i) The only-if part is the invariance of χ in a flat family. Assume that $\chi(I_\lambda/(I_\lambda)^{(2)}) = \chi(I/I^{(2)})$ if $0 < |\lambda| \ll 1$. Let J be the sheaf of defining ideals of $\mathcal{E}_{\text{all}}(u)$ in $\mathcal{X}_{\text{all}}(u)$. Then $J\mathcal{O}_X = I$, $J^{(2)}\mathcal{O}_X$ is a subsheaf of $I^{(2)}$ such that $\text{len } I^{(2)}/J^{(2)}\mathcal{O}_X < \infty$, and

$$J\mathcal{O}_{X_\lambda} = I_\lambda \quad \text{and} \quad J^{(2)}\mathcal{O}_{X_\lambda} = (I_\lambda)^{(2)} \quad \text{if } 0 < |\lambda| \ll 1.$$

Then $F = J/J^{(2)}$ is flat over $(\mathbb{C}_\lambda^1, 0)$ and $F_\lambda = F \otimes \mathcal{O}_{X_\lambda} = I_\lambda/(I_\lambda)^{(2)}$ if $0 < |\lambda| \ll 1$, and we have a surjection $F_0 \rightarrow I/I^{(2)}$. By $\chi(I_\lambda/(I_\lambda)^{(2)}) = \chi(I/I^{(2)})$, we see that $\text{Ker}[F_0 \rightarrow I/I^{(2)}]$ is a sheaf supported on a finite set such that

$$\chi(\text{Ker}[F_0 \rightarrow I/I^{(2)}]) = \chi(F_0) - \chi(I/I^{(2)}) = \chi(F_\lambda) - \chi(I_\lambda/(I_\lambda)^{(2)}) = 0,$$

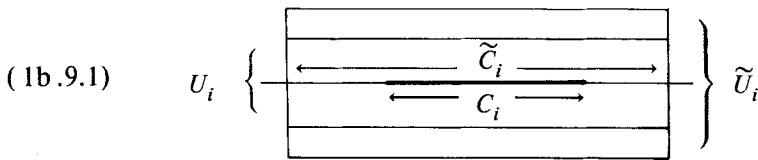
whence $F_0 \simeq I/I^{(2)}$, $J^{(2)}\mathcal{O}_X = I^{(2)}$, and (i) is proved.

(ii) Since L and $\mathcal{O}u_1(\cdot, 0)$ patch on some $X \left(\begin{smallmatrix} (\beta, \alpha) \times \gamma \\ u(\cdot, 0) < \delta \end{smallmatrix} \right)$, the trivial extension of L on $\mathcal{X}_{\text{out}}(u)$ and $\mathcal{O}u_1$ on $\mathcal{X}_{\text{in}}(u)$ patch on $\mathcal{X}_{\text{mid}}(u)$ because $\rho^*u_1(\cdot, 0) = u_1$. \square

(1b.9) **Proposition.** *Let X be an analytic space and C a compact subvariety which is a smooth irreducible curve. Then there is an open neighborhood V of C in X and an effective Cartier divisor E on V such that every irreducible compact subvariety C' in V with dimension > 0 is a curve and $(C' \cdot E) > 0$.*

Proof. Let $\varepsilon > \delta > 0$. For each $P \in C$, let \tilde{U}_P be a neighborhood of P in X such that \tilde{U}_P is identified with an analytic set of $D_\varepsilon \times D_\varepsilon^N$ such that $\tilde{C}_P = \tilde{U}_P \cap C$ and P are equal to $D_\varepsilon \times 0$ and 0×0 , respectively, where $D_\varepsilon = \{z \in \mathbb{C} \mid |z| < \varepsilon\}$. Let $U_P = \tilde{U}_P \cap D_\varepsilon \times D_\delta^N$ and $C_P = D_\delta \times 0 \subset U_P \cap C$.

Since $C = \bigcup_p C_p$ and C is compact, there are finitely many C_p 's covering C . We use the notation $\tilde{U}_i, U_i, \tilde{C}_i, C_i, P_i$ ($i = 1, \dots, r$) for these (cf. (1b.9.1)). In $D_\epsilon \times D_\epsilon^N$ containing \tilde{U}_i , the closed set $\bar{C}_i \times (\bar{D}_\delta^N - D_{\delta/2}^N)$ is compact and disjoint from \tilde{C}_i . Hence in $\bigcup \tilde{U}_i, \bigcup \bar{C}_i \times (\bar{D}_\delta^N - D_{\delta/2}^N)$ is compact and disjoint from C . Thus there is an open neighborhood $V \subset \bigcup U_i$ of C such that \bar{V} is disjoint from $\bigcup \bar{C}_i \times (\bar{D}_\delta^N - D_{\delta/2}^N)$. In particular $(\bar{V} \cap U_i) \cap \bar{C}_i \times D_\delta^N \subset \bar{C}_i \times D_{\delta/2}^N$ for all i . Since $D_{\delta/2}^N \subset \subset D_\delta^N$, $(\bar{V} \cap U_i) \cap (\bar{C}_i \times D_\delta^N)$ is compact for all i . We note that the first projections $f_i: U_i \rightarrow \tilde{C}_i$ are not necessarily compatible with each other.



By replacing V with a smaller open neighborhood of C , one may assume that $V \subset \bigcup f_i^{-1}(C_i)$, and that $E_i = f_i^{-1}(P_i) \cap V$ is closed in V and hence an effective Cartier divisor. Let $E = E_1 + \dots + E_r$. Let C' be an irreducible compact subvariety in V . Then $(C' \cap U_i) \cap f_i^{-1}(\bar{C}_i)$ is closed in the compact set $(\bar{V} \cap U_i) \cap (\bar{C}_i \times D_\delta^N)$ hence is compact. Thus the induced map $f'_i: C'_i \rightarrow C_i$ is proper for all i , where $C'_i = (C' \cap U_i) \cap f_i^{-1}(C_i)$. Since fibers of f'_i lie in $D_{\delta/2}^N$ and compact, f'_i must be finite. Thus $\dim C' = 1$ and $(C' \cdot E_i) \geq 0$ by $C' = \bigcup C'_i$ and f'_i is open and proper. Let j be such that $C'_j \neq \emptyset$. Then $f_j(C'_j) = C_j$ by connectedness of C_j and $(E_j \cdot C') = (f_j^{-1}(P_j) \cdot C') > 0$. Hence $(E \cdot C') \geq (E_j \cdot C') > 0$. \square

(1b.10) **Corollary.** *Under the notation and assumptions of (1b.9), assume that there is a line bundle L on X such that $(L \cdot C) > 0$. Then there is an open neighborhood W of C in X such that every irreducible compact subvariety C' in W with dimension > 0 is a curve and $(L \cdot C') > 0$.*

Proof. Let V and E be as in (1b.9). By [Go, Théorème 4.11.1], one has

$$\lim_{U \text{ open } \supset C} H^2(U, \mathbb{Z}_U) \simeq H^2(C, \mathbb{Z}_C) \simeq \mathbb{Z}.$$

Thus on a sufficiently small neighborhood W of C , $(L \cdot C)E_W$ and $L_W^{\otimes (E \cdot C)}$ induce the same class in $H^2(W, \mathbb{Z}_W)$, where E_W and L_W are restrictions of E and L to W , respectively. Hence for $C' \subset W$, one has $(L_W \cdot C') = (L \cdot C') > 0$ from $(E_W \cdot C') = (E \cdot C') > 0$. \square

2. NUMERICAL INVARIANTS $i_p(n)$, $w_p(0)$, AND $w_p^*(n)$

The main purpose of this section is to introduce numerical invariants which will be the main tools of our study.

We will use the results and definitions in §1a on 3-fold terminal singularities.

(2.1) Let X be an analytic normal Cohen-Macaulay 3-fold and $C \subset X$ a reduced curve such that K_X is \mathbb{Q} -Cartier and no irreducible component of C is contained in the singular locus of X .

(2.2) **Definition.** For coherent \mathcal{O}_X -modules $\mathcal{S} \supset \mathcal{T}$, we say that \mathcal{T} is saturated in \mathcal{S} (or, \mathcal{S} -saturated) iff \mathcal{S}/\mathcal{T} has no \mathcal{O}_X -submodule of finite length > 0 . For each $\mathcal{T} \subset \mathcal{S}$, there is a smallest \mathcal{S} -saturated submodule \mathcal{V} containing \mathcal{T} , and one sees that $\text{Supp}(\mathcal{V}/\mathcal{T})$ is 0-dimensional and $\text{len}_P \mathcal{V}/\mathcal{T} < \infty$ at all $P \in X$. We call such \mathcal{V} the \mathcal{S} -saturation of \mathcal{T} and denote it by $\text{Sat}_{\mathcal{S}} \mathcal{T}$.

Let I_C (resp. $I_C^{(n)}$) be the sheaf of ideals defining C in X (resp. the symbolic n th power of I_C , that is $\text{Sat}_{\mathcal{O}_X} I_C^{(n)}$). For a coherent \mathcal{O}_X -module \mathcal{S} , let $F^n \mathcal{S} = \text{Sat}_{\mathcal{S}} I_C^n \mathcal{S} = \text{Sat}_{\mathcal{S}} I_C^{(n)} \mathcal{S}$ ($n \geq 0$) and $\text{gr}_C^n(\mathcal{S}) = F^n \mathcal{S} / F^{n+1} \mathcal{S}$. Let $\text{gr}_C^n \mathcal{O} = \text{gr}_C^n(\mathcal{O}_X)$ and $\text{gr}_C^n \omega = \text{gr}_C^n(\omega_X)$. We note that $\text{gr}^{\bullet} \mathcal{O} = \bigoplus_{n \geq 0} \text{gr}_C^n \mathcal{O}$ is naturally a graded \mathcal{O}_C -algebra and $\text{gr}^{\bullet}(\mathcal{S}) = \bigoplus_{n \geq 0} \text{gr}_C^n(\mathcal{S})$ is naturally a graded $\text{gr}^{\bullet} \mathcal{O}$ -module, and one has a natural homomorphism

$$\gamma_C^n(\mathcal{S}): S^n(\text{gr}_C^1 \mathcal{O}) \otimes \text{gr}_C^0(\mathcal{S}) \rightarrow \text{gr}_C^n(\mathcal{S}) \quad (n \geq 1),$$

where $S^n(\text{gr}_C^1 \mathcal{O})$ denotes the symmetric n th power of $\text{gr}_C^1 \mathcal{O}$.

It is easy to see the following

(2.2.1) **Lemma.** Via the natural map $\Omega_X^3 \rightarrow \omega_X$, there is a map

$$\begin{aligned} \text{gr}_C^1 \mathcal{O} \times \text{gr}_C^1 \mathcal{O} \times \Omega_C^1 &\rightarrow \Omega_X^3 \otimes \mathcal{O}_C \rightarrow \text{gr}_C^0 \omega, \\ x \times y \times zdu &\mapsto zdx \wedge dy \wedge du \end{aligned}$$

which induces a homomorphism $\wedge^2(\text{gr}_C^1 \mathcal{O}) \otimes \Omega_C^1 \rightarrow \text{gr}_C^0 \omega$.

Thus one can define homomorphisms

$$\begin{aligned} \alpha_1: \wedge^2(\text{gr}_C^1 \mathcal{O}) &\rightarrow \text{Hom}_{\mathcal{O}_C}(\Omega_C^1, \text{gr}_C^0 \omega), \\ \alpha_n = \gamma_C^n(\mathcal{O}_X): S^n(\text{gr}_C^1 \mathcal{O}) &\rightarrow \text{gr}_C^n \mathcal{O} \quad (n \geq 2). \end{aligned}$$

We denote by $i_p(n)$ the length $\text{len}_P \text{Coker } \alpha_n$ of $\text{Coker } \alpha_n$ at $P \in C$. We note that $i_p(n) \in \mathbb{Z}_+$ ($P \in C$) and that $i_p(n) = 0$ if X and C are smooth at P .

Let $m \in \mathbb{N}$ be such that mK_X is a Cartier divisor, and let

$$\begin{aligned} \beta_0: (\text{gr}_C^0 \omega)^{\otimes m} &\rightarrow \omega_X^{(m)} \otimes \mathcal{O}_C, \\ \beta_n = \gamma_C^n(\omega_X): \text{gr}_C^0 \omega \otimes S^n(\text{gr}_C^1 \mathcal{O}) &\rightarrow \text{gr}_C^n \omega \quad (n \geq 1) \end{aligned}$$

be the natural homomorphisms, where $\omega_X^{(m)}$ is the double dual of $\omega_X^{\otimes m}$.

We denote by $w_P(n)$ for $n \geq 1$ (resp. $w_P(0)$) $\text{len}_P \text{Coker } \beta_n$ (resp. $(\text{len}_P \text{Coker } \beta_0)/m \in \mathbb{Q}$), and define

$$w_P^*(n) = \binom{n+1}{2} i_P(1) - w_P(n) \quad \text{for } n \geq 1.$$

We note that $w_P(n) \in \mathbb{Z}_+$, $w_P^*(n) \in \mathbb{Z}$ ($P \in C$), and $w_P(n) = w_P^*(n) = 0$ ($n \geq 1$) if X and C are smooth at P . We also note that $w_P(0)$ is independent of the choice of m if C is smooth at P .

(2.3) From the above constructions, one can see

(2.3.1) (Formulae) If $C \simeq \mathbb{P}^1$, then

$$\text{deg gr}_C^1 \mathcal{O} = 2 + \text{deg gr}_C^0 \omega - \sum_P i_P(1),$$

$$\text{deg gr}_C^n \mathcal{O} = \frac{1}{2}n(n+1) \text{deg gr}_C^1 \mathcal{O} + \sum_P i_P(n) \quad (n \geq 2),$$

$$(\omega_X \cdot C) = \text{deg gr}_C^0 \omega + \sum_P w_P(0), \quad \text{and}$$

$$\text{deg gr}_C^n \omega = \frac{1}{2}(n+1)(n+2) \text{deg gr}_C^0 \omega + n(n+1) - \sum_P w_P^*(n) \quad (n \geq 1).$$

(2.3.2) (Formulae) If $X \supset C \simeq \mathbb{P}^1$ is an extremal nbd, then $\text{gr}_C^0 \omega \simeq \mathcal{O}_C(-1)$, and

$$\text{deg gr}_C^1 \mathcal{O} = 1 - \sum_P i_P(1),$$

$$\text{deg gr}_C^n \mathcal{O} = \frac{1}{2}n(n+1) \text{deg gr}_C^1 \mathcal{O} + \sum_P i_P(n) \quad (n \geq 2),$$

$$(\omega_X \cdot C) = -1 + \sum_P w_P(0), \quad \text{and}$$

$$\text{deg gr}_C^n \omega = \frac{1}{2}(n+1)(n-2) - \sum_P w_P^*(n) \quad (n \geq 1).$$

(2.3.3) (Necessary conditions) If $X \supset C \simeq \mathbb{P}^1$ is an extremal nbd, then

$$\sum_P w_P(0) < 1,$$

$$\sum_{1 \leq i \leq n} \text{deg gr}_C^i \mathcal{O} \geq - \sum_{1 \leq i \leq n} (i+1) \quad \text{for all } n \in \mathbb{N},$$

$$\sum_{1 \leq i \leq n} \text{deg gr}_C^i \omega \geq - \sum_{1 \leq i \leq n} (i+1) \quad \text{for all } n \in \mathbb{N},$$

and when $n = 1$, these reduce to

$$\sum_P i_P(1) \leq 3 \quad \text{and} \quad \sum_P w_P^*(1) \leq 1.$$

Proof of (2.3). Indeed (2.3.1) follows immediately from the constructions of $\alpha_1, \alpha_n, \beta_0,$ and β_n . By (1.14(i)), we have $\text{gr}_C^0 \omega \simeq \mathcal{O}_C(-1)$, hence the assertions in (2.3.2) follow. For (2.3.3), one has $H^1(\mathcal{O}_X/I_C^{(n+1)}) = H^1(\omega_X/F^{n+1}\omega_X) = 0$ by (1.2.1). Then $H^1(I_C/I_C^{(n+1)}) = 0$ from the exact sequence

$$0 \rightarrow I_C/I_C^{(n+1)} \rightarrow \mathcal{O}_X/I_C^{(n+1)} \rightarrow \mathcal{O}_C \rightarrow 0$$

and $H^0(\mathcal{O}_X/I_C^{(n+1)}) \rightarrow H^0(\mathcal{O}_C) = \mathbb{C}$. Then from the exact sequences:

$$\begin{aligned} 0 &\rightarrow \text{gr}_C^n \mathcal{O} \rightarrow I_C/I_C^{(n+1)} \rightarrow I_C/I_C^{(n)} \rightarrow 0 \quad (n \geq 1), \\ 0 &\rightarrow \text{gr}_C^n \omega \rightarrow \omega_X/F^{n+1}\omega_X \rightarrow \omega_X/F^n\omega_X \rightarrow 0 \quad (n \geq 0), \end{aligned}$$

one has

$$\chi(I_C/I_C^{(n+1)}) = \sum_{i=1}^n \chi(\text{gr}_C^i \mathcal{O}) \quad \text{and} \quad \chi(\omega_X/F^{n+1}\omega_X) = \sum_{i=0}^n \chi(\text{gr}_C^i \omega).$$

Now (2.3.3) follows from these. \square

(2.3.4) *Remark.* Let $n \in \mathbb{Z}_+$. The proof shows that if $\text{deg gr}_C^d \mathcal{O} \leq -d - 1$ (resp. $\text{deg gr}_C^d \omega \leq -d - 1$) for $d \in [1, n]$, then $\text{gr}_C^d \mathcal{O} \simeq \mathcal{O}_C(-1)^{\oplus(d+1)}$ (resp. $\text{gr}_C^d \omega \simeq \mathcal{O}_C(-1)^{\oplus(d+1)}$) for $d \in [1, n]$ and $H^1(\text{gr}_C^{n+1} \mathcal{O}) = 0$ (resp. $H^1(\text{gr}_C^{n+1} \omega) = 0$).

(2.4) In the remainder of this section, we assume that (X, P) is a germ of a 3-fold analytic terminal singularity of index m and $C \subset (X, P)$ is a smooth curve such that P has subindex \bar{m} and splitting degree s (1.7). We note that $m = \bar{m} \cdot s$. Let $(X^\sharp, P^\sharp) \rightarrow (X, P)$ be the canonical μ_m -cover and $C^\sharp = (C \times_X X^\sharp)_{\text{red}} \subset (X^\sharp, P^\sharp)$. Then (C^\sharp, P^\sharp) has s irreducible components (1.7), and C^\sharp is not contained in the singular locus of X^\sharp .

(2.5) **Definition.** Let (C^\dagger, P^\dagger) be an irreducible component of the normalization of one of the irreducible components $C^\sharp(i)$ ($i \in [1, s]$) of (C^\sharp, P^\sharp) . Then μ_m naturally acts on (X^\sharp, P^\sharp) and (C^\sharp, P^\sharp) , and so does $\mu_{\bar{m}}$ on (C^\dagger, P^\dagger) (1.7), and let

$$\tau: \mathcal{O}_{X^\sharp, P^\sharp} \rightarrow \mathcal{O}_{C^\dagger, P^\dagger}$$

be the natural map. Since (X, P) and (C, P) are normal, one has

$$\mathcal{O}_{X, P} = (\mathcal{O}_{X^\sharp, P^\sharp})^{\mu_m}, \quad \text{and} \quad \mathcal{O}_{C, P} = (\mathcal{O}_{C^\dagger, P^\dagger})^{\mu_{\bar{m}}}.$$

Since μ_m acts freely on $X^\sharp - \{P^\sharp\}$, so does it on $C^\sharp - \{P^\sharp\}$ and so does $\mu_{\bar{m}}$ on $C^\dagger - \{P^\dagger\}$. Hence $\mathcal{O}_{C^\dagger, P^\dagger}$ has a uniformizing parameter t (i.e. $\mathcal{O}_{C^\dagger, P^\dagger} = \mathbb{C}\{t\}$) such that t is a μ_m -semi-invariant, and let χ be a generator of $X(\mu_m) = \text{Hom}(\mu_m, \mathbb{C}^*) = \mathbb{Z}_m$ such that the restriction $\bar{\chi}$ of χ to $\mu_{\bar{m}}$ is the character associated to t . Hence $\mathcal{O}_{C, P} = \mathbb{C}\{t^{\bar{m}}\}$, and we call such t an ℓ -coordinate of

(C, P) , and χ an ℓ -character of (C, P) . We note that $\bar{\chi} \equiv \chi \pmod{\bar{m}\mathbb{Z}_m}$ is determined independent of choice of ℓ -coordinate t .

For a semi-invariant $v \neq 0$, let C^\sharp -wt v (or simply wt v if there is no danger of confusion) be $n \in \mathbb{Z}_m$ such that $n\chi$ (or χ^n if written multiplicatively) is the character associated to v . We note that changing ℓ -character χ to $a\chi$ ($a \in \mathbb{Z}_m$ is a unit such that $a \equiv 1 \pmod{\bar{m}}$) changes C^\sharp -wt v to $a^{-1}C^\sharp$ -wt v .

If C^\dagger dominates $C^\sharp(i)$, then for $z \in \mathcal{O}_{\chi^\sharp, P^\sharp}$, let

$$C^\sharp(i)\text{-ord}(z) = \sup\{n \in \mathbb{Z}_+ \mid \tau(z) \in t^n \mathbb{C}\{t\}\}.$$

If z is a μ_m -semi-invariant, then $C^\sharp(i)\text{-ord}(z) = C^\sharp(j)\text{-ord}(z)$ for all i, j , since μ_m acts transitively on $C^\sharp(i)$'s. In this case, we may write $C^\sharp(i)\text{-ord}(z)$ as $C^\sharp\text{-ord}(z)$ or even as $\text{ord}(z)$, if it does not cause confusion.

For a μ_m -semi-invariant $z \in \mathcal{O}_{\chi^\sharp, P^\sharp}$, we define $C^\sharp\text{-ow}(z)$ (or $\text{ow}(z)$) as $\text{ow}(z) = (\text{ord}(z), \text{wt}(z)) \in (\mathbb{Z}_+ \cup \{\infty\}) \times \mathbb{Z}_m$. We note that $\text{ord}(z) \equiv \text{wt}(z) \pmod{\bar{m}}$ if $\text{ord}(z) < \infty$. We define semigroups

$$\begin{aligned} \text{ord}(C^\sharp) &= \{\text{ord}(w) \mid w \in \mathcal{O}_{C^\sharp, P^\sharp} - I_{C^\sharp, P^\sharp}\} \subset \mathbb{Z}_+, \quad \text{and} \\ \text{ow}(C^\sharp) &= \{\text{ow}(w) \mid w \in \mathcal{O}_{C^\sharp, P^\sharp} - I_{C^\sharp, P^\sharp}\} \subset \mathbb{Z}_+ \times \mathbb{Z}_m. \end{aligned}$$

Assume that μ_m acts on a \mathbb{C} -algebra R via \mathbb{C} -algebra automorphisms, and that M is a μ_m - R -module. Let $n \in \mathbb{Z}_m$. Let $\mathbb{C}[n]$ be the 1-dimensional \mathbb{C} -vector space on which μ_m acts by $n\chi$, where χ is an ℓ -character of (C, P) , and let $M[n] = M \otimes_{\mathbb{C}} \mathbb{C}[n]$ be the R -module with the naturally induced μ_m -action. Let

$$\begin{aligned} M_{\{n\}} &= \text{Hom}_{\mu_m}(\mathbb{C}[n], M) \\ &= \{x \in M \mid \gamma(x) = \chi(\gamma)^n x \text{ for all } \gamma \in \mu_m\}. \end{aligned}$$

Then $M_{\{n\}} = M[-n]_{\{0\}}$ is an $R_{\{0\}}$ -module.

(2.6) **Definition.** Let (x) be an ℓ -coordinate system for (X, P) (1a.5). We note that $\text{wt}(x)$ up to permutation does not depend on choice of (x) . Let $a_i = \text{ord}(x_i)$ ($i \in [1, 4]$, $0 < a_i \leq \infty$). We say that (x) is *minimal* if a_i is the smallest positive integer such that $(a_i, \text{wt}x_i) \in \text{ow}(C^\sharp)$ (in particular $a_i < \infty$) for all i . Thus $\text{ord}(x)$ and $\text{ow}(x)$ up to permutation do not depend on choice of minimal (x) . We say that (x) and an ℓ -coordinate t of (C, P) are *normalized ℓ -coordinates* if (x) is minimal and $\tau(x_i) = t^{a_i}$ for all i (cf. (2.7(iv))). Then we have

(2.7) **Lemma.** Let $(X, P) \supset C$ be as in (2.4), and χ an ℓ -character. Then (i) if (x) and t are an ℓ -coordinate system of (X, P) and an ℓ -coordinate of (C, P) , then there exists an ℓ -coordinate system (y) of (X, P) such that

$$\text{ow}(y_i) = \text{ow}(x_i) \quad \text{and} \quad \tau(y_i) = t^{a_i} \quad \text{for all } i,$$

(ii) for (y) and t as in (i), let $\Lambda = \{i \mid \text{ord}(y_i) \neq \infty\}$. Then y_i ($i \in \Lambda$) give an embedding $(C^\sharp, P^\sharp) \subset (\mathbb{C}^\Lambda, 0)$ such that the images of s irreducible components are parametrized by t as

$$(\chi(g)^{\text{wt}(y_i)} t^{\text{ord}(y_i)})_{i \in \Lambda} \quad (g \in \mu_m / \mu_{\overline{m}} \simeq \mu_s),$$

ow induces an isomorphism of semigroups

$$\text{ow}_y : \left\{ \begin{array}{l} \text{functions} \neq 0 \text{ on } (C^\sharp, P^\sharp) \text{ which are} \\ \text{monomials in } y \text{ with coefficients } 1 \end{array} \right\} \xrightarrow{\sim} \text{ow}(C^\sharp),$$

(iii) $(\overline{m}, 0) \in \text{ow}(C^\sharp)$, and $(u, v) \in \text{ow}(C^\sharp)$ for all large enough $u \in \mathbb{N}$ and all $v \in \mathbb{Z}_m$ such that $u \equiv v \pmod{\overline{m}}$, and

(iv) $(X, P) \supset C$ have normalized ℓ -coordinates.

Proof. Since τ is μ_m -equivariant, we have $\tau(x_i) = t^{a_i} g_i$ for some units $g_i \in \mathbb{C}\{t^{\overline{m}}\}$. Since

$$\text{Im}(\mathcal{O}_{X,P} \rightarrow \mathcal{O}_{C^\sharp,P^\sharp}) = \text{Im}(\mathcal{O}_{C,P} \rightarrow \mathcal{O}_{C^\sharp,P^\sharp}) = \mathbb{C}\{t^{\overline{m}}\},$$

we can choose units $h_i \in \mathcal{O}_{X,P}$ such that $\tau(h_i) = g_i$ for all i . Then it is enough to take $y_i = x_i h_i^{-1}$ for (i). Since μ_m acts transitively on irreducible components of C^\sharp , the first assertion of (ii) follows easily. Whence ow_y is injective, and we see that, for each μ_m -semi-invariant u , there exists a monomial v in y such that $\text{ow}(u) = \text{ow}(v)$. Hence follows (ii). As for (iii), $(\overline{m}, 0) \in \text{ow}(C^\sharp)$ follows from $\mathcal{O}_{C,P} \xrightarrow{\sim} \mathbb{C}\{t^{\overline{m}}\}$, and we see $(\text{wt } y_i, m) = 1$ for some $i \in \Lambda$ given in (ii) because the μ_m -action on $C^\sharp - \{P^\sharp\}$ is free. Whence the second assertion follows because $\mathbb{Z}_+(\overline{m}, 0) + \mathbb{Z}_+ \text{ow}(y_i)$ contains (u, v) given in (iii). For (iv), let (x) and t be as in (i). For each i , let $f_i \in \mathbb{C}\{x\}$ be a μ_m -semi-invariant with $\text{wt}(f_i) \equiv \text{wt}(x_i) \pmod{m}$, minimizing $\text{ord}(f_i)$. For some suitable $\varepsilon \in \mathbb{C}$, $(x^\sharp) = (x_1 + \varepsilon f_1, \dots, x_4 + \varepsilon f_4)$ is an ℓ -coordinate system such that $\text{ord}(x^\sharp)$ is the smallest. Then (i) gives normalized ℓ -coordinates. \square

(2.8) **Definition.** Under the notation and assumptions of (2.4), let $z = (x, w) \in \mathbb{Z}_+ \times \mathbb{Z}_m$ be such that $x \equiv w \pmod{\overline{m}}$.

(i) We define $U(z), R(w) \in \mathbb{Z}$ as

$$\begin{aligned} U(z) &= \min\{k \in \mathbb{Z} \mid k(\overline{m}, 0) - z \in \text{ow}(C^\sharp)\} \\ &= 1 + \max\{k \in \mathbb{Z} \mid k(\overline{m}, 0) - z \notin \text{ow}(C^\sharp)\}, \quad \text{and} \\ R(w) &= \min\{u \in \mathbb{Z}_+ \mid (u, -w) \in \text{ow}(C^\sharp)\}. \end{aligned}$$

We may write $R(w)$ as $R(z)$. These are well defined since second projection $\text{ow}(C^\sharp) \rightarrow \mathbb{Z}_m$ is surjective and the equality for $U(z)$ follows by $(\overline{m}, 0) \in \text{ow}(C^\sharp)$ (2.7(iii)). For a μ_m -semi-invariant v , we denote $(R(\text{wt } v), -\text{wt } v) \in \text{ow}(C^\sharp)$ by $C^\sharp\text{-ow}^*(v)$ (or $\text{ow}^*(v)$),

(ii) we see $\overline{m} \cdot U(z) = x + R(z)$ and $U(z + (\overline{m}, 0)) = U(z) + 1$. We may write $U(x, x \bmod m)$ as $U(x)$ for $x \in \mathbb{Z}$. In particular,

(iii) if P is a primitive point, then $s = 1$ and $\overline{m} = m$, whence $U(z)$ and $R(z)$ reduce to

$$\begin{aligned} U(x) &= \min\{k \in \mathbb{Z} \mid mk - x \in \text{ord}(C^\sharp)\} \\ &= 1 + \max\{k \in \mathbb{Z} \mid mk - x \notin \text{ord}(C^\sharp)\}, \\ R(x) &= R(x \bmod m) = \min\{u \in \text{ord}(C^\sharp) \mid u \equiv -x \pmod{m}\}, \end{aligned}$$

with the properties $m \cdot U(x) = x + R(x)$, $U(x + m) = U(x) + 1$.

(2.9) **Lemma.** *One has*

$$I_C^{(n)} = (I_{C^\sharp}^{(n)})_{\{0\}} \quad \text{and} \quad F^n \omega_X = (F^n \omega_{X^\sharp})_{\{0\}},$$

whence

$$\text{gr}_C^n(\mathcal{O}_X) = \text{gr}_{C^\sharp}^n(\mathcal{O}_{X^\sharp})_{\{0\}} \quad \text{and} \quad \text{gr}_C^n(\omega_X) = \text{gr}_{C^\sharp}^n(\omega_{X^\sharp})_{\{0\}}.$$

Proof. Since $(I_{C^\sharp}^{(n)})_{\{0\}} = \mathcal{O}_{X,P} \cap I_{C^\sharp}^{(n)}$, one sees that $(I_{C^\sharp}^{(n)})_{\{0\}}$ is a primary ideal associated to C . Since X^\sharp is étale over X at general points of C^\sharp , $I_C^{(n)} = (I_{C^\sharp}^{(n)})_{\{0\}}$ at general points of C^\sharp , whence $I_C^{(n)} = (I_{C^\sharp}^{(n)})_{\{0\}}$. Since $\omega_X = (\omega_{X^\sharp})_{\{0\}}$ (via the trace map), the exact sequence

$$0 \rightarrow F^n \omega_{X^\sharp} \rightarrow \omega_{X^\sharp} \rightarrow \text{gr}_{C^\sharp}^n \omega_{X^\sharp} \rightarrow 0$$

gives the exact sequence

$$0 \rightarrow (F^n \omega_{X^\sharp})_{\{0\}} \rightarrow \omega_X \rightarrow (\text{gr}_{C^\sharp}^n \omega_{X^\sharp})_{\{0\}} \rightarrow 0.$$

Since X^\sharp is étale over X at general points of C^\sharp , one has $I_C^{(n)} \omega_X = (F^n \omega_{X^\sharp})_{\{0\}}$ at general points of C . Since $\omega_X / (F^n \omega_{X^\sharp})_{\{0\}}$ is torsion-free, one has $F^n \omega_X = (F^n \omega_{X^\sharp})_{\{0\}}$. \square

Let x be an ℓ -coordinate system and ϕ an ℓ -equation (1a.5).

(2.10) **Corollary.** *One has $\omega_{X^\sharp} \otimes \mathcal{O}_{C^\sharp} \simeq \mathcal{O}_{C^\sharp}[k]$, and $w_p(0) = R(k)/\overline{m}$, where $k = C^\sharp\text{-wt}(x_1 x_2 x_3 x_4 / \phi)$.*

Proof. By the residue formula, $\omega_{X^\sharp} \otimes \mathcal{O}_{C^\sharp}$ is an invertible $\mathcal{O}_{C^\sharp, P^\sharp}$ -module whose generator is the image $\overline{\omega}$ of

$$\text{Res}_{X^\sharp} \frac{dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4}{\phi}$$

which has $C^\sharp\text{-wt} \equiv k$, whence $\omega_{X^\sharp} \otimes \mathcal{O}_{C^\sharp} \simeq \mathcal{O}_{C^\sharp}[k]$. We see that $\text{gr}_C^0(\omega) = u\mathbb{C}\{\overline{t}^m\}\overline{\omega}$ for some semi-invariant $u \in \mathcal{O}_{X^\sharp, P^\sharp}$ with the minimal $\text{ord} u$ under the condition that $\text{wt} u + k \equiv 0 \pmod{m}$. Hence $w_p(0) = R(k)/\overline{m}$ by definition of $R(k)$. \square

(2.11) We introduce the following construction to compute $i_p(1)$ and $w_p^*(1)$ for the terminal singularity (X, P) in (2.4).

Let x and ϕ be an ℓ -coordinate system and an ℓ -equation at (X, P) (1a.5), and t an ℓ -coordinate of (C, P) (2.5). We will use the symbols introduced in (2.5) and (2.6). Let $h_i(t) = \tau(x_i) \in \mathbb{C}\{t\}$ and $\phi_i = \partial\phi/\partial x_i$ ($i \in [1, 4]$). By symmetry among x_1, \dots, x_4 , we will assume that $h_4(t) \neq 0$.

Let $(X^\sharp, P^\sharp) \subset (Z^\sharp, P^\sharp) = (x_1x_2x_3x_4\text{-space}, 0)$ be an embedding by the ℓ -coordinate system x . Let $(C^\dagger, P^\dagger) \subset (Z^\dagger, P^\dagger) = (y_1y_2y_3y_4\text{-space}, 0)$ be the embedding $(t) \mapsto (h_1(t), h_2(t), h_3(t), t)$. Then $(y) \mapsto (y_1, y_2, y_3, h_4(y_4))$ defines a morphism $(Z^\dagger, P^\dagger) \mapsto (Z^\sharp, P^\sharp)$ fitting in the commutative diagram

$$\begin{array}{ccccc} Z^\dagger & \supset & X^\dagger & \supset & C^\dagger \\ \downarrow & & \downarrow & & \downarrow \\ Z^\sharp & \supset & X^\sharp & \supset & C^\sharp \end{array}$$

where $X^\dagger \subset Z^\dagger$ is defined by $\psi(y) \equiv \phi(y_1, y_2, y_3, h_4(y_4)) = 0$.

We note that $\mu_{\bar{m}}$ naturally acts on $Z^\dagger, X^\dagger, C^\dagger$, and that C^\dagger is not contained in the singular locus of X^\dagger because $X^\dagger \rightarrow X^\sharp$ is étale outside $\{y_4 = 0\}$ and $C^\dagger = \{\bar{y}_1 = \bar{y}_2 = \bar{y}_3 = 0\}$, where $\bar{y}_i = y_i - h_i(y_4)$ for $i = 1, 2, 3$.

Let J^\sharp (resp. J^\dagger) be the ideal defining C^\sharp in Z^\sharp (resp. C^\dagger in Z^\dagger). Let $J^{\sharp(n)}$ and $J^{\dagger(n)}$ be the symbolic n th power of J^\sharp and J^\dagger , respectively. Then $J^\dagger = (\bar{y}_1, \bar{y}_2, \bar{y}_3)$, and

$$J^\dagger/J^{\dagger(2)} \simeq \mathbb{C}\{t\}\bar{y}_1 \oplus \mathbb{C}\{t\}\bar{y}_2 \oplus \mathbb{C}\{t\}\bar{y}_3$$

(we identify $y_4 = t$ in $\mathcal{O}_{C^\dagger, P^\dagger}$) and let

$$\sigma: J^\sharp \rightarrow J^\dagger/J^{\dagger(2)}$$

be the natural map. For $g_1, g_2, g_3 \in J^\dagger/J^{\dagger(2)}$, let $[g_1, g_2, g_3] \in \mathbb{Z}_+ \cup \{\infty\}$ be the length of

$$(J^\dagger/J^{\dagger(2)})/\mathbb{C}\{t\}g_1 + \mathbb{C}\{t\}g_2 + \mathbb{C}\{t\}g_3,$$

and for subsets $A, B, C \subset J^\dagger/J^{\dagger(2)}$, let

$$[A, B, C] = \inf\{[a, b, c] \mid a \in A, b \in B, c \in C\}.$$

By abuse of notation, one may denote $[\dots, \sigma a, \dots]$ by $[\dots, a, \dots]$ when $a \in J^\sharp$ (resp. $[\dots, \sigma A, \dots]$ by $[\dots, A, \dots]$ when $A \subset J^\sharp$). We note that $[\ast, \ast, \ast]$ has semiadditivity

$$[g_1, g_2, g_3 + g'_3] \geq \min\{[g_1, g_2, g_3], [g_1, g_2, g'_3]\}$$

because $[g_1, g_2, g_3] = \text{ord } g_1 \wedge g_2 \wedge g_3$.

(2.12) **Proposition.** *Under the notation of (2.11), one has*

$$\begin{aligned} i_p(1) &= \bar{m}^{-1} \cdot (\bar{m} - R(k) - \text{ord}(x_4) + [\phi, J^\sharp_{\{0\}}, J^\sharp_{\{0\}}]) \quad \text{and} \\ w_p^*(1) &= \bar{m}^{-1} \cdot (\bar{m} - 3R(k) - \text{ord}(x_4) + [\phi, J^\sharp_{\{-k\}}, J^\sharp_{\{-k\}}]) \end{aligned}$$

if $\text{ord}(x_4) < \infty$, where $k = C^\sharp\text{-wt}(x_1x_2x_3x_4/\phi)$.

Proof. By symmetry among x_1, x_2, x_3 , one may assume

$$\tau(\phi_2), \tau(\phi_3) \in \tau(\phi_1)\mathbb{C}\{t\}.$$

In this proof, we denote tensor products over \mathcal{O}_X (resp. $\mathcal{O}_{X^\sharp}, \mathcal{O}_{X^\dagger}$) by \otimes (resp. $\otimes^\sharp, \otimes^\dagger$).

We put α_1 and β_1 in the commutative diagrams of natural homomorphisms

$$\begin{CD} (\wedge^2 \text{gr}_C^1 \mathcal{O}) \otimes \omega_C \otimes \mathcal{O}_{C^\dagger} @>\alpha_1 \otimes \mathcal{O}_{C^\dagger}>> \text{gr}_C^0 \omega \otimes \mathcal{O}_{C^\dagger} \\ @VV(\wedge^2 f_1) \otimes^\dagger f_2 V @VVf_3 V \\ (\wedge^2 \text{gr}_{C^\dagger}^1 \mathcal{O}) \otimes^\dagger \omega_{C^\dagger} @>f_4>> \omega_{X^\dagger} \otimes^\dagger \mathcal{O}_{C^\dagger} \end{CD}$$

and

$$\begin{CD} (\text{gr}_C^0 \omega) \otimes (\text{gr}_C^1 \mathcal{O}) \otimes \mathcal{O}_{C^\dagger} @>\beta_1 \otimes \mathcal{O}_{C^\dagger}>> (\text{gr}_C^1 \omega) \otimes \mathcal{O}_{C^\dagger} \\ @VVg_2 \otimes^\dagger f_1 V @VVg_1 V \\ \omega_{X^\sharp} \otimes^\sharp \text{gr}_{C^\dagger}^1 \mathcal{O} @>\sim>> \text{gr}_{C^\dagger}^1(\omega_{X^\sharp} \otimes^\sharp \mathcal{O}_{C^\dagger}) \end{CD}$$

where

$$\begin{aligned} f_1 &: \text{gr}_C^1 \mathcal{O} \otimes \mathcal{O}_{C^\dagger} \rightarrow \text{gr}_{C^\dagger}^1 \mathcal{O}, \\ f_2 &: \omega_C \otimes \mathcal{O}_{C^\dagger} \rightarrow \omega_{C^\dagger}, \\ g_2 &: (\text{gr}_C^0 \omega) \otimes \mathcal{O}_{C^\dagger} \rightarrow \omega_{X^\sharp} \otimes^\sharp \mathcal{O}_{C^\dagger}, \end{aligned}$$

and f_4 is defined similarly to α_1 . Thus

$$\bar{m} \cdot i_p(1) = \sum_{i=1,2,4} \text{len Coker } f_i - \text{len Coker } f_3,$$

and

$$\bar{m} \cdot w_p(1) = 2 \cdot \text{len Coker } g_2 + \text{len Coker } f_1 - \text{len Coker } g_1,$$

because $(C^\dagger, P^\dagger) \rightarrow (C, P)$ is of degree \bar{m} . We note

$$\begin{aligned} \sigma(\phi) &= \phi(y_1, y_2, y_3, h_4(y_4)) \pmod{J^{\dagger(2)}} \\ &= \phi(h_1 + \bar{y}_1, h_2 + \bar{y}_2, h_3 + \bar{y}_3, h_4) \pmod{J^{\dagger(2)}} \\ &= \tau(\phi_1)\bar{y}_1 + \tau(\phi_2)\bar{y}_2 + \tau(\phi_3)\bar{y}_3. \end{aligned}$$

Since $C^\sharp \not\subset \text{Sing } X^\sharp$ (2.4), one has $\sigma(\phi) \neq 0$. Hence $\tau(\phi_1) \neq 0$, and $\sigma(\phi)/\tau(\phi_1)$ is a part of a free basis of $J^\dagger/J^{\dagger(2)}$. Then we claim

$$(2.12.1) \quad \text{gr}_{C^\dagger}^1(\mathcal{O}_{X^\dagger}) \simeq (J^\dagger/J^{\dagger(2)})/\mathbb{C}\{t\}\sigma(\phi)/\tau(\phi_1).$$

Indeed, since $\sigma(\phi) = (\sigma(\phi)/\tau(\phi_1)) \cdot \tau(\phi_1) \in J^\dagger/J^{\dagger(2)}$ is sent to 0 in $\text{gr}_{C^\dagger}^1(\mathcal{O}_{X^\dagger})$ and $\tau(\phi_1) \neq 0$ in \mathcal{O}_{C^\dagger} , one sees that $\sigma(\phi)/\tau(\phi_1)$ is also sent to 0. Thus it induces a surjective homomorphism

$$(J^\dagger/J^{\dagger(2)})/\mathbb{C}\{t\}\sigma(\phi)/\tau(\phi_1) \twoheadrightarrow \text{gr}_{C^\dagger}^1(\mathcal{O}_{X^\dagger}),$$

which is an isomorphism because both are free of rank 2, and (2.12.1) is checked. Since we have \mathcal{O}_C -module surjections

$$\mathrm{gr}_C^1 \mathcal{O} \simeq (\mathrm{gr}_{C^\#}^1 \mathcal{O})_{\{0\}} \leftarrow J_{\{0\}}^\#$$

and

$$\mathrm{gr}_C^1 \omega \simeq (\mathrm{gr}_{C^\#}^1 \mathcal{O} \otimes^\# \omega_{X^\#})_{\{0\}} \simeq (\mathrm{gr}_{C^\#}^1 \mathcal{O})_{\{-k\}} \leftarrow J_{\{-k\}}^\#$$

by (2.9) and (2.10), we can use (2.12.1) to calculate the cokernels of f_1 and g_1 as

$$\mathrm{Coker} f_1 \simeq (J^\dagger/J^{\dagger(2)})/\mathbb{C}\{t\}\sigma(J_{\{0\}}^\#) + \mathbb{C}\{t\}(\sigma(\phi)/\tau(\phi_1)),$$

and

$$\mathrm{Coker} g_1 \simeq (J^\dagger/J^{\dagger(2)})/\mathbb{C}\{t\}\sigma(J_{\{-k\}}^\#) + \mathbb{C}\{t\}(\sigma(\phi)/\tau(\phi_1)).$$

Hence we have

$$\mathrm{len}(\mathrm{Coker} f_1) = [\phi, J_{\{0\}}^\#, J_{\{0\}}^\#] - \mathrm{ord} \tau(\phi_1),$$

and

$$\mathrm{len}(\mathrm{Coker} g_1) = [\phi, J_{\{-k\}}^\#, J_{\{-k\}}^\#] - \mathrm{ord} \tau(\phi_1),$$

because $\sigma(\phi)/\tau(\phi_1)$ is a part of a free basis of $J^\dagger/J^{\dagger(2)}$.

It is clear that $\mathrm{len} \mathrm{Coker} f_2 = \bar{m} - 1$, because $C^\dagger \rightarrow C$ is given by $t \mapsto t^{\bar{m}}$. One sees that the cokernel of

$$f_3: \mathrm{gr}_C^0 \omega \otimes_{\mathcal{O}_{C^\dagger}} \rightarrow \omega_{X^\dagger} \otimes^\# \mathcal{O}_{C^\dagger} \rightarrow \omega_{X^\dagger} \otimes^\dagger \mathcal{O}_{C^\dagger}$$

has length $R(k) + a_4 - 1$ because the cokernel of $g_2: \mathrm{gr}_C^0 \omega \otimes_{\mathcal{O}_{C^\dagger}} \rightarrow \omega_{X^\dagger} \otimes^\# \mathcal{O}_{C^\dagger}$ has length $R(k)$ by the proof of (2.10) and

$$\begin{aligned} \omega_{X^\dagger} \otimes^\# \mathcal{O}_{C^\dagger} &= \mathcal{O}_{C^\dagger} \frac{dx_2 \wedge dx_3 \wedge dx_4}{\phi_1} \\ &= t^{a_4-1} \mathcal{O}_{C^\dagger} \frac{d\bar{y}_2 \wedge d\bar{y}_3 \wedge dy_4}{\psi_1} = t^{a_4-1} \omega_{X^\dagger} \otimes^\dagger \mathcal{O}_{C^\dagger}. \end{aligned}$$

One sees that $\mathrm{len} \mathrm{Coker} f_4 = \mathrm{ord}(\tau\phi_1)$ because we have

$$\mathrm{Im} f_4 = \mathcal{O}_{C^\dagger} d\bar{y}_2 \wedge d\bar{y}_3 \wedge dy_4 = \psi_1 \omega_{X^\dagger} \otimes^\dagger \mathcal{O}_{C^\dagger} = \tau\phi_1 \omega_{X^\dagger} \otimes^\dagger \mathcal{O}_{C^\dagger}$$

by $\mathrm{gr}_{C^\dagger}^1 \mathcal{O} = \mathcal{O}_{C^\dagger} \bar{y}_2 \oplus \mathcal{O}_{C^\dagger} \bar{y}_3$ (2.12.1). Thus one has

$$\bar{m} \cdot i_p(1) = \bar{m} - R(k) - a_4 + [\phi, J_{\{0\}}^\#, J_{\{0\}}^\#].$$

Since $\mathrm{len} \mathrm{Coker} g_2 = R(k)$ as above, one also has

$$\bar{m} \cdot w_p(1) = 2 \cdot R(k) + [\phi, J_{\{0\}}^\#, J_{\{0\}}^\#] - [\phi, J_{\{-k\}}^\#, J_{\{-k\}}^\#],$$

whence

$$\bar{m} \cdot w_p^*(1) = \bar{m} - 3 \cdot R(k) - a_4 + [\phi, J_{\{-k\}}^\#, J_{\{-k\}}^\#]. \quad \square$$

(2.13) **Definition.** For $b \in \mathbb{Z}_+^4$, let

$$x^b = x_1^{b_1} x_2^{b_2} x_3^{b_3} x_4^{b_4}, \quad \text{where } b = (b_1, b_2, b_3, b_4).$$

For normalized ℓ -coordinates (x) and t , we define

(2.13.1) for $b, c \in \mathbb{Z}_+^4$ such that $\text{ow}(x^b) = \text{ow}(x^c)$, let $P(b, c) = x^b - x^c$, $\|P(b, c)\| = \text{ord}(x^b) \in \mathbb{Z}_+$ if $b \neq c$ (∞ if $b = c$), and let us define $\text{ex}_4(P(b, c)) \in \mathbb{Z}^3$ by $\text{ex}_4(P(b, c))_i = b_i - c_i$ ($i \in [1, 3]$). Then all such $P(b, c)$ generate the ideal J^\sharp , and

$$(2.13.2) \quad P(b, c) = P(b, d) - P(c, d),$$

$$(2.13.3) \quad P(b + b', c + c') = P(b, c)x^{b'} - P(b', c')x^c$$

if $b, c, d, b', c' \in \mathbb{Z}_+^4$ satisfy $\text{ow}(x^b) = \text{ow}(x^c) = \text{ow}(x^d)$ and $\text{ow}(x^{b'}) = \text{ow}(x^{c'})$.

The following allows us to calculate $[\ast, \ast, \ast]$ in (2.11).

(2.14) **Proposition.** Assume that (x) and t are normalized and let us use the notation of (2.11). Then for P as in (2.13.1), one has

$$\sigma(P) = t^{\|P\|} \sum_{i=1}^3 \text{ex}_4(P)_i Y_i \quad (Y_i = \bar{y}_i / t^{a_i}),$$

and hence

$$[P_1, P_2, P_3] = \begin{cases} \infty & \text{if } \text{ex}_4(P_1), \text{ex}_4(P_2), \text{ex}_4(P_3) \text{ are dependent,} \\ \sum_{i=1,2,3} \|P_i\| - \sum_{i=1,2,3} a_i & \text{otherwise,} \end{cases}$$

for such P_1, P_2, P_3 as in (2.13.1).

Proof. One sees

$$\begin{aligned} \sigma(P) &\equiv t^{a_4 b_4} \prod_{i=1}^3 (t^{a_i} + \bar{y}_i)^{b_i} - t^{a_4 c_4} \prod_{i=1}^3 (t^{a_i} + \bar{v}_i)^{c_i} \quad (J^{\dagger(2)}) \\ &\equiv t^{\|P\|} \left(\prod_{i=1}^3 (1 + Y_i)^{b_i} - \prod_{i=1}^3 (1 + Y_i)^{c_i} \right) \\ &\equiv t^{\|P\|} \sum_{i=1}^3 (b_i - c_i) Y_i = t^{\|P\|} \sum_{i=1}^3 \text{ex}_4(P)_i Y_i. \quad \square \end{aligned}$$

(2.15) **Corollary.** Under the notation and assumptions of (2.4), assume that (X, P) is singular. Then $i_P(1) \geq 1$.

Proof. If $m = 1$, then $C^\sharp \simeq C$ is smooth and the assertion follows from (2.16). Let us assume $m > 1$ and choose normalized ℓ -coordinates (x) and t (2.7)

and an ℓ -equation ϕ (1a.5) so that $\text{wt } x_4 \equiv \text{wt } \phi$ and $\text{wt } x_1 x_2 \equiv 0 \pmod{m}$ by the classification of terminal singularities (1a.5), and we will follow (2.11). By (1a.6), we see that $\text{wt } x_1, \text{wt } x_2, \text{wt } x_3$ are prime to m , and treat two cases.

Case where $C^\sharp\text{-wt } \phi \equiv 0 \pmod{m}$. One has $a_4 = \bar{m}$ since (x) and t are normalized and $(\bar{m}, 0) \in \text{ow}(C^\sharp)$. One has

$$(2.15.1) \quad \bar{m} \cdot i_p(1) \geq [J_{\{0\}}^\sharp, J_{\{0\}}^\sharp, J_{\{0\}}^\sharp] - R(\text{wt } x_3)$$

by (2.12). Hence we see $\bar{m} \cdot i_p(1) \geq [P_1, P_2, P_3] - R(\text{wt } x_3)$ for suitable invariant P_1, P_2, P_3 in (2.13.1). Hence $\text{wt}(P_i) \equiv 0 \pmod{m}$ for all i . Since $\text{ex}_4(P_1), \dots, \text{ex}_4(P_3)$ are independent, one sees that there is a permutation $\gamma \in \mathfrak{S}_3$ such that $\text{ex}_4(P_{\gamma(i)})_i \neq 0$ for all $i \in [1, 3]$ by expanding the determinant. Thus one has monomials v_i in x such that $\text{ow}(v_i x_i) = (\|P_{\gamma(i)}\|, 0)$, and we have $v_i \neq 1$ by $\text{wt } x_i \not\equiv 0 \pmod{m}$ for all $i \in [1, 3]$. Hence $\|P_{\gamma(i)}\| > a_i$ for $i = 1, 2$ and $\|P_{\gamma(3)}\| \geq a_3 + R(\text{wt } x_3)$ (2.8). By (2.14), one has

$$\bar{m} \cdot i_p(1) \geq \sum_{i=1,2} \{\|P_{\gamma(i)}\| - a_i\} + \|P_{\gamma(3)}\| - a_3 - R(\text{wt } x_3) > 0.$$

Case where $m = 4$ and $C^\sharp\text{-wt } \phi \equiv 2 \pmod{4}$. Then $(4, 0) \in \text{ow}(C^\sharp)$ by (2.7(iii)), and $(4, 0) \in (\text{ow}(C^\sharp) - \{0\}) + (\text{ow}(C^\sharp) - \{0\})$ by (2.7(ii)) because no x_i has $\text{wt} \equiv 0 \pmod{4}$, whence $(1, \pm 1), (2, 0)$, or $(2, 2) \in \text{ow}(C^\sharp)$ by $\bar{m} = 2, 4$ (1.16). If $(2, 0) \in \text{ow}(C^\sharp)$, then we can repeat the same argument to get $(1, \pm 1) \in \text{ow}(C^\sharp)$. If $(1, \pm 1) \in \text{ow}(C^\sharp)$, then we have $(2, 2) = 2(1, \pm 1) \in \text{ow}(C^\sharp)$. Thus we have $(2, 2) \in \text{ow}(C^\sharp)$ anyway, and hence $a_4 = \text{ord } x_4 = 2$ by normalizedness. Since $x_4 \phi \in J_{\{0\}}^\sharp$, (2.12) implies the same formula as (2.15.1):

$$\begin{aligned} m \cdot i_p(1) &\geq m - R(\text{wt } x_3) - a_4 + [J_{\{0\}}^\sharp, J_{\{0\}}^\sharp, J_{\{0\}}^\sharp] - a_4 \\ &= [J_{\{0\}}^\sharp, J_{\{0\}}^\sharp, J_{\{0\}}^\sharp] - R(\text{wt } x_3), \end{aligned}$$

where we used that $m = 4 = 2a_4$. The rest is the same as the previous case. \square

(2.16) **Lemma.** Assume that (C^\sharp, P^\sharp) is $smno$ l -coordinate system (x) and assumptions of (2.4). Then there exists notation and assumptions of (2.4). Then there exists an l -coordinate system (x) and an ℓ -equation ϕ (1a.5) of (X, P) such that C^\sharp is the x_4 -axis and $\phi \equiv x_4^r x_1 \pmod{(x_1, x_2, x_3)^2}$ ($r \geq 0$). Furthermore we have

- (i) $\text{len}_{P^\sharp} I_{C^\sharp}^{(2)} / I_{C^\sharp}^2 = r$,
- (ii) if $\text{wt } \phi \equiv 0 \pmod{m}$ then

$$i_p(1) = \begin{cases} r & \text{if } m = 1, \\ [r/m] + 1 & \text{if } m > 1 \end{cases}$$

and

- (iii) if $m = 4$ and $\text{wt } \phi \equiv 2 \pmod{4}$ then $i_p(1) = [(r + 6)/4]$.

Proof. We note that $s = 1$ and $\bar{m} = m$ by smoothness of C^\sharp . The first assertion is obvious, and (i) follows from $I_{C^\sharp}^{(2)}/I_{C^\sharp}^2 \simeq (\mathcal{O}_{C^\sharp}/(x_4^r)) \cdot x_1 \simeq \mathbb{C}\{x_4\}/(x_4^r)$. From (2.12) and the definition of $[\ast, \ast, \ast]$ follows

$$\begin{aligned} i_P(1) &= \frac{1}{m} \left\{ m - R \left(\sum_{i=1}^4 \text{wt } x_i - \text{wt } \phi \right) - 1 + [x_4^r x_1, x_4^{R(\text{wt } x_2)} x_2, x_4^{R(\text{wt } x_3)} x_3] \right\} \\ &= \frac{1}{m} \left\{ \sum_{i=1}^4 R(\text{wt } x_i) + r - R(\text{wt } x_1) - R \left(\sum_{i=1}^4 \text{wt } x_i - \text{wt } \phi \right) \right\}. \end{aligned}$$

If $m = 1$, then we are done because $R(z) = 0$ for $z \in \mathbb{Z}$. Assume that $m > 1$ and $\text{wt } \phi = 0 \pmod{m}$. By (1a.6), there exists $\sigma \in \mathfrak{S}_4$ such that $R(\text{wt } x_{\sigma(1)}) = 0$, $R(\text{wt } x_{\sigma(2)}) + R(\text{wt } x_{\sigma(3)}) = m$, whence

$$\sum_{1 \leq i \leq 4} R(\text{wt } x_i) - R \left(\sum_{1 \leq i \leq 4} \text{wt } x_i - \text{wt } \phi \right) = m,$$

and

$$i_P(1) = \frac{1}{m} \{ m + r - R(\text{wt } x_1) \} = [r/m] + 1,$$

because $R(\text{wt } x_1) \in [0, m)$. This proves (ii). Assume that $m = 4$ and $\text{wt } \phi \equiv 2 \pmod{4}$. By (1a.6), there exists $\sigma \in \mathfrak{S}_4$ such that $R(\text{wt } x_{\sigma(1)}) = 2$, $R(\text{wt } x_{\sigma(2)}) + R(\text{wt } x_{\sigma(3)}) = 4$, whence $\sum_{1 \leq i \leq 4} R(\text{wt } x_i) - R(\sum_{1 \leq i \leq 4} \text{wt } x_i - \text{wt } \phi) = 6$, and

$$i_P(1) = \frac{1}{4} \{ r + 6 - R(\text{wt } x_1) \} = [(r + 6)/4],$$

because $R(\text{wt } x_1) \in [0, 4)$. \square

3. EMBEDDING DIMENSION OF (C^\sharp, P^\sharp)

The main purpose of this section is to prove that the pull back C^\sharp (2.4) of C by the canonical cover $(X^\sharp, P^\sharp) \rightarrow (X, P)$ has only planar singularities if $X \supset C \simeq \mathbb{P}^1$ is an extremal nbd. We treat primitive points and imprimitive points separately.

Part I. Primitive point P . The following is the main theorem.

(3.1) **Theorem.** *If $X \supset C \simeq \mathbb{P}^1$ is an extremal nbd and P a primitive point of index m , then $\text{emb dim}_{P^\sharp} C^\sharp \leq 2$, where (C^\sharp, P^\sharp) is the pull back of C to the canonical cover (X^\sharp, P^\sharp) .*

(3.2) Under the above notation and assumptions, let us choose normalized ℓ -coordinates (x) and t (2.7), and an ℓ -equation ϕ so that $C^\sharp\text{-wt } \phi \equiv C^\sharp\text{-wt } x_4$, $C^\sharp\text{-wt } x_2 x_3 \equiv 0 \pmod{m}$ (1a.5). (We note that we may permute x_2 and x_3 .)

We note that $\text{emb dim}_{P^\sharp} C^\sharp \leq 2$ is equivalent to “the semigroup $\text{ord}(C^\sharp)$ is generated by two elements (2.7).” It is clear that $\text{ord}(C^\sharp)$ is generated by 2 and

the smallest odd number if $2 \in \text{ord}(C^\sharp)$. So we assume that $m, a_1, \dots, a_4 \geq 3$ and use the notation of §2 in Part I of this section.

First we treat the exceptional terminal singularities (1a.6).

(3.3) **Lemma.** *If $m = 4$ and $C^\sharp\text{-wt}(\phi) \equiv 2 \pmod{4}$, then one has $\text{emb dim}_{P^\sharp} C^\sharp \leq 2$.*

Proof. One has $a_4 \equiv 2 \pmod{4}$. Then $a_1, a_2, a_3, a_4 \not\equiv 0 \pmod{4}$ and

$$4 \in (\text{ord}(C^\sharp) - \{0\}) + (\text{ord}(C^\sharp) - \{0\}) \pmod{4} \quad (2.7).$$

Whence $2 \in \text{ord}(C^\sharp)$, and we are done by the remark in (3.2). \square

(3.4) **Lemma.** *Assume that $w_p(0) < 1$, $i_p(1) \leq 3$, and $\text{emb dim}_{P^\sharp} C^\sharp \geq 3$. Then $i_p(1) = 3$, $a_1 > m$, $a_2 = 3$, $a_4 = m$, $R(a_1) = 3n$ for some integer $n \in [1, m/3]$ modulo permutation of x_2 and x_3 .*

Proof. By normalizedness of (x) and t , one has $a_4 = m$ and $(a_1 a_2 a_3, m) = 1$. Since $m \cdot w_p(0) = R(a_1)$ (2.10), one has $R(a_1) < m$ by (2.3.3). From $R(a_1) \in \text{ord}(C^\sharp)$, one has $R(a_1) \in \mathbb{Z}_+ a_1 + \mathbb{Z}_+ a_\alpha$ ($\alpha = 2$ or 3), because $a_2 + a_3, a_4 \equiv 0 \pmod{m}$.

(3.4.1) We claim that $a_1 > m$ and thus $R(a_1) \in \mathbb{Z} a_\alpha$. To see this, let us assume $a_1 \leq m$. Then $a_1 < m$ by $(a_1, m) = 1$. Thus $m = a_1 + R(a_1)$ by $R(a_1)$, $a_1 < m$, and $a_1 + R(a_1) \equiv 0 \pmod{m}$. Whence $m \in \mathbb{Z}_+ a_1 + \mathbb{Z}_+ a_\alpha$. Since $a_1 > 1$ and $(m, a_1) = 1$, one sees $m \notin \mathbb{Z}_+ a_1$ and hence $a_4 = m \in \mathbb{Z}_+ a_1 + \mathbb{Z}_+ a_\alpha$. Thus $a_2 + a_3 \in \mathbb{Z}_+ a_1 + \mathbb{Z}_+ a_\alpha$ by $a_2 + a_3 \equiv 0 \pmod{m}$, whence $a_{5-\alpha} \in \mathbb{Z}_+ a_1 + \mathbb{Z}_+ a_\alpha$. Thus a_1 and a_α generate $\text{ord}(C^\sharp)$, which is a contradiction, and (3.4.1) is proved.

(3.4.2) We claim that $m \geq 4$, and that a_2 or $a_3 = 3$ implies (3.4) if $i_p(1) = 3$. First by a remark in (3.2), $m > R(a_1) \geq 3$, which implies $m \geq 4$. If one has $a_3 \geq a_2 = 3$ modulo permutation, then $2a_3 \geq a_2 + a_3 \geq m$, whence $R(a_1) = a_3$ ($\alpha = 3$) or $R(a_1) \in 3\mathbb{Z}_+$ ($\alpha = 2$). If $R(a_1) = a_3$, then $a_3 < m$. Thus one sees $a_3 = m - 3$, $a_1 \equiv 3 \pmod{m}$, and hence $a_1, a_4 \in \mathbb{Z}_+ a_2 + \mathbb{Z}_+ a_3$. This is a contradiction, and $R(a_1) \in 3\mathbb{Z}_+$. Thus (3.4.2) is proved.

(3.4.3) Let S be the set of $b = (b_1, b_2, b_3) \in \mathbb{Z}_+^3 - \{0\}$ such that $a_1 b_1 + a_2 b_2 + a_3 b_3 \equiv 0 \pmod{m}$. We say that $b \in S$ is *reduced* if $b \notin S + S$. Let $\|b\| = a_1 b_1 + a_2 b_2 + a_3 b_3$.

Let $\lambda = (0, 1, 1)$. We note that $b = \lambda$ if $b \in S$ is reduced and satisfies $b_2, b_3 > 0$. For $b = (b_1, b_2, b_3) \in S$, let

$$Q(b) = P((b_1, b_2, b_3, 0), (0, 0, 0, \|b\|/m)),$$

with the notation of (2.13.1). Then $\text{ex}_4(Q(b)) = b$ and $\|Q(b)\| = \|b\|$. One sees that $J_{\{0\}}^\sharp$ is generated by $Q(b)$ ($b \in S$) by (2.13.2) and that by $Q(b)$'s such that $b \in S$ is reduced because (2.13.3) reduces to

$$Q(b + b') = Q(b)x^{b'} - Q(b')x_4^{\|b\|/m} \quad (b, b' \in S).$$

By subadditivity of $[*, *, *]$ (2.11), (2.12) implies

$$3m \geq m \cdot i_p(1) = -R(a_1) + [\phi, J_{\{0\}}^\#, J_{\{0\}}^\#] \geq -R(a_1) + [Q(b^1), Q(b^2), Q(b^3)]$$

for some reduced $b^1, b^2, b^3 \in S$. Throughout Part I of this section, b^i denotes such a vector *not* a power of b . By (2.14), one has

$$(3.4.4) \quad 3m \geq \sum_{i=1}^3 (\|b^i\| - a_i) - R(a_1)$$

and that b^1, b^2, b^3 are independent. Let $b^i = (b_j^i)$. We note

$$(3.4.5) \quad \|b^i\| \geq (b_1^i - 1)m + a_1 + R(a_1) \quad \text{if } b_1^i \geq 1.$$

Indeed one sees $\|b^i\| \geq (b_1^i - 1)a_1 + a_1 \geq (b_1^i - 1)m + a_1$ by (3.4.1), whence $\|b^i\| - (b_1^i - 1)m \geq a_1$. Thus $\|b^i\| - (b_1^i - 1)m \geq a_1 + R(a_1)$ because $\|b^i\| \equiv 0 \pmod{m}$ and $a_1 + R(a_1) = m \cdot [a_1/m]$ by $R(a_1) < m$.

(3.4.6) We treat three cases.

$\begin{pmatrix} + & 0 \\ & + & 0 \\ & 0 & + \end{pmatrix}$	$\begin{pmatrix} + & 0 \\ + & & \\ 0 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} + & & \\ 0 & + & 0 \\ 0 & 1 & 1 \end{pmatrix}$	<p>b^i is the ith row. + at (i, j) means $b_j^i > 0$.</p>
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FIGURE (3.4.A) FIGURE (3.4.B) FIGURE (3.4.C)

Case (A). None of b^1, b^2, b^3 is λ . We will derive a contradiction in this case. Expanding $\det(b_j^i)$, one sees that there is a $\sigma \in \mathfrak{S}_3$ such that $b_i^{\sigma(i)} \neq 0$ for all i . Since $b^{\sigma(1)} \neq \lambda$, one has $b_{\tau(3)}^{\sigma(1)} = 0$ for some $\tau \in \mathfrak{S}_3$ fixing 1 (3.4.3). Modulo such permutations $\sigma \circ \tau$ (resp. τ) of b^i 's (resp. x_2 and x_3), one may assume that $b_i^i \neq 0$ for all i and $b_3^1 = 0$. Since b^2 and b^3 are reduced and $b^2, b^3 \neq \lambda$, one sees that $b_3^2 = b_2^3 = 0$ (3.4.3). Now (b_j^i) is as in Figure (3.4.A) above.

By (3.4.5), we rewrite (3.4.4) as

$$(3.4.6.1) \quad 3m \geq (\|b^1\| - a_1 - R(a_1)) + (\|b^2\| - a_2) + (\|b^3\| - a_3) \geq (b_1^1 - 1)m + (\|b^2\| - a_2) + (\|b^3\| - a_3).$$

Let $i \in [2, 3]$. If $b_1^i = 0$, then $b_i^i = m$ by $b_3^2 = b_2^3 = 0$, whence $\|b^i\| - a_i = (m - 1)a_i \geq 3(m - 1) > 2m$ by $a_i \geq 3$ and $m \geq 4$. If $b_1^i \geq 1$, then $\|b^i\| - a_i \geq b_1^i a_1 > b_1^i m$. Hence from (3.4.6.1), one has $b_1^2 = b_1^3 = 1$. Then (3.4.6.1) implies $3m > (b_1^1 - 1)m + 2m$, and one has $b_1^1 = 1$. Since b^1 and b^2 are reduced and $b_1^1 = b_1^2 = 1$, one has $b_2^1, b_2^2 \in [0, m - 1]$. By $b^1, b^2 \in S$ and $b_1^1 = b_1^2 = 1$ and $b_3^1 = b_3^2 = 0$, one has $b_2^1 \equiv b_2^2 \pmod{m}$, whence $b_2^1 = b_2^2$ and $b^1 = b^2$, whence b^1, b^2, b^3 are dependent. This is a contradiction.

Case (B). $b^3 = \lambda$ and $b_1^1, b_1^2 > 0$ modulo permutation of b^1, b^2, b^3 . We will also derive a contradiction in this case. Modulo permutation of b^1 and b^2 one may assume $\|b^1\| \leq \|b^2\|$, and modulo further permutation of x_2 and x_3 one may also assume $b_3^1 = 0$ because b^1 is reduced and $b^1 \neq \lambda$ (3.4.3). Now (b_j^i) is as in (3.4.B).

Since $a_1 > m$, one has $2m \leq a_1 + R(a_1)$, whence $m \geq 3m - a_1 - R(a_1)$. Then (3.4.4) gives

$$m \geq 3m - (a_1 + R(a_1)) \geq \sum_{i=1,2} \{\|b^i\| - (a_1 + R(a_1))\}.$$

Since $\|b^i\| - (a_1 + R(a_1)) \in m\mathbb{Z}_+$ (3.4.5), one has $\|b^1\| = a_1 + R(a_1)$ by $\|b^1\| \leq \|b^2\|$. Then the inequality gives $\|b^2\| \leq 3m$. Hence one has $b_1^1 = 1, b_1^2 = 1, 2$ by (3.4.5). Since $b_1^1 = 1$ and $b_3^1 = 0$, one has $b_2^1 > 0$. Since $\|b^1\| = a_1 + R(a_1)$, one has $b_2^1 = R(a_1)/a_2 < m/3$. Since $b_1^2 = 1$ or 2 and since b^2 is reduced, one easily sees $b^2 = (b_1^2, b_1^2 \cdot b_2^1, 0)$ or $(b_1^2, 0, m - b_1^2 \cdot b_2^1)$ by $b_2^2 \cdot b_3^2 = 0$ and $0 < b_1^2 \cdot b_2^1 < m$. We have the second case since b^2 is independent of b^1 . Then

$$\begin{aligned} \|b^2\| &= b_1^2 \cdot a_1 + (m - b_1^2 \cdot b_2^1)a_3 \geq b_1^2 \cdot a_1 + 3(m - b_1^2 \cdot b_2^1) \\ &= 3m + b_1^2(a_1 - 3b_2^1) > 3m, \end{aligned}$$

because $a_1 > m > 3b_2^1$. This contradicts $\|b^2\| \leq 3m$ obtained above.

Case (C). $b^3 = \lambda, b_1^1 > 0$, and $b_1^2 = 0$ modulo permutation of b^1, b^2, b^3 . Modulo permutation of x_2 and x_3 , one may further assume $b_3^2 = 0$. Now (b_j^i) is as in (3.4.C). Then $b^2 = (0, m, 0)$ since it is reduced. By (3.4.5) applied to b^1 and (3.4.4), one has $3m \geq (b_1^1 - 1)m + a_2m$. Thus $a_2 = 3$ and $i_P(1) = 3$ from the inequality. Thus the proof of (3.4) is finished. \square

(3.5) We now prove (3.1). We will derive a contradiction assuming that there is a point $P \in C$ at which $\text{emb dim}_{P^*} C^\# \geq 3$. By (2.3.3), one sees $w_P(0) < 1$ and $i_P(1) \leq 3$, since $w_Q(0) \geq 0$ and $i_Q(1) \geq 0$ for all $Q \in C$. By (3.4), one has $i_P(1) = 3$. One has $\sum_{Q \neq P} i_Q(1) = 0$ by (2.3.3), whence P is the only singular point of X on C by (2.15). Thus by (2.3.3), one has $w_P^*(1) \leq 1$. We will work at P using the notation of (3.2), and estimate $w_P^*(1)$ by an argument similar to (3.4). By (3.4), one has $a_2 = 3, a_4 = m, R(a_1) = 3n$ for some $n \in [1, m/3)$, and $a_1 > m$. By $x_2^n \phi \in J_{\{3n\}}^\#$, one sees from (2.12) that

$$\begin{aligned} (3.5.1) \quad m \geq m \cdot w_P^*(1) &= -9n + [\phi, J_{\{3n\}}^\#, J_{\{3n\}}^\#] \\ &= -12n + [x_2^n \phi, J_{\{3n\}}^\#, J_{\{3n\}}^\#] \\ &\geq [J_{\{3n\}}^\#, J_{\{3n\}}^\#, J_{\{3n\}}^\#] - 12n. \end{aligned}$$

Let $\nu = (0, n, 0)$ and, following (3.4.3), let T be the set of $b \in \mathbb{Z}_+^3 - \{\nu\}$ such that $\|b\| \equiv 3n \pmod{m}$. We say that $b \in T$ is reduced if $b \notin T + S$ (3.4.3).

For $b = (b_1, b_2, b_3) \in T$, let

$$\overline{Q}(b) = P((b_1, b_2, b_3, 0), (0, n, 0, (\|b\| - 3n)/m)).$$

Then $\text{ex}_4(\overline{Q}(b)) = b - \nu$ and $\|\overline{Q}(b)\| = \|b\|$. One sees that $J_{\{3n\}}^\sharp$ is generated by $\overline{Q}(b)$ ($b \in T$) by (2.13.2) because $3n < m$, and that by $\overline{Q}(b)$'s such that $b \in T$ is reduced, because (2.13.3) reduces to

$$\overline{Q}(b + b') = \overline{Q}(b)x^{b'} - \overline{Q}(b' + \nu)x_4^{(\|b\| - 3n)/m} \quad (b \in T, b' \in S).$$

By subadditivity of $[*, *, *]$ (2.11), (2.14) and (3.5.1) imply

$$(3.5.2) \quad m \geq \sum_{i=1}^3 (\|b^i\| - a_i) - 12n$$

for some reduced $b^1, b^2, b^3 \in T$ such that $b^1 - \nu, b^2 - \nu, b^3 - \nu$ are independent. Let $b^i = (b_j^i)$. As in case (A) of (3.4.6), we may assume $(b^i - \nu)_i \neq 0$ for all i .

(3.5.3) We note that an integer $z \in (0, m)$ belongs to $\text{ord}(C^\sharp)$ iff $z = a_3 = m - 3$ or $z \in 3\mathbb{Z}_+$, and there is only one way to express such z as sum of a_1, a_2, a_3, a_4 .

(3.5.4) We claim $\|b^1\| \geq a_1 + 6n$. One sees $6n - m \notin \text{ord}(C^\sharp)$ by (3.5.3), because $6n - m < m$ by $m > 3n$, $6n - m \neq 0$ (3) by $(3, m) = 1$, and $6n - m \neq m - 3$. On the other hand, one sees that $\|b^1\| - a_1 \equiv 6n - m \pmod{m}$ by $\|b^1\| \equiv 3n \pmod{m}$, and that $\|b^1\| - a_1 \in \text{ord}(C^\sharp)$ by $(b^1 - \nu)_1 = b_1^1 > 0$. Thus $\|b^1\| - a_1 \geq 6n$ (cf. (2.8)), which is (3.5.4).

(3.5.5) We claim $\|b^3\| \geq a_3 + 3 + 3n$ and that the equality holds only when $b^3 = (0, n + 1, 1)$. Let us see that $a_1z_1 + a_2z_2 + a_3z_3 = 3 + 3n$ ($z \in \mathbb{Z}_+^3$) has a unique solution $z = (0, n + 1, 0)$. We claim

$$(3.5.5.1) \quad 3 + 3n \neq a_1, ia_3 \quad (i = 1, 2).$$

Indeed $3n + 3 = a_1$ or ia_3 implies $2n + 1$ or $n + 1 + i \equiv 0 \pmod{m}$ because $a_1 \equiv -3n, ia_3 \equiv -3i \pmod{m}$ and $(3, m) = 1$. By $m > 3n$, one sees that this is possible only when $i = 2, m = 4, n = 1$, whence $a_3 = 6/2 = 3$, which implies a contradiction $a_2 + a_3 \neq 0 \pmod{m}$. Hence (3.5.5.1) is proved. Thus one sees $3 + 3n - a_1 \notin \text{ord}(C^\sharp)$ because $3 + 3n - a_1 \leq 1$ by $a_1 > m > 3n$. Hence $z_1 = 0$. If $z_2 > 0$, then (3.5.3) implies $z = (0, n + 1, 0)$, because $a_2(z_2 - 1) + a_3z_3 = 3n < m$. Hence we may assume $z_2 = 0$. Then $z_3 \geq 3$ by (3.5.5.1). Hence by $a_3 > m/2$ ($a_3 \geq 3, a_3 + 3 \geq m$, and $(a_3, m) = 1$), one sees that $z_3a_3 > 3m/2 \geq m + 2 \geq 3 + 3n$. Thus $z = (0, n + 1, 0)$ is the only solution. Let us prove (3.5.5). Since $(b^3 - \nu)_3 = b_3^3 \geq 1, \|b^3\| - a_3 \in \text{ord}(C^\sharp)$. From $\|b^3\| - a_3 \equiv 3 + 3n$ and $3 + 3n - m \notin \text{ord}(C^\sharp)$ ($3 + 3n - m \leq 2$), one sees $\|b^3\| - a_3 \geq 3 + 3n$. The rest follows from the above.

(3.5.6) We claim that $\|b^2\| \geq m + 3n$, and that the equality holds only if $b^2 = (0, n + 1, 1)$. By $a_1 + 6n > m + 3n$ and $a_3 + 3 + 3n \geq m + 3n$, one may reduce (3.5.6) to (3.5.4) or (3.5.5) if b_1^2 or $b_3^2 > 0$. Hence we assume $b_1^2 = b_3^2 = 0$. Then by $b^2 \neq \nu$, one sees $b_2^2 \geq n + m$ and $\|b^2\| \geq 3n + 3m$, which proves (3.5.6).

(3.5.7) From (3.5.4)–(3.5.6), one sees

$$\sum_{i=1}^3 (\|b^i\| - a_i) \geq m + 12n$$

and that the equality holds only if $b^2 = b^3 = (0, n + 1, 1)$. This contradicts (3.5.2). Thus (3.1) is proved. \square

Part II. Imprimitve point P . The main result is (3.8).

(3.6) Let $X \supset C \simeq \mathbb{P}^1$ be an extremal nbd with an imprimitive point P of index m , subindex $\bar{m} (> 1)$, and splitting degree s . Let $X^b \supset C^b = C^b(1) \cup \dots \cup C^b(s) \ni P^b$ be the splitting cover of $X \supset C$ associated to P (1.12). We denote by $X^b(i) \supset C^b(i) \ni P^b(i)$ the extremal nbd obtained from $X^b \supset C^b \supset P^b$ by restriction to a small nbd of $C^b(i)$. Let $(X^\sharp, P^\sharp) \supset C^\sharp = C^\sharp(1) \cup \dots \cup C^\sharp(s)$ be the canonical cover of (X, P) and the irreducible components of the pull back C^\sharp of C (with reduced structure) such that $C^\sharp(i)$ dominates $C^b(i)$ for each i . We note that $(X^\sharp, P^\sharp) \rightarrow (X^b, P^b)$ is a μ_m -canonical cover. Let (x) and t be normalized ℓ -coordinates for $X \supset C$ at P and ϕ an ℓ -equation such that $\text{wt } x_1 + \text{wt } x_3 \equiv 0$ and $\text{wt } x_4 \equiv \text{wt } \phi \pmod{m}$. By (1.14) and (2.9), we have

$$\text{gr}_C^0 \omega \simeq (\mathcal{O}_{C^\sharp} \bar{\omega})^{\mu_{\bar{m}}} \xrightarrow{\sim} \bigoplus_i \text{gr}_{C^b(i)}^0 \omega \simeq \bigoplus_i (\mathcal{O}_{C^\sharp(i)} \bar{\omega})^{\mu_{\bar{m}}},$$

where $\bar{\omega}$ is a μ_m -semi-invariant generator of ω_{X^\sharp} at P^\sharp .

Hence we have $\mu_{\bar{m}}$ -semi-invariants $u_i \in \mathcal{O}_{X^\sharp, P^\sharp}$ for $i \in [1, s]$ such that $u_i|_{C^\sharp(j)} = 0$ for $j \in [1, s] \setminus \{i\}$ and $(u_i|_{C^\sharp(i)})\bar{\omega}$ is a generator of $\text{gr}_{C^b(i)}^0 \omega$. We also have a μ_m -semi-invariant $u \in \mathcal{O}_{X^\sharp, P^\sharp}$ such that $u\bar{\omega}$ is a generator of $\text{gr}_C^0 \omega$ at P . It is easy to see that we may further assume that u is a monomial in x with coefficient 1. For each i , we note that $\bar{m} \cdot w_{P^b(i)}(0)$, $\bar{m} \cdot w_P(0)$ are nonnegative integers such that $\bar{m} \cdot w_{P^b(i)}(0) \equiv \bar{m} \cdot w_P(0) \pmod{\bar{m}}$ by (2.10) and $\bar{m} \cdot w_{P^b(i)}(0)$, $\bar{m} \cdot w_P(0) < \bar{m}$ by (2.3.3). Whence we have

(3.7) **Proposition.** $w_{P^b(i)}(0) = w_P(0)$ for all i .

Hence we see $C^\sharp(i)\text{-ord}(u_i) = \text{ord}(u) = \bar{m} \cdot w_P(0) < \bar{m}$ (2.10). Thus if we consider the induced map

$$\Phi: (\mathcal{O}_{C^\sharp, P^\sharp} \bar{\omega})^{\mu_{\bar{m}}} \rightarrow \bigoplus_i \text{gr}_{C^b(i)}^0 \omega \otimes \mathbb{C}(P^b(i)) = \bigoplus_i \mathbb{C}(P^b) u_i \bar{\omega},$$

we see $\Phi(v\bar{\omega}) = 0$ for μ_m -semi-invariants v with $\text{ord}(v) > \bar{m} \cdot w_P(0)$.

(3.8) **Proposition.** *Modulo permutations of x_1, \dots, x_4 and modulo changes of ℓ -character χ (2.5), we have*

$$\text{ow}(C^\sharp) = \mathbb{Z}_+ \text{ow}(x_1) + \mathbb{Z}_+ \text{ow}(x_2)$$

(in particular (C^\sharp, P^\sharp) is a planar singularity (2.7)) and one of the following holds:

(3.8.1) $\text{wt } \phi \equiv 0 \pmod{m}$, $\text{wt } x_1 + \text{wt } x_3 \equiv 0 \pmod{m}$, $\text{ow}(x_4) = (\bar{m}, 0)$, $(\bar{m}, 0) \in \mathbb{Z}_{++} \text{ow}(x_1) + \mathbb{Z}_{++} \text{ow}(x_2)$, and $w_p(0) \geq 1/2$,

(3.8.2) $\text{wt } \phi \equiv 0 \pmod{m}$, $s = 2$, \bar{m} is an even integer ≥ 4 , and a list of wt and ord for (x) :

	x_1	x_2	x_3	x_4	
wt	1	-1	0	$\bar{m} + 1$	\pmod{m} ,
ord	1	$\bar{m} - 1$	\bar{m}	$\bar{m} + 1$	

(3.8.3) $s = \bar{m} = 2$, $\text{wt } \phi \equiv 2 \pmod{4}$, and

	x_1	x_2	x_3	x_4	
wt	1	3	3	2	$\pmod{4}$.
ord	1	1	1	2	

Proof. We treat three cases.

(3.8.4) *Case $\text{wt } \phi \equiv 0 \pmod{m}$ and $\text{ord } x_2 < \bar{m}$.* Then we claim that u is not a power of x_2 . Indeed if $u = x_2^\alpha$ for some α , then $C^\sharp \cdot \text{wt } x_2^{\alpha+1} \equiv 0 \pmod{m}$, whence $\alpha + 1 \geq m \geq 2\bar{m}$ by $(\text{wt } x_2, m) = 1$ (1a.5), and $\text{ord } u \geq \bar{m}$. This is a contradiction. Thus up to a permutation of x_1 and x_3 , we may assume $u = x_1 v$ for some monomial v . Then $\text{wt } x_2 u \equiv 0 \pmod{m}$, whence $\text{ord } x_2 u \equiv 0 \pmod{\bar{m}}$ and $\text{ord } x_2 u < 2\bar{m}$. Hence $\text{ord } x_2 u = \bar{m}$, i.e. $\text{ow}(x_2 u) = (\bar{m}, 0)$. By normalizedness, $\text{ow}(x_4) = \text{ow}(x_2 u)$. We also see $\text{wt } x_2 v \equiv \text{wt } x_2 u - \text{wt } x_1 \equiv \text{wt } x_3 \pmod{m}$ and $\text{ord } x_2 v < \text{ord } x_2 u = \bar{m}$, whence $\text{ow}(x_2 v) = \text{ow}(x_3)$ by normalizedness. Thus we have $\text{ow}(X^\sharp) = \mathbb{Z}_+ \text{ow}(x_1) + \mathbb{Z}_+ \text{ow}(x_2)$.

We claim $(\bar{m}, 0) \in \mathbb{Z}_{++} \text{ow}(x_1) + \mathbb{Z}_{++} \text{ow}(x_2)$. Indeed if otherwise, we have $(\bar{m}, 0) = \alpha \cdot \text{ow}(x_1)$ ($\alpha \in \mathbb{Z}$) up to permutation of x_1 and x_2 , whence $\alpha \equiv 0 \pmod{m}$ by $(\text{wt } x_1, m) = 1$, which contradicts $\bar{m} = \alpha \cdot \text{ord } x_1$ and $m > \bar{m}$. Thus the claim is proved.

We also claim that $w_p(0) \geq 1/2$. Indeed if otherwise, we have $\bar{m}/2 > \bar{m} \cdot w_p(0) = \text{ord } u$. Hence $\text{ord } x_2 = \bar{m} - \text{ord } u > \bar{m} \cdot w_p(0)$ and $\text{ord } x_3 \geq \bar{m} - \text{ord } x_1 \geq \bar{m} - \text{ord } u > \bar{m} \cdot w_p(0)$, because $x_1 \mid u$. Thus Φ factors through

$$(\mathcal{O}_{C^\sharp, P^\sharp} \bar{\omega} / (x_1^{\bar{m}}, x_2, x_3, x_4))^{\mu_{\bar{m}}} \simeq \mathbb{C}(P^b) x_1^\lambda \bar{\omega}$$

for a unique $\lambda \in (0, \bar{m})$ such that $\lambda \text{wt } x_1 \equiv -\text{wt } \bar{\omega} \pmod{\bar{m}}$. This contradicts the surjectivity of Φ and $s \geq 2$. Thus $w_p(0) \geq 1/2$ as claimed, and we have case (3.8.1).

(3.8.5) *Case $\text{ord } x_2 \geq \bar{m}$ and $\text{wt } \phi \equiv 0 \pmod{m}$.* We note first that $\text{ord } x_2 > \bar{m}$ by $(\text{ord } x_2, \bar{m}) = (\text{wt } x_2, \bar{m}) = 1$ and $\bar{m} > 1$. We may assume $\text{ord } x_1 \leq \text{ord } x_3$ up

to a permutation of x_1 and x_3 , whence $\text{ord}(x_3^2) \geq \bar{m}$. Since $x_1^{\bar{m}}, x_2, x_1x_3, x_3^2, x_4$ have $\text{ord} \geq \bar{m}$, we see that Φ factors through

$$(\mathcal{O}_{C^\sharp, P^\sharp} \bar{\omega} / (x_1^{\bar{m}}, x_1x_3, x_3^2, x_2, x_4))^{\mu_{\bar{m}}} \simeq \mathbb{C}(P^b)x_1^\lambda \bar{\omega} \oplus (\mathbb{C}(P^b)x_3 \bar{\omega})^{\mu_{\bar{m}}}$$

for a unique $\lambda \in (0, \bar{m})$ such that $\lambda \text{ wt } x_1 + \text{wt } \bar{\omega} \equiv 0 \pmod{\bar{m}}$. Hence by surjectivity of Φ , we have $s = 2$ and $\text{wt } x_3 + \text{wt } \bar{\omega} \equiv 0 \pmod{\bar{m}}$. Since $\text{wt } x_1 \equiv -\text{wt } x_3 \pmod{m}$, we see $\lambda \equiv -1 \pmod{\bar{m}}$, whence $\lambda = \bar{m} - 1$ and $\text{ord } x_1 = 1$ by $\text{ord } x_1^\lambda < \bar{m}$. Thus $\text{ord } x_3 = \bar{m} - 1$ by $\text{wt } x_3 \equiv -1$ and $\text{ord } x_3 < \bar{m}$ because $\Phi(x_3) \neq 0$. Hence $\text{ow}(x_1) + \text{ow}(x_3) = (\bar{m}, 0)$, whence it is equal to $\text{ow}(x_4)$ by the normalizedness as before. Since Φ factors through $\mathbb{C}x_1^{\bar{m}-1} \bar{\omega} \oplus \mathbb{C}x_3 \bar{\omega}$, we see that $u = x_1^{\bar{m}-1}$ or x_3 .

We claim that $u = x_1^{\bar{m}-1}$. Indeed if $u = x_3$, then $\text{wt } x_2 + \text{wt } x_3 \equiv 0 \pmod{m}$, and $\text{wt } x_2 \equiv \text{wt } x_1 \pmod{m}$ which contradicts normalizedness by $\text{ord } x_2 > 1$ and $\text{ord } x_1 = 1$. Hence $u = x_1^{\bar{m}-1}$.

Thus we have $\text{wt } x_2 \equiv -(\bar{m}-1) \text{ wt } x_1 \equiv (\bar{m}+1) \text{ wt } x_1 \pmod{m}$. Since $\text{ord } x_2 > \bar{m}$ and $\text{ord } x_1^{\bar{m}+1} = \bar{m} + 1$, we have $\text{ow}(x_2) = (\bar{m} + 1) \text{ow}(x_1)$ by normalizedness, whence $\text{ow}(C^\sharp) = \mathbb{Z}_+ \text{ow}(x_1) + \mathbb{Z}_+ \text{ow}(x_3)$. By $(\text{wt } x_2, m) = ((\bar{m} + 1) \text{ wt } x_1, m) = 1$, we see $\bar{m} \equiv 0 \pmod{2}$. By changing ℓ -character χ with $(1 + \bar{m})\chi$ if necessary, we may assume $\text{ow}(x_1) = (1, 1)$, $\text{ow}(x_2) = (\bar{m} + 1, \bar{m} + 1)$, $\text{ow}(x_4) = (\bar{m}, 0)$. Since $\text{ow}(x_3) \notin \mathbb{Z}_+ \text{ow}(x_1)$ (otherwise (C^\sharp, P^\sharp) is smooth (2.7)), we have $\text{ow}(x_3) = (\bar{m} - 1, -1)$. We note that if $\bar{m} = 2$, then permuting x_2 and x_3 , we have case (3.8.1). Hence we may assume $\bar{m} \geq 4$, and by permuting x_i 's, we have case (3.8.2).

(3.8.6) *Case* $m = 4, \bar{m} = s = 2, \text{wt } \phi \equiv 2 \pmod{4}$. Then we have $\text{wt } x \equiv (1, 3, 3, 2) \pmod{4}$ up to permutation of x_1, \dots, x_4 and a choice of ℓ -character χ (1a.5). Then we have $\text{wt } u \equiv 1 \pmod{4}$ and $\text{ord } u = 1$. Hence $u = x_1$ and $\text{ord } x_1 = 1$, whence $\text{ow}(x_4) = \text{ow}(x_1^2)$. Since $\bar{m} \cdot w_p(0) = 1$, Φ factors through $\mathbb{C}x_1 \bar{\omega} + \mathbb{C}x_2 \bar{\omega} + \mathbb{C}x_3 \bar{\omega}$. Thus $\text{ord } x_2 = 1$ or $\text{ord } x_3 = 1$ by surjectivity of Φ , and $\text{ord } x_2 = \text{ord } x_3 = 1$ by normalizedness. Hence $\text{ow}(C^\sharp) = \mathbb{Z}_+ \text{ow}(x_1) + \mathbb{Z}_+ \text{ow}(x_2)$, and we have case (3.8.3). \square

4. CLASSIFICATION OF $X \supset C$ AT P INTO CASES

After classifying $X \supset C \ni P$ into several cases, we will study the deformation processes using results in §§1a and 1b. These processes allow us to treat fewer cases with only ordinary singular points (4.5).

(4.1) Let $X \supset C \simeq \mathbb{P}^1$ be an extremal nbd, and $P \in C$ a singular point of X of index $m \geq 1$, subindex \bar{m} , splitting degree s . We note that $m = \bar{m} \cdot s$ and that $s = 1$ iff P is primitive. We consider the germs $(X, P) \supset (C, P)$, and use the notation in §2 like μ_m -cover $(X^\sharp, P^\sharp) \supset (C^\sharp, P^\sharp)$ of $X \supset C$, etc. One has $\text{emb dim}_{P^\sharp} C^\sharp \leq 2$ by (3.1) and (3.8). So, let $(x) = (x_1, \dots, x_4)$ and t be

normalized ℓ -coordinates (2.6) and ϕ an ℓ -equation of $X \supset C \ni P$ (1a.5) such that x_1 and x_2 form coordinates for (C^\sharp, P^\sharp) . Then (a_1, w_1) and (a_2, w_2) generate $\text{ow}(C^\sharp)$, where $\text{ow}_i = (a_i, w_i) = \text{ow}(x_i)$ ($i \in [1, 4]$).

We note that $U(s \cdot a_1 a_2) = U(sa_1 \cdot \text{ow}_2) = U(sa_2 \cdot \text{ow}_1)$ (cf. (2.8)) depends only on $\text{ow}(C^\sharp)$ and m . Indeed if $\text{ow}(C^\sharp) \simeq \mathbb{Z}_+$, then a_1 or $a_2 = 1$ and $a_1, a_2 \leq m$, and $U(a_1 a_2) = 1$; if $\text{ow}(C^\sharp) \neq \mathbb{Z}_+$, then

$$\{\text{ow}_1, \text{ow}_2\} = (\text{ow}(C^\sharp) - \{0\}) \setminus ((\text{ow}(C^\sharp) - \{0\}) + (\text{ow}(C^\sharp) - \{0\}))$$

and $U(s \cdot a_1 a_2)$ depends only on $\text{ow}(C^\sharp)$ and m .

(4.1.1) **Definition.** We call $U(s \cdot a_1 a_2)$ the *size* of $X \supset C \ni P$ (or simply *size* of P) and denote it by siz_P (cf. (4.10)).

(4.2) **Proposition-Definition.** *Modulo permutations of x_i 's (and changes of ℓ -characters in case of imprimitive points) which preserve the condition*

$$\text{ow}(C^\sharp) = \mathbb{Z}_+ \text{ow}_1 + \mathbb{Z}_+ \text{ow}_2,$$

exactly one of the following cases holds. This division is independent of the choice of normalized ℓ -coordinates (x) and t .

(Primitive point P)

Case (I) $m > 1$, $C^\sharp\text{-wt } \phi \equiv 0 \pmod{m}$,

(IA) $a_1 + a_3 \equiv 0 \pmod{m}$, $a_4 = m$, $m \in \mathbb{Z}_{++} a_1 + \mathbb{Z}_{++} a_2$,
 where we may still permute x_1 and x_3 if $a_2 = 1$,

(IB) $a_1 + a_3 \equiv 0 \pmod{m}$, $a_2 = m$, $2 \leq a_1$,

(IC) $a_1 + a_2 = a_3 = m$, $a_4 \neq a_1, a_2$, $2 \leq a_1 < a_2$.

Case (II) $m = 4$, $C^\sharp\text{-wt } \phi \equiv 2 \pmod{4}$, $\phi \in (x)^2$,

(IIA) $(a_1, \dots, a_4) = (1, 1, 3, 2)$,

(IIB) $(a_1, \dots, a_4) = (3, 2, 5, 5)$, and

Case (III) $m = 1$. In this case, $X^\sharp = X$ and $C^\sharp = C$.

(Imprimitive point P)

Case (I^V) \bar{m} , $s > 1$, $C^\sharp\text{-wt } \phi \equiv 0 \pmod{m}$,

(IA^V) $w_1 + w_3 \equiv 0 \pmod{m}$, $\text{ow}(x_4) = (\bar{m}, 0) \in \mathbb{Z}_{++} \text{ow}_1 + \mathbb{Z}_{++} \text{ow}_2$, and $w_p(0) \geq 1/2$,

(IC^V) $s = 2$, \bar{m} is an even integer ≥ 4 , and

	x_1	x_2	x_3	x_4	
wt	1	-1	0	$\bar{m} + 1$	\pmod{m} ,
ord	1	$\bar{m} - 1$	\bar{m}	$\bar{m} + 1$	

Case (II^V) $s = \bar{m} = 2$, $\text{wt } \phi \equiv 2 \pmod{4}$, $\phi \in (x)^2$, and

	x_1	x_2	x_3	x_4	
wt	1	3	3	2	$\pmod{4}$.
ord	1	1	1	2	

Depending on the case occurring, we say that $X \supset C \ni P$ (or $X \supset C$ at P , or simply P (if there is no danger of confusion)) is of type (IA), ..., (III), (IA[∨]), ..., (II[∨]).

Proof. The case of imprimitive points was treated in (3.8). Now assume that P is primitive. First we show that one of cases (IA), ..., (III) occurs.

In case (I), one has $a_i + a_j \equiv 0 \pmod{m}$, $a_k = m$ for some distinct $i, j, k \in [1, 4]$ by normalizedness of (x) and t and (1a.6). If $\text{ord}(C^\sharp) = \mathbb{Z}_+$, then $(a_{\sigma(1)}, \dots, a_{\sigma(4)}) = (1, b, m-1, m)$ or $(b, 1, m-b, m)$ for some $b \in [1, m-1]$ and $\sigma \in \mathfrak{S}_4$, where $\mathbb{Z}_+ a_{\sigma(1)} + \mathbb{Z}_+ a_{\sigma(2)} = \mathbb{Z}_+$ is maintained. Thus we may assume $a_1, a_2 \geq 2$, and treat three cases modulo (12) and (34) $\in \mathfrak{S}_4$.

(4.2.1) *Case* $a_1 + a_3 \equiv 0$, a_2 or $a_4 = m$. If $a_4 = m$, then $m \notin \mathbb{Z}_+ a_1 \cup \mathbb{Z}_+ a_2$ by $(m, a_1 a_2) = 1$, and we have (IA). Otherwise, we have (IB).

(4.2.2) *Case* $a_1 + a_2 \equiv 0$, $a_3 = m$, and $2 \leq a_1 \leq a_2$. If $a_4 \equiv a_1$ or a_2 , this belongs to (4.2.1) modulo (34) $\in \mathfrak{S}_4$. Otherwise, we have (IC), because $a_1 \neq a_2$ by $(m, a_1) = 1$ and $m \in (a_1, a_2)$.

(4.2.3) *Case* $a_3 + a_4 \equiv 0$, $a_2 = m$. Since $a_2 = m$ and $a_1 \not\equiv 0 \pmod{m}$, we have $R(a_1) \in \mathbb{Z}_{++} a_1$. By (2.3.3) and (2.10), one has $R(a_1) = m \cdot w_p(0) < m$, whence $a_1 < m$. Thus $a_1 + R(a_1) = m$, because $a_1 + R(a_1) (< 2m)$ belongs to $\mathbb{Z}_+ m$. Hence $m \in \mathbb{Z}_+ a_1$, which contradicts $(m, a_1) = 1$. Thus (4.2.3) does not occur, and case (I) is finished.

(4.2.4) In case (II), one has $a_1, \dots, a_4 \not\equiv 0 \pmod{4}$ (1a.6) and $4 \in (\text{ord}(C^\sharp) - \{0\}) + (\text{ord}(C^\sharp) - \{0\})$, whence $2 \in \text{ord}(C^\sharp)$. Since $4w_p(0) (\in \mathbb{N})$ is prime to 4 and $4w_p(0) < 4$ by (2.3.3) and (2.10), one has $3 \in \text{ord}(C^\sharp)$. Thus $\text{ord}(C^\sharp) = \mathbb{Z}_+$ or $\mathbb{Z}_+ - \{1\}$. Hence by $C^\sharp\text{-wt}(x) \equiv (1, 2, 3, 1), (1, 2, 3, 3) \pmod{4}$ and modulo permutation, one has $C^\sharp\text{-ord}(x) = (1, 2, 3, 1), (3, 2, 5, 5), (1, 2, 3, 3)$, or $(3, 2, 5, 3)$ modulo permutation. The last case is impossible by (2.3.3), because one has $w_p(0) = 5/4 > 1$ by (2.10). In the third case, one sees that x_1^2 must appear in the power series expansion of ϕ by (1a.6) and $\text{wt} x_1 \neq \text{wt} x_3, \text{wt} x_4$ (4). Thus by $\phi \in (x_2 - x_1^2, x_3 - x_1^3, x_4 - x_1^3)$, one has $\partial\phi/\partial x_2(0) \neq 0$, which contradicts the assumption $\phi \in (x)^2$ of case (II). From the first two cases, one has (IIA) or (IIB).

(4.2.5) We will show that there are no overlaps in cases and the division does not depend on choice of (x) and t . By normalizedness, a_1, \dots, a_4 (modulo permutation) is independent of choice of (x) and t (2.6). In case (II), we have (IIA) iff $\text{ord}(C^\sharp) = \mathbb{Z}_+$. We only need to consider case (I). If $\text{ord}(C^\sharp) = \mathbb{Z}_+$, then we have (IA), and we are done. Assume that $\text{ord}(C^\sharp) \neq \mathbb{Z}_+$, where $\text{ord}(C^\sharp)$ needs exactly two generators a_1 and a_2 . One has (IB) iff m is one of the generators. In case (IC), there is exactly one solution $(i, j) = (1, 2)$ for $a_i + a_j \equiv 0 \pmod{m}$ ($1 \leq i < j \leq 4$), and it is given by the generators a_1 and a_2 . Therefore (IA) and (IC) do not occur at the same time. \square

(4.3) Let f_3 and f_4 be monomials in x_1 and x_2 such that

$$x_i = f_i(x_1, x_2) \quad \text{on } C^\sharp \quad (i = 3, 4).$$

Then

$$F_2 = x_1^{sa_2} - x_2^{sa_1}, \quad F_3 = x_3 - f_3, \quad F_4 = x_4 - f_4$$

generate the defining ideal J^\sharp of (C^\sharp, P^\sharp) in $(\mathbb{C}^4, 0)$. Then

$$\phi = g_2 \cdot F_2 + g_3 \cdot F_3 + g_4 \cdot F_4$$

for some semi-invariant g_i 's $\in \mathbb{C}\{x\}$ with $\text{wt } g_i \equiv \text{wt } \phi - \text{wt } F_i$.

(4.4) **Lemma.** *Under the notation of (4.2), one has*

- (i) (X^\sharp, P^\sharp) is smooth and $J^\sharp = (F_2, F_4, \phi)$ if P is of type (IC) or (IC^\vee) , and
- (ii) $J^\sharp = (F_3, F_4, \phi)$ if P is of type (IIB) or (II^\vee) .

Proof. (i) One sees that f_3 must be x_1x_2 , and

$$\phi = g_2 \cdot (x_1^{sa_2} - x_2^{sa_1}) + g_3 \cdot (x_3 - x_1x_2) + g_4 \cdot (x_4 - f_4).$$

Since P is of type (IC) or (IC^\vee) , one sees that $m = s(a_1 + a_2) > 4$, and $g_2 \in (x)$, and that $g_4, f_4 \in (x)^2$ by $\text{wt } x_4 \neq 0, \pm \text{wt } x_1, \pm \text{wt } x_2$ (m). Since $m \geq 5$, one sees that x_1x_2 or x_3 must appear in the power series expansion of ϕ (1a.5). By $sa_1, sa_2 \geq 2$, this is only possible if g_3 is a unit. Thus (X^\sharp, P^\sharp) is smooth and $(F_2, F_3, F_4) = (F_2, F_3, \phi)$.

(ii) By (1a.6), x_1^2 must appear in the power series expansion of ϕ , since $\text{wt } x_1 \neq \text{wt } x_2, \text{wt } x_3, \text{wt } x_4$ (4). One sees that f_3 and f_4 must be x_1x_2 if P is of type (IIB) (resp. $f_3 = x_2$ and $f_4 \in (x)^2$ if P is of type (II^\vee)). Thus similarly in

$$\phi = g_2 \cdot (x_1^2 - x_2^{sa_1}) + g_3 \cdot (x_3 - f_3) + g_4 \cdot (x_4 - f_4),$$

we see that $g_4 \in (x)$ by $\phi \in (x)^2$, and hence that g_2 is a unit, whence $(F_2, F_3, F_4) = (F_3, F_4, \phi)$. \square

(4.5) **Definition.** Let $X \supset C \simeq \mathbb{P}^1$ be an extremal nbd, and $P \in C$ be a singular point of X .

- (i) We say $X \supset C$ is *ordinary* at P (or P is an *ordinary point*) if (X, P) is an ordinary double point or a cyclic quotient singularity (i.e. the canonical cover (X^\sharp, P^\sharp) is smooth), and
- (ii) We say that P is *almost ordinary* iff P is of type (IB) and (C^\sharp, P^\sharp) is a complete intersection in (X^\sharp, P^\sharp) .

(4.5.1) **Remark.** (i) P is an ordinary point iff either

- (a) P is of type (IA) or (IA^\vee) , and (X, P) is a cyclic quotient singularity ($\Leftrightarrow J^\sharp = (\phi, F_2, F_3) \Leftrightarrow (C^\sharp, P^\sharp)$ is a complete intersection in (X^\sharp, P^\sharp)),

- (b) P is of type (IC) or (IC^\vee) , or
- (c) P is of type (III), and (X, P) is an ordinary double point (i.e. $(X, P) \supset C$ is given as $(\{xy = zu\}, 0) \supset x$ -axis for suitable coordinates (x, y, z, u)),
- (ii) a type (IB) point P is almost ordinary iff $J^\sharp = (\phi, F_3, F_4)$.

Proof. We treat (i) first. If P is of type (IA) or (IA^\vee) , then $F_4 = x_4 - f_4(x_1, x_2)$ and $\text{wt } \phi \neq \text{wt } F_2, \text{wt } F_3 \ (m)$. Therefore the equivalence in case (a) is clear. If P is of type (IB), then $\phi \in (x)^2$ by $\phi \in (F_2, F_3, F_4), F_2 \in (x)^2$ and $\text{wt } \phi \neq \text{wt } F_3, \text{wt } F_4 \ (m)$. Thus P is not ordinary. (4.4(i)) treats points of type (IC), (IC^\vee) . Points of type (IIA), (IIB), (II^\vee) are never ordinary by $\phi \in (x)^2$. (ii) follows from $\text{wt } \phi \neq \text{wt } F_3, \text{wt } F_4 \ (m)$. \square

Points of type (IA) or (IA^\vee) require delicate analysis. For these, we state the following technical lemma to be used in §9.

(4.5.2) **Lemma.** *If P is an ordinary point of type (IA) or (IA^\vee) such that $\text{siz}_P = 1$ and (C^\sharp, P^\sharp) is singular, then $\{F_2, F_3\}$ is a free basis for $\text{gr}_{C^\sharp}^1 \mathcal{O}$ at P^\sharp such that $\text{ow}^*(F_2) = (\bar{m}, 0) - sa_1 \cdot \text{ow}_2, \text{ow}^*(F_3) = \text{ow}_1$ (2.8), $R(\text{wt } F_2) > R(\text{wt } F_3)$, and*

$$\text{ow}^*(F_2) - \text{ow}^*(F_3), (\bar{m}, 0) + \text{ow}^*(F_3) - \text{ow}^*(F_2) \in \text{ow}(C^\sharp).$$

Proof. By $\text{wt } F_2 \equiv sa_1 a_2$ and $\text{wt } F_3 \equiv -\text{wt } x_1 \ (m)$, we see $\text{ow}^*(F_2) = (\bar{m}, 0) - sa_1 \cdot \text{ow}_2$ and $\text{ow}^*(F_3) = \text{ow}_1$ by $\text{siz}_P = U(sa_1 \cdot \text{ow}_2) = 1$ and $a_1 < \bar{m}$. We have $\text{ow}^*(F_2) \in \mathbb{Z}_+ \text{ow}_1 + \mathbb{Z}_+ \text{ow}_2$ (2.8), and claim further that $\text{ow}^*(F_2) \in \mathbb{Z}_{++} \text{ow}_1 + \mathbb{Z}_{++} \text{ow}_2$. Indeed if otherwise, we have $(\bar{m}, 0) = \alpha \cdot \text{ow}_i$ for some $\alpha \in \mathbb{Z}_+, i \in [1, 2]$, whence $\alpha \equiv 0 \ (m)$ by $(\text{wt } x_1, m) = (\text{wt } x_2, m) = 1$ and thus $\bar{m} = m$ and $a_i = 1$ by $\bar{m} = \alpha a_i$, which implies that P is a type (IA) point and (C^\sharp, P^\sharp) is smooth. This contradicts our original assumption. Thus we have $\text{ow}^*(F_2) \in \mathbb{Z}_{++} \text{ow}_1 + \mathbb{Z}_{++} \text{ow}_2$, hence $R(\text{wt } F_2) - R(\text{wt } F_3) > 0$, and $\text{ow}^*(F_2) - \text{ow}^*(F_3) \in \text{ow}(C^\sharp)$. Then it is obvious that $(\bar{m}, 0) + \text{ow}^*(F_3) - \text{ow}^*(F_2) = sa_1 \cdot \text{ow}_2 + \text{ow}_1 \in \text{ow}(C^\sharp)$. \square

(4.6) Let $X \supset C \simeq \mathbb{P}^1$ be an extremal nbd with a singular point $P \in C$ of X of index m , and let P_1, \dots, P_r be all the other singular points of X on C . Let $(X^\sharp, P^\sharp) \supset (C^\sharp, P^\sharp)$ be the μ_m -canonical cover of (X, P) and the preimage of C . Let $x = (x_1, \dots, x_4)$ and t be normalized ℓ -coordinates giving embedding $(X^\sharp, P^\sharp) \subset (\mathbb{C}^4, 0)$ such that $a_i = C^\sharp\text{-ord}(x_i) \ (i \in [1, 4])$ are as in (4.2). Let $\phi \in \mathbb{C}\{x\}$ be an ℓ -equation of (X, P) , and $J^\sharp \subset \mathbb{C}\{x\}$ the ideal defining C^\sharp in $(\mathbb{C}^4, 0)$, and let F_2, F_3, F_4 be as in (4.3). The following is the deformation process.

Let $\theta = x_2$ if P is an almost ordinary type (IB) point or a type (IIB) point, and $\theta = 0$ otherwise. Let $C_\lambda^\sharp \subset (\mathbb{C}^4, 0)$ be defined by $F_2 + \lambda\theta = F_3 = F_4 = 0$

for $\lambda \in \mathbb{C}$. Then we follow step (L) in §1b. We will choose semi-invariant $\psi \in (x)\mathbb{C}\{x\}$ with $\text{wt } \psi \equiv \text{wt } \phi \pmod{m}$ such that $\phi_\lambda = \phi + \lambda\psi$ induces 0 on $C_\lambda^\#$ and such that $X_{\lambda,\varepsilon} (\ni P = 0)$ has only terminal singularities if $|\lambda| \ll \varepsilon \ll 1$, where we introduce $B_\varepsilon = \{(x) \mid |x_i| < \varepsilon (\forall i)\}$, $X_{\lambda,\varepsilon}^\# = X_\lambda^\# \cap B_\varepsilon$, $X_{\lambda,\varepsilon} = X_{\lambda,\varepsilon}^\# / \mu_m$, $C_{\lambda,\varepsilon}^\# = C_\lambda^\# \cap B_\varepsilon$, $C_{\lambda,\varepsilon} = C_{\lambda,\varepsilon}^\# / \mu_m$ (this part will be studied more closely case by case later in the proof). Then step (LG) embeds the deformation $X_{\lambda,\varepsilon} \supset C_{\lambda,\varepsilon} \ni P$ to a deformation $X_\lambda \supset C_\lambda \simeq \mathbb{P}^1$ which is trivial outside $X_{\lambda,\varepsilon}$ for $|\lambda| \ll \varepsilon \ll 1$ (1b.7). Thus by (G) in §1b, one has nearby extremal nbds $X_\lambda^\circ \supset C_\lambda \simeq \mathbb{P}^1$ such that $X_\lambda^\circ \subset X_\lambda$ for $|\lambda| \ll 1$.

(4.7) Proposition-Definition. *For suitable choice of ψ , each nearby extremal nbd $X_\lambda^\circ \supset C_\lambda \simeq \mathbb{P}^1$ ($|\lambda| \ll \varepsilon \ll 1$) contains P, P_1, \dots, P_r in the natural way mentioned above so that $(X_\lambda^\circ, P_i) \supset (C_\lambda, P_i)$ is naturally isomorphic to $(X, P_i) \supset (C, P_i)$ for $i \in [1, r]$ and $X_{\lambda,\varepsilon} \supset C_{\lambda,\varepsilon}$ contains all the singularities ($\in C_\lambda$) of $X_\lambda \supset C_\lambda$ other than P_1, \dots, P_r . If P is a primitive (resp. an imprimitive) point, then $X_{\lambda,\varepsilon} \supset C_{\lambda,\varepsilon}$ is locally primitive (resp. P is an imprimitive point of $X_{\lambda,\varepsilon} \supset C_{\lambda,\varepsilon}$ with the same subindex and splitting degree as $X \supset C \ni P$). Depending on the type of $X \supset C \ni P$, one has*

Type (IA) (resp. (IB) but not almost ordinary, (IA^V)). P is the only singularity of $X_{\lambda,\varepsilon}$ on $C_{\lambda,\varepsilon}$, and it is an ordinary type (IA) (resp. an almost ordinary type (IB), an ordinary type (IA^V)) point, and one can use the same (x) , t , ow_i 's, F_i 's, except for the ℓ -equation which is $\phi_\lambda = 0$, whence size, $w(0)$ remain the same.

Type (IIA) (resp. (II^V)). P is the only singularity of $X_{\lambda,\varepsilon}$ on $C_{\lambda,\varepsilon}$, and it is an ordinary type (IA) (resp. (IA^V)) point, with the same index, size, and $w(0)$ as $X \supset C$ at P .

Type (IB) and almost ordinary. $X_{\lambda,\varepsilon}$ has exactly a_1 singular points (including P) on $C_{\lambda,\varepsilon}$, which are ordinary points of type (IA) with index m , and the invariant $w(0)$ for $X_{\lambda,\varepsilon} \supset C_{\lambda,\varepsilon}$ takes the same value for these a_1 points.

Type (IIB). $X_{\lambda,\varepsilon}$ has exactly two singular points P and Q on $C_{\lambda,\varepsilon}$, which are ordinary points of type (IA), with indices 4 and 2, respectively, and

Type (III). $X_{\lambda,\varepsilon}$ has exactly n singular points, and they are on $C_{\lambda,\varepsilon}$ and of type (III), where n is the invariant $i_P(1)$ for $X \supset C$ at P . If $n = 1$, then the singular point on $C_{\lambda,\varepsilon}$ is ordinary.

We call all these processes (or $X_\lambda^\circ \supset C_\lambda$, by abuse of language) L -deformations at P . The L -deformation at a type (III) point P is also called a separating L -deformation.

In case of type (III), one can also make $X_{\lambda,\varepsilon}$ smooth when $0 < |\lambda| \ll \varepsilon \ll 1$ by choosing some other suitable ψ . We call this process an L -smoothing at P . (L -deformation will mean separating L -deformation for type (III) points unless otherwise mentioned.)

Proof. First we note that, if P is an almost ordinary point of type (IB) or a point of type (IIB), $F_2 + \lambda x_2$ is semi-invariant and C_λ^\sharp is smooth in a small enough nbd of $(\mathbb{C}^4, 0)$ when $0 < |\lambda| \ll 1$ (1b.4). Thus the assertion on local primitiveness (or imprimitiveness) follows from (1b.3.1) if other parts are done. We treat several cases depending on the type of P .

(4.7.1) (IA), (IIA), (IA^\vee) , (II^\vee) (resp. (IB) but not almost ordinary). We choose semi-invariant general $\psi \in F_4 + \mathbb{C}\{x\}F_2 + \mathbb{C}\{x\}F_3$ (resp. $F_2 + (\mathbb{C}^*x_1 + (x)^2\mathbb{C}\{x\})F_3 + \mathbb{C}\{x\}F_4$) with $\text{wt } \psi \equiv \text{wt } \phi \pmod{m}$. We note that $\{\phi = \psi = 0\}$ is a normal surface by Bertini's Theorem because $((x)^2J^\sharp)^{\mu_m} \cdot \mathbb{C}\{x\}$ defines a closed subscheme $\subset (\mathbb{C}^4, 0)$ which is equal to C^\sharp except at a finite number of points. Thus (1b.3(i)) is satisfied, and (1b.5(ii')) (resp. (1b.5(ii''))) is satisfied by the choice of ψ . Thus $X_{\lambda,\varepsilon}$ has only terminal singularities and these come from points in $X_{\lambda,\varepsilon}^\sharp \cap x_a$ -axis for $a = 4$ (resp. $a = 2$) (1b.3). Since $C^\sharp \cap x_a$ -axis $= \{0\}$, P is the only singular point of $X_{\lambda,\varepsilon}$ lying on C . We note $J^\sharp = (\phi_\lambda, F_2, F_3)$ (resp. $J^\sharp = (\phi_\lambda, F_3, F_4)$). The assertions on size, $w(0)$ follow from definition and (2.10).

(4.7.2) Almost ordinary and (IB), or (IIB). We write ϕ as $\phi = g_2F_2 + g_3F_3 + g_4F_4$, where g_2, g_3, g_4 are semi-invariants such that $\text{wt } g_i \equiv \text{wt } \phi - \text{wt } F_i$. We note that g_2 is a unit by the hypothesis or (4.4(ii)). We choose semi-invariant general $\psi \in g_2x_2 + \mathbb{C}\{x\}F_3 + \mathbb{C}\{x\}F_4$ with $\text{wt } \psi \equiv \text{wt } \phi \pmod{m}$. Thus $\phi_\lambda \in g_2(F_2 + \lambda x_2) + \mathbb{C}\{x\}F_3 + \mathbb{C}\{x\}F_4$, and $\phi_\lambda \equiv 0$ on C_λ^\sharp . Then $\psi \notin (x)^2$ since g_2 is a unit. One can check (1b.3(i)) similarly to the previous case. By (1b.3) and $\psi \notin (x)^2$, $X_{\lambda,\varepsilon}$ has only terminal cyclic quotient singularities and they are images of $X_{\lambda,\varepsilon}^\sharp \cap x_2$ -axis. If P is of type (IB), $C_\lambda^\sharp \cap x_2$ -axis $= \{(0, \xi, 0, 0) \mid \xi^{a_1} = \lambda\xi\}$ and x_1, x_3, x_4 form coordinates for X_λ^\sharp at all these a_1 points by $\partial\phi_\lambda/\partial x_2 \neq 0$. Since x_1 is a uniformizant for C_λ^\sharp at all these points, μ_m is the stabilizer of each point, and $\text{wt}(x)$ are the same at these points, one sees that they have index m and the same $w(0)$ by (2.10). If P is of type (IIB), one has $C_\lambda^\sharp \cap x_2$ -axis $= (0)$ or $(0, \pm\sqrt{\lambda}, 0, 0)$. Stabilizer of (0) is μ_4 , and stabilizers of the conjugate $(0, \pm\sqrt{\lambda}, 0, 0)$ are μ_2 . So we have two points with indices 4 and 2.

(4.7.3) (III). In this case, we choose coordinates (z_1, \dots, z_4) so that C is z_1 -axis and $\phi \equiv z_1^n z_2 (z_2, z_3, z_4)^2$ (2.16). Then $n = i_P(1)$ by (2.16). Let $\psi \in z_1 z_2 + (z_2, z_3, z_4)^2$ be general enough so that $(\{\psi = 0\}, 0)$ is an ordinary double point and $\{\phi = \psi = 0\}$ is a surface smooth outside z_1 -axis by Bertini's Theorem. Let $\phi_\lambda = \phi + \lambda\psi$, $X_{\lambda,\varepsilon} = \{(z) \mid \phi_\lambda(z) = 0, |z_i| < \varepsilon (\forall i)\}$ and $C_{\lambda,\varepsilon} = X_{\lambda,\varepsilon} \cap C$ when $|\lambda| \ll \varepsilon \ll 1$. By (1b.4), $X_{\lambda,\varepsilon}$ has singularities only on $C_{\lambda,\varepsilon}$ if $|\lambda| \ll \varepsilon \ll 1$. Since $\phi_\lambda \equiv (z_1^{n-1} + \lambda)z_1 z_2 (z_2, z_3, z_4)^2$, one sees that $X_{\lambda,\varepsilon}$ has exactly n singular points $\{(\xi, 0, 0, 0) \mid \xi^n + \lambda\xi = 0\}$ in $C_{\lambda,\varepsilon}$ by solving

$\partial\phi_\lambda/\partial z_2 = 0$ on x_1 -axis. Again by $\phi_\lambda \equiv (z_1^{n-1} + \lambda)z_1z_2 - (z_2, z_3, z_4)^2$, $X_{\lambda,\varepsilon}$ has cA -points at these points, whence terminal (1a.2(ii)). If $n = 1$, then $(X_{\lambda,\varepsilon}, 0)$ is an ordinary double point if $0 < |\lambda| \ll 1$ because so is $(\{\psi = 0\}, 0)$.

(4.7.4) To construct an L -smoothing for type (III) P , one may choose general $\psi \in z_3 + (z_2, z_3, z_4)^2$ with the notation of (4.7.3), and the rest is similar to (4.7.3). \square

By using L -deformation (4.7), we can reduce many problems on extremal nbds to the ones on extremal nbds with only ordinary singularities. To derive formulae for ordinary singularities, we need an easy

(4.8) **Lemma.** *Under the notation of (4.1), assume that*

$$I_{C^\sharp}/I_{C^\sharp}^2 = \mathcal{O}_{C^\sharp}v_1 \oplus \mathcal{O}_{C^\sharp}v_2 \quad \text{and} \quad \omega_{X^\sharp} \otimes \mathcal{O}_{C^\sharp} = \mathcal{O}_{C^\sharp}\Omega$$

near P^\sharp , where I_{C^\sharp} is the defining ideal of C^\sharp in X^\sharp , and v_1, v_2 and Ω are semi-invariants with C^\sharp -wts b_1, b_2, k . Then

$$\text{gr}_C^d \mathcal{O} = \bigoplus_{i,j \geq 0, i+j=d} \mathcal{O}_C t^{R(ib_1+jb_2)} v_1^i v_2^j,$$

$$\text{gr}_C^d \omega = \bigoplus_{i,j \geq 0, i+j=d} \mathcal{O}_C t^{R(ib_1+jb_2+k)} v_1^i v_2^j \Omega,$$

$$i_P(d) = \frac{1}{m} \sum_{i,j \geq 0, i+j=d} \{iR(b_1) + jR(b_2) - R(ib_1 + jb_2)\} \quad (d \geq 2), \quad \text{and}$$

$$w_P(d) = \frac{1}{m} \sum_{i,j \geq 0, i+j=d} \{iR(b_1) + jR(b_2) + R(k) - R(ib_1 + jb_2 + k)\} \quad (d \geq 1).$$

Proof. By the assumptions, one has $I_{C^\sharp} = (v_1, v_2)$. Hence $I_{C^\sharp}^{(d)} = (v_1, v_2)^d$, $\text{gr}_{C^\sharp}^d \mathcal{O} \simeq \bigoplus \mathcal{O}_{C^\sharp} v_1^i v_2^j$, and $\text{gr}_{C^\sharp}^d \omega \simeq \bigoplus \mathcal{O}_{C^\sharp} v_1^i v_2^j \Omega$. The lemma follows from (2.8) and (2.9). \square

Then we have

(4.9) **Theorem.** *Let $X \supset C \simeq \mathbb{P}^1$ be an extremal nbd, and let $P \in C$ be a singular point of X of index $m \geq 1$, subindex \bar{m} , and splitting degree s . Let (x) and t be normalized ℓ -coordinates as in (4.2), $a_i = \text{ord}(x_i)$, $\text{ow}_i = \text{ow}(x_i)$ ($i \in [1, 4]$). Then*

(i) *the value of $w_P(0)$ is given depending on type of P :*

$w_P(0)$	$(\bar{m} - a_2)/\bar{m}$	$R(a_4)/\bar{m}$	3/4	1/2	0
type	(IA), (IA $^\vee$)	(IB), (IC), (IC $^\vee$)	(IIA), (IIB)	(II $^\vee$)	(III)

(ii) if P is ordinary, then depending on type of P , one has
 Type (IA):

$$\begin{aligned}
 i_p(1) &= U(a_1 a_2), & w_p(0) &= (m - a_2)/m, \\
 i_p(d) &= \sum_{i,j \geq 0, i+j=d} \{iU(a_1 a_2) - U(ia_1 a_2 - ja_1)\} & (d \geq 2), \\
 w_p^*(d) &= \sum_{i,j \geq 0, i+j=d} U(ia_1 a_2 - ja_1 + a_2) - (d + 1) & (d \geq 1),
 \end{aligned}$$

Type (IA[∨]):

$$\begin{aligned}
 i_p(1) &= U(s \cdot a_1 a_2), & w_p(0) &= (\bar{m} - a_2)/\bar{m}, \\
 i_p(d) &= \sum_{i,j \geq 0, i+j=d} \{iU(sa_2 \cdot ow_1) - U(isa_2 \cdot ow_1 - j \cdot ow_1)\} & (d \geq 2), \\
 w_p^*(d) &= \sum_{i,j \geq 0, i+j=d} U(isa_2 \cdot ow_1 - j \cdot ow_1 + ow_2) - (d + 1) & (d \geq 1),
 \end{aligned}$$

Type (IC) or (IC[∨]):

$$\begin{aligned}
 i_p(1) &= U(sa_1 a_2), & w_p(0) &= R(a_4)/\bar{m}, \\
 i_p(d) &= \sum_{i,j \geq 0, i+j=d} \{iU(sa_1 a_2) + jU(a_4) - U(isa_1 a_2 + ja_4)\} & (d \geq 2), \\
 w_p^*(d) &= \sum_{i,j \geq 0, i+j=d} \{U(isa_1 a_2 + (j + 1)a_4) - (j + 1)U(a_4)\} & (d \geq 1),
 \end{aligned}$$

and

Type (III):

$$\begin{aligned}
 i_p(1) &= 1, & w_p(0) &= 0, \\
 i_p(d) &= [d^2/4] & (d \geq 2), \\
 w_p^*(d) &= [(d + 1)^2/4] & (d \geq 1).
 \end{aligned}$$

Proof. We omit the proof of (i) since it follows immediately from (2.10) for the case of type (IA) (resp. (IA[∨])), where $R(a_2) = \bar{m} - a_2$ by $(\bar{m}, 0) \in \mathbb{Z}_{++} \text{ow}_1 + \mathbb{Z}_{++} \text{ow}_2$. We assume that P is ordinary, and consider type (I) or (I[∨]) case first. Let F_2, F_3, F_4, J^\sharp be as in (4.3) (hence as in (2.11)). We use the notation of §2. We note $C^\sharp\text{-wt}(F_i) \equiv \|F_i\| \pmod{\bar{m}}$ ($i \in [2, 4]$) (cf. (2.13)). Since ϕ is a part of basis for $J^\sharp/J^{\sharp 2}$, one has

$$\begin{aligned}
 [\phi, J_{\{0\}}^\sharp, J_{\{0\}}^\sharp] &= [J_{\{0\}}^\sharp, J_{\{0\}}^\sharp, J_{\{0\}}^\sharp] \\
 &= [t^{R(\text{wt } F_2)} F_2, t^{R(\text{wt } F_3)} F_3, t^{R(\text{wt } F_4)} F_4] \\
 &= \bar{m} \sum_{i=2}^4 U(\|F_i\|, \text{wt } F_i) - \sum_{i=1}^3 a_i
 \end{aligned}$$

by (2.8(ii)) and (2.14). Whence by (2.12), one has

$$i_p(1) = 1 + \sum_{i=2}^4 U(\|F_i\|, \text{wt } F_i) - \frac{1}{\bar{m}} \left\{ R(k) + \sum_{i=1}^4 a_i \right\},$$

where $k = \text{wt } x_2$ (resp. $\text{wt } x_4$) if P is of type (IA) or (IA^\vee) (resp. (IC) or (IC^\vee)).

If P is of type (IA) or (IA^\vee) , we have $U(\|F_2\|, \text{wt } F_2) = U(\text{sa}_2 \cdot \text{ow}_1) = U(\text{sa}_1 a_2)$, $U(\|F_4\|, \text{wt } F_4) = U(\bar{m}, 0) = 1$, and $U(\|F_3\|, \text{wt } F_3) = U(\text{ow } x_3) = (a_1 + a_3)/\bar{m}$ by $\text{wt } x_1 + \text{wt } x_3 \equiv 0 \pmod{m}$ and $a_1 < \bar{m}$. Since $(\bar{m}, 0) \in \mathbb{Z}_{++} \text{ow}_1 + \mathbb{Z}_{++} \text{ow}_2$, we have $(\bar{m}, 0) - \text{ow}_2 \in \text{ow}(C^\sharp)$, and one has $R(k) + a_2 = U(\text{ow}_2) = \bar{m}$, and thus $i_P(1) = U(\text{sa}_1 a_2)$. The rest follows from (4.8) ($v_1 = F_2, v_2 = F_3$) because $R(z) = U(z)\bar{m} - x$ ($z = (x, y) \in \text{ow}(C^\sharp)$) (2.8).

If P is of type (IC) or (IC^\vee) , we have $U(\|F_2\|, \text{wt } F_2) = U(\text{sa}_1 a_2)$, $U(\|F_3\|, \text{wt } F_3) = U(\bar{m}, 0) = 1$, and $U(\|F_4\|, \text{wt } F_4) = U(a_4)$ (we note $a_4 \equiv \text{wt } x_4 \pmod{m}$ for type (IC^\vee) points). Whence we have $i_P(1) = U(\text{sa}_1 a_2)$. The rest follows similarly from (4.8) and $I_C = (F_2, F_4)$ (4.4(i)).

We consider type (III). Since (X, P) is Gorenstein, $w_P(1) = 0$ and $w_P(d) = i_P(d)$ ($d \geq 2$), whence $w_P^*(1) = i_P(1)$ and $w_P^*(d) = \binom{d+1}{2} i_P(1) - i_P(d)$ ($d \geq 2$). We see $i_P(1) = 1$ (2.16), and we need to calculate $i_P(d)$ ($d \geq 2$). We may choose coordinates (z_1, \dots, z_4) so that C is z_1 -axis and $X = \{z_1 z_2 = z_3 z_4\}$ in $(\mathbb{C}^4, 0)$. Then $\mathcal{O}_{X,P} = \mathbb{C}\{z\}/(z_1 z_2 - z_3 z_4)$, and let $I_C = (z_2, z_3, z_4)$, and $I_C^{(d)}$ the symbolic d th power of I_C . We note $z_2 \in I_C^{(2)}$ by $z_1 z_2 = z_3 z_4 \in I_C^2$. Let J_d ($d \in \mathbb{N}$) be the ideal generated by $z_2^i z_3^j z_4^k$ for $i, j, k \in \mathbb{Z}_+$ such that $2i + j + k \geq d$. Then $I_C J_d \subset J_{d+1}$, and $J_d \subset I_C^{(d)}$ by $z_2 \in I_C^{(2)}$. By $z_3 z_4 = z_1 z_2$, one sees that $v_{jk} := z_2^{\min\{j,k\}} z_3^{j-\min\{j,k\}} z_4^{k-\min\{j,k\}} \in \mathcal{O}_{X,P}$ satisfies $z_3^j z_4^k = z_1^{\min\{j,k\}} v_{jk}$ for $j, k \in \mathbb{Z}_+$. Thus one easily sees that $J_d = (v_{d,0}, v_{d-1,1}, \dots, v_{0,d})$ and

$$\bigoplus_{0 \leq i \leq d} \mathcal{O}_{C,P} v_{i,d-i} \rightarrow J_d / J_{d+1}.$$

Since $J_1 = I_C$, we will see $J_d = I_C^{(d)}$ by induction on d . Indeed if $J_d = I_C^{(d)}$, then

$$(*) \quad \bigoplus_{i=0}^d \mathcal{O}_{C,P} v_{i,d-i} \rightarrow (I_C^{(d)} / J_{d+1} \rightarrow I_C^{(d)} / I_C^{(d+1)}) = \text{gr}_C^d \mathcal{O}$$

is a surjection between free $\mathcal{O}_{C,P}$ -modules of the same rank $d + 1$, and hence it is an isomorphism and $J_{d+1} = I_C^{(d+1)}$. By (*), one has

$$\text{gr}_C^d \mathcal{O} / S^d \text{gr}_C^1 \mathcal{O} \simeq \bigoplus_{i=0}^d (\mathcal{O}_{C,P} / (z_1^{\min\{i,d-i\}})) \cdot v_{i,d-i}$$

by $\text{gr}_C^1 \mathcal{O} = \mathcal{O}_{C,P} z_3 \oplus \mathcal{O}_{C,P} z_4$ and $z_3^i z_4^{d-i} = z_1^{\min\{i,d-i\}} v_{i,d-i}$. Hence one sees $i_p(d) = [d^2/4]$ and $w_p^*(d) = [(d+1)^2/4]$ ($d \geq 2$) by

$$i_p(d) = \sum_{i=0}^d \min\{i, d-i\} = \begin{cases} 2 \sum_{i=0}^{(d-1)/2} i = (d^2 - 1)/4 & \text{if } d \text{ odd,} \\ 2 \sum_{i=0}^{d/2} i - d/2 = d^2/4 & \text{if } d \text{ even.} \end{cases}$$

We note that the formula for $w_p^*(d)$ works for $d = 1$ as well. \square

(4.10) *Remark.* We note that $i_p(1) = \text{siz}_P$ for ordinary P by (4.9(ii)) (cf. (4.1.1)).

We can consider two slightly more delicate deformations than in (4.7).

(4.11) **Lemma.** *Under the notation of (4.2), assume that (X, P) is a cD -type terminal singularity of index 3 (1a.2) and that P is a type (IA) point. Then $(a_1, \dots, a_4) = (1, 1, 2, 3)$ or $(2, 1, 1, 3)$ and $w_p(0) = 2/3$. Thus modulo permutation of x_1 and x_2 (4.2), we “may” assume $(a_1, \dots, a_4) = (1, 1, 2, 3)$.*

Proof. By $m = 3 \in \mathbb{Z}_{++} a_1 + \mathbb{Z}_{++} a_2$, we see $a_1 = 1$ or $a_2 = 1$. If $a_2 = 1$, then $a_1 + a_3 = m = 3$ by $\text{ord}(C^\sharp) = \mathbb{Z}_+$ and normalizedness, whence $(a_1, a_3) = (1, 2)$ or $(2, 1)$. We can also see $a_2 = 1, 2$ and that $(a_1, a_2) = (1, 2)$ if $a_1 = 1$. Thus it is enough to disprove $(a) = (1, 2, 2, 3)$. Indeed it means that $x_1|_{C^\sharp} = t$, $x_2|_{C^\sharp} = x_3|_{C^\sharp} = t^2$, and $x_4|_{C^\sharp} = t^3$. By (1a.6), $\phi \in (x_4)^2 + x_4(x_1, x_2, x_3)^2 + (x_1, x_2, x_3)^3$ and x_1^3 appears in ϕ because $\text{wt } x_1 \not\equiv \text{wt } x_2, \text{wt } x_3$. Hence $\phi|_{C^\sharp}$ is of order 3, which is a contradiction. Thus $(a) = (1, 1, 2, 3)$ or $(2, 1, 1, 3)$, whence $w_p(0) = 2/3$ by (4.9(i)). \square

(4.12) Under the notation and assumptions of (4.2), assume that P is a type (IA) point with $a_1 = 1$ or a type (IIA) point. Then there exists an ℓ -coordinate system $(z) = (z_1, \dots, z_4)$ for (X, P) and an ℓ -coordinate for (C, P) such that C^\sharp is the z_1 -axis, $\text{wt } z_i \equiv a_i$ ($i \in [1, 4]$), and $\phi \equiv z_1^n z_q$ (z_2, z_3, z_4)², where $n = \text{len}_{P^\sharp} I_{C^\sharp}^{(2)} / I_{C^\sharp}^2 \geq 0$ and $q \in [2, 4]$ (2.16).

We note that if P is of type (IA) and $n \geq 2$, then $\text{wt } z_2 z_3 \equiv 0$ (m). Because otherwise, we see that $m \geq 3$ and $\phi \in \mathbb{C} z_4^2 + (z)^3$, hence (X, P) is a cD -type terminal singularity of index 3 (1a.6) and $\text{wt } z_2 z_3 \equiv 0$ by (4.11), which is a contradiction. Thus we have

(4.12.1) **Sublemma.** *If P is a type (IA) point with $a_1 = 1$ and $\text{len}_{P^\sharp} I_{C^\sharp}^{(2)} / I_{C^\sharp}^2 \geq 2$, then $a_2 = 1$.*

If P is of type (IA) and $n \geq m$ (resp. P is of type (IIA)), then we can choose a general semi-invariant

$$\psi \in \mathbb{C}^* z_1^{m-a_q} z_q + \mathbb{C}^* z_2 z_3 + I_{C^\sharp}^3 \quad (\text{resp. } \psi \in \mathbb{C}^* z_1 z_2 + \mathbb{C}^* z_4^3 + I_{C^\sharp}^4)$$

such that $\{\phi = \psi = 0\}$ is a surface which is smooth outside z_1 -axis by Bertini's Theorem. Then as in (4.6) and (4.7), $\phi_\lambda = \phi + \lambda\psi$ induces a deformation $X_{\lambda,\varepsilon} \supset C_{\lambda,\varepsilon} \ni P$ of nbd of P in $X \supset C$ such that $X_{\lambda,\varepsilon} \supset C_{\lambda,\varepsilon}$ is locally primitive, and nearby extremal nbds $X_\lambda^\circ \supset C_\lambda$.

(4.12.2) **Proposition-Definition.** (i) If P is a type (IA) point with $a_1 = 1$ and $n = \text{len}_{P^\#} I_{C^\#}^{(2)} / I_{C^\#}^2 \geq m$, then $X_{\lambda,\varepsilon} \supset C_{\lambda,\varepsilon}$ has a type (IA) point with index m and at least one type (III) point, and

(ii) if P is of type (IIA), then $X_\lambda \supset C_\lambda \ni P$ has an ℓ -coordinate system (y) (depending on λ) such that $\text{wt } y_i \equiv a_i \pmod{m}$ ($i \in [1, 4]$); $C_\lambda^\#$ is the y_1 -axis, $\phi_\lambda = y_1 y_2 + y_3^2 + y_4^3$ for each small enough $\lambda > 0$.

We call all these processes (or $X_\lambda^\circ \supset C_\lambda$, by abuse of language) L' -deformations at P .

Proof. Since $S^\# = \text{Sing}\{\phi = \psi = 0\}$ is contained in z_1 -axis, $S^\# \cap F^\# \subset \{0\}$, where $F^\#$ is the fixed axis, i.e. z_4 -axis (1a.5).

(i) Since $z_2 z_3$ appears in ψ , the conditions in (1b.3) are satisfied by (1b.5), and (1b.3) applies. The argument is similar to (4.7), and we only show existence of an extra type (III) point. By

$$\phi_\lambda \equiv (z_1^{n-(m-a_1)} + \lambda) z_1^{m-a_1} z_4 \pmod{(z_2, z_3, z_4)^2},$$

we see a moving singular point by $n - (m - a_1) > 0$. This gives a type (III) point since it is not on the fixed axis (1b.3).

(ii) By (1a.5), z_3^2 appears in ϕ , whence the quadratic part of ϕ_λ is of the form $\alpha z_1 z_2 + \beta z_2^2 + \gamma z_3^2$ ($\alpha, \beta, \gamma \in \mathbb{C}$, $\alpha\gamma \neq 0$). Thus using the Weierstrass Preparation Theorem, one can easily obtain an ℓ -coordinate system (y) such that $\text{wt } y_i \equiv a_i \pmod{m}$ ($i \in [1, 4]$), $C_\lambda^\#$ is the y_1 -axis, $\phi_\lambda = y_1 y_2 + y_3^2 + y_4^{2r+1}$ ($r \geq 1$). By construction of ψ , $\phi|_{F^\#}$ has exactly a triple root at the origin, and we note that $F^\# = y_4$ -axis in terms of the coordinates y . Thus $r = 1$ (1a.5(iii)). By (1a.7(ii)), $(\{\phi_\lambda = 0\}, 0) / \mu_m$ is terminal. Thus the conditions in (1b.3) are satisfied and the rest is the same. \square

5. NUMERICAL CALCULATIONS FOR (IA), (IC), (IA[∨]), AND (IC[∨])

The main purpose of this section is to make formulae (4.9) more explicit for later use. Throughout this section, we will use the notation of (4.9).

(5.1) **Proposition.** Assume that P is an ordinary point of type (IA) with index m such that $i_P(1) = 1$, $w_P(0) < 1/2$. Then

- (i) one has $a_1 = 1$, and

(ii) if $d \in \mathbb{N}$ satisfies $w_p(0) < 1/d$, then one has (we note $d \leq m - 1$ and $1/(d + 1) \leq (m - 1)/md < 1/d$)

$$(i_p(d + 1), w_p^*(d)) = \begin{cases} \left(0, \frac{1}{2}d(d + 1)\right) & \text{if } w_p(0) < \frac{1}{d + 1}, \\ \left(1, \frac{1}{2}d(d + 1) - 1\right) & \text{if } w_p(0) \geq \frac{1}{d + 1}, w_p(0) \neq \frac{m - 1}{m \cdot d}, \\ \left(2, \frac{1}{2}d(d + 1) - 2\right) & \text{if } w_p(0) = \frac{m - 1}{m \cdot d}, d \neq m - 1, \\ \left(d + 2, \frac{1}{2}(d - 2)(d + 1)\right) & \text{if } w_p(0) = \frac{1}{m}, d = m - 1. \end{cases}$$

Proof. One has $a_1 a_2 \leq m$ by $U(a_1 a_2) = 1$ and $a_2 > m/2$ by $w_p(0) = (m - a_2)/m < 1/2$ (4.9). Thus $a_1 = 1$ and $\text{ord}(C^\sharp) = \mathbb{Z}_+$, which proves (i). Since $w_p(0) \geq 1/m$, one has $d < m$ and $1/(d + 1) \leq (m - 1)/md < 1/d$ is immediately checked. We note that $U(n) = [n/m]$ for $n \in \mathbb{Z}$ by $\text{ord}(C^\sharp) = \mathbb{Z}_+$. Then $r = m - a_2 \in \mathbb{N}$ satisfies $dr < m$ by $w_p(0) = r/m < 1/d$. Let us consider arbitrary $i, j \in \mathbb{Z}_+$ such that $i + j = d + 1$. Then we claim

$$U(ia_2 - j) = \begin{cases} i - 1 & \text{if } \begin{cases} j = 0 & \text{and } w_p(0) \geq \frac{1}{d + 1}, \\ j = 1 & \text{and } w_p(0) = \frac{m - 1}{m \cdot d}, \text{ or} \\ j > 1 & \text{and } w_p(0) = \frac{1}{m} \text{ and } d = m - 1, \end{cases} \\ i & \text{otherwise.} \end{cases}$$

Indeed from

$$mi - (ia_2 - j) = (m - a_2)i + j > 0,$$

and

$$\begin{aligned} (ia_2 - j) - m(i - 2) &= 2m - j - i(m - a_2) \\ &= 2m - j - ir \\ &\geq 2m - (d + 1)r > 0, \end{aligned}$$

one has $i \geq U(ia_2 - j) \geq i - 1$ and $U(ia_2 - j) = i - 1$ iff $X(i, j) \leq 0$, where $X(i, j) = i(a_2 - j) - m(i - 1) = m - j - ir$. It is clear that $X(d + 1, 0) \leq 0 \Leftrightarrow m \leq (d + 1)r \Leftrightarrow w_p(0) \geq 1/(d + 1)$. If $j \geq 1$, then

$$\begin{aligned} X(i, j) &= m - 1 - (j - 1) - ir \\ &\geq m - 1 - (j - 1)r - ir \\ &= m - 1 - dr \geq 0. \end{aligned}$$

One sees that $X(d, 1) \leq 0$ iff $m - 1 = dr$, that is $w_p(0) = (m - 1)/md$. If $j > 1$, then $X(i, j) \leq 0$ iff $m - 1 = dr$ and $r = 1$, that is $w_p(0) = 1/m$ and

$d = m - 1$. Thus the claim is proved. Since (4.9) implies

$$\begin{aligned} i_p(d + 1) &= \sum \{i - U(ia_2 - j)\}, \quad \text{and} \\ w_p^*(d) &= \sum_{i \geq 1} U(ia_2 - j) - (d + 1) \\ &= \sum i - \sum \{i - U(ia_2 - j)\} - U(-d - 1) - (d + 1) \\ &= \frac{1}{2}d(d + 1) - i_p(d + 1) - U(-d - 1), \end{aligned}$$

the above claim implies (5.1). \square

(5.2) **Proposition.** *Assume that P is an ordinary point of type (IA) and index m . One has*

- (i) $i_p(1) = 1$ and $a_2 > 1 \Rightarrow i_p(2) \leq 1$,
- (ii) $a_2 > 1 \Rightarrow w_p^*(1) \geq i_p(1) - 1$, and
- (iii) $w_p^*(1) \geq -1$, where “=” holds iff $a_2 = 1$ ($\Leftrightarrow w_p(0) = (m - 1)/m$). If furthermore $a_2 = 1$, then

$$\begin{aligned} i_p(d) &= \frac{1}{2}d(d + 1) - \lceil \frac{d}{2} \rceil + \#S(d) \quad (d \geq 2), \\ w_p^*(d) &= -\lceil \frac{d}{2} \rceil + \#S(d) \quad (d \geq 1), \end{aligned}$$

where $S(d) = \{i \in \mathbb{Z} \mid 0 < i \leq d, i \equiv d \pmod{2}, i \equiv 0 \pmod{m}\}$ (we note $\#S(d) = \delta_{d,m}$ if $d \leq m$).

Proof. Assume that $a_2 > 1$. Let $m = \alpha a_1 + \beta a_2$ ($\alpha, \beta \in \mathbb{N}$). By $(a_1, m) = 1$, one sees $(a_1, a_2) = 1$. Thus $a_1 - a_2 \notin \mathbb{Z}_+ a_1 + \mathbb{Z}_+ a_2$ and $m + a_1 - a_2 = (\alpha + 1)a_1 + (\beta - 1)a_2 \in \mathbb{Z}_+ a_1 + \mathbb{Z}_+ a_2$, whence $U(a_2 - a_1) = 1$. Hence

$$\begin{aligned} w_p^*(1) &= U(a_1 a_2 + a_2) + U(a_2 - a_1) - 2 \\ &= U(a_1 a_2 + a_2) - 1 \geq U(a_1 a_2) - 1 \geq i_p(1) - 1, \end{aligned}$$

which proves (ii). One sees $-m - (-2a_1) = 2a_1 - m = (2 - \alpha)a_1 - \beta a_2 \notin \mathbb{Z}_+ a_1 + \mathbb{Z}_+ a_2$ by $2 - \alpha \leq 1$ and $\beta > 0$ (note $(a_1, a_2) = 1$ and $a_2 > 1$). Thus $U(-2a_1) = 0$. Let us assume $i_p(1) = U(a_1 a_2) = 1$ to prove (i). Then $U(a_1 a_2 - a_1) = 1$ by $a_1 a_2 - a_1 > 0$ and $U(a_1 a_2) \geq U(a_1 a_2 - a_1) \geq 1$. One has $2 = 2U(a_1 a_2) \geq U(2a_1 a_2) \geq U(a_1 a_2) = 1$. Thus one has $i_p(2) = 2U(a_1 a_2) - U(2a_1 a_2) + U(a_1 a_2) - U(a_1 a_2 - a_1) - U(-2a_1) = 0$ or 1 , which proves (i). By (ii) and (2.15), one sees $w_p^*(1) \geq 0$ if $a_2 > 1$. The formula in (iii) implies $w_p^*(1) = -1$ if $a_2 = 1$. We now need to prove the formulae in (iii) assuming $a_2 = 1$. Then $\text{ord}(C^\sharp) = \mathbb{Z}_+$, $U(n) = \lceil n/m \rceil$ for $n \in \mathbb{Z}$, and $U(a_1, a_2) = 1$. Hence (4.9) implies

$$\begin{aligned} i_p(d) &= \sum_{i, j \geq 0, i+j=d} \left(i - \left\lceil \frac{(i-j)a_1}{m} \right\rceil \right) \\ &= \frac{1}{2}d(d + 1) - \sum_{0 < i \leq d, i \equiv d(2)} \left(\left\lceil \frac{ia_1}{m} \right\rceil + \left\lceil -\frac{ia_1}{m} \right\rceil \right) \end{aligned}$$

for $d \geq 2$. By

$$2 = 2 + \frac{x}{m} + \frac{-x}{m} > \left\lceil \frac{x}{m} \right\rceil + \left\lceil \frac{-x}{m} \right\rceil \geq \frac{x}{m} + \frac{-x}{m} = 0,$$

one sees

$$\left\lceil \frac{x}{m} \right\rceil + \left\lceil \frac{-x}{m} \right\rceil = \begin{cases} 1 & \text{if } x \not\equiv 0 \pmod{m}, \\ 0 & \text{otherwise} \end{cases}$$

whence formula for $i_p(d)$ follows. One also has

$$\begin{aligned} w_p^*(d) &= \sum_{i,j \geq 0, i+j=d} \left(\left\lceil \frac{(i-j)a_1 + 1}{m} \right\rceil - 1 \right) \\ &= \sum_{0 < i \leq d, i \equiv d(2)} \left(\left\lceil \frac{ia_1 + 1}{m} \right\rceil + \left\lceil \frac{-ia_1 + 1}{m} \right\rceil - 2 \right) \end{aligned}$$

for $d \geq 1$. It is clear that

$$1 + \frac{x}{m} = \frac{m-1}{m} + \frac{x+1}{m} \geq \left\lceil \frac{x+1}{m} \right\rceil \geq \frac{x+1}{m} \quad (x \in \mathbb{Z})$$

and that the first inequality is an inequality iff $x \equiv 0 \pmod{m}$. Hence from

$$2 = 2 + \frac{x}{m} + \frac{-x}{m} \geq \left\lceil \frac{1+x}{m} \right\rceil + \left\lceil \frac{1-x}{m} \right\rceil \geq \frac{1+x}{m} + \frac{1-x}{m} = \frac{2}{m},$$

one sees that

$$2 \geq \left\lceil \frac{1+x}{m} \right\rceil + \left\lceil \frac{1-x}{m} \right\rceil \geq 1$$

and the first inequality is an equality iff $x \equiv 0 \pmod{m}$. Then the formula for $w_p^*(d)$ follows. \square

There is another case of (IA) with $i_p(1) = 1$, in which we need to calculate $i_p(d)$ for large d .

(5.3) **Proposition.** *Assume that P is an ordinary point of type (IA) with index m . If $w_p(0) = 1/m$, then $a_1 = 1$, $a_2 = m - 1$, $i_p(1) = 1$, and*

$$\begin{aligned} i_p(d) &= (d+1) \left\lceil \frac{d}{m} \right\rceil \quad (d \geq 2), \\ w_p^*(d) &= \frac{1}{2}(d+1) \left\{ d - 2 \left\lceil \frac{d+1}{m} \right\rceil \right\} \quad (d \geq 1). \end{aligned}$$

Proof. By $w_p(0) = 1/m$, one has $a_2 = m - 1$ (4.9). Thus by $m \in \mathbb{Z}_{++}a_1 + \mathbb{Z}_{++}a_2$, one has $a_1 = 1$. Hence $i_p(1) = 1$, $\text{ord}(C^\sharp) = \mathbb{Z}_+$, and $U(n) = \lceil n/m \rceil$ for $n \in \mathbb{Z}$. Then (4.9) implies

$$\begin{aligned} i_p(d) &= \sum \{i - U(im - d)\} = \sum \left\lceil \frac{d}{m} \right\rceil = (d+1) \left\lceil \frac{d}{m} \right\rceil \quad (d \geq 2), \\ w_p^*(d) &= \sum U((i+1)m - (d+1)) - (d+1) \\ &= \sum \left(i - \left\lceil \frac{d+1}{m} \right\rceil \right) = \frac{1}{2}(d+1) \left\{ d - 2 \left\lceil \frac{d+1}{m} \right\rceil \right\} \quad (d \geq 1). \quad \square \end{aligned}$$

The following provides us with necessary information for our calculation in (6.5).

(5.4) **Proposition.** *Assume that P is an ordinary point of type (IA) with index m such that $i_p(1) = 2$ and $w_p(0) < 1/2$. Let a_1, a_2 be as in the definition of type (IA) in (4.2). Then*

- (i) $a_1 = 2, a_2 \geq (m + 1)/2, a_2, m \equiv 1 \pmod{2}$, and
- (ii)

$$w_p^*(1) \geq 2, \quad w_p^*(2) \geq \begin{cases} 5 & \text{if } w_p(0) < 2/5, \\ 4 - \delta_{a_2, (m+1)/2} & \text{if } w_p(0) \geq 2/5, \end{cases}$$

and furthermore if $w_p^*(2) = 3$, then $U((2i + 1)a_2 - 2(2 - i)) = i + 1$ for $i \in [0, 2]$ (cf. (2.8)).

Proof. By $w_p(0) = (m - a_2)/m < 1/2$ (4.9), one has $a_2 > m/2$. From $i_p(1) = U(a_1 a_2) = 2$ (4.9), one sees $2m \geq a_1 a_2 > a_1 m/2$, whence $a_1 \leq 3$. Since $a_1 \geq 2$ by $U(a_1 a_2) = 2$, one will see $a_1 = 2$. Indeed if $a_1 = 3$, then $2m - 3a_2 (< a_2)$ belongs to $3\mathbb{Z}_+ + a_2\mathbb{Z}_+$, whence $2m - 3a_2 \equiv 0 \pmod{3}$, which however contradicts $(m, 3) = 1$. Hence $a_1 = 2$ and $m, a_2 \equiv 1 \pmod{2}$. This proves (i).

By $m \in \mathbb{Z}_+ + 2 + \mathbb{Z}_+ a_2$ and $a_2 > m/2$, one sees $m \geq 5$ and $m = a_2 + 2r$ for some $r \in \mathbb{N}$ such that $2r < a_2$. Let $i, j \in \mathbb{Z}_+$ be such that $i + j \leq 2$. Then we claim

(5.4.1) $U((2i + 1)a_2 - 2j) \geq i + 1$, where

$$\text{“} = \text{”} \Leftrightarrow \begin{cases} i = 0, & \text{or} \\ (i + 1)r + j \geq a_2. \end{cases}$$

To see this, let $X(\alpha, i, j) = (i + \alpha)m - \{(2i + 1)a_2 - 2j\}$ for $\alpha \in \mathbb{Z}_+$. We note that $U((2i + 1)a_2 - 2j) \leq i + \alpha$ iff $X(\alpha, i, j) \in \mathbb{Z}_+ + 2 + \mathbb{Z}_+ a_2$ by definition of $U(\)$, and that

$$X(0, i, j) = -(i + 1)a_2 + 2(ir + j) = (1 - i)a_2 + 2(ir + j - a_2)$$

by $m = a_2 + 2r$. Since $(a_2, 2) = 1$ and $ir + j - a_2 \leq 2r - a_2 < 0$, this means that $X(0, i, j) \notin \mathbb{Z}_+ a_2 + 2\mathbb{Z}_+$ and $U((2i + 1)a_2 - 2j) \geq i + 1$. Then

(5.4.2)
$$\begin{aligned} X(1, i, j) &= X(0, i, j) + m = -ia_2 + 2((i + 1)r + j) \\ &= (2 - i)a_2 + 2((i + 1)r + j - a_2) \end{aligned}$$

by $m = a_2 + 2r$. Then $X(1, 0, j) = 2(r + j) \in \mathbb{Z}_+ + 2$ and thus $U((2i + 1)a_2 - 2j) = i + 1$ if $i = 0$. Assume that $i > 0$, i.e. $i = 1, 2$. Then one similarly sees by (5.4.2) that $X(1, i, j) \in \mathbb{Z}_+ a_2 + 2\mathbb{Z}_+$ iff $(i + 1)r + j - a_2 \geq 0$. Thus our claim (5.4.1) is proved. By (5.4.1) and $a_2 > 2r$, one sees $U(3a_2) \geq 3$ and $U(a_2 - 2) = 1$, whence

$$w_p^*(1) = U(3a_2) + U(a_2 - 2) - 2 \geq 2$$

by (4.9(ii)). Similarly, one sees $U(a_2 - 4) = 1$, and

$$U(3a_2 - 2) \geq 2, \text{ where " = " } \Leftrightarrow 2r + 1 = a_2, \text{ i.e. } a_2 = (m + 1)/2,$$

$$U(5a_2) \geq 3, \text{ where " = " } \Leftrightarrow 3r \geq a_2, \text{ i.e. } w_p(0) \geq 2/5$$

because $2r + 1 \leq a_2$ and $a_2 = m - 2r$. By (4.9(ii)), one has

$$w_p^*(2) = U(5a_2) + U(3a_2 - 2) + U(a_2 - 4) - 3.$$

If $w_p(0) < 2/5$, then $a_2 > 3m/5 \geq (m + 1)/2$ by $m \geq 5$. Hence $w_p^*(2) \geq 5$ if $w_p(0) < 2/5$, and the estimate for $w_p^*(2)$ follows immediately. If $w_p^*(2) = 3$, then the above argument shows $U(5a_2) = 3$, $U(3a_2 - 2) = 2$, $U(a_2 - 4) = 1$. \square

(5.5) **Proposition.** *Assume that P is of type (IC) with index m . Then one has*

(i) $i_p(1) = a_1$ (≥ 2 by (4.2)),

(ii)

$$w_p^*(1) = \begin{cases} i_p(1) & \text{if } w_p(0) \neq (m - 1)/m, \\ i_p(1) - \delta_{a_1, 2} - \delta_{a_1, (m-1)/2} & \text{otherwise,} \end{cases}$$

(iii) if $i_p(1) = 2$ and $w_p(0) \neq (m - 1)/m$, then $i_p(2) = 0$,

(iv) if $d \in \mathbb{N}$ satisfies $w_p(0) < 1/d \leq 1/2$, then $i_p(d' + 1) = 0$ and $w_p^*(d') = a_1 d'(d' + 1)/2$ for all $d' \in [1, d]$, and

(v) if $w_p(0) = 4/5$ and $m = 5$, then $w_p^*(2) = 0$ and $w_p^*(3) = 4$.

Proof. By $w_p(0) = R(a_4)/m < 1$, $R(a_4)$ ($< m$) belongs to $\mathbb{Z}_+ a_1 + \mathbb{Z}_+ a_2$. By $a_4 \not\equiv a_1, a_2 \pmod{m}$, $a_1 + a_2 = m$, $2 \leq a_1 < a_2$, one sees $m \geq 5$ and $R(a_4) = ra_1$ for some $r \in [2, m/a_1]$. Let $u = U(a_4)$, whence $a_4 = um - ra_1$. Let us choose $d \in \mathbb{N}$ such that $w_p(0) < 1/d$, i.e. $ra_1 d < m$. For $i, j \in \mathbb{Z}_+$ such that $i + j \leq d + 1$, we claim

$$(5.5.1) \quad U(ia_1 a_2 + ja_4) = ia_1 + ju - \text{Max}\{0, ia_1 + rj - a_2\}.$$

First we note

$$(5.5.2) \quad \begin{aligned} ia_1 + rj - m &< ia_1 + rj - ra_1 d \\ &\leq (d + 1) \cdot \text{Max}\{a_1, r\} - ra_1 d \\ &= \text{Max}\{a_1, r\}(d + 1 - d \cdot \text{Min}\{a_1, r\}) \leq 0 \end{aligned}$$

by $a_1, r \geq 2$. For $\alpha \geq 0$, let $X(\alpha) = (ia_1 + ju - \alpha)m - (ia_1 a_2 + ja_4)$ so that $U(ia_1 a_2 + ja_4) \leq ia_1 + ju - \alpha$ iff $X(\alpha) \in \mathbb{Z}_+ a_1 + \mathbb{Z}_+ a_2$. Then

$$\begin{aligned} X(\alpha) &= ia_1(m - a_2) + j(um - a_4) - \alpha m \\ &= ia_1^2 + jra_1 - \alpha(a_1 + a_2) \\ &= (ia_1 + rj - \alpha)a_1 - \alpha a_2 \\ &= (ia_1 + rj - a_2 - \alpha)a_1 + (a_1 - \alpha)a_2. \end{aligned}$$

Thus $X(0) \in \mathbb{Z}_+ a_1$ and $U(ia_1 a_2 + ja_4) \leq ia_1 + ju$. For $\alpha > 0$, one has $ia_1 + rj - a_2 - \alpha \leq m - a_2 - \alpha = a_1 - \alpha < a_1 < a_2$ by (5.5.2). Thus by $(a_1, a_2) = 1$, one sees $X(\alpha) \in \mathbb{Z}_+ a_1 + \mathbb{Z}_+ a_2$ iff $ia_1 + rj - a_2 - \alpha \geq 0$. Hence

$\alpha = \text{Max}\{0, ia_1 + rj - a_2\}$ is the largest $\alpha \in \mathbb{Z}_+$ such that $X(\alpha) \in \mathbb{Z}_+a_1 + \mathbb{Z}_+a_2$, whence (5.5.1) follows. Setting $(i, j) = (1, 0)$, one has $U(a_1a_2) = a_1$ and (i) follows. Using $m \geq ra_1 + 1$ and setting $d = 1$, one sees

$$\begin{aligned} ia_1 + rj - a_2 &= (i + 1)a_1 + rj - m \\ &\leq (i + 1)a_1 + rj - 1 - ra_1 \\ &= -(a_1 - j)(r - i - 1) + j(i + 1) - 1. \end{aligned}$$

For $(i, j) = (1, 1)$ (resp. $(0, 2)$), one sees $j(i + 1) = 2$, $a_1 \geq j$, $r \geq i + 1$. Thus

$$ia_1 + rj - a_2 \leq 1$$

and the equality holds iff $m = ra_1 + 1$ and $r = 2$ (resp. $a_1 = 2$), i.e. iff $a_4 \equiv 1 (m)$ and $a_1 = (m - 1)/2$ (resp. $a_1 = 2$). This is because $r = 2 \Leftrightarrow a_1 = (m - 1)/2$ when $m = ra_1 + 1$. Thus by (4.9) and (5.5.1), one has

$$\begin{aligned} w_p^*(1) &= \sum_{(i,j)=(1,1),(0,2)} \{U(ia_1a_2 + ja_4) - ju\} \\ &= \begin{cases} a_1 - \delta_{a_1,2} - \delta_{a_1,(m-1)/2} & \text{if } a_4 \equiv 1 (m), \\ a_1 & \text{otherwise,} \end{cases} \end{aligned}$$

whence follows (ii). By (5.5.1) with $(i, j) = (2, 0)$, one has $U(2a_1a_2) = 2a_1 - \text{Max}\{0, 3a_1 - m\}$, whence one similarly obtains

$$i_p(2) = \begin{cases} \text{Max}\{0, 3a_1 - m\} + \delta_{a_1,2} + \delta_{a_1,(m-1)/2} & \text{if } a_4 \equiv 1 (m), \\ \text{Max}\{0, 3a_1 - m\} & \text{otherwise.} \end{cases}$$

If $i_p(1) = 2$ and $w_p(0) \neq (m - 1)/m$, then $a_1 = 2$ and $a_4 \not\equiv 1 (m)$ by (i), whence $m \geq 2r + 2 \geq 6$ and $i_p(2) = 0$. Hence follows (iii). When $d \geq 2$, one sees

$$\begin{aligned} ia_1 + rj - a_2 &\leq ia_1 + rj - m + \text{Max}\{a_1, r\} \\ &< \text{Max}\{a_1, r\}(d + 2 - d \cdot \text{Min}\{a_1, r\}) \leq 0 \end{aligned}$$

by $a_1, r \geq 2$ and (5.5.2). Thus by (5.5.1), one sees

$$i_p(d' + 1) = 0 \quad \text{and} \quad w_p^*(d') = a_1d'(d' + 1)/2 \quad \text{for } d' \in [1, d]$$

if $d \geq 2$. This proves (iv). Assume that $m = 5$ and $a_4 \equiv 1 (5)$. Then $a_1 = r = 2$, $a_2 = 3$, $u = 2$, and $a_4 = 6$. Thus $\mathbb{Z}_+a_1 + \mathbb{Z}_+a_2 = \mathbb{Z}_+ - \{1\}$. Hence for $x, y, z \in \mathbb{Z}_+$ such that $x + y = z$, one has $U(xa_1a_2 + (y + 1)a_4) = U(6z + 6)$. Thus by (4.9), one has

$$\begin{aligned} w_p^*(z) &= (z + 1) \cdot U(6z + 6) - (z + 1)(z + 2) = (z + 1) \cdot U(z - 4) \\ &= (z + 1) \left(\left\lceil \frac{z - 4}{5} \right\rceil + \delta_{(z \bmod 5), 3} \right), \end{aligned}$$

where $\delta_{(x \bmod 5), y}$ denotes 1 if $x \equiv y (5)$ and 0 otherwise. This proves (v). \square

The calculation needed for imprimitive points is much simpler.

(5.6) **Proposition.** *Let P be an ordinary point of type (IA^\vee) . Then we have*

- (i) $w_p^*(1) \geq i_p(1) - 1$, and
- (ii) $i_p(1) \geq 2$ if $w_p(0) < 2/3$.

Proof. Let the notation be as in (4.9). Since C^\sharp is not smooth, $\text{ow}(C^\sharp)$ is not generated by ow_2 (2.7). Hence $\text{ow}_1 - \text{ow}_2 \notin \text{ow}(C^\sharp)$ by $\text{ow}(C^\sharp) = \mathbb{Z}_+ \text{ow}_1 + \mathbb{Z}_+ \text{ow}_2$. Hence $U(\text{ow}_2 - \text{ow}_1) \geq 1$ (2.8). By (4.9(ii)), we have

$$\begin{aligned} w_p^*(1) &= U(sa_2 \text{ow}_1 + \text{ow}_2) + U(\text{ow}_2 - \text{ow}_1) - 2 \\ &= U(sa_2 \text{ow}_1 + \text{ow}_2) - 1. \end{aligned}$$

By definition of U (2.8), we have $U(sa_2 \text{ow}_1 + \text{ow}_2) \geq U(sa_2 \text{ow}_1) = i_p(1)$. Whence $w_p^*(1) \geq i_p(1) - 1$ and (i) is proved. Assume now that $i_p(1) = U(sa_1 \text{ow}_2) \leq 1$ and $w_p(0) = (\bar{m} - a_2)/\bar{m} < 2/3$ (4.9) to prove (ii). Then $(\bar{m}, 0) - sa_1 \cdot \text{ow}_2 \in \text{ow}(C^\sharp)$ and $3a_2 > \bar{m}$. Comparing the order part, we see $s = 2$, $a_1 = 1$, and $(\bar{m}, 0) - 2\text{ow}_2 = \alpha \text{ow}_1$ for some $\alpha \in \mathbb{Z}_+$. Thus $\bar{m} = 2a_2 + \alpha$ and $2a_2 + \alpha \text{wt} x_1 \equiv 0 \pmod{2\bar{m}}$. Hence $\text{wt} x_1 \not\equiv 1 \pmod{2\bar{m}}$, and $\text{wt} x_1 \equiv 1 + \bar{m} \pmod{2\bar{m}}$. Since $(\text{wt} x_1, 2\bar{m}) = 1$, \bar{m} must be even, whence $\alpha = \bar{m} = 2a_2$ is even. Then $\bar{m} = 2a_2 + \alpha \equiv 2a_2 + \alpha \text{wt} x_1 \equiv 0 \pmod{2\bar{m}}$, which is a contradiction. \square

(5.7) **Proposition.** *Let P be a type (IC^\vee) point. Then $i_p(1) = 2$, $w_p^*(1) = 2$, and $w_p(0) \geq 3/4$.*

Proof. Since $\text{ow}(C^\sharp) = \mathbb{Z}_+(1, 1) + \mathbb{Z}_+(\bar{m} - 1, -1)$, we can easily see

$$U(x) = \begin{cases} 1 & \text{if } x = 1, \\ 2 & \text{if } 2 \leq x \leq 2\bar{m} \end{cases}$$

for integers $x \in [1, 2\bar{m}]$. Thus

$$i_p(1) = U(2\bar{m} - 2) = 2,$$

$$w_p(0) = R(\bar{m} + 1)/\bar{m} = U(\bar{m} + 1) - (\bar{m} + 1)/\bar{m} = (\bar{m} - 1)/\bar{m} \geq 3/4,$$

and

$$\begin{aligned} w_p^*(1) &= U(2\bar{m} - 2 + \bar{m} + 1) - U(\bar{m} + 1) + U(2\bar{m} + 2) - 2U(\bar{m} + 1) \\ &= 2 + U(\bar{m} - 1) - U(\bar{m} + 1) + 2 + U(2) - 2U(\bar{m} + 1) \\ &= 2 \end{aligned}$$

by $\bar{m} \geq 4$ (4.9(ii)). \square

6. POSSIBLE SINGULARITIES ON AN EXTREMAL NBD $X \supset C \simeq \mathbb{P}^1$

The main purpose of this section is to limit the possible combination of singularities of types (IA)–(III), (IA^\vee) – (II^\vee) using invariants $i_p(d)$, $w_p(0)$, and $w_p^*(d)$.

The results are summarized in (6.7). Easy cases (6.7.1)–(6.7.3) will be treated in §7 and the delicate case (6.7.4) will be treated in §9.

We consider imprimitive points first, since they are much easier to treat.

(6.1) **Theorem.** *Let $X \supset C \simeq \mathbb{P}^1$ be an extremal nbd with an imprimitive point P . Then*

- (i) P is of type (IA^\vee) or (II^\vee) (i.e. not of type (IC^\vee)),
- (ii) $X \supset C$ has at most one more singular point, and
- (iii) if $w_P(0) < 2/3$ or $\text{siz}_P \geq 2$ (in particular if P is of type (II^\vee)), then P is the only singular point of X on C .

Proof. By (1.13), $X \supset C$ has no other imprimitive points. Since L -deformation (4.7) at a type (IA^\vee) or (II^\vee) point P produces an extremal nbd $Y \supset D$ with an ordinary type (IA^\vee) point Q such that $\text{siz}_Q = \text{siz}_P$, $w_Q(0) = w_P(0)$, and $Y \supset D$ is isomorphic to $X \supset C$ outside small nbds of Q and P , we may replace $X \supset C$ with an L -deformation at P . Also by replacing $X \supset C$ with an L -deformation at other points, we may assume that $X \supset C$ has only ordinary points (4.7), hence has only points of type (IA) , (IA^\vee) , (IC) , (IC^\vee) , (III) .

(i) Assume that $X \supset C$ has a type (IC^\vee) point P . Then by $w_P^*(1) = 2$ (5.7), $X \supset C$ has another singular point Q such that $w_Q^*(1) \leq -1$ (2.3.3). By $i_P(1) = 2$ (5.7), we have $i_Q(1) = 1$ by (2.3.3) and (2.15). Thus Q is not of type (IC) by (5.5(i)), and not of type (III) by (4.9(ii)), hence of type (IA) . By $w_Q^*(1) \leq -1$, we have $w_Q(0) \geq 1/2$ (5.2(iii)), which contradicts $w_P(0) + w_Q(0) < 1$ (2.3.3) because $w_P(0) \geq 1/2$ (5.7). Hence P cannot be a type (IC^\vee) point.

(ii) Assume that $X \supset C$ has two more singular points Q and R . Then by (2.3.3) and (2.15), we have $i_P(1) = i_Q(1) = i_R(1) = 1$ and no other singular points. Hence Q and R are not of type (IC) by (5.5(i)). By $w_P(0) \geq 1/2$ (4.2), we see $w_Q(0), w_R(0) < 1/2$ by (2.3.3). Then we see $w_Q^*(1) = w_R^*(1) = 1$, by (4.9(i)) for type (III) points, and by (5.1(ii)) with $d = 1$ for type (IA) points. We also have $w_P^*(1) \geq i_P(1) - 1 = 0$ by (5.6(i)). Hence $w_P^*(1) + w_Q^*(1) + w_R^*(1) \geq 2$, which contradicts (2.3.3).

(iii) Assume that $w_P(0) < 2/3$ or $\text{siz}_P \geq 2$ and that $X \supset C$ has another singular point Q . Then $i_P(1) \geq 2$ by (5.6(ii)) or (4.10), and $w_P^*(1) \geq i_P(1) - 1 \geq 1$ (5.6(i)). Thus by (2.3.3) and (2.15), one sees that P and Q are the only singular points, $i_Q(1) = 1$ and $w_Q^*(1) \leq 0$. Thus Q is a type (IA) point as in (i). Then by $w_Q(0) < 1 - w_P(0) \leq 1/2$ (4.2), we have $w_Q^*(1) = 1$ by (5.1(ii)) with $d = 1$. This is a contradiction. \square

(6.2) **Theorem.** *Let $X \supset C \simeq \mathbb{P}^1$ be an extremal nbd. Then*

- (i) X has at most three singular points on C and at least one of them has index > 1 , and
- (ii) if X has three singular points on C , then they consist of a type (III) point, a type (IA) point of index 2, and a type (IA) point (say, P) of odd index (say, $m \geq 3$) and size 1 such that $w_P(0) = (m - 1)/(2 \cdot m)$.

Proof. By (6.1), we may assume that $X \supset C$ is locally primitive. By (2.3.3), we have $\sum_P i_P(1) \leq 3$. Since $i_P(1) \geq 0$ for all $P \in C$ and $i_P(1) \geq 1$ for singular P by (2.15), one sees that X has at most three singular points on C . For (i), it is enough to derive a contradiction assuming that $X \supset C$ has only type (III) singular points. Applying L -smoothing (4.7) repeatedly at type (III) points of $X \supset C$, one obtains a locally primitive extremal nbd $Y \supset D \simeq \mathbb{P}^1$ such that Y is smooth. By (2.3.2), one has $\text{deg gr}_D^1 \mathcal{O} = 1$ because $i_P(1) = 0$ for smooth points P . Thus $\text{deg } N_{D/Y} = -\text{deg gr}_D^1 \mathcal{O} = -1$ and the space of embedded deformation of D in Y has dimension $\geq \chi(N_{D/Y}) = 2 - 1 = 1$. This means that D moves inside Y while it is the only reduced compact subvariety of dimension ≥ 1 in its small enough neighborhood. This is a contradiction and (i) is proved.

We will prove (ii). Let P, Q, R be the singular points with indices k, m, n , respectively. As in the proof of (i), one has $i_P(1) = i_Q(1) = i_R(1) = 1$ and $\text{deg gr}_C^1 \mathcal{O} = -2$. We note that $X \supset C$ has no points of type (IC) by (5.5(i)). Thus P, Q, R are of type (IA), (IB), (IIA), (IIB), or (III). We treat four cases (modulo permutation of P, Q, R).

(6.2.1) *Case ordinary P, Q, R of type (IA), (III), (III), respectively.* We will derive a contradiction in this case. By $w_P^*(1) + w_Q^*(1) + w_R^*(1) \leq 1$ (2.3.3) and $w_Q^*(1) = w_R^*(1) = 1$ (4.9(ii)), one has $w_P^*(1) \leq -1$. Thus by (5.2(iii)), one has $w_P^*(d) = -[d/2] + \delta_{d,k}$ for $d \in [1, k]$. Let $d \in [1, k]$ and let $r \in [0, 1]$ be such that $r \equiv d \pmod{2}$. By $w_Q^*(d) = w_R^*(d) = [(d+1)^2/4]$ (4.9(ii)), one has

$$1 + d + \text{deg gr}_C^d \omega = d + 1 + \frac{1}{2}(d+1)(d-2) + \left\lfloor \frac{d}{2} \right\rfloor - \delta_{d,k} - 2 \left\lfloor \frac{(d+1)^2}{4} \right\rfloor \\ = \frac{1}{2}d(d+1) + \frac{d+r}{2} - \delta_{d,k} - \frac{(d+1)^2 - (1-r)}{2} = -\delta_{d,k},$$

by (2.3.2). Thus we have

$$\sum_{d=1}^k (1 + d + \text{deg gr}_C^d \omega) = -1 < 0,$$

which contradicts (2.3.3).

(6.2.2) *Case ordinary P, Q, R , all of type (IA) with $w_P(0) \geq w_Q(0) \geq w_R(0)$.* We will also derive a contradiction in this case. By $w_P(0) + w_Q(0) + w_R(0) < 1$ (2.3.3), one has $w_Q(0), w_R(0) < 1/2$. Hence by (5.1(ii)) with $d = 1$, one has $i_Q(2) = i_R(2) = 0$. Hence by $\text{deg gr}_C^1 \mathcal{O} = -2$ and (2.3.3), one has $\text{deg gr}_C^2 \mathcal{O} \geq -3$ and whence $-6 + i_P(2) + i_Q(2) + i_R(2) \geq -3$, i.e. $i_P(2) \geq 3$ by (2.3.2). Applying (5.2(i)) to P , one has $a_2 = 1$ under the notation of (5.2), whence $3 \leq i_P(2) = 2 + \delta_{2,k}$ by (5.2(iii)). Thus $k = 2$ and $\text{deg gr}_C^2 \mathcal{O} = -3$, whence $w_P(0) = 1/2$ and $i_P(d) = (d+1)[d/2]$ for $d \geq 2$ by (5.3). One

has $w_R(0) < (1 - w_P(0))/2 = 1/4$, and $i_R(3) = 0$ follows from (5.1(ii)) with $d = 2$. From $\text{deg gr}_C^d \mathcal{O} = -d - 1$ ($d = 1, 2$), one has $-4 \leq \text{deg gr}_C^3 \mathcal{O} = -3 \cdot 4 + i_P(3) + i_Q(3) + i_R(3) = -8 + i_Q(3)$ by (2.3.2) and (2.3.3). Hence $w_Q(0) < 1 - w_P(0) = 1/2$ and $i_Q(3) \geq 4$, whence $m = 3$ and $w_Q(0) = 1/3$ by (5.1(ii)) with $d = 2$. Thus $i_Q(d) = (d + 1)[d/3]$ for $d \geq 2$ by (5.3). From $w_R(0) < 1 - w_P(0) - w_Q(0) = 1/6$, one has $i_R(4) = i_R(5) = 0$. Thus by (2.3.2), one has

$$\text{deg gr}_C^4 \mathcal{O} = -4 \cdot 5 + i_P(4) + i_Q(4) = -20 + 10 + 5 = -5,$$

$$\text{deg gr}_C^5 \mathcal{O} = -5 \cdot 6 + i_P(5) + i_Q(5) = -30 + 12 + 6 = -12,$$

which contradicts $\text{deg gr}_C^4 \mathcal{O} + \text{deg gr}_C^5 \mathcal{O} \geq -5 - 6$ (2.3.3). So this case does not occur.

(6.2.3) *Case ordinary P, Q, R of type (IA), (IA), (III), respectively, with $w_P(0) \geq w_Q(0)$. We will prove $k = 2$, $m \equiv 1 \pmod{2}$, and $w_Q(0) = (m - 1)/2m$ in this case. By $w_Q(0) \leq w_P(0)$ and $w_P(0) + w_Q(0) < 1$ (2.3.3), one has $w_Q(0) < 1/2$ and hence $w_Q^*(1) = 1$ by (5.1(ii)) with $d = 1$. By $w_R^*(1) = 1$ (4.9(ii)) and $w_P^*(1) \leq 1 - w_Q^*(1) - w_R^*(1) = -1$ (2.3.3), one has $w_P^*(1) = -1$, $w_P(0) = (k - 1)/k$, $w_P^*(2) = -1 + \delta_{2,k}$ by (5.2(iii)). Hence $\text{deg gr}_C^1 \omega = -1 - w_P^*(1) - w_Q^*(1) - w_R^*(1) = -2$ (2.3.2) and $\text{deg gr}_C^2 \omega \geq -3$ (2.3.3), whence $-3 \leq \text{deg gr}_C^2 \omega = -w_P^*(2) - w_Q^*(2) - w_R^*(2)$ (2.3.2). By $w_R^*(2) = 2$ (4.9(ii)), one has $w_Q^*(2) \leq 2 - \delta_{2,k}$. We claim $k = 2$. Indeed if $k \geq 3$, then $w_Q(0) < 1 - w_P(0) = 1/k \leq 1/3$ and one sees $w_Q^*(2) = 3$ by (5.1(ii)) with $d = 2$, which contradicts $w_Q^*(2) \leq 2 - \delta_{2,k}$. Thus $k = 2$ is proved, and hence $w_Q^*(2) \leq 1$ and $w_Q(0) < 1/2$. By (5.1(ii)) with $d = 2$, one sees $w_Q(0) = (m - 1)/2m$ and $m \equiv 1 \pmod{2}$.*

(6.2.4) *General case. We claim that $X \supset C$ has no points of type (IB) or (IIB) (resp. (IIA)). Indeed if otherwise, we may apply L-deformation (4.7) (repeatedly) at such points to produce a locally primitive extremal nbd $Y \supset D$ with only ordinary singular points which has at least four singular points (resp. one point with index 4 and at least two more singular points), which contradicts (i) (resp. (6.2.1)–(6.2.3)). Thus the claim is proved and $X \supset C$ has only points of type (IA) or (III) with $i(1) = 1$. When we apply L-deformation (4.7) repeatedly at these points to produce a locally primitive extremal nbd with only ordinary singular points, each singular point deforms to exactly one singular point of the same type with the same index, size, and $w(0)$. Hence we are done by (6.2.1)–(6.2.3). \square*

(6.3) **Theorem.** *Let $X \supset C \simeq \mathbb{P}^1$ be an extremal nbd. Then $X \supset C$ has no type (IB) points.*

Proof. We will derive a contradiction assuming that $X \supset C$ contains a type (IB) point P . We may apply L-deformation (4.7) repeatedly to $X \supset C$, and obtain

an extremal nbd $Y \supset D \simeq \mathbb{P}^1$ with only ordinary singular points which has at least two points P and Q of type (IA) with the same index $m (> 1)$ such that $w_P(0) = w_Q(0)$. By (6.1) and (6.2), P and Q are the only singular points of Y on D . By $w_P(0) = w_Q(0) < 1/2$ (2.3.3), one has $w_P^*(1) = w_Q^*(1) = 1$ by (5.1(ii)) with $d = 1$, which contradicts (2.3.3). Hence $X \supset C$ has no type (IB) points. \square

(6.4) **Theorem.** *Let $X \supset C \simeq \mathbb{P}^1$ be an extremal nbd. If $X \supset C$ has a point P of type (IIB), then P is the only singular point of X on C .*

Proof. If $X \supset C$ has singular points other than P , then one may apply L -deformation (4.7) at type (IIB) point P to $X \supset C$ to get an extremal nbd $Y \supset D \simeq \mathbb{P}^1$ with at least three singular points, two of which have indices 2 and 4. This contradicts (6.2). \square

(6.5) **Theorem.** *Let $X \supset C \simeq \mathbb{P}^1$ be an extremal nbd. Assume that $X \supset C$ has a type (IC) point P with index m . Then P is the only singular point of X on C , and one has $i_P(1) = 2$ and $w_P(0) = (m - 1)/m$.*

Proof. If $X \supset C$ has an imprimitive point $Q (\neq P)$, then we may pass to the splitting cover (1.12) $Y \supset D$ because the point over Q has index > 1 (1.16) and Y is locally primitive along an arbitrary irreducible component of D (1.13). Hence we may assume that $X \supset C$ is locally primitive.

Using L -deformation (4.7), one may assume that $X \supset C$ has only ordinary singularities. We note $i_P(1) \geq 2$ (5.5(i)). We treat three cases.

(6.5.1) *Case $i_P(1) \geq 3$.* We will derive a contradiction in this case. One has $i_P(1) = 3$ and P is the only singular point of X on C by (2.3.3) and (2.15), whence $w_P^*(1) \leq 1$ by (2.3.3). This is a contradiction, because (5.5(i)) and (5.5(ii)) show $w_P^*(1) \geq 3 - 1 = 2$.

(6.5.2) *Case $i_P(1) = 2$ and $w_P(0) \neq (m - 1)/m$.* We will derive a contradiction in this case. By (5.5(ii)) and (5.5(iii)), one has $w_P^*(1) = 2$ and $i_P(2) = 0$. Hence by (2.3.3) and $i_P(1) = 2$, one sees that there exists exactly one more singular point Q of X (say, with index n) on C , $i_Q(1) = 1$, $\deg \text{gr}_C^1 \mathcal{O} = -2$ and $w_Q^*(1) \leq -1$. We see that Q is not of type (IC) by (5.5(i)) and hence Q is of type (IA) by (4.9(ii)). By (5.2(iii)), one has $w_Q(0) = (n - 1)/n$, $i_Q(2) = 2 + \delta_{2,n}$, and $i_Q(3) = 4 + \delta_{3,n}$. By $\deg \text{gr}_C^1 \mathcal{O} = -2$, one sees

$$-3 \leq \deg \text{gr}_C^2 \mathcal{O} = -6 + i_P(2) + i_Q(2) = -4 + \delta_{2,n}$$

by (2.3.2) and (2.3.3). Thus $n = 2$, $\deg \text{gr}_C^2 \mathcal{O} = -3$, and $w_P(0) < 1 - w_Q(0) = 1/2$. By (5.5(iv)) with $d = 2$, one has $i_P(3) = 0$, and hence $\deg \text{gr}_C^3 \mathcal{O} = -12 + i_P(3) + i_Q(3) = -8$ by (2.3.2). This contradicts $\deg \text{gr}_C^3 \mathcal{O} \geq -4$ (2.3.3).

(6.5.3) *Case $i_P(1) = 2$ and $w_P(0) = (m - 1)/m$.* We will derive a contradiction assuming that X has another singular point Q on C (Q is unique by (2.3.3)

and (2.15)). Since P is of type (IC), one has $m \geq 5$, whence $w_Q(0) < 1 - w_P(0) \leq 1/5$ (2.3.3). Since $i_Q(1) = 1$ by (2.3.3), Q is of type (IA) or (III). Let Q be of type (IA) (resp. (III)). Then by $w_Q(0) \leq 1/5$ and (5.1(ii)) (resp. by (4.9(ii))) with $d \in [1, 3]$, one has $w_Q^*(d) = d(d+1)/2$ (resp. $[(d+1)^2/4]$) for $d \in [1, 3]$. By (5.5(ii)), one has $w_P^*(1) = 1 - \delta_{m,5}$. Thus by (2.3.3), one has $1 \geq w_P^*(1) + w_Q^*(1) = 2 - \delta_{m,5}$, whence $m = 5$. Hence by (5.5(v)) and (2.3.2), one has

$$\sum_{d=1}^3 (1 + d + \text{deg gr}_C^d \omega) = \sum_{d=1}^3 \frac{d(d+1)}{2} - 4 - \sum_{d=1}^3 \frac{d(d+1)}{2} = -4 < 0$$

$$\left(\text{resp. } \sum_{d=1}^3 \frac{d(d+1)}{2} - 4 - \sum_{d=1}^3 \left\lceil \frac{(d+1)^2}{4} \right\rceil = -1 < 0 \right).$$

This contradicts (2.3.3), and we are done. \square

(6.6) **Theorem.** *Let $X \supset C \simeq \mathbb{P}^1$ be an extremal nbd. Assume that $X \supset C$ has a type (IA) point P of size > 1 . Then P is the only singular point of X on C , $\text{siz}_P = 2$, and $w_P(0) > 1/2$.*

Proof. As in the proof of (6.5), we may assume that $X \supset C$ is locally primitive. By using L -deformation (4.7), one may assume that $X \supset C$ has only ordinary singular points of type (IA) or (III) (6.5). Then $i_P(1) = \text{siz}_P > 1$ (4.10). Let m be the index of P , and a_1, \dots, a_4 as in the definition of type (IA) in (4.2). We treat three cases.

(6.6.1) *Case $i_P(1) \geq 3$.* We will derive a contradiction in this case. By $i_P(1) \geq 3$, one has $\text{ord}(C^{\sharp}) \neq \mathbb{Z}_+$ and $a_2 > 1$, whence $w_P^*(1) \geq 2$ by (5.2(ii)). Thus P is the only singular point of $X \supset C$ by $\sum_{Q \neq P} i_Q(1) + 3 \leq 3$ (2.3.3) and (2.15), while $X \supset C$ must have some other singular point by $\sum_{Q \neq P} w_Q^*(1) + 2 \leq 1$ (2.3.3). This is a contradiction.

(6.6.2) *Case $i_P(1) = 2$ and $w_P(0) \geq 1/2$.* We will first derive a contradiction assuming that $X \supset C$ has another singular point Q . By $\sum_{R \neq P, Q} i_R(1) + 2 + i_Q(1) \leq 3$ (2.3.3) and (2.15), P and Q are the only singular points of $X \supset C$ and $i_Q(1) = 1$. One has $w_Q(0) < 1 - w_P(0) \leq 1/2$ by (2.3.3). Then one sees that $w_Q^*(1) = 1$ by (5.1(ii)) if Q is of type (IA), or by (4.9(ii)) if Q is of type (III). Since $i_P(1) = 2$, one has $w_P^*(1) \geq i_P(1) - 1 = 1$ by (5.2(i)), as in (6.6.1). Thus we have $w_P^*(1) + w_Q^*(1) \geq 2$, which contradicts (2.3.3). Hence P is the only singular point in this case. We can see $w_P(0) = (m - a_2)/m \neq 1/2$ (4.9(i)). Indeed if otherwise, one sees $m = 2a_2$ and $a_2 = 1$ by $(a_2, m) = 1$, whence $i_P(1) = 1$ (4.9(ii)), which contradicts the hypothesis (6.6.2). Hence $w_P(0) > 1/2$, which was to be proved in this case.

(6.6.3) *Case $i_P(1) = 2$ and $w_P(0) < 1/2$.* In this case, we will also derive a contradiction. By (5.4), we have $a_1 = 2$ and $w_P^*(1) \geq 2$. Hence by

$\sum_{Q \neq P} w_Q^*(1)+2 \leq 1$ (2.3.3), $X \supset C$ has another singular point Q (say, of index n) such that $w_Q^*(1) \leq -1$. By $\sum_{Q \neq P, Q} i_R(1)+2+i_Q(1) \leq 3$ (2.3.3) and (2.15), we see that $X \supset C$ has no other singular points, $i_Q(1) = 1$, $\text{deg gr}_C^1 \mathcal{O} = -2$. Now we claim

$$(6.6.4) \quad n = 2, \quad a_2 = (m + 1)/2, \quad w_P^*(2) = 3, \quad \text{deg gr}_C^d \omega = -d - 1 \quad (d = 1, 2).$$

Since $w_Q^*(1) \leq -1$, Q is a type (IA) point by (4.9(ii)), whence

$$w_Q(0) = (n - 1)/n, \quad w_Q^*(1) = -1, \quad w_Q^*(2) = -1 + \delta_{2,n}$$

by (5.2(iii)). By $w_P^*(1) \geq 2$ (5.4(ii)) and $w_P^*(1) + w_Q^*(1) \leq 1$ (2.3.3), one has $w_P^*(1) = 2$ and $\text{deg gr}_C^1 \omega = -2$ (2.3.3). Hence $\text{deg gr}_C^2 \omega \geq -3$ by (2.3.3). If $n \geq 3$, then $w_P(0) < 1 - w_Q(0) = 1/n \leq 1/3$ ($< 2/5$), whence $w_P^*(2) \geq 5$ (5.4(ii)) and we have a contradiction $-3 \leq -w_P^*(2) - w_Q^*(2) \leq -5 + 1 - \delta_{2,n} \leq -4$ from (2.3.2). Hence $n = 2$ and $w_Q^*(2) = 0$, and (2.3.2) implies $-3 \leq \text{deg gr}_C^2 \omega = -w_P^*(2)$, i.e. $w_P^*(2) \leq 3$. Thus by (5.4(ii)), one sees $w_P^*(2) = 3$, $a_2 = (m + 1)/2$, and $\text{deg gr}_C^2 \omega = -3$. This proves our claim (6.6.4).

By (2.3.4), one has $\text{gr}_C^1 \mathcal{O} \simeq \mathcal{O}(-1)^{\oplus 2}$ and $\text{gr}_C^d \omega \simeq \mathcal{O}(-1)^{\oplus (d+1)}$ for $d \in [0, 2]$. Applying (4.8) to $X \supset C \ni Q$ (with $b_1 = b_2 = k = 1$ and index 2), one sees that

$$\beta_2: S^2(\text{gr}_C^1 \mathcal{O}) \otimes \text{gr}_C^0 \omega \hookrightarrow \text{gr}_C^2 \omega$$

(cf. (2.2)) is described at Q (by notation of (4.8)) as

$$(\beta_2)_Q: \sum_{i+j=2} \mathcal{O} t^3 v_1^i v_2^j \Omega \hookrightarrow \sum_{i+j=2} \mathcal{O} t v_1^i v_2^j \Omega,$$

where t^2 is the uniformizing parameter of $\mathcal{O}_{C,Q}$. Hence it induces an injection

$$\gamma_2: S^2(\text{gr}_C^1 \mathcal{O}) \otimes \text{gr}_C^0 \omega \otimes \mathcal{O}(Q) \hookrightarrow \text{gr}_C^2 \omega$$

which is an isomorphism at Q and hence an isomorphism outside P . We will study γ_2 near P to derive a contradiction. Let (x) and t be normalized ℓ -coordinates at P , and we will use (X^\sharp, P^\sharp) , (C^\sharp, P^\sharp) , and wt ($= C^\sharp$ - wt) in the sense of §2. From now on we use a instead of $a_2 = (m + 1)/2$ for simplicity of notation. We note $a, m \equiv 1 \pmod{2}$ by (5.4(i)) and $a \geq 3$. One has $\mathcal{O}_{C^\sharp, P^\sharp} \simeq \mathbb{C}\{t^2, t^a\}$ (2.7) and $\text{gr}_{C^\sharp}^1 \mathcal{O}$ (resp. $\text{gr}_{C^\sharp}^0 \omega$) has μ_m -semi-invariant free $\mathcal{O}_{C^\sharp, P^\sharp}$ -basis A and B (resp. C) near P^\sharp so that $\text{wt } A \equiv 2a \equiv 1$, $\text{wt } B \equiv a_3 \equiv -2 \pmod{m}$ (resp. $\text{wt } C \equiv a \pmod{m}$) by (2.10)). This is because (C^\sharp, P^\sharp) is a complete intersection $F_2 = F_3 = 0$ in (X^\sharp, P^\sharp) , where $\text{wt } F_2 \equiv 2a$, $\text{wt } F_3 \equiv a_3 \pmod{m}$ (4.5.1). Hence $\text{gr}_C^1 \mathcal{O} = \mathcal{O}_C(t^{m-1}A) \oplus \mathcal{O}_C(t^2B)$ near P by (4.8), where $t^{m-1} = (t^2)^{a-1}$ and $t^2 \in \mathcal{O}_{C^\sharp, P^\sharp}$. We recall that $\mathcal{O}_{C,P} = \mathbb{C}\{t^m\}$ is the

subring of μ_m -invariants of \mathcal{O}_{C^*, P^*} . Since $\text{gr}_C^1 \mathcal{O} \simeq \mathcal{O}(-1)^{\oplus 2}$, one can choose a splitting into invertible sheaves

$$\text{gr}_C^1 \mathcal{O} = L \oplus M$$

so that $L \simeq \mathcal{O}(-1)$, $M \simeq \mathcal{O}(-1)$, $L \otimes \mathbb{C}(P) = \mathbb{C}(P)(t^{m-1}A)$, and $M \otimes \mathbb{C}(P) = \mathbb{C}(P)(t^2B)$, where $\mathbb{C}(P) = \mathcal{O}/t\mathcal{O}$. Thus L and M have bases \bar{L} and \bar{M} near P so that

$$(6.6.5) \quad \bar{L} = t^{m-1}A + \lambda t^{m+2}B \quad \text{and} \quad \bar{M} = \mu t^{2m-1}A + t^2B$$

for some $\lambda, \mu \in \mathbb{C}\{t^m\}$. Since $t^{2m-3} = t^m \cdot (t^2)^{(m-3)/2} \in \mathcal{O}_{C^*, P^*}$, we will replace B with $\mu t^{2m-3}A + B$ so that one may set $\mu = 0$ in (6.6.5). Let \bar{N} be an \mathcal{O}_C -basis of $N = \text{gr}_C^0 \omega (\simeq \mathcal{O}(-1))$ near P so that $\bar{N} = t^{m-a}C$ near P , where $t^{m-a} = (t^2)^{(m-a)/2} \in \mathcal{O}_{C^*, P^*}$. We will show that γ_2 induces an isomorphism

$$(6.6.6) \quad L^2 \otimes N(2P + Q) \oplus L \otimes M \otimes N(P + Q) \oplus M^2 \otimes N(Q) \xrightarrow{\sim} \text{gr}_C^2 \omega$$

which will contradict $H^1(\text{gr}_C^2 \omega) = 0$ because the source is isomorphic to $\mathcal{O} \oplus \mathcal{O}(-1) \oplus \mathcal{O}(-2)$. By (4.8) and $a \geq 3$ above, one has

$$\begin{aligned} \text{gr}_C^2 \omega &= \bigoplus_{i, j \geq 0, i+j=2} \mathcal{O}_C^{R((2i+1)a-2j)} A^i B^j C \\ &= \mathcal{O}_C t^{a-3} A^2 C \oplus \mathcal{O}_C t^a ABC \oplus \mathcal{O}_C t^{a+3} B^2 C \end{aligned}$$

near P by direct calculation, or by

$$\begin{aligned} R((2i + 1)a - 2j) &= m \cdot U((2i + 1)a - 2j) - ((2i + 1)a - 2j) \\ &= (i + 1)m - (2i + 1)a + 2j \\ &= m - a - i + 2j = a - 1 - i + 2j, \end{aligned}$$

where we used $U((2i + 1)a - 2j) = i + 1$ by (5.4(ii)) and $2a = m + 1$. Then

$$\begin{aligned} \bar{L}^2 \bar{N} &= t^{2m}(t^{m-a-2}A^2C + 2\lambda t^a ABC + \lambda^2 t^{a+3}B^2C), \\ \bar{L} \cdot \bar{M} \cdot \bar{N} &= t^m(t^a ABC + \lambda t^{a+3}B^2C), \\ \bar{M}^2 \bar{N} &= t^{a+3}B^2C, \end{aligned}$$

and $t^{-2m}\bar{L}^2\bar{N}$, $t^{-m}\bar{L} \cdot \bar{M} \cdot \bar{N}$, $\bar{M}^2\bar{N}$ form an \mathcal{O}_C -basis for $\text{gr}_C^2 \omega$ near P . Since $\bar{L}^i \bar{M}^j \bar{N}$ is a basis at P of components $L^i \otimes M^j \otimes N(Q)$ of $S^2(\text{gr}_C^1 \mathcal{O}) \otimes \text{gr}_C^1 \omega \otimes \mathcal{O}(Q)$, γ_2 induces homomorphisms $L^i \otimes M^j \otimes N(iP + Q) \rightarrow \text{gr}_C^2 \omega$ ($i, j \in \mathbb{Z}_+, i + j = 2$), which give splitting (6.6.6).

Thus we have a contradiction and case (6.6.3) does not occur. This proves (6.6). \square

Summarizing the results in this section, we have

(6.7) **Theorem.** *Let $X \supset C \simeq \mathbb{P}^1$ be an extremal nbd. Then we have three cases:*

(i) *Case X has only one point P with index > 1 . Then*

(6.7.1) *P is of type (IA), (IIA), (IA^\vee) , or (II^\vee) , and $X \supset C$ has at most one more singular point, which is a type (III) point, or*

(6.7.2) *P is of type (IC) or (IIB), $X \supset C$ has no other singular points,*

(ii) *Case X has P of index > 1 and Q of index 2. Then*

(6.7.3) *P, Q are of type (IA), $\text{siz}_P = 1$, $X \supset C$ has at most one more singular point, which is a type (III) point,*

(iii) *Case X has points P, Q of index > 2 . Then*

(6.7.4) *P, Q are of type (IA), (IIA), or (IA^\vee) , and are of size 1, and $X \supset C$ has no other singular points.*

Proof. First, $X \supset C$ has a point with index > 1 by (6.2), no type (IC^\vee) points by (6.1) and no type (IB) points by (6.3).

Case (i). If P is of type (IA), (IIA), (IA^\vee) , or (II^\vee) , then $X \supset C$ has at most one more type (III) point by (6.2), whence we have (6.7.1). If P is of type (IIB) or (IC), then $X \supset C$ has no other singular points by (6.4) or (6.5), whence we have (6.7.2).

In other cases, $X \supset C$ has no type (IC) (resp. (IIB), (II^\vee)) points by (6.5) (resp. (6.4), (6.1(iii)) and $w_P(0) = 1/2$ (4.9(i))).

Case (ii). $X \supset C$ has at most one more type (III) point by (6.2), and Q is of type (IA) because it is of index 2 (4.2). Thus $w_Q(0) = 1/2$ (4.9(i)) and $w_P(0) < 1 - w_Q(0) = 1/2$ by (2.3.3). Hence P cannot be of type (IIA) or (IA^\vee) and it is a type (IA) point with size 1 by (6.6).

Case (iii). X has no other singular points by (6.2). One sees that type (IIA) points have size 1 by definition and type (IA) (resp. (IA^\vee)) points have size 1 by (6.6) (resp. (6.1)). \square

7. EXISTENCE OF “GOOD” ANTI(BI)CANONICAL DIVISOR (EASY CASE).

The main purpose of this section is to prove the existence of “good” members in $|-K_X|$ or $|-2K_X|$ when extremal nbd $X \supset C \simeq \mathbb{P}^1$ has at most one point of index > 2 (i.e. cases (6.7.1)–(6.7.3)). We start with quoting

(7.1) **Theorem** (Reid [R4, (6.3)]). *Let (X, P) be a 3-fold terminal singularity. Then general members of $|-K_X|$ passing through P have only rational double points at P .*

Following the same idea, one can show the following by a simple explicit calculation for cA type points. We note that one can reduce (7.2) for arbitrary 3-fold terminal singularities to (7.1) by Kawamata’s method in [Ka2, §8].

(7.2) **Proposition.** *Let (X, P) be a 3-fold cA -type terminal singularity of index $m > 2$. Let T be a general member of $|-2K_X|$, and $(Y, Q) \rightarrow (X, P)$ the double cover with branch locus T . Then (Y, Q) has only a canonical singularity.*

(7.2.1) *Remark.* We note that a normal Gorenstein singularity (Y, Q) is a canonical singularity iff it is a rational singularity [EI2, Fl, SB].

Proof. Let (x, y, z, u) (resp. ϕ) be an ℓ -coordinate system (resp. ℓ -equation) for (X, P) such that $\text{wt } x + \text{wt } y \equiv 0, \text{wt } u \equiv \text{wt } \phi \pmod{m}$ (1a.5). Then the preimage of T has a semi-invariant equation $\psi = \lambda z^2 + \dots$, and (Y, Q) is the μ_m -quotient of (Y^\sharp, Q^\sharp) : $\phi = v^2 - \psi = 0 \subset (\mathbb{C}^5, 0)$, where v is an extra coordinate with $\text{wt } v \equiv \text{wt } z \pmod{m}$ (cf. (2.10)). Since (Y, Q) is an isolated singularity by Bertini's Theorem and Gorenstein by the adjunction formula, we only have to show that (Y^\sharp, Q^\sharp) has only canonical singularities. We treat three cases.

(7.2.2) *Case $m = 4$ and $\text{wt } \phi \equiv 2 \pmod{4}$.* We may assume that $\text{wt}(x, y, z, u, \phi, v) \equiv (3, 1, 1, 2, 2, 1)$, $\phi = x^2 + z^2 - f(y, u)$, where $f(y, u) \in (y^2, y^2u, u^3)$ by consideration of wt (1a.6). Since $\psi = \lambda x^2 + u + \dots$ ($\lambda \in \mathbb{C}$), (Y^\sharp, Q^\sharp) is an isolated hypersurface singularity $x^2 + z^2 - f(y, v^2 - \lambda z^2 - \dots) = 0$ in $xyzv$ -space. Thus it is a cA point, and hence is canonical (1a.2(ii)).

(7.2.3) *Case $\phi = xy + f(z, u)$, $\text{wt } z^2 \equiv \text{wt } x \pmod{m}$.* We note $f \in (z^m, u)$ by consideration of wt and $f(0, u) \neq 0$ by u -axis $\not\subset (X^\sharp, P^\sharp)$ (1a.6). Since $\text{wt } z^2 \not\equiv 0, \text{wt } y, \text{wt } z$ by $m > 2$, we may write $\psi = h \cdot \{x - g(y, z, u)\}$ for some g and h such that $h(0) \neq 0, g \in (y, z)^2$, and z^2 appears in g by Weierstrass Preparation Theorem. Since g and h are unique by Weierstrass Preparation Theorem and since ψ is semi-invariant, g and h are also semi-invariant. Then $h^{-1}v^2 + g = x$, and replacing v by $vh^{1/2}$, one may write (Y^\sharp, Q^\sharp) as a hypersurface $F(y, z, u, v) = (v^2 - g)y + f(z, u) = 0$ in $yzuv$ -space. Choosing a general ψ , one may assume that the plane curve

$$c(F)(y, z, u) \equiv \{v^2 - g_2(y, z, 0)\}y - f_3(z, 0) = 0 \subset \mathbb{P}(y, z, v)$$

has only ordinary double points, where (e.g.) g_n denotes the degree n part of g . Since $f(0, u) \neq 0$, it is enough to prove

(7.2.3.1) **Sublemma.** *Assume that a hypersurface singularity*

$$(V, R): F(y, z, u, v) \equiv (v^2 - g(y, z, u))y + f(z, u) = 0 \subset (\mathbb{C}^4, 0)$$

such that $g \in (y, z)^2, f \in \mathbb{C} + (z^3, u)$, and $f(0, u) \neq 0$, has only canonical singularities outside $\{R\}$. If $c(F) = 0 \subset \mathbb{P}(y, z, v)$ is a plane curve with only ordinary double points, then V has only canonical singularities.

We prove this by induction on $k = \text{ord } f(0, u)$. If $k = 0, 1$, then $F = 0$ is smooth, and if $k = 2$, then R is a cA or cD point by its normality and hence canonical (1a.2(ii)). Assume $k \geq 3$. The exceptional divisor D

of the blow-up $\sigma: \bar{V} \rightarrow V$ of V at R lying in $\mathbb{P}(y, z, u, v)$ is defined by $c(F)(y, z, u) + \lambda u^3 = 0$ ($\lambda \in \mathbb{C}$), and it is easy to see that D has at most A_∞ -type singularity outside $(0 : 0 : 1 : 0)$. Thus \bar{V} is normal and has only rational singularities outside the point \bar{R} corresponding to $(0 : 0 : 1 : 0)$ above. Let $\bar{y} = y/u, \bar{z} = z/u, \bar{v} = v/u$ and $\bar{g} = g(\bar{y}u, \bar{z}u, u)u^{-2}, \bar{f} = f(zu, u)u^{-3}$, and $\bar{F} = (\bar{v}^2 - \bar{g}) + \bar{f}$. Then $\bar{g} \in (\bar{y}, \bar{z})^2, \bar{f} \in \mathbb{C} + (\bar{z}^3, u)$, $\text{ord } \bar{f}(0, u) = k - 3$, and $c(\bar{F})(\bar{y}, \bar{z}, \bar{v}) = c(F)(\bar{y}, \bar{z}, \bar{v})$. Thus \bar{V} has only rational singularities by the induction hypothesis, and (V, R) is a canonical singularity because $K_{\bar{V}} = \sigma^* K_V$ as is easily seen, for example, by

$$\sigma^* \text{Res} \frac{dy \wedge dz \wedge du \wedge dv}{F} = \text{Res} \frac{d\bar{y} \wedge d\bar{z} \wedge du \wedge d\bar{v}}{\bar{F}},$$

and the claim (7.2.3.1) is proved.

(7.2.4) Case $\phi = xy + f(z, u)$, $\text{wt } z^2 \neq \text{wt } x, \text{wt } y$ (m). This case is quite similar to (7.2.3). It is easily reduced to the following

(7.2.4.1) **Sublemma.** Assume that a 3-fold $(V, R) \subset (\mathbb{C}^5, 0)$ defined by $F \equiv xy + f(z, u) = 0$ and $G \equiv v^2 + g(x, y, z, u) = 0$ such that $g \in (x, y, z)^2, f \in \mathbb{C} + (z^3, u)$, and $f(0, u) \neq 0$, has only canonical singularities outside $\{R\}$. Then V has only canonical singularities if we assume the following condition $(*)_\lambda$ for $\lambda = 0$ and 1 .

Condition $(*)_\lambda: xy + \lambda u^2 = 0$ and $c(G)(x, y, z, v) \equiv v^2 + g_2(x, y, z, 0) = 0$ define a surface S_λ in $\mathbb{P}(x, y, z, u, v)$ with at most A_∞ -type singularities outside $(0 : 0 : 0 : 1 : 0)$.

The proof is similarly done by induction on $k = \text{ord } f(0, u)$.

(7.2.4.2) It remains to see that $(*)_0$ and $(*)_1$ are satisfied by general G (or $T \in |-2K|$). Since S_0 is the cone over the curve $Z \subset \mathbb{P}(x, y, z, v)$ defined by $xy = 0$ and $c(G) = 0, (*)_0$ is equivalent to the assertion that Z has at most ordinary double points. Thus $(*)_0$ is certainly satisfied by general G because it is satisfied by $G = v^2 + \mu z^2$ ($\mu \in \mathbb{C} - \{0\}$).

On the other hand, S_1 is a double cover of the surface $W \subset \mathbb{P}(x, y, z, v)$ defined by $c(G) = 0$ with branch locus Z . Thus one can see that S_1 has at most A_∞ -type singularities for general G by the fact that W has at most A_∞ -type singularities when z^2 appears in G , as follows. If $\xi \in S_1$ does not lie over a point in Z , then $S_1 \rightarrow W$ is étale at ξ , whence S_1 has at most A_∞ -type singularity at ξ because so does W everywhere. If $\xi \in S_1$ lies over $\zeta \in Z$, then the Cartier divisor $(\{v = 0\}, \xi)$ of (S_1, ξ) is the ramification divisor and isomorphic to (Z, ζ) . Since (Z, ζ) has at most an ordinary double point as above, (S_1, ξ) has at most an A_∞ -type singular point. Thus the case (7.2.4) is finished, and (7.2) is proved. \square

(7.3) **Theorem.** Let $X \supset C \simeq \mathbb{P}^1$ be an extremal nbd. Assume that $X \supset C$ has at most one point of index > 2 ($\Leftrightarrow X \supset C$ is in one of (6.7.1), (6.7.2),

(6.7.3)). We have three cases by (6.7) :

(7.3.1) a point P is of type (IA), (IIA), (IA^\vee) , or (II^\vee) , and no other point of index > 1 (\Leftrightarrow (6.7.1)),

(7.3.2) a point P is of type (IC) or (IIB), and no other singular point (\Leftrightarrow (6.7.2)), and

(7.3.3) P, Q are of type (IA), index $Q = 2$, $\text{siz}_P = 1$, and no other points of index > 1 (\Leftrightarrow (6.7.3)).

Then (i) in case (7.3.1), general members (D, P) of $|-K_{(X,P)}|$ satisfy $D \cap C = \{P\}$, $D \in |-K_X|$ and that D has only rational double points as singularities, and

(ii) in case (7.3.2) or (7.3.3), general members (D, P) of $|-2K_{(X,P)}|$ satisfy $D \cap C = \{P\}$, $D \in |-2K_X|$ and that the double cover Y of X with branch locus D has at most canonical singularities.

Proof. Let $(x) = (x_1, \dots, x_4)$ and t be normalized ℓ -coordinates and ϕ an ℓ -equation of $X \supset C$ at P as in (4.2). Let a_1, \dots, a_4 be as in (4.2). Then elements of $\mathcal{O}(-K_X)$ (resp. $\mathcal{O}(-2K_X)$) at P are of the form $f\Omega^{-1}$ (resp. $f\Omega^{-2}$), where f is a semi-invariant in \mathcal{O}_{X^*, P^*} such that $\text{wt } f \equiv \text{wt } \Omega$ (resp. $2 \text{wt } \Omega$) and $\Omega = \text{Res } dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 / \phi$.

In case (7.3.1), P is of type (IA), (IIA), (IA^\vee) , or (II^\vee) , whence $\text{wt } \Omega \equiv \text{wt } x_2$ (m) by $\text{wt } x_1 x_3 \equiv 0$ and $\text{wt } x_4 \equiv \text{wt } \phi$ (4.2). Then a general f with $\text{wt } f \equiv \text{wt } \Omega$ (m) is $\lambda x_2 + \dots$ and $(D, P) \in |-K_{(X,P)}|$ defined by f satisfies $(mD) \cdot C = a_2 P$, and D is a \mathbb{Q} -Cartier Weil divisor of X . One has $D \sim -K_X$ as elements of $\text{Cl}(X, P)$ because $\text{wt } \Omega^{-1} \equiv \text{wt } f^{-1} \pmod{m}$, whence $D + K_X$ is in $\text{Pic } X$ with $(D + K_X \cdot C) = 0$ because $(K_X \cdot C) = -1 + w_P(0) = -a_2/m$ (2.3.2). Thus $D \in |-K_X|$ (1.3(ii)), whence we are done by (7.1).

We note that, in the remaining cases, $X \supset C$ is locally primitive. In case (7.3.2), let P be of type (IC) (resp. (IIB)). Then one has $a_1 = 2$, $a_2 = m - 2$, $a_3 = m$, $a_4 \equiv 1$ (m) (resp. $(a_1, \dots, a_4) = (3, 2, 5, 5)$) by (4.9(ii)), (5.5(i)) and (6.5) (resp. definition). Hence $\text{wt } \Omega \equiv a_4 \equiv 1$ (m), and a general f with $\text{wt } f \equiv 2 \text{wt } \Omega \equiv 2$ (m) is $\lambda x_1 + \dots$ by $a_2 = 2$ (resp. $\lambda x_2 + \dots$ by $a_2 = 2$) with $\lambda \in \mathbb{C}^*$. Thus $(D, P) \in |-2K_{(X,P)}|$ defined by f satisfies $(mD) \cdot C = 2P$, and D is a \mathbb{Q} -Cartier Weil divisor of X . One has $D \sim -2K_X$ as elements of $\text{Cl}(X, P)$, whence $D + 2K_X$ is in $\text{Pic } X$ with $(D + 2K_X \cdot C) = 0$ because $(K_X \cdot C) = -1 + w_P(0) = -1/m$ by (2.3.2) and (6.5) (resp. (4.9(ii))). Thus $D \in |-2K_X|$ (1.3(ii)). We note that P is of cA -type, by (1a.6) and $m \geq 5$ (part of definition) if P is of type (IC), and by definition otherwise. Hence we are done by (7.2).

Let us consider (7.3.3). First we note that $w_Q(0) = 1/2$ by (4.9(i)) and index $Q = 2$. Then $w_P(0) = (m - a_2)/m < 1 - w_Q(0) = 1/2$ by (2.3.3) and (4.9(i)), whence $2a_2 > m$. This means $m \geq 3$ since $m > a_2$ (4.2). On the other hand, one has $a_1 a_2 \leq m$ by $\text{siz}_P = U(a_1 a_2) = 1$. Hence $a_1 = 1$. One has $\text{wt } \Omega \equiv a_2$ by (2.10), and a general f with $\text{wt } f \equiv 2 \text{wt } \Omega \equiv 2a_2 - m$ (m) is

$\lambda x_2^2 + \mu x_1^{2a_2-m} + \dots$ ($\lambda, \mu \in \mathbb{C}^*$). Thus $(D, P) \in |-2K_{(X,P)}|$ defined by f satisfies $(mD) \cdot C = (2a_2 - m)P$, and D is a \mathbb{Q} -Cartier Weil divisor of X . We note that D and $-2K_X$ are Cartier divisors outside $\{P\}$ because index $Q = 2$. Thus by $D \sim -2K_X$ as elements of $\text{Cl}(X, P)$, one sees $D \sim -2K_X$ in $\text{Pic } X$ by

$$\begin{aligned} (-2K_X \cdot C) &= -2(-1 + w_P(0) + w_Q(0)) \\ &= -2(-1 + (m - a_2)/m + 1/2) \\ &= (2a_2 - m)/m. \end{aligned}$$

Hence $D \in |-2K_X|$. We need to show that P is of cA -type (1a.6). Indeed if P is not of cA -type, then by $m \geq 3$ we see that P is of cD -type and $m = 3$ (1a.6). Thus by (4.11), we have $w_P(0) = 2/3$, which contradicts $w_P(0) < 1/2$. Thus P is of cA -type. \square

8. J-FILTRATION AND LIFTING STRUCTURE

We need to introduce several notions for the arguments in the next section. The main ingredients are rather simple commutative algebra lemmas (8.3) and (8.11), and other parts are formal arguments to obtain the main (8.12) from these.

We note that Reid’s notion of width in [R1, (5.3)] is different from our width in (8.2).

(8.1) Let X be an analytic normal 3-fold and $C \subset X$ a reduced curve such that no irreducible components of C are contained in the singular locus of X .

(8.1.1) **Lemma.** Let $\mathcal{S} = \mathcal{S}_0 \supset \mathcal{S}_1 \supset \dots \supset \mathcal{S}_n$ be a chain of coherent sheaves on X . Then we have equivalent conditions:

- (a) \mathcal{S}_i is \mathcal{S}_{i-1} -saturated for all $i \in [1, n]$,
- (b) \mathcal{S}_i is \mathcal{S}_0 -saturated for all $i \in [1, n]$, and
- (c) \mathcal{S}_i is \mathcal{S}_j -saturated for all $i, j \in [0, n]$ with $j < i$.

Proof. First it is obvious that (c) \Rightarrow (a), (b). When $n = 2$, (a) \Rightarrow (c) and (b) \Rightarrow (c) amount to saying the following:

(*) When \mathcal{S}_1 is \mathcal{S}_0 -saturated, \mathcal{S}_2 is \mathcal{S}_1 -saturated iff \mathcal{S}_2 is \mathcal{S}_0 -saturated.

Thus $n = 2$ case is easily checked. Indeed for every closed point $P \in X$, we have

$$\text{Hom}(\mathbb{C}(P), \mathcal{S}_0/\mathcal{S}_2) \simeq \text{Hom}(\mathbb{C}(P), \mathcal{S}_1/\mathcal{S}_2)$$

from the exact sequence

$$0 \rightarrow \mathcal{S}_1/\mathcal{S}_2 \rightarrow \mathcal{S}_0/\mathcal{S}_2 \rightarrow \mathcal{S}_0/\mathcal{S}_1 \rightarrow 0$$

and $\text{Hom}(\mathbb{C}(P), \mathcal{S}_0/\mathcal{S}_1) = 0$.

(a) \Rightarrow (c) This is proved by induction on $j - i$ by virtue of the exact sequence

$$0 \rightarrow \mathcal{S}_{j+1}/\mathcal{S}_i \rightarrow \mathcal{S}_j/\mathcal{S}_i \rightarrow \mathcal{S}_j/\mathcal{S}_{j+1} \rightarrow 0$$

and the above (*).

(b) \Rightarrow (c) This is proved by the above (*) and the exact sequence

$$0 \rightarrow \mathcal{S}_j/\mathcal{S}_i \rightarrow \mathcal{S}_0/\mathcal{S}_i \rightarrow \mathcal{S}_0/\mathcal{S}_j \rightarrow 0. \quad \square$$

(8.2) **Definition.** Let $\mathcal{S} = \mathcal{S}_0 \supset \mathcal{S}_1 \supset \dots \supset \mathcal{S}_n$ be a chain of coherent sheaves on X . We say that it is *saturated* if the equivalent conditions in (8.1.1) hold. We call the chain $\mathcal{S}_0 \supset \text{Sat}_{\mathcal{S}} \mathcal{S}_1 \supset \dots \supset \text{Sat}_{\mathcal{S}} \mathcal{S}_n$ the *saturation* of the chain $\mathcal{S}_0 \supset \dots \supset \mathcal{S}_n$.

Let $J \subset \mathcal{O}_X$ be a sheaf of ideals. We say that J is *C-laminal* (or *laminal*, if there is no danger of confusion) if J has no embedded primary components, $I_C = \sqrt{J}$, and $I_D^{(2)} \not\supset J$ for an arbitrary irreducible component D of C (cf. (2.2)). The *width* of J is the smallest $d \in \mathbb{N}$ such that $J \supset I_C^d$, and J is said to be of *pure width* d if the D -primary component of J is a D -laminal ideal of width d for each irreducible component D of C .

Let J be a C -laminal ideal of pure width d and let \mathcal{S} be a coherent \mathcal{O}_X -module. For $n \in \mathbb{Z}_+$, let $q = [n/d]$ and $r = n - dq$ (note $r \in [0, d - 1]$). Let $F^n(\mathcal{S}, J) = \text{Sat}_{\mathcal{S}}(J^q I_C^r + J^{q+1})\mathcal{S}$, $\text{gr}^n(\mathcal{S}, J) = F^n(\mathcal{S}, J)/F^{n+1}(\mathcal{S}, J)$. One easily sees that

(8.2.1) **Lemma.** $F^n(\mathcal{O}_X, J) \cdot F^{n'}(\mathcal{S}, J) \subset F^{n+n'}(\mathcal{S}, J)$.

Proof. Let $q = [n/d]$, $r = n - dq$, and $q' = [n'/d]$, $r' = n' - dq'$. Then $q'' = [(n + n')/d]$ is given as $q'' = q + q' + \varepsilon$ for some $\varepsilon = 0, 1$, and $r'' = (n + n') - dq''$ is given as $r'' = r + r' - \varepsilon d$. It is enough to see

$$(J^q I_C^r + J^{q+1}) \cdot (J^{q'} I_C^{r'} + J^{q'+1}) \subset J^{q''} I_C^{r''} + J^{q''+1}.$$

If $\varepsilon = 0$, then this is obvious because $q'' = q + q'$ and $r'' = r + r'$. If $\varepsilon = 1$, then we use $I_C^d \subset J$. Indeed $J^q I_C^r \cdot J^{q'} I_C^{r'} = J^{q''-1} I_C^{r''+d} \subset J^{q''} I_C^{r''}$, $J^q I_C^r \cdot J^{q'+1} = J^{q''} I_C^r \subset J^{q''} I_C^{r''}$ by $r'' = r + r' - d < r$, and the rest is similar. \square

(8.2.2) Then

$$\mathcal{S} = F^0(\mathcal{S}, J) \supset \dots \supset F^n(\mathcal{S}, J) \supset F^{n+1}(\mathcal{S}, J) \supset \dots$$

is called the *J-filtration*, and $\text{gr}^{\cdot}(\mathcal{O}, J) = \bigoplus_{n \geq 0} \text{gr}^n(\mathcal{O}, J)$ is a graded \mathcal{O}_C -algebra and $\text{gr}^{\cdot}(\mathcal{S}, J) = \bigoplus_{n \geq 0} \text{gr}^n(\mathcal{S}, J)$ is a graded $\text{gr}^{\cdot}(\mathcal{O}, J)$ -module.

For $s \in \mathcal{S}$, the *width* of s with respect to J (denoted by $\text{width}_J s$) is given by $\text{width}_J s = \inf\{i \in \mathbb{Z}_+ \mid s \in F^i(\mathcal{S}, J)\}$. It is easy to see but worthwhile to note that $F^1(\mathcal{O}_X, J) = I_C$, $F^d(\mathcal{O}_X, J) = J$, $\text{gr}^d(\mathcal{O}, J) = \text{gr}_C^0 J$, and in particular the usage of $\text{width}_J s$ is compatible with width of J .

For $n = qd + r$ as above, one has a natural homomorphism

$$\gamma^n(\mathcal{S}, J): S^q(\text{gr}^d(\mathcal{O}, J)) \otimes (\text{gr}^1(\mathcal{O}, J))^{\otimes r} \otimes \text{gr}_C^0 \mathcal{S} \rightarrow \text{gr}^n(\mathcal{S}, J).$$

We construct a 0-sequence

$$E(J): 0 \rightarrow \text{gr}^1(\mathcal{O}, J)^{\otimes d} \xrightarrow{\Delta_J} \text{gr}_C^0(J) \xrightarrow{\alpha_J} \text{Ker} \left[\text{gr}_C^1 \mathcal{O} \xrightarrow{\beta_J} \text{gr}^1(\mathcal{O}, J) \right] \rightarrow 0,$$

together with the remarks: If $d = 1$, then $J = I_C$, $F^n(\mathcal{S}, J) = F^n \mathcal{S}$, $\gamma^n(\mathcal{S}, J) = \gamma_C^n(\mathcal{S})$ (§2), and we set $\alpha_J = 0$ and $\beta_J = \Delta_J = \text{id}$, and $E(J)$ is exact. If $d \geq 2$, then $\alpha_J, \beta_J, \gamma_J$ are naturally induced homomorphisms, $E(J)$ is a 0-sequence by $I_C^d \subset J$ and $I_C^{(2)} + J \subset F^2(\mathcal{O}, J)$, and we note that $\text{Ker } \beta_J$ and $\text{gr}^1(\mathcal{O}, J)$ are of rank 1.

(8.3) **Proposition.** *Under the notation and assumptions of (8.1), assume that $(u, v) = I_{C,P}$ ($= I_C \otimes_{\mathcal{O}_{X,P}}$) and let $d \geq 1$. Then*

- (i) $J = (u^d, v)$ is a C -laminal ideal of pure width d in a nbd of P , and one has the following for all $n \geq 0$:

$$F^n(\mathcal{O}_X, J)_P = \sum_{i,j \geq 0, i+dj \geq n} (u^i v^j),$$

and

$$\text{gr}^n(\mathcal{O}, J)_P = \bigoplus_{i,j \geq 0, i+dj=n} \mathcal{O}_{C,P} u^i v^j.$$

Thus $E(J)$ is exact and $\gamma^n(\mathcal{O}_X, J)$ is isomorphic at P , and

- (ii) let J be a C -laminal ideal of pure width d in a nbd of $P \in C$. Then $(u^d, v) = J_P$ ($= J \otimes_{\mathcal{O}_{X,P}}$) iff $v \in J_P$.

Proof. (i) Since (C, P) is Cohen-Macaulay, so is (X, P) . Hence $(u, v): (X, P) \rightarrow (\mathbb{C}^2, 0)$ is flat since it is equidimensional. Since $K_n = \sum_{i+dj \geq n} (u^i v^j) \mathbb{C}\{u, v\}$ contains some power of maximal ideal (u, v) , it is (u, v) -primary. Thus by flatness, $K_n \mathcal{O}_{X,P}$ has no embedded primes. Then following the definition, one easily sees $F^n(\mathcal{O}_X, J)_P = K_n \mathcal{O}_{X,P}$. It is clear that $K_n/K_{n+1} = \bigoplus_{i+dj=n} \mathbb{C} u^i v^j$, whence one sees the assertion on $\text{gr}^n(\mathcal{O}, J)_P$ by flatness of $(X, P) \rightarrow (\mathbb{C}^2, 0)$, and $\gamma^n(\mathcal{O}_X, J)_P$ is an isomorphism for all $n \geq 0$. If $d \geq 2$, then one sees $\text{gr}^1(\mathcal{O}, J)_P = \mathcal{O}_{C,P} u$, $\text{gr}_C^0(J) = \mathcal{O}_{C,P} u^d \oplus \mathcal{O}_{C,P} v$, $\text{gr}_C^1 \mathcal{O} = \mathcal{O}_{C,P} u \oplus \mathcal{O}_{C,P} v$, and $\text{Ker } \beta_J = \mathcal{O}_{C,P} v$, whence $E(J)$ is exact.

(ii) The only-if part is obvious. Assume that $v \in J_P$. By (i), (u^d, v) has no embedded primes at P , and $(u^d, v) \subset J_P$ by the definition of width. Thus it is enough to prove $(u^d, v)_Q = J_Q$ for close enough points $Q \in C - \{P\}$. Let Q be a point such that X and C are smooth at Q , whence u and v form a part of coordinates at Q . Then $J_Q/(v)_Q$ is a divisorial ideal of a smooth surface $(\{v = 0\}, Q)$ and one easily sees $J_Q/(v)_Q = \{(u, v)/(v)\}_Q^d = (u^d, v)_Q/(v)_Q$, whence $J_Q = (u^d, v)_Q$. \square

(8.4) **Definition-Corollary.** *Let the notation and assumptions be as in (8.1). Let J be a C -laminal ideal of pure width d . We say that J is a nested complete*

intersection (nested c.i., in short) at P if $(u, v) = I_{C,P}$ and $(u^d, v) = J_P$ for some u and v , and that J is locally a nest c.i. on a subset if J is a nested c.i. at each point of the set. Then

- (i) let $P \in C$. Then J is a nested c.i. at P iff (C, P) is a complete intersection in (X, P) , $\text{Im } \alpha_J = \text{Ker } \beta_J$, and $\text{gr}^1(\mathcal{O}, J)$ is a locally free \mathcal{O}_C -module at P , and
- (ii) J is locally a nested c.i. on an open dense subset of C . In particular, $\text{Ker } \beta_J$ is the $\text{gr}^1 \mathcal{O}$ -saturation of $\text{Im } \alpha_J$.

Proof. (i) If $d = 1$ then this is obvious; if $d \geq 2$ then the only-if part is done in (8.3(i)). If (C, P) is a c.i. in (X, P) , then $\text{gr}_C^1 \mathcal{O} \simeq I_C/I_C^2$ is locally free of rank 2 at P . From the other assumptions, $\text{Im } \alpha_J \simeq \text{Ker } \beta_J$ and $\text{Im } \beta_J$ are invertible at P . Thus there exist $v \in J_P$ and $u \in I_{C,P}$ such that $\text{Im } \alpha_J = \mathcal{O}_C v$ and $\text{Im } \beta_J = \mathcal{O}_C u$ in a nbd of P . Then $(u, v) = I_{C,P}$ by Nakayama's lemma, and (8.3(ii)) proves our claim.

(ii) Let $d \geq 2$. Since $\text{gr}_C^1 \mathcal{O}$ (resp. $\text{Im } \alpha_J, \text{Ker } \beta_J, \text{gr}^1(\mathcal{O}, J)$) is of rank 2 (resp. 1, 1, 1) on each irreducible component of C , one sees that $E(J)$ is exact on an open dense subset. \square

(8.5) **Corollary.** *Let the notation and assumptions be as in (8.1). Let J be a C -laminal ideal of pure width d and \mathcal{S} a torsion-free \mathcal{O}_X -module which is locally free of rank ρ on an open set of X containing an open dense subset of C . Let $n \geq 0$. Then*

- (i) $\gamma^n(\mathcal{S}, J)$ is injective at smooth points of C , and
- (ii) $\gamma^n(\mathcal{S}, J)$ is isomorphic on an open dense subset of C consisting of points at which \mathcal{S} is locally free and J is a nested c.i. In particular, $\text{gr}^n(\mathcal{S}, J)$ is pure of rank $\rho[n/d] + \rho$ (i.e. of the same rank $\rho[n/d] + \rho$ on each irreducible component of C).

Proof. The assertion (i) follows from (ii) since the source of $\gamma^n(\mathcal{S}, J)$ is free at smooth points. The first part in (ii) follows from (8.3(i)) and (8.4(ii)), and the second from the first. \square

(8.6) **Definition-Corollary.** *Under the notation and assumptions of (8.5), assume that $d \geq 2$, and let $q = [n/d]$, and $r = n - qd$. Then $E(J)$ induces a saturated filtration (or chain)*

$$\text{gr}^n(\mathcal{S}, J) = \Phi^0 \text{gr}^n(\mathcal{S}, J) \supset \dots \supset \Phi^i \text{gr}^n(\mathcal{S}, J) \supset \dots \supset \Phi^{q+1} \text{gr}^n(\mathcal{S}, J) = 0$$
 such that $\text{gr}^{n,i}(\mathcal{S}, J) = \Phi^i \text{gr}^n(\mathcal{S}, J) / \Phi^{i+1} \text{gr}^n(\mathcal{S}, J)$ is a torsion-free \mathcal{O}_C -module of pure rank ρ for $i \in [0, q]$ and there is a naturally induced map ($i \in [0, q]$)

$$\gamma^{n,i}(\mathcal{S}, J): \text{gr}^1(\mathcal{O}, J)^{\otimes (di+r)} \otimes (\text{Im } \alpha_J)^{\otimes (q-i)} \otimes \text{gr}_C^0 \mathcal{S} \rightarrow \text{gr}^{n,i}(\mathcal{S}, J).$$

For an arbitrary $i \in [0, q]$, one has

- (i) $\gamma^{n,i}(\mathcal{S}, J)$ is injective at smooth points of C , and

(ii) $\text{Im } \alpha_J = \text{Ker } \beta_J$ and $\gamma^{n,i}(\mathcal{S}, J)$ is an isomorphism on the open dense subset of C consisting of the points P at which \mathcal{S} is a locally free \mathcal{O}_X -module and J is a nested c.i. Furthermore, if u, v are the elements in $\mathcal{O}_{X,P}$ as in (8.4), then we have

$$\Phi^i \text{gr}^n(\mathcal{S}, J) = u^{di} \text{gr}^{n-\text{id}}(\mathcal{S}, J) \text{ for } i \in [0, q].$$

Proof. By (8.4(ii)), $\text{Ker } \alpha_J$ and $\text{Im } \alpha_J$ are pure rank 1 \mathcal{O}_C -modules fitting in the exact sequence $0 \rightarrow \text{Ker } \alpha_J \rightarrow \text{gr}_C^0 J \rightarrow \text{Im } \alpha_J \rightarrow 0$. Let Φ^i be the subsheaf of $\text{gr}^n(\mathcal{S}, J)$ given as the saturation of the image of the map

$$(\text{Ker } \alpha_J)^{\otimes i} \otimes S^{q-i}(\text{gr}^d(\mathcal{O}, J)) \otimes \text{gr}^1(\mathcal{O}, J)^{\otimes r} \otimes \text{gr}_C^0 \mathcal{S} \rightarrow \text{gr}^n(\mathcal{S}, J)$$

induced by $\gamma^n(\mathcal{S}, J)$ for $i \in [0, q]$. Then Φ^i is a saturated \mathcal{O}_C -submodule of pure rank $(q + 1 - i)\rho$ by (8.5). Since the sequence

$$(\text{Ker } \alpha_J) \otimes S^{q-i-1} \text{gr}^d(\mathcal{O}, J) \xrightarrow{\delta(i)} S^{q-i} \text{gr}^d(\mathcal{O}, J) \rightarrow S^{q-i}(\text{Im } \alpha_J)/\text{Sat}(0) \rightarrow 0$$

constructed in the obvious way is a 0-sequence ($\text{Sat}(0)$ is the saturation of 0) and exact on an open dense subset of C , one has

$$S^{q-i}(\text{gr}^d(\mathcal{O}, J))/\text{Sat}(\text{Im}(\delta(i))) \simeq S^{q-i}(\text{Im } \alpha_J)/\text{Sat}(0).$$

Thus by the right exactness of the tensor functor, one has

$$(\text{Ker } \alpha_J)^{\otimes i} \otimes S^{q-i}(\text{Im } \alpha_J) \otimes \text{gr}^1(\mathcal{O}, J)^{\otimes r} \otimes \text{gr}_C^0 \mathcal{S} \rightarrow \Phi^i/\Phi^{i+1}.$$

Using the map $\Delta_J: \text{gr}^1(\mathcal{O}, J)^{\otimes d} \rightarrow \text{Ker } \alpha_J$ induced by $E(J)$, we have $\gamma^{n,i}(\mathcal{S}, J)$. The rest follows from (8.4) as in (8.5), except that the last equality follows from (8.3(i)). \square

(8.7) *Remark.* If $X \subset C \simeq \mathbb{P}^1$ is an extremal nbd and J is a C -laminal ideal of width d , then for $\mathcal{S} = \mathcal{O}_X, \omega_X$, one sees that $\text{gr}^n(\mathcal{S}, J)$ is a locally free \mathcal{O}_C -module of rank $[n/d] + 1$. Let $n \in \mathbb{Z}_+$. As in (2.3.3) and (2.3.4), one sees that

$$\chi(F^1(\mathcal{S}, J)/F^{n+1}(\mathcal{S}, J)) = \sum_{1 \leq e \leq n} (\text{deg } \text{gr}^e(\mathcal{S}, J) + [e/d] + 1) \geq 0,$$

and that if $\text{deg } \text{gr}^e(\mathcal{S}, J) \leq -[e/d] - 1$ for all $e \in [1, n]$, then

$$\text{gr}^e(\mathcal{S}, J) \simeq \mathcal{O}(-1)^{\oplus([e/d]+1)}$$

for $e \in [1, n]$ and $\text{deg } \text{gr}^e(\mathcal{S}, J) \geq -[e/d] - 1$ for $e = n + 1$.

(8.8) Let (X, P) be the germ of a 3-fold terminal singularity of index m and $C \subset (X, P)$ a reduced curve. Let $\pi^\sharp: (X^\sharp, P^\sharp) \rightarrow (X, P)$ be the μ_m -canonical cover and $C^\sharp = \pi^{\sharp-1}(C)_{\text{red}} \subset (X^\sharp, P^\sharp)$. Meanings of these symbols will be fixed in paragraphs (8.8)–(8.8.6).

Let \mathcal{S} be a coherent sheaf (on X , or X^\sharp). A subquotient of \mathcal{S} is a coherent sheaf \mathcal{L} endowed with an identification $\mathcal{L} = \mathcal{M}|\mathcal{N}$, where \mathcal{M} and \mathcal{N} are coherent subsheaves of \mathcal{S} . The subquotient \mathcal{L} is said to be \mathcal{S} -saturated if $\mathcal{S} \supset \mathcal{M} \supset \mathcal{N}$ is saturated. An isomorphism theorem says that an arbitrary subquotient \mathcal{K} of \mathcal{L} is a subquotient of \mathcal{S} , and if furthermore \mathcal{L} is \mathcal{S} -saturated, then \mathcal{K} is \mathcal{L} -saturated $\Leftrightarrow \mathcal{K}$ is \mathcal{S} -saturated (8.1.1). Therefore we may simply say, for instance, \mathcal{L} is saturated if there is no danger of confusion.

Let $\text{SSQ}(\mathcal{S})$ be the small category of saturated subquotients of \mathcal{S} , where $\text{Hom}(\mathcal{M}_1|\mathcal{N}_1, \mathcal{M}|\mathcal{N})$ consists only of the natural map if $\mathcal{M}_1 \subset \mathcal{M}$ and $\mathcal{N}_1 \subset \mathcal{N}$ (or \emptyset if otherwise). If \mathcal{S} is a μ_m -sheaf on X^\sharp , then $\text{SSQ}(\mathcal{S}, \mu_m)$ is the full subcategory of $\text{SSQ}(\mathcal{S})$ consisting of $\mathcal{M}|\mathcal{N}$ such that \mathcal{M} and \mathcal{N} are μ_m -stable subsheaves of \mathcal{S} .

Let \mathcal{S} be a coherent sheaf on X such that $\mathcal{S} \supset 0$ is saturated. By an ℓ -structure of \mathcal{S} at P (or, of $\mathcal{S}_P = \mathcal{S} \otimes_{\mathcal{O}_{X,P}}$), we mean a coherent sheaf \mathcal{S}^\sharp on X^\sharp with μ_m -action endowed with an isomorphism $\mathcal{S}^\sharp_{\{0\}} \simeq \mathcal{S}$ (cf. (2.5)) such that $\mathcal{S}^\sharp \supset 0$ is saturated. We note that \mathcal{O}_X has a trivial ℓ -structure $\mathcal{O}_{X^\sharp} \supset \mathcal{O}_X$.

(8.8.1) **Lemma-Definition.** *Let \mathcal{S} be a coherent sheaf on X with an ℓ -structure $\mathcal{S}^\sharp \supset \mathcal{S}$. Then there is a 1-1 correspondence between $\text{SSQ}(\mathcal{S})$ and $\text{SSQ}(\mathcal{S}^\sharp, \mu_m)$:*

$$\text{SSQ}(\mathcal{S}) \overset{\text{lif}}{\underset{\text{quo}}{\rightleftarrows}} \text{SSQ}(\mathcal{S}^\sharp, \mu_m),$$

where *lif* and *quo* are exact functors such that $\text{lif} \circ \text{quo} = \text{id}$ and $\text{quo} \circ \text{lif} = \text{id}$, and are defined for subsheaves \mathcal{M} and \mathcal{M}^\sharp as $\text{lif}(\mathcal{M}) = \text{Sat}_{\mathcal{S}^\sharp}(\mathcal{O}_{X^\sharp} \mathcal{M})$ and $\text{quo}(\mathcal{M}^\sharp) = \mathcal{M}^\sharp_{\{0\}} = \mathcal{M}^\sharp \cap \mathcal{S}$, where $\mathcal{O}_{X^\sharp} \mathcal{M}$ is the subsheave of \mathcal{S} generated by $\pi^{\sharp*} \mathcal{M}$. For each saturated subquotient \mathcal{M} of \mathcal{S} , $\text{lif}(\mathcal{M})$ and $\text{lif}(\mathcal{M}) \supset \mathcal{M}$ are called the canonical lifting of \mathcal{M} and the induced ℓ -structure of \mathcal{M} . (If there is a danger of confusion, \mathcal{S} - $\text{lif}(\mathcal{M})$ will be used.)

Proof. It is clear that *lif* and *quo* for subsheaves satisfy $\text{lif} \circ \text{quo} = \text{id}$ and $\text{quo} \circ \text{lif} = \text{id}$, because π^\sharp is étale outside P^\sharp . Therefore the definition of *lif* (similarly of *quo*) extends to subquotients $\mathcal{M}|\mathcal{N}$ by exactness: $\text{lif}(\mathcal{M}|\mathcal{N}) = \text{lif}(\mathcal{M})/\text{lif}(\mathcal{N})$. \square

(8.8.2) **Proposition-Definition.** *Let \mathcal{L} , \mathcal{M} and \mathcal{N} be coherent sheaves on X with ℓ -structures $\mathcal{L} \subset \mathcal{L}^\sharp$, $\mathcal{M} \subset \mathcal{M}^\sharp$, and $\mathcal{N} \subset \mathcal{N}^\sharp$. Then*

(i) *the restriction map*

$$\text{res}: \text{Hom}_{\mu_m - \mathcal{O}_{X^\sharp}}(\mathcal{M}^\sharp, \mathcal{N}^\sharp) \rightarrow \text{Hom}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N})$$

defined by $\text{res}(\phi^\sharp) = \phi|_{\mathcal{M}}$ is an injection. If an \mathcal{O}_X -homomorphism $\phi: \mathcal{M} \rightarrow \mathcal{N}$ extends to a $\mu_m - \mathcal{O}_{X^\sharp}$ -homomorphism $\phi^\sharp: \mathcal{M}^\sharp \rightarrow \mathcal{N}^\sharp$ in this sense, we say that

ϕ is an ℓ -homomorphism and that ϕ^\sharp is the canonical lifting of ϕ , since it is unique,

(ii) we say that an ℓ -homomorphism ϕ is an ℓ -surjection (resp. ℓ -isomorphism, ℓ -injection) if the canonical lifting ϕ^\sharp is a surjection (resp. isomorphism, injection). Then ϕ is an injection iff it is an ℓ -injection, and ϕ is a surjection (resp. isomorphism) if it is an ℓ -surjection (resp. ℓ -isomorphism),

(iii) if $\mathcal{M} = \mathcal{A} \oplus \mathcal{B}$ is a splitting by \mathcal{O}_X -submodules $\mathcal{A}, \mathcal{B} \subset \mathcal{M}$, then it is called an ℓ -splitting and written as $\mathcal{M} = \mathcal{A} \tilde{\oplus} \mathcal{B}$ if $\mathcal{M}^\sharp = \text{lif}(\mathcal{A}) \oplus \text{lif}(\mathcal{B})$,

(iv) let $\psi: \mathcal{L} \rightarrow \mathcal{M}$ and $\phi: \mathcal{M} \rightarrow \mathcal{N}$ be ℓ -homomorphisms. Then $\mathcal{E}: 0 \rightarrow \mathcal{L} \xrightarrow{\psi} \mathcal{M} \xrightarrow{\phi} \mathcal{N} \rightarrow 0$ is a 0-sequence iff its canonical lifting $\mathcal{E}^\sharp: 0 \rightarrow \mathcal{L}^\sharp \xrightarrow{\psi^\sharp} \mathcal{M}^\sharp \xrightarrow{\phi^\sharp} \mathcal{N}^\sharp \rightarrow 0$ is a 0-sequence. We say that \mathcal{E} is ℓ -exact (resp. ℓ -split) if \mathcal{E}^\sharp is exact (resp. split) as a sequence of μ_m - \mathcal{O}_{X^\sharp} -homomorphisms. Then \mathcal{E} is exact (resp. split) if it is ℓ -exact (resp. ℓ -split). We note that these are compatible with the definitions in (ii) and (iii), and

(v) let $\phi: \mathcal{M} \rightarrow \mathcal{N}$ be an ℓ -homomorphism. If ϕ is an ℓ -surjection (resp. ℓ -injection and $\mathcal{N} \supset \phi(\mathcal{M})$ is saturated), then $\text{Ker } \phi$ (resp. $\text{Coker } \phi$) has an induced ℓ -structure and $0 \rightarrow \text{Ker } \phi \rightarrow \mathcal{M} \rightarrow \mathcal{N} \rightarrow 0$ (resp. $0 \rightarrow \mathcal{M} \rightarrow \mathcal{N} \rightarrow \text{Coker } \phi \rightarrow 0$) is ℓ -exact.

Proof. (i) The map res is well defined because $\phi^\sharp(\mathcal{M}) \subset \mathcal{N}_{\{0\}}^\sharp = \mathcal{N}$. It is injective because $\mathcal{M}^\sharp \supset 0$ is saturated and $\phi^\sharp|_{X^\sharp - \{P^\sharp\}} = (\pi^\sharp|_{X - \{P\}})^* \phi$ if $\phi = \text{res}(\phi^\sharp)$, which follows from the fact that $X^\sharp - \{P^\sharp\} \rightarrow X - \{P\}$ is a finite étale μ_m -Galois covering.

(ii) If $\text{Ker } \phi^\sharp \neq 0$ then $\text{Supp}(\text{Ker } \phi^\sharp) \not\subset \{P^\sharp\}$ by $P^\sharp \notin \text{Ass}(\mathcal{M}^\sharp)$, whence $\text{Supp}(\text{Ker } \phi) \not\subset \{P\}$ by flat descent. Thus ϕ is an ℓ -injection if it is an injection. Other assertions are obvious because μ_m is linearly reductive.

(iv) Since $\text{res}(\phi^\sharp \circ \psi^\sharp) = \phi \circ \psi$, one sees $\phi \circ \psi = 0 \Leftrightarrow \phi^\sharp \circ \psi^\sharp = 0$ by (i). This proves the first assertion, and others follow from the linear reductivity of μ_m .

(v) By the assumptions, $\text{Ker } \phi$ (resp. $\text{Coker } \phi$) is a saturated subquotient of \mathcal{M} whence has an induced ℓ -structure. Since $0 \rightarrow \text{Ker } \phi^\sharp \rightarrow \mathcal{M}^\sharp \rightarrow \mathcal{N}^\sharp \rightarrow 0$ (resp. $0 \rightarrow \mathcal{M}^\sharp \rightarrow \mathcal{N}^\sharp \rightarrow \text{Coker } \phi^\sharp \rightarrow 0$) is exact, the assertion follows by (iv). \square

(8.8.3) **Definition.** Let \mathcal{M} be a coherent sheaf on X with ℓ -structure $\mathcal{M} \subset \mathcal{M}^\sharp$. If $s_1^\sharp, \dots, s_r^\sharp \in \mathcal{M}^\sharp$ are μ_m -semi-invariants generating \mathcal{M}^\sharp as \mathcal{O}_{X^\sharp} -module at P^\sharp , we say that $\{s_1^\sharp, \dots, s_r^\sharp\}$ is an ℓ -basis of \mathcal{M} at P (or, of \mathcal{M}_P).

Let Y be a closed subscheme of X such that $P \notin \text{Ass}(\mathcal{O}_Y)$ and let $Y^\sharp = \text{lif}(Y)$. Then it is easy to see that \mathcal{M} is an \mathcal{O}_Y -module iff \mathcal{M}^\sharp is an \mathcal{O}_{Y^\sharp} -module. We say that \mathcal{M} is an ℓ -free \mathcal{O}_Y -module at P iff \mathcal{M}^\sharp is a free \mathcal{O}_{Y^\sharp} -module at P^\sharp . If \mathcal{M} is an ℓ -free \mathcal{O}_Y -module at P , then an ℓ -basis of \mathcal{M} at P is said to be ℓ -free if it is a free \mathcal{O}_{Y^\sharp} -basis. We say that an ℓ -free \mathcal{O}_Y -module is ℓ -trivial

if it has an l -free l -basis consisting of μ_m -invariants. We note that l -trivial l -free \mathcal{O}_Y -modules are free, but in general, there is no implication between freeness and l -freeness in either direction. Therefore, we call a free l -free \mathcal{O}_Y -module at P a ql -free \mathcal{O}_Y -module. We call l -free (resp. ql -free) \mathcal{O}_Y -modules at P of rank 1 l -invertible (resp. ql -invertible) \mathcal{O}_Y -modules at P .

(8.8.4) Let Y, Y^\sharp be as above, and Z a closed subscheme of Y such that $P \notin \text{Ass}(Z)$. Let $\mathcal{L}, \mathcal{M}, \mathcal{N}$ be \mathcal{O}_Y -modules at P with l -structures $\mathcal{L} \subset \mathcal{L}^\sharp, \mathcal{M} \subset \mathcal{M}^\sharp, \mathcal{N} \subset \mathcal{N}^\sharp$. We list the following obvious operations together with easy comments.

($\tilde{\oplus}$) $\mathcal{M} \tilde{\oplus} \mathcal{N} \subset (\mathcal{M} \tilde{\oplus} \mathcal{N})^\sharp$ is an \mathcal{O}_Y -module at P with l -structure defined by

$$(\mathcal{M} \tilde{\oplus} \mathcal{N})^\sharp = \mathcal{M}^\sharp \oplus \mathcal{N}^\sharp.$$

We note that this is compatible with l -splitting in (8.8.2(iii)), and we have natural l -isomorphisms

$$(\tilde{\oplus}.1) \quad (\mathcal{L} \tilde{\oplus} \mathcal{M}) \tilde{\oplus} \mathcal{N} \simeq \mathcal{L} \tilde{\oplus} (\mathcal{M} \tilde{\oplus} \mathcal{N}),$$

$$(\tilde{\oplus}.2) \quad \mathcal{M} \tilde{\oplus} \mathcal{N} \simeq \mathcal{N} \tilde{\oplus} \mathcal{M}.$$

Let χ be an l -character (2.5) for $X \supset C$ at P . Let $\mathcal{L}_Y[\nu]$, for $\nu \in \mathbb{Z}_m$, be the l -invertible \mathcal{O}_Y -module with an l -basis e_ν on which μ_m acts by $\zeta(e_\nu) = \chi(\zeta)^\nu e_\nu$ ($\zeta \in \mu_m$). Then l -free \mathcal{O}_Y -module \mathcal{M} of rank r is determined by wt (say, $\nu_1, \dots, \nu_r \pmod m$) of its l -free l -basis, that is there is an l -isomorphism $\mathcal{M} \simeq \mathcal{L}_Y[\nu_1] \tilde{\oplus} \dots \tilde{\oplus} \mathcal{L}_Y[\nu_r]$. Hence there are at most $\binom{m}{r}$ l -free rank r \mathcal{O}_Y -modules. We note that \mathcal{M} is ql -free iff $\mathcal{L}_Y[\nu_i]$ is ql -free for all $i \in [1, r]$ by (8.8.2(ii)), and that \mathcal{M} is l -trivial iff $\nu_i \equiv 0 \pmod m$ for all $i \in [1, r]$.

($\tilde{\otimes}$) $\mathcal{M} \tilde{\otimes} \mathcal{N} \subset (\mathcal{M} \tilde{\otimes} \mathcal{N})^\sharp$ is an \mathcal{O}_Y -module at P with l -structure defined by

$$(\mathcal{M} \tilde{\otimes} \mathcal{N})^\sharp = (\mathcal{M}^\sharp \otimes_{\mathcal{O}_X^\sharp} \mathcal{N}^\sharp) / \text{Sat}_{\mathcal{M}^\sharp \otimes_{\mathcal{O}_X^\sharp} \mathcal{N}^\sharp}(0).$$

It is a biadditive functor in the obvious sense, and we have natural l -isomorphisms.

$$(\tilde{\otimes}.0) \quad \mathcal{M} \tilde{\otimes} \mathcal{O}_Y \simeq \mathcal{M} \quad (\mathcal{O}_Y \text{ denotes the } l\text{-trivial } \mathcal{O}_Y),$$

$$(\tilde{\otimes}.1) \quad (\mathcal{L} \tilde{\otimes} \mathcal{M}) \tilde{\otimes} \mathcal{N} \simeq \mathcal{L} \tilde{\otimes} (\mathcal{M} \tilde{\otimes} \mathcal{N}),$$

$$(\tilde{\otimes}.2) \quad \mathcal{M} \tilde{\otimes} \mathcal{N} \simeq \mathcal{N} \tilde{\otimes} \mathcal{M},$$

$$(\tilde{\otimes}.3) \quad \mathcal{L} \tilde{\otimes} (\mathcal{M} \tilde{\oplus} \mathcal{N}) \simeq (\mathcal{L} \tilde{\otimes} \mathcal{M}) \tilde{\oplus} (\mathcal{L} \tilde{\otimes} \mathcal{N}),$$

and easy comments

$$(\tilde{\otimes}.4) \quad \mathcal{L} \tilde{\otimes} \alpha \text{ is } l\text{-surjective if } \alpha: \mathcal{M} \rightarrow \mathcal{N} \text{ is } l\text{-surjective,}$$

$$(\tilde{\otimes}.5) \quad \mathcal{L} \tilde{\otimes} \mathcal{E} \text{ is } l\text{-exact if } \mathcal{L} \text{ is } l\text{-free and } \mathcal{E} \text{ is } l\text{-exact,}$$

($\tilde{\otimes}.6$) $\mathcal{M} \tilde{\otimes} \mathcal{N}$ is an l -free \mathcal{O}_Y -module at P of rank $r \cdot s$ if \mathcal{M} and \mathcal{N} are l -free \mathcal{O}_Y -modules at P of rank r and s ,

($\tilde{\otimes}.7$) $\mathcal{M} \tilde{\otimes} \mathcal{O}_Z$ is an l -free \mathcal{O}_Z -module at P if \mathcal{M} is an l -free \mathcal{O}_Y -module at P ,

($\tilde{\otimes}$.8) $\mathcal{M} \tilde{\otimes} \mathcal{N} \simeq \mathcal{M} \otimes \mathcal{N}$ as \mathcal{O}_Y -modules if \mathcal{M} is ℓ -trivial, and
 ($\tilde{\otimes}$.9) the natural \mathcal{O}_X -homomorphism $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N} \rightarrow \mathcal{M} \tilde{\otimes} \mathcal{N}$ is an isomorphism outside P .

($\tilde{\otimes}$. $\tilde{\otimes}$) ℓ -invertible \mathcal{O}_Y -module $\mathcal{M}^{\tilde{\otimes} n} \subset (\mathcal{M}^{\tilde{\otimes} n})^\sharp$ at P is defined for $n \in \mathbb{Z}$ and ℓ -invertible \mathcal{O}_Y -module \mathcal{M} at P by $(\mathcal{M}^{\tilde{\otimes} n})^\sharp = (\mathcal{M}^\sharp)^{\otimes n}$. We have natural ℓ -isomorphisms

$$(\tilde{\otimes}\tilde{\otimes}.1) \quad \mathcal{L}_Y[\nu]^{\tilde{\otimes} n} \simeq \mathcal{L}_Y[n\nu] \quad (n \in \mathbb{Z}, \nu \in \mathbb{Z}_m),$$

$$(\tilde{\otimes}\tilde{\otimes}.2) \quad \mathcal{M}^{\tilde{\otimes} a} \tilde{\otimes} \mathcal{M}^{\tilde{\otimes} b} \simeq \mathcal{M}^{\tilde{\otimes}(a+b)} \quad (a, b \in \mathbb{Z}),$$

whence ℓ -invertible \mathcal{O}_Z -modules $(\mathcal{O}_Z \tilde{\otimes} \mathcal{M}^{\tilde{\otimes} n})$ at P ($\tilde{\otimes}$.7) satisfy

$$(\tilde{\otimes}\tilde{\otimes}.3) \quad (\mathcal{O}_Z \tilde{\otimes} \mathcal{M})^{\tilde{\otimes} n} \simeq \mathcal{O}_Z \tilde{\otimes} (\mathcal{M}^{\tilde{\otimes} n}) \quad (n \in \mathbb{Z})$$

by ($\tilde{\otimes}$.1), ($\tilde{\otimes}$.2) and $\mathcal{O}_Z \tilde{\otimes} \mathcal{O}_Z \simeq \mathcal{O}_Z$ ($\tilde{\otimes}$.0). We note by ($\tilde{\otimes}\tilde{\otimes}$.1) that

$$(\tilde{\otimes}\tilde{\otimes}.4) \quad \mathcal{M}^{\tilde{\otimes} a} \text{ is } \ell\text{-trivial if } a \equiv 0 \pmod{m},$$

whence we see by ($\tilde{\otimes}$.8) that

($\tilde{\otimes}\tilde{\otimes}$.5) $(\mathcal{M} \tilde{\otimes} \mathcal{N})^{\tilde{\otimes} mn} \simeq (\mathcal{M}^{\tilde{\otimes} m})^{\otimes n} \otimes (\mathcal{N}^{\tilde{\otimes} m})^{\otimes n}$ as \mathcal{O}_Y -modules for $n \in \mathbb{Z}$, if \mathcal{N} is also an ℓ -invertible \mathcal{O}_Y -module at P .

($\tilde{\det}$) ℓ -invertible \mathcal{O}_Y -module $\tilde{\det} \mathcal{M} \subset (\tilde{\det} \mathcal{M})^\sharp$ at P is defined for ℓ -free \mathcal{O}_Y -module \mathcal{M} at P by $(\tilde{\det} \mathcal{M})^\sharp = \det_{\mathcal{O}_{X^{\sharp}}} \mathcal{M}^\sharp$.

(\tilde{S}) ℓ -free \mathcal{O}_Y -module $\tilde{S}^n(\mathcal{M}) \subset \tilde{S}^n(\mathcal{M})^\sharp$ at P is defined for $n \in \mathbb{N}$ and ℓ -free \mathcal{O}_Y -module \mathcal{M} at P by $\tilde{S}^n(\mathcal{M})^\sharp = S^n_{\mathcal{O}_{X^{\sharp}}}(\mathcal{M}^\sharp)$ (symmetric n th power). If \mathcal{M} and \mathcal{N} are ℓ -free \mathcal{O}_Y -modules of rank 2 and 1 respectively, then we have a natural ℓ -isomorphism for $n \in \mathbb{N}$

$$(\tilde{S}.1) \quad \tilde{\det}(\tilde{S}^n(\mathcal{M}) \tilde{\otimes} \mathcal{N}) \simeq (\tilde{\det} \mathcal{M})^{\tilde{\otimes} n(n+1)/2} \tilde{\otimes} \mathcal{N}^{\tilde{\otimes}(n+1)}.$$

(8.8.5) If furthermore $\dim Y = 1$ and \mathcal{M} is ql -free, then one can define local degree $ql \deg(\mathcal{M}, P) \in \mathbb{Z}_+$ as follows: let s_1, \dots, s_r (resp. $s_1^\sharp, \dots, s_r^\sharp$) be a free basis (resp. an ℓ -free basis) of \mathcal{M} at P . Then $s_i = \sum_j f_{ij} s_j^\sharp$ ($i \in [1, r]$) and $g = \det(f_{ij})$ is a semi-invariant by $s_1 \wedge \dots \wedge s_r = g(s_1^\sharp \wedge \dots \wedge s_r^\sharp)$, whence $g^m \in \mathcal{O}_{Y,P}$. Then $gl \deg(\mathcal{M}, P) = \text{len}_P(\mathcal{O}_{Y,P}/(g^m))$, which is obviously independent of choice of ℓ -basis and basis.

If we set $B(Y, P) = \text{Max}\{ql \deg(\mathcal{L}_Y[\nu], P) \mid \nu \in \mathbb{Z}_m, \mathcal{L}_Y[\nu]: ql\text{-free}\}$, then $ql \deg(\mathcal{M}, P) \leq B(Y, P) \cdot r$ for arbitrary ql -free \mathcal{M} of rank r .

We note that if Y is a smooth curve at P and if \mathcal{M} is a ql -free (i.e. ℓ -free) \mathcal{O}_C -module at P with an ℓ -free ℓ -basis $(s_1^\sharp, \dots, s_r^\sharp)$, then

$$ql \deg(\mathcal{M}, P) = \sum_i R(Y^\sharp\text{-wt } s_i^\sharp)$$

(cf. (2.8)).

As a corollary to these definitions, we have

(8.8.6) **Corollary.** *Let J be a C -laminal ideal of width d and $\mathcal{S} \subset \mathcal{S}^\sharp$ an ℓ -free \mathcal{O}_X -module at P . Then*

(i) $\mathcal{O}_{C^\sharp} = \mathcal{O}_X\text{-lif}(\mathcal{O}_C)$ (that is $C^\sharp = \text{lif}(C)$), $J^\sharp = \mathcal{O}_X\text{-lif}(J)$ is a C^\sharp -laminal ideal of width d , we have equalities of subquotients

$$F^n(\mathcal{S}^\sharp, J^\sharp) = \mathcal{S}\text{-lif } F^n(\mathcal{S}, J), \quad \text{gr}^n(\mathcal{S}^\sharp, J^\sharp) = \mathcal{S}\text{-lif } \text{gr}^n(\mathcal{S}, J)$$

and hence ℓ -isomorphisms

$$F^n(\mathcal{S}, J) \simeq F^n(\mathcal{O}, J) \tilde{\otimes} \mathcal{S} \quad \text{and} \quad \text{gr}^n(\mathcal{S}, J) \simeq \text{gr}^n(\mathcal{O}, J) \tilde{\otimes} \mathcal{S}$$

for all $n \geq 0$. If furthermore $d \geq 2$, then similarly we have

$$\text{gr}^{n,i}(\mathcal{S}^\sharp, J^\sharp) = \mathcal{S}\text{-lif } \text{gr}^{n,i}(\mathcal{S}, J), \quad \text{gr}^{n,i}(\mathcal{S}, J) \simeq \text{gr}^{n,i}(\mathcal{O}, J) \tilde{\otimes} \mathcal{S}$$

for all n, i (8.6), and

(ii) homomorphisms $\alpha_{J^\sharp}, \beta_{J^\sharp}, \Delta_{J^\sharp}$ (8.2.2), $\gamma^n(\mathcal{S}^\sharp, J^\sharp)$ (and $\gamma^{n,i}(\mathcal{S}^\sharp, J^\sharp)$ if $d \geq 2$) are μ_m -homomorphisms “lifting” $\alpha_J, \beta_J, \Delta_J, \gamma^n(\mathcal{S}, J)$ (and $\gamma^{n,i}(\mathcal{S}, J)$ if $d \geq 2$) in the obvious sense. In particular, α_J is an ℓ -homomorphism, β_J is an ℓ -surjection, $\text{Ker } \beta_J \in \text{SSB}(\mathcal{O}_X)$, $\text{Ker } \beta_{J^\sharp} = \text{lif}(\text{Ker } \beta_J)$, $\gamma^n(\mathcal{S}, J) = \gamma^n(\mathcal{O}, J) \tilde{\otimes} \mathcal{S}$ (and $\gamma^{n,i}(\mathcal{S}, J) = \gamma^{n,i}(\mathcal{O}, J) \tilde{\otimes} \mathcal{S}$ if $d \geq 2$).

Proof. Equalities in (i) are for saturated subquotients and they hold outside P^\sharp ((8.3(i)) and (8.6(ii))) where π^\sharp is étale. Therefore they hold in a nbd of P^\sharp . The meaning of “lifting” in (ii) is the following commutative diagram

$$\begin{array}{ccccccc} 0 \rightarrow & \text{gr}^1(\mathcal{O}, J)^{\otimes d} & \xrightarrow{\Delta_J} & \text{gr}_C^0(J) & \xrightarrow{\alpha_J} & \text{Ker} \left[\text{gr}_C^1 \mathcal{O} \xrightarrow{\beta_J} \text{gr}^1(\mathcal{O}, J) \right] & \rightarrow 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \rightarrow & \text{gr}^1(\mathcal{O}^\sharp, J^\sharp)^{\otimes d} & \xrightarrow{\Delta_{J^\sharp}} & \text{gr}_{C^\sharp}^0(J^\sharp) & \xrightarrow{\alpha_{J^\sharp}} & \text{Ker} \left[\text{gr}_{C^\sharp}^1 \mathcal{O}^\sharp \xrightarrow{\beta_{J^\sharp}} \text{gr}^1(\mathcal{O}^\sharp, J^\sharp) \right] & \rightarrow 0 \end{array}$$

where β_{J^\sharp} is surjective. The rest follows from the construction and definition. \square

(8.9) **Definition.** Let $X \supset C$ be an extremal nbd and $Y \subset X$ a closed subspace without embedded or isolated points. Let $P_1, \dots, P_t \in X$ be points of indices m_1, \dots, m_t (≥ 1) such that $X - \{P_1, \dots, P_t\}$ has no points of indices > 1 , and Y is given the induced ℓ -structure at these points. A coherent \mathcal{O}_Y -module is said to be ℓ -coherent if it is given ℓ -structure at each of P_1, \dots, P_t . If an ℓ -coherent \mathcal{O}_Y -module \mathcal{M} is locally free of rank r on $Y^\circ = Y - \{P_1, \dots, P_t\}$ and ℓ -free (resp. $q\ell$ -free) at each P_i , then we say that \mathcal{M} is a locally ℓ -free (resp. $q\ell$ -free) \mathcal{O}_Y -module of rank r , and we may also say that \mathcal{M} is an ℓ -invertible (resp. $q\ell$ -invertible) \mathcal{O}_Y -module if $r = 1$. Local definitions of $\tilde{\otimes}$ and $\tilde{\otimes}$ for ℓ -coherent \mathcal{O}_Y -modules (cf. ($\tilde{\otimes}$.9) in (8.8.4)), powers for ℓ -invertible sheaves, \tilde{S}^n and $\tilde{\det}$ for ℓ -free \mathcal{O}_Y -modules (8.8.4) patch with the corresponding operations on Y° , and thus these operations give ℓ -coherent \mathcal{O}_Y -modules. Formulae in (8.8.4) extending the usual ones to \mathcal{O}_Y -modules with ℓ -structures at P_i 's extend further to ℓ -coherent \mathcal{O}_Y -modules since they coincide with the classical ones on the punctured nbds. Therefore we will quote the local

formulae in (8.8.4) in global setting by abuse of language. For example, $\text{gr}_C^0 \omega \simeq \mathcal{O}_C \tilde{\otimes} \omega_X$ is an l -invertible \mathcal{O}_C -module by $(\tilde{\otimes}.7)$ in (8.8.4) and (8.8.6(i)).

The notion of l -homomorphism, l -injection, l -surjection, l -isomorphism, l -splitting, and l -exactness for l -coherent sheaves are defined in the obvious way since these are local properties. An l -exact sequence $0 \rightarrow \mathcal{L} \rightarrow \mathcal{M} \rightarrow \mathcal{N} \rightarrow 0$ is (globally) l -split if \mathcal{M} has a coherent subsheaf \mathcal{H} giving an l -splitting $\mathcal{M} = \text{Im}(\mathcal{L} \rightarrow \mathcal{M}) \tilde{\otimes} \mathcal{H}$.

We note that l -structure (l -coherence, ...) does not depend on choice of above P_1, \dots, P_t as long as $X - \{P_1, \dots, P_t\}$ has no points of indices > 1 .

(8.9.1) **Proposition-Definition.** Assume further that Y is a complete pure 1-dimensional closed subscheme of X . Then

(i) if \mathcal{M} is a locally l -free \mathcal{O}_Y -module, then $(\widetilde{\det} \mathcal{M})^{\tilde{\otimes} \mu}$ is ql -invertible and

$$(\widetilde{\det} \mathcal{M})^{\tilde{\otimes} \mu n} \simeq ((\widetilde{\det} \mathcal{M})^{\tilde{\otimes} \mu})^{\otimes n}$$

as \mathcal{O}_Y -modules for all $n, \mu \in \mathbb{N}$ such that $\text{lcm}\{m_1, \dots, m_r\} \mid \mu$. Then we define

$$l \deg_Y \mathcal{M} = \frac{1}{\mu} \deg_Y (\widetilde{\det} \mathcal{M})^{\tilde{\otimes} \mu} \in \mathbb{Q}$$

which is independent of choice of such μ , where \deg_Y is the usual degree for \mathcal{O}_Y -invertible sheaves. We note that $l \deg_C(\text{gr}_C^0 \omega) = (\omega_X \cdot C)$, and

$$l \deg_Y(\mathcal{M} \tilde{\otimes} \mathcal{N}) = l \deg_Y \mathcal{M} + l \deg_Y \mathcal{N}$$

if \mathcal{M} and \mathcal{N} are of rank 1, and

$$l \deg_Y(\mathcal{N} \tilde{\otimes} S^n \mathcal{M}) = \frac{1}{2} n(n+1) l \deg_Y \mathcal{M} + (n+1) l \deg_Y \mathcal{N} \quad (n \geq 0)$$

if \mathcal{M} is of rank 2 and \mathcal{N} is of rank 1,

(ii) let $QL(Y) = \mathbb{Z} \oplus (\bigoplus_{1 \leq i \leq t} \mathbb{Z} P_i^\#) / (\sum_{1 \leq i \leq t} \mathbb{Z}(1 - m_i P_i^\#))$, then

$$\text{deg}: QL(Y) \rightarrow \mathbb{Q}; \quad \text{deg} \left(z + \sum z_i P_i^\# \right) = z + \sum z_i / m_i$$

is a well-defined homomorphism. For a locally ql -free \mathcal{O}_Y -module \mathcal{M} , let

$$ql_Y(\mathcal{M}) = \deg_Y \mathcal{M} + \sum_{i=1}^t ql \deg(\mathcal{M}, P_i) P_i^\# \in QL(Y),$$

then $l \deg_Y \mathcal{M} = \text{deg}(ql_Y(\mathcal{M}))$; and $ql_Y(\mathcal{M} \tilde{\otimes} \mathcal{N}) = ql_Y(\mathcal{M}) + ql_Y(\mathcal{N})$ for ql -invertible \mathcal{M} and \mathcal{N} such that $\mathcal{M} \tilde{\otimes} \mathcal{N}$ is ql -invertible; and

$$0 \leq l \deg_Y \mathcal{M} - \deg_Y \mathcal{M} \leq r \sum_{1 \leq i \leq t} B(Y, P_i) / m_i$$

(cf. (8.8.5)) if \mathcal{M} is of rank r , and

(iii) if $Y \simeq \mathbb{P}^1$ and $X \supset Y$ is locally primitive, then

$$TL: QL(Y) \rightarrow \mathbb{Z}; \quad TL \left(z + \sum z_i P_i^\# \right) = z - \sum U_i(-z_i)$$

is well defined, where U_i is defined for $(X, P_i) \supset (Y, P_i)$ as in (2.8). Furthermore, if \mathcal{M} and \mathcal{N} are ℓ -invertible \mathcal{O}_Y -modules, then $\deg_Y \mathcal{M} = TL(ql_Y(\mathcal{M}))$, whence

$$\deg_Y \left(\mathcal{M}^{\otimes a} \otimes \mathcal{N}^{\otimes b} \right) = TL(a \, ql_Y(\mathcal{M}) + b \, ql_Y(\mathcal{N})) \quad (a, b \in \mathbb{Z}).$$

We say that an expression $\zeta = z + \sum_i z_i P_i^\sharp$ of $\zeta \in QL(Y)$ is normalized if $U_i(-z_i) = 0$ for all i .

Proof. (i) The first assertion follows from $(\tilde{\otimes}\tilde{\otimes}.5)$ in (8.8.4). By ℓ -isomorphism $gr_C^0 \omega \simeq \mathcal{O}_C \tilde{\otimes} \omega_X$ in (8.9), one also has

$$l \deg_C(gr_C^0 \omega) = \frac{1}{\mu} \deg_C(\mathcal{O}_C \tilde{\otimes} \omega_X)^{\otimes \mu} = \frac{1}{\mu} \deg_C \mathcal{O}_C \otimes (\omega_X^{\otimes \mu}) = (\omega_X \cdot C).$$

The last two assertions follow from the definition by $(\mathcal{M} \tilde{\otimes} \mathcal{N})^{\otimes \mu} \simeq (\mathcal{M}^{\otimes \mu}) \otimes (\mathcal{N}^{\otimes \mu})$, and $(\tilde{\otimes}\tilde{\otimes}.5)$ and $(\tilde{S}.1)$ in (8.8.4).

(ii) The definition of $ql \deg(\mathcal{M}, P_i)$ shows

$$(*)_{\mathcal{M},i} \quad \frac{\mu}{m_i} \cdot ql \deg(\mathcal{M}, P_i) = \text{len}_{P_i} \text{Coker} \left[(\det \mathcal{M})^{\otimes \mu} \rightarrow (\tilde{\det} \mathcal{M})^{\otimes \mu} \right]$$

for locally ql -free \mathcal{M} if $m_i \mid \mu$. By summing up $(*)_{\mathcal{M},1}, \dots, (*)_{\mathcal{M},i}$, one has $l \deg_Y \mathcal{M} = \deg(ql_Y(\mathcal{M}))$. Let \mathcal{M} and \mathcal{N} be ql -invertible \mathcal{O}_Y -modules such that $\mathcal{M} \tilde{\otimes} \mathcal{N}$ is ql -invertible. By $(\tilde{\otimes}\tilde{\otimes}.5)$ in (8.8.4) and $(*)_{\mathcal{M},i} + (*)_{\mathcal{N},i} - (*)_{\mathcal{M} \tilde{\otimes} \mathcal{N},i}$, one has

$$\begin{aligned} & \frac{\mu}{m_i} (ql \deg(\mathcal{M}, P_i) + ql \deg(\mathcal{N}, P_i) - ql \deg(\mathcal{M} \tilde{\otimes} \mathcal{N}, P_i)) \\ &= \text{len}_{P_i} \text{Coker}[(\mathcal{M} \otimes \mathcal{N})^{\otimes \mu} \rightarrow (\mathcal{M} \tilde{\otimes} \mathcal{N})^{\otimes \mu}]. \end{aligned}$$

Let a_i be the length of $\mathcal{M} \tilde{\otimes} \mathcal{N} | \mathcal{M} \otimes \mathcal{N}$ at P_i . Then

$$\deg_Y \mathcal{M} \tilde{\otimes} \mathcal{N} = \deg_Y \mathcal{M} \otimes \mathcal{N} + a_1 + \dots + a_i,$$

and

$$ql \deg(\mathcal{M}, P_i) + ql \deg(\mathcal{N}, P_i) = ql \deg(\mathcal{M} \tilde{\otimes} \mathcal{N}, P_i) + a_i m_i.$$

Thus

$$\begin{aligned} ql_Y(\mathcal{M}) + ql_Y(\mathcal{N}) &= \deg_Y \mathcal{M} \otimes \mathcal{N} + \sum_i (ql \deg(\mathcal{M}, P_i) + ql \deg(\mathcal{N}, P_i)) P_i^\sharp \\ &= \deg_Y \mathcal{M} \otimes \mathcal{N} + \sum_i (ql \deg(\mathcal{M} \tilde{\otimes} \mathcal{N}, P_i) + a_i m_i) P_i^\sharp \\ &= \deg_Y \mathcal{M} \otimes \mathcal{N} + \sum_i a_i + \sum_i ql \deg(\mathcal{M} \tilde{\otimes} \mathcal{N}, P_i) P_i^\sharp \\ &= ql_Y(\mathcal{M} \tilde{\otimes} \mathcal{N}). \end{aligned}$$

The last assertion follows from (8.8.5).

(iii) TL is well defined by $U_i(m_i + z) = 1 + U_i(z)$ ($z \in \mathbb{Z}$) for all i (2.8(iii)). By (8.8.5) and the definition of R_i for $(X, P_i) \supset (Y, P_i)$, (2.8) shows that

$$U_i(-ql \deg(\mathcal{M}, P_i)) = 0$$

for all i , whence $TL(ql_Y(\mathcal{M})) = \deg_Y \mathcal{M}$. The rest follows from (ii). \square

(8.10) **Definition-Corollary.** Let $(X, P) \supset C$ be the germ of a 3-fold terminal singularity of index $m \geq 1$ and a reduced curve, and let (X^\sharp, P^\sharp) be the canonical cover. Let J be a C -laminal ideal of pure width d , and J^\sharp the canonical lifting of J at P . If an ℓ -basis $\{s_1, s_2\}$ of I_C at P satisfies $(s_1^d, s_2) = J^\sharp$ at P^\sharp , we say that the ordered ℓ -basis (s_1, s_2) is a $(1, d)$ -monomializing ℓ -basis of $I_C \supset J$ at P , and that J is $(1, d)$ -monomializable at P if such (s_1, s_2) exists. We note that $\text{width}_{J^\sharp} s_1 = 1$ and $\text{width}_{J^\sharp} s_2 = d$ by (8.3(i)).

Assume that J is $(1, d)$ -monomializable at P and let \mathcal{S} be an ℓ -free \mathcal{O}_X -module at P . Let $n \in \mathbb{Z}_+$, $q = [n/d]$, $r = n - qd$, and let i be an arbitrary integer in $[0, q]$. Then

- (i) $\text{gr}^n(\mathcal{S}, J)$, $\text{gr}_C^0 \mathcal{S}$, $\text{gr}_C^1 \mathcal{O}$, $\text{gr}^1(\mathcal{O}, J)$, $\text{gr}^d(\mathcal{O}, J)$, $\text{Ker } \beta_J$ are ℓ -free \mathcal{O}_C -modules at P with their induced ℓ -structures (8.8.6(ii)), and

$$\begin{aligned} \text{gr}^n(\mathcal{S}, J)^\sharp &\simeq \left(\bigoplus_{i+dj=n} \mathcal{O}_{C^\sharp} s_1^i s_2^j \right) \otimes_{\mathcal{O}_{C^\sharp}} \mathcal{S}^\sharp, \\ (\text{gr}_C^1 \mathcal{O})^\sharp &= \mathcal{O}_{C^\sharp} s_1 \oplus \mathcal{O}_{C^\sharp} s_2, \\ \text{gr}^1(\mathcal{O}, J)^\sharp &= \mathcal{O}_{C^\sharp} s_1 \quad \text{if } d \geq 2, \\ \text{gr}^d(\mathcal{O}, J)^\sharp &= \mathcal{O}_{C^\sharp} s_1^d \oplus \mathcal{O}_{C^\sharp} s_2, \\ (\text{Ker } \beta_J)^\sharp &= \begin{cases} \mathcal{O}_{C^\sharp} s_2 & \text{if } d \geq 2, \\ 0 & \text{if } d = 1, \end{cases} \end{aligned}$$

- (ii) $\gamma^n(\mathcal{S}, J)$ induces an ℓ -isomorphism

$$\tilde{\gamma}^n(\mathcal{S}, J): \tilde{S}^q(\text{gr}_C^d(\mathcal{O}, J)) \tilde{\otimes} (\text{gr}^1(\mathcal{O}, J)) \tilde{\otimes}^r \tilde{\otimes} \text{gr}_C^0 \mathcal{S} \rightarrow \text{gr}^n(\mathcal{S}, J),$$

- (iii) if $d \geq 2$, then we have an isomorphism

$$\text{gr}^{n,i}(\mathcal{S}, J)^\sharp \simeq \mathcal{O}_{C^\sharp} s_1^{di+r} s_2^{q-i} \otimes_{\mathcal{O}_{C^\sharp}} \mathcal{S}^\sharp,$$

whence $\gamma^{n,i}(\mathcal{S}, J)$ induces an ℓ -isomorphism

$$\tilde{\gamma}^{n,i}(\mathcal{S}, J): \text{gr}^1(\mathcal{O}, J) \tilde{\otimes}^{(di+r)} \tilde{\otimes}(\text{Ker } \beta_J) \tilde{\otimes}^{(q-i)} \tilde{\otimes} \text{gr}_C^0 \mathcal{S} \rightarrow \text{gr}^{n,i}(\mathcal{S}, J).$$

Indeed (i) and (ii) are immediate corollaries to (8.3) and (8.8.6) via the definitions in (8.8). For (iii), we also use (8.6).

(8.10.1) *Remark.* If $m = 1$ in (8.10), then J is $(1, d)$ -monomializable at P iff J is a nested c.i. at P .

(8.11) **Definition-Proposition.** Let $(X, P) \supset C$ be the germ of a 3-fold terminal singularity of index $m \geq 1$ and a reduced curve defined by ideal I_C and let (X^\sharp, P^\sharp) be the canonical cover. Let C^\sharp, I_{C^\sharp} be the canonical liftings of C, I_C at P . Let $\{s_1, s_2, e\}$ be an ℓ -basis of I_C at P , b and d integers ≥ 2 , and $f \in \mathcal{O}_{X^\sharp}$ a semi-invariant such that $\bar{f} = f \bmod I_{C^\sharp}$ is a nonzero divisor of \mathcal{O}_{C^\sharp} . For $n \geq 0$, let

$$\begin{aligned} \mathcal{K}_n &= \sum_{i+bj+dk \geq n} (s_1^i e^j s_2^k) \mathcal{O}_{X^\sharp} \supset \mathcal{K}_n^\circ = \sum_{\substack{i+bj+dk \geq n \\ i+j+k \geq 2}} (s_1^i e^j s_2^k) \mathcal{O}_{X^\sharp}, \\ \mathcal{K}'_n &= \sum_{\substack{i+bj+dk \geq n \\ 0 \leq i < b}} (s_1^i e^j s_2^k) \mathcal{O}_{X^\sharp}. \end{aligned}$$

Assume that $fe - s_1^b \in \mathcal{K}_{b+1}$ at P^\sharp . Then one has

(i) $J^\sharp = \mathcal{K}_d$ is a C^\sharp -laminal ideal of pure width d , $\text{wt } fe \equiv b \text{ wt } s_1 \pmod{m}$, $\text{width}_{J^\sharp} s_1 = 1$, $\text{width}_{J^\sharp} s_2 = d$, $\text{width}_{J^\sharp} e = b$, and furthermore for all $n \geq 0$, we have $F^n(\mathcal{O}_{X^\sharp}, J^\sharp) = \mathcal{K}_n = \mathcal{K}'_n$,

$$\delta_{J^\sharp}^n : \text{gr}^n(\mathcal{O}_{X^\sharp}, J^\sharp) \simeq \bigoplus_{\substack{i+bj+dk=n \\ 0 \leq i < b, 0 \leq j, k}} \mathcal{O}_{C^\sharp} s_1^i e^j s_2^k,$$

and an isomorphism of graded $\mathcal{O}_{C^\sharp, P^\sharp}$ -algebras with μ_m -action

$$\delta_{J^\sharp, P^\sharp} : \text{gr}(\mathcal{O}_{X^\sharp}, J^\sharp)_{P^\sharp} \xrightarrow{\sim} \mathcal{O}_{C^\sharp, P^\sharp}[S_1, S_2, E]/(\bar{f}E - S_1^b),$$

sending s_1, s_2, e to S_1, S_2, E , respectively. In particular,

$$\text{Im } \alpha_{J^\sharp} = \mathcal{O}_{C^\sharp} s_2, \quad \text{gr}^1(\mathcal{O}_{X^\sharp}, J^\sharp) = \mathcal{O}_{C^\sharp} s_1$$

(cf. (8.8.6(ii)) for α_{J^\sharp}),

(ii) if we further assume

(a) $d > b$ and $fe - s_1^b - hs_2 \in \mathcal{K}_{b+1}^\circ$ for some unit h (resp. (b) $fe - s_1^b \in \mathcal{K}_{b+1}^\circ$), then $\text{gr}_{C^\sharp}^1 \mathcal{O}$ is a free \mathcal{O}_{C^\sharp} -module with a free basis $\{s_1, e\}$ (resp. $\{s_1, s_2\}$) and $\text{Ker } \beta_{J^\sharp}$ is an invertible sheaf (cf. (8.8.6(ii))) and

$$\text{Im } \alpha_{J^\sharp} = \bar{f} \cdot \text{Ker } \beta_{J^\sharp} \quad (\text{resp. } \text{Ker } \beta_{J^\sharp}),$$

i.e. $\text{Ker } \beta_{J^\sharp} = \mathcal{O}_{C^\sharp} e$ (resp. $\mathcal{O}_{C^\sharp} s_2$),

(iii) let J be a C -laminal ideal of width d and let $J^\sharp = \text{lif } J$. Then $J^\sharp = \mathcal{K}_d$ iff $s_2 \in J^\sharp$.

If the condition in (iii) holds and if (a) (resp. (b)) in (ii) holds, then we say that the ordered ℓ -basis (s_1, s_2, e) is a $(1, d, b)$ -monomializing ℓ -basis of the first (resp. second) kind of $I_C \supset J$ at P , and that f is the attached semi-invariant. We say that J is $(1, d, b)$ -monomializable at P and $I_C \supset J$ at P

(or simply P) is of the first kind (resp. the second kind) if such an ℓ -basis exists.

Proof. (i) Since $s_1^b \equiv fe \pmod{\mathcal{K}_{b+1}}$ and $\mathcal{K}_{b+1} \cdot \mathcal{K}_{n-b} \subset \mathcal{K}_{n+1}$, we have a natural surjection

$$\gamma_n: \bigoplus_{\substack{i+bj+dk=n \\ 0 \leq i < b, 0 \leq j, k}} \mathcal{O}_{C^\sharp} s_1^i e^j s_2^k \rightarrow \mathcal{K}_n / \mathcal{K}_{n+1}$$

from a free \mathcal{O}_{C^\sharp} -module of rank $[n/d] + 1$. Let us fix an arbitrary $N \in \mathbb{N}$ for the moment. If $Q^\sharp \in C^\sharp - \{P^\sharp\}$ is a point close to P^\sharp such that $f(Q^\sharp) \neq 0$, then $e \in \sum_{i+dk \geq b} (s_1^i s_2^k)$ by Nakayama's lemma. Thus $I_{C^\sharp, Q^\sharp} = (s_1, s_2)$ and $\mathcal{K}_{n, Q^\sharp} = \sum_{i+dk \geq n} (s_1^i s_2^k)$. By (8.3(i)), $\mathcal{K}_n / \mathcal{K}_{n+1}$ is free of rank $[n/d] + 1$ at Q^\sharp , \mathcal{K}_d is a C^\sharp -laminal ideal of pure width d at Q^\sharp , and $\mathcal{K}_n = F^n(\mathcal{O}_{X^\sharp}, \mathcal{K}_d)$ for all $n \in [0, N]$ if Q^\sharp is close enough to P^\sharp . Hence $(\text{Ker } \gamma_n)_{Q^\sharp} = 0$, and $\text{Ker } \gamma_n = 0$ by $\text{Ker } \gamma_n \subset \text{Sat}(0) = 0$ ($n \in [0, N]$). Hence γ_n is an isomorphism for all $n \in [0, N]$ and we note $\sqrt{\mathcal{K}_n} = I_{C^\sharp}$ for all $n \geq 1$. Hence $\mathcal{O}_{X^\sharp} / \mathcal{K}_n$ has no embedded primes for all $n \in [0, N]$, whence $\mathcal{K}_d = \text{Sat}(\mathcal{K}_d) (= J^\sharp)$ is a C^\sharp -laminal ideal of pure width d and $\mathcal{K}_n = \text{Sat}(\mathcal{K}_n) = F^n(\mathcal{O}_{X^\sharp}, J^\sharp)$ for all $n \in [0, N]$ because $(\mathcal{K}_n)_{Q^\sharp} = F^n(\mathcal{O}_{X^\sharp}, \mathcal{K}_d)_{Q^\sharp}$ at nearby points $Q^\sharp (\neq P^\sharp)$. Hence it holds for all $n \geq 0$ and $\delta_{J^\sharp}^n$ is a μ_m -isomorphism for all $n \geq 0$, because N was chosen arbitrarily. Hence $\mathcal{K}_n = \mathcal{K}_{n+1} + \mathcal{K}_n'$ for all $n > 0$, and $\mathcal{K}_n = \mathcal{K}_{n+n'} + \mathcal{K}_n'$ for all $n, n' > 0$. If n' is large enough, then $\mathcal{K}_{n+n'} \subset I_C \mathcal{K}_n$ and $\mathcal{K}_n = \mathcal{K}_n'$ by Nakayama's lemma. By $\delta_{J^\sharp}^b(s_1^b) = \bar{f} \delta_{J^\sharp}^b(e)$, we have $\text{wt } fe \equiv b \text{ wt } s_1 \pmod{m}$. Since $d \geq 2$ and $\text{gr}_{C^\sharp}^0 J^\sharp = \text{gr}^d(\mathcal{O}, J^\sharp)$ has a free basis $\{s_2, s_1^{d-b[d/b]} \cdot e^{[d/b]}\}$ by $\delta_{J^\sharp}^d$, we see that $\text{Im } \alpha_{J^\sharp} = \mathcal{O}_{C^\sharp} s_2$ if $d \neq b$ by $d - b[d/b] + [d/b] \geq 2$. If $d = b$, then $fe - s_1^b \in \mathcal{K}_{b+1}$ implies $e \in I^{(2)}$ by $b \geq 2$, whence $\alpha_{J^\sharp}(e) = 0$ and $\text{Im } \alpha_{J^\sharp} = \mathcal{O}_{C^\sharp} s_2$ anyway. The rest is obvious.

(ii) The assertion on the free basis is straightforward. By (8.4), $\text{Ker } \beta_{J^\sharp} = \text{Sat}(\text{Im } \alpha_{J^\sharp})$ and (ii) follows from (i).

(iii) The only-if part is obvious by definition. By (8.8.6), J^\sharp is a C^\sharp -laminal ideal of width d . Assume that $s_2 \in J^\sharp$. Let $Q^\sharp \in C^\sharp - \{P^\sharp\}$ be a point close to P^\sharp such that $f(Q^\sharp) \neq 0$. Then one has $I_{C^\sharp, Q^\sharp} = (s_1, s_2)$, and the same argument as in the proof of (8.3(ii)) can be used to get $J^\sharp_{Q^\sharp} = (\mathcal{K}_d)_{Q^\sharp}$. Then $J^\sharp = \text{Sat}(J^\sharp) = \text{Sat}(\mathcal{K}_d) = \mathcal{K}_d$. \square

(8.11.1) *Remark.* Let the notation and assumptions be as in (8.11(i)). Let \mathcal{S} be a locally ℓ -free \mathcal{O}_X -module and $J = \text{quo}(J^\sharp)$ (8.8.1). Then

(i) if \bar{f} is a unit then (s_1, s_2) is a $(1, d)$ -monomializing ℓ -basis of $I_C \supset J$ at P ,

(ii) using (8.6) and (8.8.6), one sees that $\text{gr}^n(\mathcal{S}, J)$, $\text{gr}_C^0 \mathcal{S}$, $\mathcal{M} = \text{gr}^1(\mathcal{O}, J)$, $\text{gr}^d(\mathcal{O}, J)$, $\mathcal{L} = \text{Ker } \beta_J$, $\text{gr}^{n,i}(\mathcal{S}, J)$ are ℓ -free \mathcal{O}_C -modules at P with the descriptions

$$\begin{aligned} \mathcal{L}^\sharp &= \begin{cases} \mathcal{O}_{C^\sharp} e & (P \text{ first kind}), \\ \mathcal{O}_{C^\sharp} s_2 & (P \text{ second kind}), \end{cases} \\ \mathcal{M}^\sharp &= \mathcal{O}_{C^\sharp} s_1, \\ \text{gr}^d(\mathcal{O}, J)^\sharp &= \mathcal{O}_{C^\sharp} s_2 \oplus \mathcal{O}_{C^\sharp} e^{[d/b]} s_1^{d-b[d/b]}, \\ \text{gr}^{n,i}(\mathcal{O}, J)^\sharp &= \mathcal{O}_{C^\sharp} s_2^{q-i} e^{[(di+r)/b]} s_1^{(di+r)-b[(di+r)/b]}, \end{aligned}$$

and that $\gamma^{n,i}(\mathcal{S}, J)$ ($n \geq 0$) induces an ℓ -isomorphism

$$\tilde{\gamma}^{n,i}(\mathcal{S}, J): \mathcal{M}^{\tilde{\otimes}(di+r)} \tilde{\otimes} \mathcal{D}_P^{\tilde{\otimes} e_j} \tilde{\otimes} \mathcal{L}^{\tilde{\otimes}(q-i)} \tilde{\otimes} \text{gr}_C^0 \mathcal{S} \rightarrow \text{gr}^{n,i}(\mathcal{S}, J)$$

for $i \in [0, q]$, where $q = [n/d]$ and $r = n - qd$, $e_j = [(di+r)/b] - (q-i)$ (resp. $e_j = [(di+r)/b]$) if P is of the first (resp. second) kind, and \mathcal{D}_P is an ℓ -invertible \mathcal{O}_C -module at P given by

$$\mathcal{D}_P = (\mathcal{O}_{C^\sharp} \bar{f}^{-1})^{\mu_m} \subset \mathcal{D}_P^\sharp = \mathcal{O}_{C^\sharp} \bar{f}^{-1}.$$

This is because s_1, s_2 are ℓ -free ℓ -bases of $\mathcal{M}, \mathcal{L} \tilde{\otimes} \mathcal{D}^{\tilde{\otimes}(-1)}$ (resp. \mathcal{M}, \mathcal{L}), and $s_1^{di+r} \equiv (f \cdot e)^{[(di+r)/b]} \cdot s_1^{di+r-b[(di+r)/b]}$, and

(iii) if there is a global curve C extending (C, P) (cf. (8.9)), one can trivially extend \mathcal{D}_P to an ℓ -invertible \mathcal{O}_C -module \mathcal{D} on global C so that the unit section 1 of \mathcal{D}_P extends to a global section of \mathcal{D} and generates \mathcal{D} outside P . By construction, one sees $l \deg_C \mathcal{D} = m^{-1} \text{len}_P(\mathcal{O}_C / (\bar{f}^m) \mathcal{O}_C)$.

(8.12) **Theorem.** *Let $X \supset C \simeq \mathbb{P}^1$ be an extremal nbd and let J be a C -laminal ideal of pure width d and let $P_1, \dots, P_t \in C$ be such that X is smooth and J is a nested c.i. at arbitrary points of $C - \{P_1, \dots, P_t\}$. We assume that there exists $s \in [0, d]$ ($s = 0$ if $d = 1$) such that J is $(1, d, b_j)$ -monomializable for some b_j (resp. $(1, d)$ -monomializable) at P_j for all $j \leq s$ (resp. $j \geq s + 1$). We also assume that there exists $s' \in [0, s]$ such that P_j is of the first (resp. second) kind if $j \in [1, s']$ (resp. $[s' + 1, s]$). For each $j \leq s$, let \mathcal{D}_j be the ℓ -invertible \mathcal{O}_C -module obtained as \mathcal{D} at P_j in (8.11(iii)). Let \mathcal{S} be a locally ℓ -free \mathcal{O}_X -module of rank ρ . Let $n \in \mathbb{Z}_+$, $q = [n/d]$, $r = n - qd$, and let i be an arbitrary integer in $[0, q]$. Then $\text{gr}^n(\mathcal{S}, J)$, $\text{gr}_C^0 \mathcal{S}$, $\text{gr}^d(\mathcal{O}, J)$ are locally ℓ -free \mathcal{O}_C -modules of rank $qp + \rho$, ρ , 2, respectively, and*

(i) if $d = 1$, then $J = I_C$, $\text{gr}^n(\mathcal{S}, J) = \text{gr}_C^n \mathcal{S}$, $\text{gr}^1(\mathcal{O}, J) = \text{gr}_C^1 \mathcal{O}$, and $\gamma^n(\mathcal{S}, I_C)$ induces an ℓ -isomorphism

$$\tilde{\gamma}^n(\mathcal{S}, I_C): \tilde{\mathcal{S}}^n(\text{gr}_C^1 \mathcal{O}) \tilde{\otimes} \text{gr}_C^0 \mathcal{S} \xrightarrow{\sim} \text{gr}_C^n \mathcal{S},$$

and one has $l \deg_C \text{gr}_C^1 \mathcal{O} \geq 0$, and

(ii) if $d \geq 2$, then $\mathcal{M} = \text{gr}^1(\mathcal{O}, J)$, $\mathcal{L} = \text{Ker } \beta_J$ (8.2.2), and $\text{gr}^{n,i}(\mathcal{S}, J)$ (8.6) are ℓ -invertible \mathcal{O}_C -modules, β_J induces an ℓ -exact sequence

$$0 \rightarrow \mathcal{L} \rightarrow \text{gr}_C^1 \mathcal{O} \rightarrow \mathcal{M} \rightarrow 0,$$

and $\gamma^{n,i}(\mathcal{S}, J)$ (8.6) induces an ℓ -isomorphism

$$\tilde{\gamma}^{n,i}(\mathcal{S}, J): \mathcal{M}^{\otimes (di+r)} \tilde{\otimes} \left(\bigotimes_{j=1}^s \mathcal{D}_j^{\otimes e_j} \right) \tilde{\otimes} \mathcal{L}^{\otimes (q-i)} \tilde{\otimes} \text{gr}_C^0 \mathcal{S} \rightarrow \text{gr}^{n,i}(\mathcal{S}, J),$$

where $e_j = [(di+r)/b_j] - (q-i)$ (resp. $[(di+r)/b_j]$) for $j \in [1, s']$ (resp. $j \in [s'+1, s]$), and one has

$$l \deg_C \mathcal{M} + \frac{1}{d} l \deg_C \mathcal{L} + \sum \left(\frac{1}{b_j} - \frac{1}{d} \right) l \deg_C \mathcal{D}_j + \sum' \frac{1}{b_j} l \deg_C \mathcal{D}_j \geq 0,$$

where \sum (resp. \sum') is over P_j of the first (resp. second) kind.

Proof. The assertions follow mostly from (8.10), (8.11), (8.11.1). For (i), we only need to show $l \deg_C \text{gr}_C^1 \mathcal{O} \geq 0$. By (8.9.1) and $\tilde{\gamma}^n(\mathcal{O}_X, I_C)$, one has

$$\deg_C \text{gr}_C^n \mathcal{O} \leq l \deg_C \text{gr}_C^n \mathcal{O} = \frac{1}{2} n(n+1) l \deg_C \text{gr}_C^1 \mathcal{O}.$$

Thus $l \deg_C \text{gr}_C^1 \mathcal{O} \geq 0$ by (8.7). For (ii), we need to prove the last inequality. We note that $l \deg_C \mathcal{D}_j \geq \deg_C \mathcal{D}_j \geq 0$ (8.9.1) since \mathcal{D}_j has a nonzero global section (8.11(iii)). Thus by (8.9.1) and $\tilde{\gamma}^{n,i}(\mathcal{O}_X, J)$, one has

$$\begin{aligned} \deg_C \text{gr}^{n,i}(\mathcal{O}_X, J) &\leq l \deg_C \text{gr}^{n,i}(\mathcal{O}_X, J) \\ &\leq (di+r) l \deg_C \mathcal{M} + (q-i) l \deg_C \mathcal{L} \\ &\quad + \sum_{j=1}^s \frac{di+r}{b_j} l \deg_C \mathcal{D}_j - \sum_{j=1}^{s'} (q-i) l \deg_C \mathcal{D}_j, \end{aligned}$$

and summing these up for $i = 0, \dots, q (= [n/d])$, one has

$$\begin{aligned} \deg_C \text{gr}^n(\mathcal{O}, J) &\leq \frac{n^2}{2d} \left(l \deg_C \mathcal{M} + \frac{1}{d} l \deg_C \mathcal{L} + \sum_{j=1}^s \frac{1}{b_j} l \deg_C \mathcal{D}_j - \sum_{j=1}^{s'} \frac{1}{d} l \deg_C \mathcal{D}_j \right) + O(n), \end{aligned}$$

where $O(n)$ is a term bounded by $(\text{constant}) \cdot n$ as $n \rightarrow \infty$. Thus again by (8.7), we are done. \square

(8.13) Let X be an analytic 3-fold with only terminal singularities and C a smooth curve in X defined by ideal I . Let J be a C -laminal ideal of width $d \geq 2$. Let

$$0 \rightarrow \text{gr}^{d,1}(\mathcal{O}, J) \rightarrow \text{gr}_C^0 J \rightarrow \text{gr}^{d,0}(\mathcal{O}, J) \rightarrow 0$$

be the ℓ -exact sequence given in (8.6).

Assume that it has a splitting

$$(8.13.1) \quad \text{gr}_C^0 J = \text{gr}^{d,1}(\mathcal{O}, J) \oplus \mathcal{L}_J$$

into invertible sheaves on C , where \mathcal{L}_J is a subbundle of $\text{gr}_C^0 J$ of rank 1 and hence \mathcal{L}_J has an induced ℓ -structure as a $\text{gr}_C^0 J$ -saturated subsheaf of $\text{gr}_C^0 J$ (8.8.1). We note that (8.13.1) is not necessarily an ℓ -splitting. Let K be a C -primary ideal of \mathcal{O}_X such that $J \supset K \supset F^1 J$ and $K/F^1 J = \mathcal{L}_J$. Let P be a point on C , (X^\sharp, P^\sharp) the canonical cover of germ (X, P) , and $C^\sharp \subset (X^\sharp, P^\sharp)$ the canonical lifting of C . Then we can study K in the following four cases.

(8.14) **Proposition.** *Assume that J is a nested c.i. at P (of width $d \geq 2$ as above). Then K is a nested c.i. at P of width $d + 1$ such that $F^i(\mathcal{O}, K)_P = F^i(\mathcal{O}, J)_P$ for $i \leq d$. Thus K is a C -laminal ideal of width $d + 1$ such that $F^i(\mathcal{O}, K) = F^i(\mathcal{O}, J)$ for $i \leq d$, whence $\beta_K = \beta_J$ and $\text{Ker } \beta_K = \text{Ker } \beta_J$.*

Proof. The ideal I has a basis $\{u, v\}$ at P such that $J_P = (u^d, v)$. Then \mathcal{L}_J is generated by $w = v + \alpha u^d$ for some $\alpha \in \mathcal{O}_{X,P}$ by (8.3(i)), whence $I_P = (u, w)$, $J_P = (u^d, w)$, and $K_P = (u^{d+1}, w)$ (8.3(ii)), and the equality holds by (8.3(i)). Thus K_P is a C -laminal ideal of width $d + 1$. Since $d \geq 2$, $\text{gr}^1(\mathcal{O}, J) = \text{gr}^1(\mathcal{O}, K) = (u, v)/(u^2, v)$, whence $\beta_J = \beta_K: \text{gr}_C^1 \mathcal{O} \rightarrow \text{gr}^1(\mathcal{O}, J)$ because they are naturally induced maps (8.2.2). \square

Learning that K is a C -laminal ideal of width $d + 1$, we see

(8.14.1) **Corollary.** $F^1 J/F^1 K = \text{gr}^{d+1,1}(\mathcal{O}, K)$ and the natural map $K \rightarrow J$ induces an ℓ -isomorphism $\text{gr}^{d+1,0}(\mathcal{O}, K) = K/F^1 J \xrightarrow{\sim} \mathcal{L}_J$.

Proof. Since $\text{gr}^{d+1,1}(\mathcal{O}, K)$ is a $\text{gr}_C^0 K$ -saturation of $(I^{d+1} + F^1 K)/F^1 K$ in $\text{gr}_C^0 K$ via $\gamma^{d+1,1}(\mathcal{O}, K)$ (8.6(ii)) and since $I^{d+1} \subset F^1 J$ by width $J = d$, we see $\text{gr}^{d+1,1}(\mathcal{O}, K) \subset F^1 J/F^1 K$, which are both $\text{gr}_C^0 K$ -saturated submodules of $\text{gr}_C^0 K$ of rank 1. Hence $\text{gr}^{d+1,1}(\mathcal{O}, K) = F^1 J/F^1 K$. Since $J, K, F^1 J$ are \mathcal{O}_X -saturated submodules of \mathcal{O}_X and since $\mathcal{L}_J = K + F^1 J/F^1 J$ is a $J/F^1 J$ -saturated submodule of $J/F^1 J$, the two expressions $K/F^1 J$ and $K + F^1 J/F^1 J$ induce the same ℓ -structure on \mathcal{L}_J by (8.8.1). \square

(8.15) **Proposition.** *Assume that (C^\sharp, P^\sharp) is smooth and that $I \supset J$ has a $(1, d)$ -monomializing ℓ -basis (s_1, s_2) at P . There exists a semi-invariant $\alpha \in \mathcal{O}_{X^\sharp, P^\sharp}$ such that $\text{wt } \alpha \equiv \text{wt } s_2 - d \text{ wt } s_1$, or $-\text{wt } s_2 + d \text{ wt } s_1 \pmod{m}$ and such that*

(8.15.1) *if $\text{gr}_C^0 J = \text{gr}^{d,1}(\mathcal{O}, J) \oplus \mathcal{L}_J$ (8.13.1) is an ℓ -splitting at P , then $(s_1, s_2 + \alpha s_1^d)$ is a $(1, d + 1)$ -monomializing ℓ -basis for $I \supset K$ at P , and*

(8.15.2) if $\text{gr}_C^0 J = \text{gr}^{d,1}(\mathcal{O}, J) \oplus \mathcal{L}_J$ (8.13.1) is not an ℓ -splitting at P , then $0 < \text{ord } \alpha < \infty$, and $(s_1, \alpha s_2 + s_1^d, s_2)$ is a $(1, d + 1, d)$ -monomializing ℓ -basis of the first kind for $I \supset K$ at P with attached semi-invariant $-\alpha$ and relation :

$$(-\alpha)s_2 - s_1^d = -(s_2 + \alpha s_1^d).$$

Proof. Since C^\sharp is smooth, we see that $\text{lif}(\mathcal{L}_J) = \mathcal{O}_{C^\sharp}(\bar{\beta}s_1 + \bar{\gamma}s_1^d)$ for some semi-invariants $\bar{\beta} (\neq 0)$, $\bar{\gamma} \in \mathcal{O}_{C^\sharp}$ by (8.10(iii)), and that \mathcal{L}_J defines an ℓ -splitting iff $\text{ord } \bar{\beta} \leq \text{ord } \bar{\gamma}$. In case of (8.15.1) (resp. (8.15.2)), we take a semi-invariant representative $\alpha \in \mathcal{O}_{X^\sharp, P^\sharp}$ of $\bar{\gamma} \cdot \bar{\beta}^{-1}$ (resp. $\bar{\beta} \cdot \bar{\gamma}^{-1}$). Since $s_2 + \alpha s_1^d$ (resp. $\alpha s_2 + s_1^d$) belongs to K , (8.10) (resp. (8.11)) applies. \square

(8.16) **Proposition.** Assume that (C^\sharp, P^\sharp) is smooth, that $I \supset J$ has a $(1, d, b)$ -monomializing ℓ -basis (s_1, s_2, e) at P with attached semi-invariant f , and that $\text{gr}_C^0 J = \text{gr}^{d,1}(\mathcal{O}, J) \oplus \mathcal{L}_J$ (8.13.1) is an ℓ -splitting at P . Let $q = [d/b]$ and $r = d - qb$. Assume that $I \supset J$ is of the first (resp. second) kind at P .

Then there exists a semi-invariant $\alpha \in \mathcal{O}_{X^\sharp, P^\sharp}$ such that $\text{wt } \alpha \equiv \text{wt } s_2 - r \text{ wt } s_1 - q \text{ wt } e \pmod{m}$ and $(s_1, s_2 + \alpha s_1^r e^q, e)$ is a $(1, d + 1, b)$ -monomializing ℓ -basis of the first (resp. second) kind for $I \supset K$ at P with attached semi-invariant f .

Proof. By (8.11(i)), there exists $\bar{\alpha} \in \mathcal{O}_{C^\sharp}$ such that $\mathcal{L}_J = \mathcal{O}_{C^\sharp}(s_2 + \bar{\alpha}s_1^r e^q)$. Let $s_3 = s_2 + \alpha s_1^r e^q$ for some semi-invariant representative α of $\bar{\alpha}$, whence $s_3 \in K$. Let $fe - s_1^b = g(s_1, s_2, e)$ be the relation given in (8.11(ii)). Then $fe - s_1^b = g(s_1, s_3 - \alpha s_1^r e^q, e)$ is the relation for a $(1, d + 1, b)$ -monomializing ℓ -basis (s_1, s_3, e) . If (s_1, s_2, e) is of the first kind, then $q + r \geq 2$ by $d > b$, whence (s_1, s_3, e) is of the first kind. If (s_1, s_2, e) is of the second kind, then $g(s_1, s_2, e) \in (s_1, s_2, e)^2$ and $g(s_1, s_3 - \alpha s_1^r e^q, e) \in (s_1, s_3, e)^2$. The rest is similar. \square

9. EXISTENCE OF “GOOD” ANTICANONICAL DIVISOR (DELICATE CASE).

The main purpose of this section is to treat the delicate case (6.7.4), and our main result is (9.10).

Unless otherwise mentioned, we will work under the notation and assumptions of the following (9.1) throughout this section.

(9.1) Let $X \supset C \simeq \mathbb{P}^1$ be an extremal nbd with 2 points P, P' of indices $m, m' \geq 3$. Then by (6.7), this is case (6.7.4);

(*) $X \supset C$ has no other singular points, and P, P' are of size 1 and of type (IA), (IA^V), or (IIA).

Since $X \supset C$ has only one imprimitive point (1.13), we will assume that P' is a primitive point. Let \bar{m} and s be the subindex and the splitting degree of P . We note that $m = s\bar{m}$, and that $s = 1$ iff P is primitive.

Let $I \subset \mathcal{O}_X$ be the sheaf of defining ideals of C in X . Let $\pi^\sharp: (X^\sharp, P^\sharp) \rightarrow (X, P)$ (resp. $\pi^b: (X^b, P^b) \rightarrow (X, P')$) be the canonical cover and $(C^\sharp, P^\sharp) = \pi^{-1}(C)_{\text{red}}$ (resp. $(C^b, P^b) = \pi^{b^{-1}}(C)_{\text{red}}$). Let I^\sharp (resp. I^b) be the canonical lifting of I at P (resp. P'). Let $x = (x_1, \dots, x_4)$ and t (resp. $x' = (x'_1, \dots, x'_4)$ and t') be normalized ℓ -coordinates (2.6) at P (resp. P') such that $a_i = C^\sharp\text{-ord } x_i$ (resp. $a'_i = C^b\text{-ord } x'_i$) with $i \in [1, 4]$ as in (4.2). We will use $\text{ord} = C^\sharp\text{-ord}$, $\text{wt} = C^\sharp\text{-wt}$, $\text{ow} = C^\sharp\text{-ow}$, R, U for $X \supset C$ at P (resp. $\text{ord}' = C^b\text{-ord}$, $\text{wt}' = C^b\text{-wt}$, $\text{ow}' = C^b\text{-ow}$, R', U' for $X \supset C$ at P') in the sense introduced in (2.5) and (2.8).

We note that if P (resp. P') is of type (IA) and $a_2 = 1$ (resp. $a'_2 = 1$), then one may permute x_1 and x_3 (resp. x'_1 and x'_3) (cf. (4.2)).

We make a preliminary observation at P , which applies also to P' modulo obvious changes.

(9.1.1) **Lemma.** *Under the above notation and assumptions, assume that (C^\sharp, P^\sharp) is smooth (resp. P is ordinary (4.5)). Let \mathcal{E} be a torsion-free \mathcal{O}_C -module of rank 2 with ℓ -structure (resp. $\mathcal{E} = \text{gr}_C^0 I$). Then*

- (i) \mathcal{E} is a $q\ell$ -free \mathcal{O}_C -module at P (8.8.3), and
- (ii) for an arbitrary splitting $\mathcal{E} = \mathcal{O}_C s_1 \oplus \mathcal{O}_C s_2$ near P , there exists $\mu \in \mathbb{C}$ with the following property.

For an arbitrary $s_3 \in \mathcal{E} \otimes_{\mathcal{O}_C, P}$ with $s_3(P) = \mu s_1(P) + s_2(P) \in (\mathcal{E} \otimes \mathbb{C}(P))$, there exists an ℓ -free ℓ -basis $\{u_1, u_2\}$ of \mathcal{E} at P such that $s_1 \in \mathcal{O}_{C^\sharp, P^\sharp} u_1$ and $s_3 \in \mathcal{O}_{C^\sharp, P^\sharp} u_2$.

In particular, $\mathcal{O}_C s_1$ is an ℓ -invertible \mathcal{O}_C -module at P (with its induced ℓ -structure (8.8.1)) with an ℓ -free ℓ -basis u_1 at P , and $\mathcal{E} = \mathcal{O}_C s_1 \tilde{\oplus} \mathcal{O}_C s_3$ is an ℓ -splitting at P .

Proof. (i) is obvious because \mathcal{E}^\sharp is \mathcal{O}_{C^\sharp} -free of rank 2 (8.8.6) by the assumption. Let $\{v_1, v_2\}$ be an ℓ -free ℓ -basis of \mathcal{E} at P such that $s_i = f_{i1} v_1 + f_{i2} v_2$ ($i = 1, 2$) and $\text{ord } f_{11} \leq \text{ord } f_{12}$, where f_{ij} are semi-invariants in $\mathcal{O}_{C^\sharp, P^\sharp}$. Then we have

(9.1.2) *Claim.* $R(\text{wt } v_1), R(\text{wt } v_2) < \bar{m}$, and if

$$(z, w) \in \pm(C^\sharp\text{-wt}(v_1) - C^\sharp\text{-wt}^*(v_2)) + \mathbb{Z}_+(\bar{m}, 0) \subset \mathbb{Z} \times \mathbb{Z}_m$$

satisfies $z \geq 0$ (cf. (2.8)), then $(z, w) \in \text{ow}(C^\sharp)$.

Indeed, if (C^\sharp, P^\sharp) is smooth then (9.1.2) follows from $\text{ow}(C^\sharp) \simeq \mathbb{Z}_+$ (2.5). If P is ordinary, then one sees that P is of type (IA) or (IA^\vee) and that $\text{siz}_P = 1$ by (*) in (9.1), where the claim follows from (4.5.2).

By the choice of f_{ij} , we see that

$$(9.1.3) \text{ ow } f_{ij} \in C^\sharp\text{-ow}^*(v_j) + \mathbb{Z}_+(\overline{m}, 0) \text{ for all } i, j \text{ (2.8(i))},$$

$$(9.1.4) \text{ for each } i \in [1, 2], \text{ ow } f_{ij} = C^\sharp\text{-ow}^*(v_j) \text{ for some } j.$$

This is because \mathcal{O}_{C,S_i} is an \mathcal{E} -saturated submodule of \mathcal{E} . Then we will prove

$$(9.1.5) \text{ Claim. } \text{ ow } f_{21} - \text{ ow } f_{11} \in \mathbb{Z}_+(\overline{m}, 0), \text{ and if } \text{ord } f_{21} > \text{ord } f_{11}, \text{ then } u_1 = s_1/f_{11} \text{ and } u_2 = s_2/f_{22} \text{ form an } \ell\text{-free } \ell\text{-basis for } \mathcal{E} \text{ at } P.$$

Proof of Claim. Since $\text{ord } f_{11} \leq \text{ord } f_{12}$ by the choice of v_j , we have $\text{ord } f_{11} = R(\text{wt } v_1)$ or $\text{ord } f_{11} \leq \text{ord } f_{12} = R(\text{wt } v_2)$ by (9.1.4), whence $\text{ord } f_{11} < \overline{m}$ in either case by (9.1.2), and $\text{ow } f_{11} = C^\sharp\text{-ow}^*(v_1)$ by (9.1.3). Thus $\text{ow } f_{21} - \text{ow } f_{11} \in \mathbb{Z}_+(\overline{m}, 0) \subset \text{ow}(C^\sharp)$ as claimed. Assume now $\text{ord } f_{21} > \text{ord } f_{11}$ as in (9.1.5). Then $\text{ord } f_{21} \geq \overline{m} + R(\text{wt } v_1)$ (9.1.3), whence $\text{ow } f_{22} = C^\sharp\text{-ow}^*(v_2)$ by (9.1.4). Thus $\text{ord } f_{12} - \text{ord } f_{11} = \text{ord } f_{12} - R(\text{wt } v_1) \geq 0$ and $\text{ord } f_{21} - \text{ord } f_{22} = \text{ord } f_{21} - R(\text{wt } v_2) > 0$ by $R(\text{wt } v_2) < \overline{m}$ (9.1.2), and one sees $\text{ow } f_{12} - \text{ow } f_{11}$, $\text{ow } f_{21} - \text{ow } f_{22} \in \text{ow}(C^\sharp)$ by (9.1.2) and (9.1.3). Since f_{ij} have representatives of the form

$$(\text{invariant unit}) \cdot (\text{monomial in } x)$$

in $\mathcal{O}_{X^\sharp, P^\sharp}$ (cf. (2.7)), this implies that f_{12}/f_{11} , $f_{21}/f_{22} \in \mathcal{O}_{C^\sharp, P^\sharp}$ and $(f_{21}/f_{22})(P^\sharp) = 0$ by (2.7) and $u_1 = v_1 + (f_{12}/f_{11})v_2$ and $u_2 = (f_{21}/f_{22})v_1 + v_2$ form an ℓ -free ℓ -basis of \mathcal{E} at P and the claim follows.

$$(9.1.6) \text{ Since } \text{ow } f_{21} - \text{ow } f_{11} \in \text{ow}(C^\sharp), \text{ there exists } \mu \in \mathbb{C} \text{ such that}$$

$$\text{ord}(\mu f_{11} + f_{21}) > \text{ord } f_{11} \quad (2.7).$$

If $s_3 \in \mathcal{E} \otimes_{\mathcal{O}_{C,P}} \mathcal{O}_{C,P}$ satisfies $s_3(P) = \mu s_1(P) + s_2(P)$, then $\mathcal{E} = \mathcal{O}_{C,S_1} \oplus \mathcal{O}_{C,S_3}$ near P and the coefficients f_{31} and f_{32} in $s_3 = f_{31}v_1 + f_{32}v_2$ satisfy $\text{ord } f_{31} > \text{ord } f_{11}$. Hence we can apply (9.1.5) to s_1, s_3 . \square

(9.1.7) **Corollary.** *Assume that (C^\sharp, P^\sharp) and (C^b, P^b) are smooth (resp. P and P' are ordinary). Let \mathcal{E} be a torsion-free ℓ -coherent \mathcal{O}_C -module of rank 2 (resp. $\mathcal{E} = \text{gr}_C^0 I$). Then*

- (i) \mathcal{E} is a locally ℓ -free \mathcal{O}_C -module of rank 2,
- (ii) every \mathcal{E} -saturated invertible \mathcal{O}_C -submodule \mathcal{L} of \mathcal{E} (2.2) is an ℓ -invertible \mathcal{O}_C -module with its induced ℓ -structure, and
- (iii) if $\mathcal{E} = \mathcal{L} \oplus \mathcal{M}$ is a splitting into invertible \mathcal{O}_C -modules such that $\text{deg } \mathcal{L} > \text{deg } \mathcal{M}$, then we have an ℓ -splitting $\mathcal{E} = \mathcal{L} \tilde{\oplus} \{(\text{id}_{\mathcal{M}}, f)(\mathcal{M})\}$ for some $f \in \text{Hom}(\mathcal{L}, \mathcal{M})$, where $(\text{id}_{\mathcal{M}}, f): \mathcal{M} \rightarrow \mathcal{L} \oplus \mathcal{M} = \mathcal{E}$.

Proof. (i) and (ii) follow from (9.1.1(i)) and (9.1.1(ii)), respectively. For (ii), we see that $\mathcal{M}^* \otimes \mathcal{L}$ is generated by global sections and $\dim H^0(\mathcal{M}^* \otimes \mathcal{L}) \geq 2$ by $\text{deg } \mathcal{L} > \text{deg } \mathcal{M}$, whence (iii) follows from (9.1.1(ii)). \square

(9.1.8) **Corollary.** Assume that (C^\sharp, P^\sharp) and (C^\flat, P^\flat) are smooth. Let $\mathcal{E}, \mathcal{L}, \mathcal{M}$ be ℓ -free \mathcal{O}_C -modules of rank 2, 1, 1, and let

$$0 \rightarrow \mathcal{L} \rightarrow \mathcal{E} \rightarrow \mathcal{M} \rightarrow 0$$

be an ℓ -exact sequence. Then

- (i) for a point Q such that $ql \deg(\mathcal{L}, Q) \geq ql \deg(\mathcal{M}, Q)$, an arbitrary splitting $\mathcal{E} = \mathcal{L} \oplus \mathcal{N}$ at Q is an ℓ -splitting at Q , and
- (ii) if (a) $\deg \mathcal{L} > \deg \mathcal{M}$, or
 (b) $\deg \mathcal{L} \geq \deg \mathcal{M} - 1$ and $ql \deg(\mathcal{L}, Q) \geq ql \deg(\mathcal{M}, Q)$ for $Q = P$ and P' ,

then the sequence is ℓ -split.

Proof. As for (ii), (a) follows from (9.1.7), and (b) follows from (i). As for (i), it is enough to prove that a splitting $\mathcal{E} = \mathcal{L} \oplus \mathcal{N}$ at P is an ℓ -splitting at P assuming that $ql \deg(\mathcal{L}, P) \geq ql \deg(\mathcal{M}, P)$. Let $\{u_1, u_2\}$ be an ℓ -free ℓ -basis for \mathcal{E} at P such that u_1 is an ℓ -free ℓ -basis for \mathcal{L} at P . Then $R(\text{wt } u_1) \geq R(\text{wt } u_2)$ (8.8.5) by the assumption. We note that \mathcal{N} is generated by $f_1 u_1 + f_2 u_2$ such that $f_1, f_2 \in \mathcal{O}_{C^\sharp, P^\sharp}$ are semi-invariants with $\text{ord } f_1 \geq R(\text{wt } u_1)$ and $\text{ord } f_2 = R(\text{wt } u_2)$. Thus $f_1/f_2 \in \mathcal{O}_{C^\sharp, P^\sharp}$ and $\{u_1, (f_1/f_2)u_1 + u_2\}$ is an ℓ -free ℓ -basis for \mathcal{E} at P , and $\mathcal{E} = \mathcal{L} \oplus \mathcal{N}$. \square

(9.2) Under the notation and assumptions of (9.1), assume that P and P' are ordinary. Then they are of type (IA) or (IA^\vee) and $i_P(1) = \text{siz}_P = 1$, $i_{P'}(1) = \text{siz}_{P'} = 1$ by (*) in (9.1) and (4.10). Hence $\deg \text{gr}_C^1 \mathcal{O} = -1$ (2.3.2) and $\text{gr}_C^1 \mathcal{O} \simeq \mathcal{O} \oplus \mathcal{O}(-1)$, since $H^1(\text{gr}_C^1 \mathcal{O}) = 0$ (2.3.4). We note that the subsheaf $\mathcal{L} \subset \text{gr}_C^1 \mathcal{O}$ such that $\mathcal{L} \simeq \mathcal{O}$ is unique and that \mathcal{L} is an ℓ -invertible \mathcal{O}_C -module, and let u_1 (resp. u'_1) be an ℓ -free ℓ -basis of \mathcal{L} at P (resp. P') (8.8.3).

(9.3) **Theorem.** Assume that P and P' are ordinary as in (9.2). Then (C^\sharp, P^\sharp) and (C^\flat, P^\flat) are smooth (whence P and P' are primitive), $\text{wt } u_1 \equiv -1 \pmod{m}$, and $\text{wt}' u'_1 \equiv -1 \pmod{m'}$. Furthermore, $a_1 = a'_1 = 1$ after necessary permutations are made as in (4.2).

(9.3.1) *Remark.* By (4.5.1), an ℓ -free ℓ -basis of $\text{gr}_C^1 \mathcal{O}$ at P has $\text{wt} \equiv sa_1 a_2, -a_1 \pmod{m}$. If P is of type (IA^\vee) , then we treat it as a separate case:

(9.3.ipr) $s \geq 2,$

and then consider locally primitive $X \supset C$ as follows.

We note that if $a_2 = 1$, then one may permute x_1 and x_3 (9.1), by which $a_1 a_2 \equiv a_1$ and a_3 are permuted. We also note that if $a_2 = m - 1$, then $a_1 a_2 \equiv a_3$. Thus one may assume either (i) $\text{wt } u_1 \equiv a_1 a_2$ and $2 \leq a_2 \leq m - 2$, or (ii) $\text{wt } u_1 \equiv a_3$, and the same remark applies to P' . Therefore there are three cases (modulo permutation of P and P'):

(9.3.a) $\text{wt } u_1 \equiv a_1 a_2, a_2 \not\equiv \pm 1 \pmod{m}; \quad \text{wt}' u'_1 \equiv a'_1 a'_2, a'_2 \not\equiv \pm 1 \pmod{m'},$

$$(9.3.b) \quad \text{wt } u_1 \equiv a_1 a_2, \quad a_2 \not\equiv \pm 1 \pmod{m}; \quad \text{wt}' u'_1 \equiv a'_3 \pmod{m'},$$

and

$$(9.3.c) \quad \text{wt } u_1 \equiv a_3 \pmod{m}; \quad \text{wt}' u'_1 \equiv a'_3 \pmod{m'}.$$

Proof. We will prove (9.3) by showing that only (9.3.c) is possible and that $a_1 = a'_1 = 1$. Since $\text{gr}_C^1 \mathcal{O} = \mathcal{L} \oplus \mathcal{O}(-1)$ and $\mathcal{L} \simeq \mathcal{O}$ (9.2), one may choose subbundle $\mathcal{M} \simeq \mathcal{O}(-1)$ of $\text{gr}_C^1 \mathcal{O}$ such that $\text{gr}_C^1 \mathcal{O} = \mathcal{L} \oplus \mathcal{M}$ is an ℓ -splitting (9.1.7(iii)). Let u_2 (resp. u'_2) be an ℓ -free ℓ -basis of \mathcal{M} at P (resp. P').

(9.3.2) Let J be the C -laminal ideal of width 2 such that $J/I^{(2)} = \mathcal{L}$. Since $J/I^{(2)} = \mathcal{L}$, the surjection $\alpha_J: \text{gr}_C^0 J \rightarrow J/I^{(2)} = \mathcal{L}$ induced by identity is an ℓ -surjection by (8.8.1). Thus we have equalities $\text{Im } \alpha_J = \text{Ker } \beta_J = \mathcal{L}$ with ℓ -structure by (8.4(ii)) and an ℓ -isomorphism $\text{gr}^1(\mathcal{O}, J) \simeq \mathcal{M}$. In particular J is a nested c.i. outside $\{P, P'\}$ by (8.4(i)).

Let J^\sharp (resp. J^b) be the canonical lifting of J at P (resp. P'). We may replace $u_1, u_2 \in I^\sharp/I^{\sharp(2)}$ (resp. $u'_1, u'_2 \in I^b/I^{b(2)}$) with their representatives so that we may assume $u_1 \in J^\sharp$ and $u_2 \in I^\sharp$ (resp. $u'_1 \in J^b$ and $u'_2 \in I^b$). Thus one sees $I^\sharp = (u_1, u_2)$ and $J^\sharp = (u_1, u_2^2)$ (resp. $I^b = (u'_1, u'_2)$ and $J^b = (u'_1, u'^2_2)$) by (8.3), and (u_2, u_1) (resp. (u'_2, u'_1)) is a $(1, 2)$ -monomializing ℓ -basis for $I \supset J$ at P (resp. P'). Hence

$$(9.3.3) \quad \begin{aligned} 0 &\leq l \deg_C \mathcal{M} + \frac{1}{2} l \deg_C \mathcal{L} \\ &= \left(-1 + \frac{1}{m} R(\text{wt } u_2) + \frac{1}{m'} R'(\text{wt}' u'_2)\right) \\ &\quad + \frac{1}{2} \left(\frac{1}{m} R(\text{wt } u_1) + \frac{1}{m'} R'(\text{wt}' u'_1)\right) \end{aligned}$$

by (8.12). By (2.3) and (4.9(i)), one has

$$(9.3.4) \quad 1 < a_2 \cdot \bar{m}^{-1} + a'_2 \cdot m'^{-1}.$$

(9.3.5) *Case (9.3.ipr).* We will disprove this case. Since $\text{siz}_P = i_P(1) = 1$, we have $w_P(0) \geq 2/3$ by (5.6(ii)). Whence $w_{P'}(0) < 1/3$ by (2.3.3) and $a'_1 = 1$ by (5.1(i)). Since $\text{siz}_P = \text{siz}_{P'} = 1$, we see that $\{R(\text{wt } u_1), R(\text{wt } u_2)\} = \{\bar{m} - sa_1 a_2, a_1\}$ and $a_1 \leq \bar{m} - sa_1 a_2$ by (4.5.2) and that $\{R'(\text{wt}' u'_1), R'(\text{wt}' u'_2)\} = \{m' - a'_2, 1\}$ and $1 \leq m' - a'_2$. Then (9.3.3) gives

$$\begin{aligned} 0 &\leq 2 \left(-1 + \frac{\bar{m} - sa_1 a_2}{\bar{m}} + \frac{m' - a'_2}{m'}\right) + \frac{a_1}{\bar{m}} + \frac{1}{m'} \\ &< \bar{m}^{-1}(-2sa_1 a_2 + a_1 + 3a_2), \end{aligned}$$

where $1/m' \leq (m' - a'_2)/m' < a_2 \cdot \bar{m}^{-1}$ (9.3.4) is used in the second inequality. Hence we have a contradiction:

$$\begin{aligned} 0 &< -2sa_1 a_2 + a_1 + 3a_2 \leq -4a_1 a_2 + a_1 + 3a_2 \\ &= a_1(1 - 4a_2) + 3a_2 \leq 1 \cdot (1 - 4a_2) + 3a_2 = 1 - a_2 \leq 0 \end{aligned}$$

by $s \geq 2$, $a_1, a_2 \geq 1$. Thus case (9.3.ipr) does not occur, and P is a primitive point and $\bar{m} = m$ in the rest of proof of (9.3).

(9.3.6) *Case (9.3.a).* We will disprove this case. By $U(a_1 a_2) = 1$ and $U'(a'_1 a'_2) = 1$, one has $R(\text{wt } u_1) = m - a_1 a_2$ and $R'(\text{wt}' u'_1) = m' - a'_1 a'_2$, and by definition one has $R(\text{wt } u_2) = a_1$ and $R'(\text{wt}' u'_2) = a'_1$. Thus (9.3.3) reduces to

$$0 \leq \frac{1}{2m} a_1 (2 - a_2) + \frac{1}{2m'} a'_1 (2 - a'_2),$$

which implies $a_2 = a'_2 = 2$, because $a_2, a'_2 \geq 2$. By $a_2 \not\equiv \pm 1 \pmod{m}$ and $(a_2, m) = 1$, one sees that $m \geq 5$, and similarly $m' \geq 5$. This contradicts (9.3.4).

(9.3.7) *Case (9.3.b).* We will disprove this case. We have $R(\text{wt } u_1) = m - a_1 a_2$ and $R(\text{wt } u_2) = a_1$, $R'(\text{wt}' u'_1) = a'_1$ and $R'(\text{wt}' u'_2) = m' - a'_1 a'_2$, and $m \geq 5$ as above. Thus (9.3.3) reduces to

$$(9.3.7.1) \quad 1 \geq a_1(a_2 - 2)/m + a'_1(2a'_2 - 1)/m'.$$

(9.3.7.2) *Claim $a'_1 = 1$.* First by $a_2 \not\equiv \pm 1 \pmod{m}$ and $a_2 < m$, one has $m \geq a_2 + 2$ and $1 = m/m \geq 2/m + a_2/m > 2/m + 1 - a'_2/m'$ by (9.3.4), whence $0 > 2/m - a'_2/m'$. By (9.3.7.1), one has

$$1 \geq (a_2 - 2)/m + a'_1(2a'_1 - 1)/m' > 1 - a'_2/m' - 2/m + a'_1(2a'_2 - 1)/m',$$

where we used (9.3.4) in the second inequality, whence

$$2/m - a'_2/m' > -2a'_2/m' + a'_1(2a'_2 - 1)/m'.$$

Hence one has

$$0 > 2/m - a'_2/m' > -2a'_2/m' + a'_1(2a'_2 - 1)/m' = \{(a'_1 - 1)(2a'_2 - 1) - 1\}/m'.$$

Whence $0 > (a'_1 - 1)(2a'_2 - 1) - 1 \geq (a'_1 - 1) - 1 = a'_1 - 2$. Thus $a'_1 = 1$ and (9.3.7.2) is proved.

(9.3.7.3) *Claim $a_1 = 1$.* We first consider the case $m = 5$. If $a_1 \geq 2$, then $a_1 = a_2 = 2$ by $R(\text{wt } u_1) = 5 - a_1 a_2 > 0$, which contradicts $5 \in a_1 \mathbb{Z}_+ + a_2 \mathbb{Z}_+$. Thus $a_1 = 1$ if $m = 5$. Let us assume $m \geq 6$ and $a_1 \geq 2$. Then by (9.3.7.1), one has

$$1 \geq 2(a_2 - 2)/m + (2a'_2 - 1)/m' > 2 - 4/m - 1/m',$$

where (9.3.4) is used in the second inequality. Thus by $m \geq 6$, one has $1/m' > 1 - 4/m \geq 1/3$, which contradicts our assumption $m' \geq 3$. Hence $a_1 = 1$ if $m \geq 6$, and (9.3.7.3) is proved.

(9.3.7.4) Thus (C^\sharp, P^\sharp) and (C^b, P^b) are smooth, whence $U(z) = \lceil z/m \rceil$ and $U'(z) = \lceil z'/m' \rceil$ ($z \in \mathbb{Z}$) and $TL(z + z^\sharp P^\sharp + z^b P^b) = z + \lceil z^\sharp/m \rceil + \lceil z^b/m' \rceil$ (cf. (8.9.1(iii)) for definition). One also sees $ql_C(\mathcal{L}) = (m - a_2)P^\sharp + P^b$ and $ql_C(\mathcal{M}) = -1 + P^\sharp + (m' - a'_2)P^b$ by (8.8.5) (cf. (8.9.1(ii)) for definition).

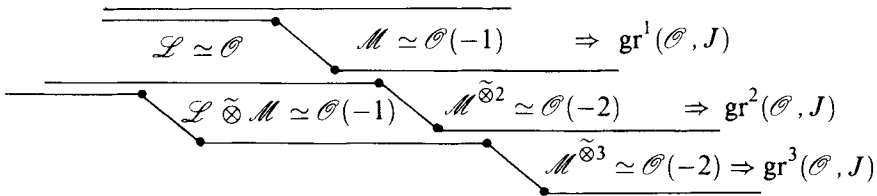
(9.3.7.5) *Claim* $2a'_2 < m'$. We will derive a contradiction assuming that $2a'_2 \geq m'$. We note that if $2a'_2 = m'$ then $m' = 2$ and $a'_2 = 1$ by $(a'_2, m') = 1$. This contradicts $m' \geq 3$, and we have $2a'_2 - 1 \geq m'$. Hence by (9.3.7.1), we have $a_2 = 2$ and $2a'_2 = m' + 1$, whence $m' \equiv 1 \pmod{2}$. Then (9.3.4) reduces to $1 < 2/m + (m' + 1)/2m'$, i.e. $1 < 4/m + 1/m'$. Since $m \geq 5$, one has $m' < 5$ by $1/m' > 1 - 4/m \geq 1/5$. Thus by $m' \equiv 1 \pmod{2}$ and $m' > 1$, one has $m' = 3$, whence $m = 5$ by $4/m > 1 - 1/3 = 2/3$. Since $a_2 = 2$ and $a'_2 = (m' + 1)/2 = 2$, one sees $ql_C(\mathcal{L}) = 3P^\# + P^b$ and $ql_C(\mathcal{M}) = -1 + P^\# + P^b$ (9.3.7.4). Hence subquotients of $F^1(\mathcal{O}, J)/F^4(\mathcal{O}, J)$ fit in exact sequences (8.6)

$$\begin{aligned} \text{gr}^1(\mathcal{O}, J) &= \text{gr}^{1,0}(\mathcal{O}, J), \\ 0 \rightarrow \text{gr}^{2,1}(\mathcal{O}, J) &\rightarrow \text{gr}^2(\mathcal{O}, J) \rightarrow \text{gr}^{2,0}(\mathcal{O}, J) \rightarrow 0, \\ 0 \rightarrow \text{gr}^{3,1}(\mathcal{O}, J) &\rightarrow \text{gr}^3(\mathcal{O}, J) \rightarrow \text{gr}^{3,0}(\mathcal{O}, J) \rightarrow 0. \end{aligned}$$

Since J is a nested c.i. outside $\{P, P'\}$ and (1, 2)-monomializable at P and P' (9.3.2), we can apply (8.12(ii)) (with $d = 2, s = s' = 0$) to get

$$\begin{aligned} \text{gr}^{1,0}(\mathcal{O}, J) &\simeq \mathcal{M} && (n = 1, q = 0, r = 1), \\ \text{gr}^{2,0}(\mathcal{O}, J) &\simeq \mathcal{L} && (n = 2, q = 1, r = 0), \\ \text{gr}^{2,1}(\mathcal{O}, J) &\simeq \mathcal{M}^{\tilde{\otimes} 2}, \\ \text{gr}^{3,0}(\mathcal{O}, J) &\simeq \mathcal{L} \tilde{\otimes} \mathcal{M} && (n = 3, q = 1, r = 1), \\ \text{gr}^{3,1}(\mathcal{O}, J) &\simeq \mathcal{M}^{\tilde{\otimes} 3}. \end{aligned}$$

These are calculated and arranged as



because $ql_C(\mathcal{L} \tilde{\otimes} \mathcal{M}) = ql_C(\mathcal{L}) + ql_C(\mathcal{M}) = -1 + 4P^\# + 2P^b, ql_C(\mathcal{M}^{\tilde{\otimes} 2}) = 2ql_C(\mathcal{M}) = -2 + 2P^\# + 2P^b$, and $ql_C(\mathcal{M}^{\tilde{\otimes} 3}) = 3ql_C(\mathcal{M}) = -2 + 3P^\#$ are normalized expressions (8.9.1). Hence $\chi(F^1(\mathcal{O}, J)/F^4(\mathcal{O}, J)) = -1 < 0$, which is a contradiction to $H^1(\mathcal{O}) = 0$ as in (8.7). Thus (9.3.7.5) is proved.

(9.3.7.6) By (9.3.7.5), one has $ql_C(\mathcal{M}^{\tilde{\otimes} 2}) = 2ql_C(\mathcal{M}) = -1 + 2P^\# + (m' - 2a'_2)P^b$ and $\deg_C \mathcal{M}^{\tilde{\otimes} 2} = -1$. Thus the \mathcal{O}_C -module exact sequence

$$0 \rightarrow \mathcal{M}^{\tilde{\otimes} 2} \rightarrow \text{gr}_C^0 J \rightarrow \mathcal{L} \rightarrow 0$$

is split, and $\text{gr}_C^0 J$ contains a unique subbundle \mathcal{L}_J such that $\mathcal{L}_J \xrightarrow{\sim} \mathcal{L}$ and $\text{gr}_C^0 J = \mathcal{M}^{\otimes 2} \oplus \mathcal{L}_J$ as \mathcal{O}_C -modules.

Let K be the C -primary ideal such that $J \supset K \supset F^1 J$ and $K/F^1 J = \mathcal{L}_J$. Then K is a C -laminal ideal of width 3 which is a nested c.i. outside $\{P, P'\}$, and we have ℓ -isomorphisms $\text{Ker } \beta_K = \text{Ker } \beta_J = \mathcal{L}$ and $\text{gr}^1(\mathcal{O}, K) = \text{gr}^1(\mathcal{O}, J) \simeq \mathcal{M}$ by (8.14). We see that $\text{gr}_C^0 J = \mathcal{L}_J \oplus \mathcal{M}^{\otimes 2}$ is an ℓ -splitting at P' by (9.1.8(i)), because $ql \deg(\mathcal{M}^{\otimes 2}, P^b) = m' - 2a'_2 \geq ql \deg(\mathcal{L}, P^b) = 1$ by (9.3.7.4) and $2a'_2 < m'$ (9.3.7.5). Hence the $(1, 2)$ -monomializing ℓ -basis (u'_2, u'_1) of $I \supset J$ at P' lifts to a $(1, 3)$ -monomializing ℓ -basis (denoted by the same (u'_2, u'_1) , by abuse of language) of $I \supset K$ at P' (8.15.1). Let

$$(\mathcal{L}_J)_P = \mathcal{L}_J \otimes_{\mathcal{O}_{C,P}} = (\mu t^{m-a_2} u_1 + t^{mk+2} u_2^2),$$

where μ is a unit and $0 \leq k \leq \infty$. If $k = 0$ (resp. $k \geq 1$), then $u_2, v = \mu x_1^{m-a_2-2} u_1 + u_2^2, u_1$ form a $(1, 3, 2)$ -monomializing ℓ -basis of the first kind (resp. (u_2, u_1) lifts to a $(1, 3)$ -monomializing ℓ -basis denoted by the same (u_2, u_1) , by abuse of language) of $I \supset K$ at P by $m - a_2 \geq 2$ and (8.15), where $x_1|_{C^*} = t$ by $a_1 = 1$ (9.1). One can see ℓ -isomorphisms

$$\mathcal{L}_J = \text{gr}^{3,0}(\mathcal{O}, K) \simeq \mathcal{L} \otimes \mathcal{D}^{\otimes(-\varepsilon)}$$

by (8.14.1) (with $d = 2$) for the first equality and by (8.12(ii)) (with $d = n = 3, q = 1, r = i = 0$) for the second, where $\varepsilon = 1$ (resp. 0) and \mathcal{D} is as defined in (8.11.1):

$$\mathcal{D}_P = \mathcal{O}_{C,P} \subset \mathcal{D}_P^\# = \mathcal{O}_{C^*,P^*} t^{-(m-a_2-2)}$$

(note $m > m - a_2 - 2 \geq 0$).

(9.3.7.7) *Claim* $a'_2 = 1$. Indeed by (9.3.7.6) and (8.12), one has

$$0 \leq l \deg_C \mathcal{M} + \frac{1}{3} l \deg_C \mathcal{L} + \varepsilon \left(\frac{1}{2} - \frac{1}{3}\right) \frac{1}{m} (m - a_2 - 2)$$

under the notation of (9.3.7.6). Thus in any case, we have

$$0 \leq (1/m - a'_2/m') + ((m - a_2)/m + 1/m')/3 + (m - a_2 - 2)/6m,$$

whence $m'(3(m - a_2) + 4) \geq 2m(3a'_2 - 1)$. Thus one sees

$$\frac{m - a_2}{a'_2} m' < m \leq \frac{3(m - a_2) + 4}{2 \cdot (3a'_2 - 1)} m',$$

where (9.3.4) was used for the first inequality. Whence follows $\{3(m - a_2) + 4\} \cdot a'_2 > 2(3a'_2 - 1)(m - a_2)$, hence $4a'_2 > (m - a_2)(3a'_2 - 2) \geq 2(3a'_2 - 2)$ by $m - a_2 \geq 2$, whence $a'_2 = 1$. Thus (9.3.7.7) is proved.

(9.3.7.8) By (8.12(ii)) (with $d = n = 3, q = i = 1, r = 0$), we see $\text{gr}^{3,1}(\mathcal{O}, K) = \mathcal{M}^{\otimes 3} \otimes \mathcal{D}^{\otimes \varepsilon}$. Hence by $\mathcal{L}_J = \text{gr}^{3,0}(\mathcal{O}, K)$ (9.3.7.6), we have an ℓ -exact sequence

$$0 \rightarrow \mathcal{M}^{\otimes 3} \otimes \mathcal{D}^{\otimes \varepsilon} \rightarrow \text{gr}_C^0 K \rightarrow \mathcal{L}_J \rightarrow 0,$$

where $\text{deg}_C \mathcal{M}^{\otimes 3} \tilde{\otimes} \mathcal{D}^{\otimes \varepsilon} = -1$ and $\text{deg}_C \mathcal{L}_J = 0$ by

$$\begin{aligned} \text{ql}_C(\mathcal{M}^{\otimes 3} \tilde{\otimes} \mathcal{D}^{\otimes \varepsilon}) &= 3\{-1 + P^\sharp + (m' - 1)P^\flat\} + \varepsilon(m - a_2 - 2)P^\sharp \\ &= \{-1 + 3P^\sharp + (m' - 3)P^\flat\} + \varepsilon(m - a_2 - 2)P^\sharp \\ &= -1 + \{3 + \varepsilon(m - a_2 - 2)\}P^\sharp + (m' - 3)P^\flat, \\ \text{ql}_C(\mathcal{L}_J) &= \text{ql}_C \mathcal{L} - \varepsilon(m - a_2 - 2)P^\sharp \\ &= \{(m - a_2) - \varepsilon(m - a_2 - 2)\}P^\sharp + P^\flat, \end{aligned}$$

where $0 < 3 + \varepsilon(m - a_2 - 2) < m$ and $0 < (m - a_2) - \varepsilon(m - a_2 - 2) < m$ by $2 \leq a_2 \leq m - 2$. Hence the exact sequence splits, and we write $\text{gr}_C^0 K = \mathcal{M}^{\otimes 3} \tilde{\otimes} \mathcal{D}^{\otimes \varepsilon} \oplus \mathcal{L}_K$ as \mathcal{O}_C -modules.

Let H be a C -primary ideal such that $K \supset H \supset F^1 K$ and $H/F^1 K = \mathcal{L}_K$. Then H is a C -laminal ideal of width 4 which is a nested c.i. outside $\{P, P'\}$, and we have ℓ -isomorphisms $\text{Ker } \beta_H = \text{Ker } \beta_K = \mathcal{L}$ and $\text{gr}^1(\mathcal{O}, H) = \text{gr}^1(\mathcal{O}, K) \simeq \mathcal{M}$ by (8.14).

(9.3.7.9) *Claim* $\text{gr}_C^0 K = \mathcal{M}^{\otimes 3} \tilde{\otimes} \mathcal{D}^{\otimes \varepsilon} \oplus \mathcal{L}_K$ is an ℓ -splitting at P if $\varepsilon = 1$.
 Indeed since $\text{ql deg}(\mathcal{M}^{\otimes 3} \tilde{\otimes} \mathcal{D}, P) = m + 1 - a_2 > 2 = \text{ql deg}(\mathcal{L}_J, P)$, it is an ℓ -splitting at P by (9.1.8(i)).

(9.3.7.10) The $(1, 3, 2)$ -monomializing ℓ -basis (u_2, v, u_1) of the first kind (resp. $(1, 3)$ -monomializing ℓ -basis (u_2, u_1)) induces a $(1, 4, 2)$ -monomializing ℓ -basis (u_2, v, u_1) of the first kind with the same attached semi-invariant (resp. $(1, 4, 3)$ -monomializing ℓ -basis of the first kind or $(1, 4)$ -monomializing ℓ -basis as in (9.3.7.6)) for $I \supset H$ at P by (8.16) (resp. by (8.15)). At P' , one has

$$(\mathcal{L}_K)_{P'} = \mathcal{L}_{P'} = \mathcal{O}_{C', P'}(\mu' t' u'_1 + t^{m'k' + m' - 3} u'_2{}^3),$$

where μ' is a unit and $0 \leq k' \leq \infty$. If $k' \geq 1$ or $m' \geq 4$, then $\text{gr}_C^0 K = \mathcal{L}_K \oplus \mathcal{M}^{\otimes 3} \tilde{\otimes} \mathcal{D}^{\otimes \varepsilon}$ is an ℓ -splitting at P' and H is $(1, 4)$ -monomializable at P' (8.15.1). If $(k', m') = (0, 3)$, then we have a $(1, 4, 3)$ -monomializing ℓ -basis $u'_2, v' = \mu' x'_1 u'_1 + u'_2{}^3, u'_1$ of the first kind for $I \supset H$ at P' (8.15.2), where we note $x'_1|_{C^v} = t'$ by $a'_1 = 1$ (9.1).

Thus we have by (8.12)

$$\begin{aligned} 0 \leq l \text{deg}_C \mathcal{M} + \frac{1}{4} l \text{deg}_C \mathcal{L} + \left(\frac{1}{2} - \frac{1}{4}\right) \frac{1}{m} (m - a_2 - 2) + \varepsilon' \left(\frac{1}{3} - \frac{1}{4}\right) \frac{1}{m'} \\ \text{(resp. } 0 \leq l \text{deg}_C \mathcal{M} + \frac{1}{4} l \text{deg}_C \mathcal{L} + \left(\frac{1}{3} - \frac{1}{4}\right) \frac{1}{m} (m - a_2 - 2) + \varepsilon' \left(\frac{1}{3} - \frac{1}{4}\right) \frac{1}{m'}), \end{aligned}$$

where $\varepsilon' = 1$ or 0 according as $(k', m') = (0, 3)$ or otherwise.

If $m' \geq 4$, then we have $\varepsilon' = 0$, and in either case, we get

$$0 \leq \left(\frac{1}{m} - \frac{1}{m'}\right) + \frac{1}{4} \left(\frac{m - a_2}{m} + \frac{1}{m'}\right) + \frac{m - a_2 - 2}{4m},$$

whence $m \leq (2m - 2a_2 + 2)m'/3$. Using (9.3.4), one sees

$$(m - a_2)m' < m \leq (2m - 2a_2 + 2)m'/3$$

whence $m - a_2 < (2m - 2a_2 + 2)/3$, i.e. $m - a_2 < 2$. This implies $m - a_2 = 1$, which contradicts $a_2 \not\equiv \pm 1 \pmod{m}$.

If $m' = 3$, then in either case, we get

$$0 \leq \left(\frac{1}{m} - \frac{1}{3}\right) + \frac{1}{4} \left(\frac{m - a_2}{m} + \frac{1}{3}\right) + \frac{m - a_2 - 2}{4m} + \frac{1}{36},$$

whence $m \leq 9(m - a_2 + 1)/4$. Thus by (9.3.4), we have

$$3(m - a_2) < m \leq 9(m - a_2 + 1)/4,$$

and $4(m - a_2) < 3(m - a_2 + 1)$, i.e. $m - a_2 < 3$. Thus $m - a_2 = 2$ by $a_2 \not\equiv \pm 1 \pmod{m}$. Then one has $6 < m \leq 27/4 = 6.75$, which is a contradiction. Thus (9.3.b) does not occur.

(9.3.8) *Case (9.3.c).* Our purpose here is to prove $a_1 = a'_1 = 1$ and smoothness of (C^\sharp, P^\sharp) and (C^b, P^b) . By symmetry of P and P' , we may assume $a'_2/m' > 1/2$ in view of (9.3.4). Since $\text{siz}_{P'} = 1$, we have $m' \geq a'_1 a'_2 > a'_1 m'/2$. Thus $a'_1 = 1$. We will prove $a_1 = 1$ by deriving a contradiction assuming $a_1 \geq 2$ till end of (9.3.8.8). We have $ql_C(\mathcal{L}) = a_1 P^\sharp + P^b$ and $ql_C(\mathcal{M}) = -1 + (m - a_1 a_2)P^\sharp + (m' - a'_2)P^b$. Thus (9.3.3) reduces to

$$(9.3.8.1) \quad 0 \leq \left(-1 + \frac{m - a_1 a_2}{m} + \frac{m' - a'_2}{m'}\right) + \frac{1}{2} \left(\frac{a_1}{m} + \frac{1}{m'}\right).$$

Using (9.3.4), we rewrite it as

$$0 < -\frac{a_1 a_2}{m} + \frac{a_2}{m} + \frac{1}{2} \left(\frac{a_1}{m} + \frac{1}{m'}\right),$$

that is

$$(9.3.8.2) \quad \frac{1}{m}(2a_1 a_2 - 2a_2 - a_1) < \frac{1}{m'}.$$

(9.3.8.3) *Claim* $a_2 = 1$. If $a_2 \geq 2$, then

$$(2a_1 a_2 - 2a_2 - a_1) - a_2 = (2a_1 - 3)a_2 - a_1 \geq 4a_1 - 6 - a_1 \geq 0,$$

whence we have $1/m' > a_2/m$. Then (9.3.4) implies $1 < a_2/m + a'_2/m' < (1 + a'_2)/m' \leq 1$, which is a contradiction and $a_2 = 1$ is proved.

(9.3.8.4) *Claim* $a_1 = 2$. By $a_2 = 1$, (9.3.4) implies $1/m > (m' - a'_2)/m' \geq 1/m'$. (9.3.8.2) implies $1/m' > (a_1 - 2)/m$, whence $1 > a_1 - 2$. By the hypothesis $a_1 \geq 2$, we have $a_1 = 2$.

(9.3.8.5) Since $a_2 = 1$ and $a'_1 = 1$, (C^\sharp, P^\sharp) and (C^b, P^b) are smooth, whence

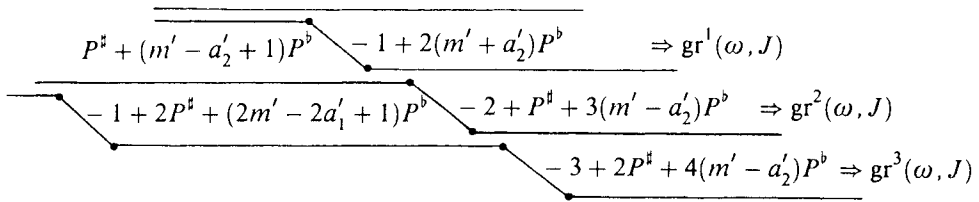
$$U(z) = [z/m] \quad \text{and} \quad U'(z) = [z/m'] \quad (z \in \mathbb{Z}),$$

and

$$TL(z + z^\sharp P^\sharp + z^b P^b) = z + [z^\sharp/m] + [z^b/m'].$$

One also sees $ql_C(\mathcal{L}) = 2P^\sharp + P^b$, $ql_C(\mathcal{M}) = -1 + (m - 2)P^\sharp + (m' - a'_2)P^b$, and $ql_C(\text{gr}_C^0 \omega) = -1 + (m - 1)P^\sharp + (m' - a'_2)P^b$ (2.10) and (8.9.1).

(9.3.8.6) *Claim* $m \geq 5$. Since $a_1 = 2$ is prime to m and $m > 1$, it is enough to derive a contradiction assuming $m = 3$. Then (9.3.4) implies $3(m' - a'_2) < m'$. Thus by (9.3.8.5) and the additivity of ql_C , we can calculate the values of ql_C for subquotients $\omega \tilde{\otimes} \mathcal{L}^{\otimes i} \tilde{\otimes} \mathcal{M}^{\otimes j}$ ($i, j \geq 0, 4 > 2i + j > 0$) of $F^1(\omega, J)/F^4(\omega, J)$ as follows:



as in (9.3.7.5). The expressions are normalized except for $ql_C(\omega \tilde{\otimes} \mathcal{M}^{\otimes 3}) = -3 + 2P^\sharp + 4(m' - a'_2)P^b$; it is not normalized only when $4(m' - a'_2) \geq m'$ and its normalized form in this case is $-2 + 2P^\sharp + (3m' - 4a'_2)P^b$. Thus $\chi(F^1(\omega, J)/F^4(\omega, J)) = -1$ or -2 , which contradicts (8.7). Hence $m \geq 5$.

(9.3.8.7) By $m \geq 5$, we see $ql_C(\mathcal{M}^{\otimes 2}) = -1 + (m - 4)P^\sharp + 2(m' - a'_2)P^b$ and $\text{deg}_C \mathcal{M}^{\otimes 2} = -1$. Thus

$$0 \rightarrow \mathcal{M}^{\otimes 2} \rightarrow \text{gr}_C^0 J \rightarrow \mathcal{L} \rightarrow 0$$

splits as \mathcal{O}_C -modules and $\text{gr}_C^0 J$ contains unique submodules isomorphic to \mathcal{L} , whence we write as $\text{gr}_C^0 J = \mathcal{M}^{\otimes 2} \oplus \mathcal{L}_J$.

Let K be the C -primary ideal such that $J \supset K \supset F^1 J$ and $K/F^1 J = \mathcal{L}_J$. Then K is a C -laminal ideal of width 3 which is a nested c.i. outside $\{P, P'\}$, and we have ℓ -isomorphisms $\text{Ker } \beta_K = \text{Ker } \beta_J = \mathcal{L}$ and $\text{gr}^1(\mathcal{O}, K) = \text{gr}^1(\mathcal{O}, J) \simeq \mathcal{M}$ by (8.14). We see that $\text{gr}_C^0 J = \mathcal{L}_J \oplus \mathcal{M}^{\otimes 2}$ is an ℓ -splitting at P' (9.1.8) because $ql \text{ deg}(\mathcal{L}, P') = 1 < ql \text{ deg}(\mathcal{M}^{\otimes 2}, P') = 2(m' - a'_2)$ by $m(m' - a'_2) < m'$ (9.3.4). Hence the $(1, 2)$ -monomializing ℓ -basis of $I \supset J$ at P' lifts to a $(1, 3)$ -monomializing ℓ -basis of $I \supset K$ at P' (8.15). Let

$$(\mathcal{L}_J)_P = \mathcal{O}_{C,P}(\mu t^2 u_1 + t^{mk+m-4} u_2^2),$$

where μ is a unit and $0 \leq k \leq \infty$. If $(k, m) = (0, 5)$ (resp. $(k, m) \neq (0, 5)$), then $u_2, v = \mu x_2 u_1 + u_2^2, u_1$ form a $(1, 3, 2)$ -monomializing ℓ -basis of the

first kind with attached semi-invariant μx_2 (resp. (u_2, u_1) lifts to a $(1, 3)$ -monomializing ℓ -basis) of $I \supset K$ at P (8.15), where we note $x_2|_{C^\sharp} = t$ by $a_2 = 1$ (9.1). Thus by (8.12(ii)) and (8.11(iii)), we have

$$\begin{aligned} 0 &\leq l \deg_C \mathcal{M} + \frac{1}{3} l \deg_C \mathcal{L} + \left(\frac{1}{2} - \frac{1}{3}\right) \frac{\varepsilon}{m} \\ &= \left(-1 + \frac{m-2}{m} + \frac{m' - a'_2}{m'}\right) + \frac{1}{3} \left(\frac{2}{m} + \frac{1}{m'}\right) + \frac{\varepsilon}{6m}, \end{aligned}$$

where $\varepsilon = 1$ (resp. $\varepsilon = 0$). Combining this with $(m' - a'_2)m < m'$ (9.3.4), we have

$$(9.3.8.8) \quad (8 - \varepsilon)(m' - a'_2)m < (8 - \varepsilon)m' \leq m\{6(m' - a'_2) + 2\}.$$

Thus $(8 - \varepsilon)(m' - a'_2) < 6(m' - a'_2) + 2$, whence $m' - a'_2 = \varepsilon = 1$ because $m' - a'_2 \geq 1$ and $\varepsilon = 0, 1$. Hence $m = 5$ by $\varepsilon = 1$. Then (9.3.8.8) implies $7 \cdot 5 < 7 \cdot m' \leq 5 \cdot 8$, which is impossible. Thus we have shown $a_1 = a'_1 = 1$ in case (9.3.c), and we are done. \square

(9.4) **Theorem.** *Let the notation and assumptions be as in (9.1). Then P and P' are type (IA) points.*

Proof. It is enough to derive a contradiction assuming that P is of type (IIA) or (IA^\vee) . Indeed it would imply that P is of type (IA) and hence P' is also of type (IA) by symmetry of P and P' . Before we start the calculation leading to a contradiction, we will replace $X \supset C$ with some nearby extremal nbd since it is enough to disprove the latter.

(9.4.1) By replacing $X \supset C$ with its L -deformation at P' (4.7), we may assume that P' is ordinary (hence of type (IA)). Then (C^b, P^b) is smooth and $a'_1 = 1$ modulo necessary permutations as in (4.2). Indeed a further L -deformation of $X \supset C$ at P would have only ordinary points, whence (C^b, P^b) is smooth and $a'_1 = 1$ modulo permutations by (9.3).

Then P is a type (IIA) point. Indeed if it is of type (IA^\vee) , then replacing $X \supset C$ with its L -deformation (4.7) at P , we have an extremal nbd $X \supset C \simeq \mathbb{P}^1$ with ordinary P, P' of indices ≥ 3 such that P is imprimitive, which contradicts (9.3).

(9.4.2) By replacing $X \supset C$ with its L' -deformation at P (4.12.2(ii)), we may take an ℓ -coordinate system $y = (y_1, \dots, y_4)$ and an ℓ -equation ψ for (X, P) such that $\text{wt } y_i \equiv a_i \pmod{\ell}$ ($i \in [1, 4]$), C^\sharp is the y_1 -axis, $\phi = y_1 y_2 + y_3^2 + y_4^3$ (we note that there exists no singularity other than P, P' by (6.2)). Hence $i_P(1) = 1$ by (2.16). Thus by $i_{P'}(1) = \text{siz}_{P'} = 1$, we have $\deg \text{gr}_C^1 \mathcal{O} = -1$ (2.3.2) and $\text{gr}_C^1 \mathcal{O} \simeq \mathcal{O} \oplus \mathcal{O}(-1)$. Then $\mathcal{L} \subset \text{gr}_C^1 \mathcal{O}$ such that $\mathcal{L} \simeq \mathcal{O}$ is unique, and \mathcal{L} is an ℓ -invertible \mathcal{O}_C -module (9.1.7).

We note that $y_2 \in I^{\sharp(2)}$, and $\{y_3, y_4\}$ is an ℓ -free ℓ -basis of $\text{gr}_C^1 \mathcal{O}$ at P such that $R(\text{wt } y_3) = R(a_3) = 1$ and $R(\text{wt } y_4) = R(a_4) = 2$ (4.2). Let

$y_1(f_3y_3+f_4y_4)$ ($f_3, f_4 \in \mathcal{O}_{C^s, P^s}$) be a basis of \mathcal{L} at P . Then $ql \deg(\mathcal{L}, P) = 1$ if f_3 is a unit (2 if otherwise) by (8.8.5).

(9.4.3) We consider the case $ql \deg(\mathcal{L}, P) \neq 1$ (otherwise this process is not needed). In this case, we see $ord f_4 = 1$ as above. Let $X_\lambda \supset C_\lambda$ be the twisted extensions (1b.8.1) of the trivial deformation

$$(X, P) \supset (C, P) \quad \text{by } (y_1(f_3y_3 + f_4y_4 + \lambda y_3), y_1y_3).$$

Then $\deg gr_{C_\lambda}^1 \mathcal{O} = -1$ by (2.3.2) since the deformation is locally trivial. Then $I^{(2)}$ deforms as $I_\lambda^{(2)}$ by (1b.8.3(i)). Since \mathcal{L}_P is generated by $y_1(f_3y_3 + f_4y_4)$, \mathcal{L} deforms as an invertible subsheaf \mathcal{L}_λ of $I_\lambda/I_\lambda^{(2)}$ such that \mathcal{L}_λ is generated by $y_1((f_3 + \lambda)y_3 + f_4y_4)$ at $P_\lambda (= P)$ and \mathcal{L}_λ is the trivial deformation of \mathcal{L} in $X_{\lambda, out} = X_{0, out}$ (1b.8.3(ii)). Then replacing $X \supset C$ by nearby extremal nbd $X_\lambda^\circ \supset C_\lambda$, we may assume $ql \deg(\mathcal{L}, P) = 1$, anyway.

(9.4.4) Thus we may assume $ql \deg(\mathcal{L}, P) = 1$. We also see $ql \deg(\mathcal{L}, P') = 1$ or $R'(a'_2)$ by (9.3.1) and (8.8.5) because $a'_1 = 1$ (9.4.1). Hence if $ql \deg(\mathcal{L}, P') > 1$, then we can apply a process similar to (9.4.3), and thus we may also assume $ql \deg(\mathcal{L}, P') = 1$.

(9.4.5) Since we have $\deg gr_C^1 \mathcal{O} = -1$ and $gr_C^1 \mathcal{O} \supset \mathcal{L} \simeq \mathcal{O}$, we have a splitting $gr_C^1 \mathcal{O} = \mathcal{L} \oplus \mathcal{O}(-1)$ and hence an ℓ -splitting $gr_C^1 \mathcal{O} = \mathcal{L} \tilde{\oplus} \mathcal{M}$ for some submodule $\mathcal{M} \simeq \mathcal{O}(-1)$ (9.1.7(iii)). By (9.4.4), we have $ql_C(\mathcal{L}) = P^\# + P^b$ and $ql_C(\mathcal{M}) = -1 + 2P^\# + (m' - a'_2)P^b$ (8.8.5) and (9.4.2). Since process (9.4.3) and (9.4.4) does not change $(X, P) \supset (C, P)$, we may use (y) given in (9.4.2). Since $ql \deg(\mathcal{L}, P) = 1$, \mathcal{L} has an ℓ -free ℓ -basis $y_3 + fy_1y_4$ at P (9.4.2), where f is an invariant in $\mathbb{C}\{y\}$. If we set $y_3'' = y_3 + fy_1y_4$ and $y_2'' = y_2 - 2fy_3y_4 - f^2y_1y_4^2$, then (y_1, y_2'', y_3'', y_4) is an ℓ -coordinate system with similar properties (e.g. $\psi = y_1y_2'' + y_3''^2 + y_4^3$). Thus choosing (y_1, y_2'', y_3'', y_4) as new (y_1, y_2, y_3, y_4) , we may simply assume that y_3 is an ℓ -free ℓ -basis of \mathcal{L} at P .

(9.4.6) Let J be a C -laminal ideal of width 2 such that $J/I^{(2)} = \mathcal{L}$. Then as in (9.3.2), J is a nested c.i. outside $\{P, P'\}$ and we have equalities $\text{Im } \alpha_j = \text{Ker } \beta_j = \mathcal{L}$ with ℓ -structure and an ℓ -isomorphism $gr^1(\mathcal{O}, J) \simeq \mathcal{M}$. Also as in (9.3.2), we can see that J is $(1, 2)$ -monomializable at ordinary point P' by the ℓ -splitting $gr_C^0 I = \text{Im } \alpha_j \tilde{\oplus} \mathcal{M}$. Then (9.4.5) shows that (y_4, y_3, y_2) is a $(1, 2, 3)$ -monomializing ℓ -basis of the second kind for $I \supset J$ at P . Let \mathcal{D} be the ℓ -invertible \mathcal{O}_C -module defined in (8.11.1(iii)) from (y_4, y_3, y_2) , then we have a normalized expression $ql_C \mathcal{D} = P^\#$. We now have an ℓ -isomorphism

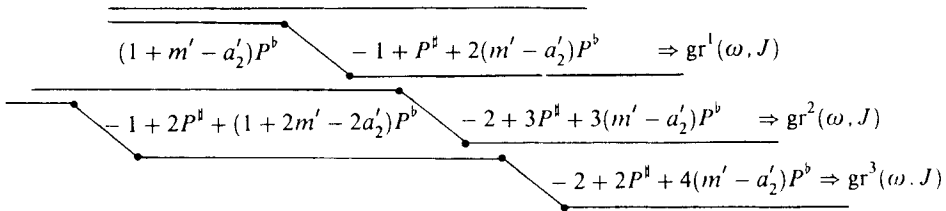
$$gr^{n,i}(\omega, J) \simeq \mathcal{M}^{\tilde{\otimes}(2i+r)} \tilde{\otimes} \mathcal{D}^{\tilde{\otimes}[(2i+r)/3]} \tilde{\otimes} \mathcal{L}^{\tilde{\otimes}(q-i)} \tilde{\otimes} gr_C^0 \omega$$

for all $n = 2q + r \geq 0$ with $r \in [0, 1]$ and $i \in [0, q]$ by (8.12) (with $d = 2, t = 2, s = 1, s' = 0$). Now the way we derive a contradiction is very similar to

(9.3.8.6). We note $ql_C(\text{gr}_C^0 \omega) = -1 + 3P^\sharp + (m' - a'_2)P^\flat$ (2.10) and (8.9.1), and calculate the values of ql_C for subquotients $\text{gr}^{n,i}(\omega, J)$ ($0 \leq i \leq [n/2]$, $n \in [1, 3]$) of $F^1(\omega, J)/F^4(\omega, J)$ as in (9.3.8.6) by (9.4.5) and additivity of ql_C . We get

$$\begin{aligned} ql_C(\text{gr}^{1,0}(\omega, J)) &= ql_C(\mathcal{M} \tilde{\otimes} \text{gr}_C^0 \omega) \\ &= ql_C(\mathcal{M}) + ql_C(\text{gr}_C^0 \omega) \\ &= -1 + P^\sharp + 2(m' - a'_2)P^\flat, \\ ql_C(\text{gr}^{2,0}(\omega, J)) &= ql_C(\mathcal{L} \tilde{\otimes} \text{gr}_C^0 \omega) \\ &= (m' - a'_2 + 1)P^\flat, \\ ql_C(\text{gr}^{2,1}(\omega, J)) &= ql_C(\mathcal{M}^{\tilde{\otimes} 2} \tilde{\otimes} \text{gr}_C^0 \omega) \\ &= -2 + 3P^\sharp + 3(m' - a'_2)P^\flat, \\ ql_C(\text{gr}^{3,0}(\omega, J)) &= ql_C(\mathcal{M} \tilde{\otimes} \mathcal{L} \tilde{\otimes} \text{gr}_C^0 \omega) \\ &= -1 + 2P^\sharp + \{2(m' - a'_2) + 1\}P^\flat, \\ ql_C(\text{gr}^{3,1}(\omega, J)) &= ql_C(\mathcal{M}^{\tilde{\otimes} 3} \tilde{\otimes} \mathcal{L} \tilde{\otimes} \text{gr}_C^0 \omega) \\ &= -2 + 2P^\sharp + 4(m' - a'_2)P^\flat, \end{aligned}$$

which are arranged as



The expressions are normalized (8.9.1) because $4(m' - a'_2) < m'$ by $w_{P'}(0) = (m' - a'_2)/m' < 1 - w_P(0) = 1/4$ (2.3.3) and (4.9(i)). Thus

$$\chi(F^1(\omega, J)/F^4(\omega, J)) < 0$$

(8.9.1(iii)), which contradicts $H^1(\omega) = 0$ as in (8.7) and we are done. \square

(9.4.7) **Corollary-Definition.** Points P and P' are type (IA) points such that $a_1 = a'_1 = 1$ (after necessary permutations are made as in (4.2)), and $\ell(P) < m$ and $\ell(P') < m'$, where we define $\ell(P) = \text{len}_{P^\sharp} I^{\sharp(2)}/I^{\sharp 2}$ and $\ell(P') = \text{len}_{P^\flat} I^{\flat(2)}/I^{\flat 2}$.

We note that P is ordinary iff $\ell(P) = 0$.

Proof. By symmetry between P and P' , we need to check the assertions only for P and P^\sharp . We will derive a contradiction assuming that $a_1 = 1$ cannot be

achieved, that is either (i) $1 < a_1 < m - 1$, or (ii) $a_1 = m - 1$ and $a_2 > 1$. Since it is enough to disprove nearby extremal nbds, we may replace $X \supset C$ with some nearby extremal nbd. So we may replace $X \supset C$ with an L -deformation $Y \supset D$ at P (4.7), and further with an L -deformation at P' of $Y \supset D$. In this process a_i 's are not changed since P and P' are of type (IA). Thus our $X \supset C$ has two ordinary points P, P' with indices $m, m' \geq 3$ such that (i) $1 < a_1 < m - 1$, or (ii) $a_1 = m - 1$ and $a_2 > 1$. This contradicts (9.3). Thus $a_1 = 1$ and similarly $a'_1 = 1$ (modulo necessary permutations). If $\ell(P) \geq m$, then an L' -deformation of $X \supset C$ at P (4.12.2) is locally primitive and has at least one point of type (III) besides P and P' of indices $m, m' \geq 3$. This contradicts (6.2), and hence $\ell(P) < m$. Thus we are done. \square

(9.4.8) **Corollary.** *We have $i_P(1) = i_{P'}(1) = 1$ and an isomorphism $\text{gr}_C^1 \mathcal{O} \simeq \mathcal{O} \oplus \mathcal{O}(-1)$.*

Proof. By (2.16), we have $i_P(1) = i_{P'}(1) = 1$. Thus $\text{deg gr}_C^1 \mathcal{O} = -1$ by (2.3.2). Hence we have $\text{gr}_C^1 \mathcal{O} \simeq \mathcal{O} \oplus \mathcal{O}(-1)$ by $H^1(\text{gr}_C^1 \mathcal{O}) = 0$ (2.3.4). \square

(9.5) Let the notation and assumptions be as in (9.1). Then P and P' are of type (IA) and we may assume $a_1 = a'_1 = 1$. Then $\text{gr}_C^1 \mathcal{O} \simeq \mathcal{O} \oplus \mathcal{O}(-1)$, hence there exists exactly one submodule \mathcal{L} of $\text{gr}_C^1 \mathcal{O}$ such that $\mathcal{L} \simeq \mathcal{O}_C$, and it is $\text{gr}_C^1 \mathcal{O}$ -saturated. We note by (9.1.7(ii)), \mathcal{L} is an ℓ -invertible \mathcal{O}_C -module.

Though the following is formulated only for P , it also applies to P' modulo obvious changes.

(9.5.1) **Definition-Lemma.** *Let $q(P) = ql \text{ deg}(\mathcal{L}, P)$. Then there exists $\sigma \in \mathfrak{S}_4$ with $\sigma(1) = 1$ such that $\ell(P) = R(a_{\sigma(4)})$, $q(P) = R(a_{\sigma(3)})$, and $\{y_{\sigma(2)}, y_{\sigma(3)}\}$ is an ℓ -free ℓ -basis of $\text{gr}_C^1 \mathcal{O}$ at P .*

Proof. There exists an ℓ -coordinate system $(y) = (y_1, \dots, y_4)$ and an ℓ -equation ϕ at P such that $C^\sharp = y_1$ -axis, $\text{wt } y_i \equiv a_i \pmod{\ell} \ (i \in [1, 4])$, and $\phi \equiv y_1^{\ell(P)} y_{\sigma(4)} \pmod{(y_2, y_3, y_4)^2}$ for some $\sigma \in \mathfrak{S}_4$ such that $\sigma(1) = 1$ (2.16). Then $\ell(P) = R(a_{\sigma(4)})$ by $\ell(P) < m$ (2.16), and $\{y_{\sigma(2)}, y_{\sigma(3)}\}$ is an ℓ -free ℓ -basis of $\text{gr}_C^1 \mathcal{O}$ at P . Thus $q(P) = R(a_{\sigma(2)})$ or $R(a_{\sigma(3)})$. \square

We will define deformation processes of $X \supset C$ which are more delicate than the ones in (4.7). We begin with a general description.

(9.6) Let the notation and assumptions be as in (9.5). We will choose an ℓ -coordinate system $(y) = (y_1, \dots, y_4)$ and an ℓ -equation ϕ at P such that $C^\sharp = y_1$ -axis and $\text{wt } y_i \equiv a_i \pmod{\ell} \ (i \in [1, 4])$, and invariant $u_1, u_2 \in (y_2, y_3, y_4)\mathbb{C}\{y\}$ such that $\text{gr}_C^1 \mathcal{O}/(u_1, u_2)$ is of finite length and \mathcal{L} is generated by u_1 at P . We will choose an invariant $\psi \in (y_2, y_3, y_4)\mathbb{C}\{y\}$ such that

$$(9.6.1) \quad \text{g.c.d.}\{\phi, \psi\} = 1 \quad \text{and} \quad \text{Sing}\{\phi = \psi = 0\} \cap y_4\text{-axis} \subset \{0\},$$

and

$$(9.6.2) \quad (X_\lambda, P_\lambda) \text{ is terminal if } 0 < |\lambda| \ll 1,$$

where $\phi_\lambda = \phi + \lambda\psi$, and $(X_\lambda, P_\lambda) = (\{\phi_\lambda = 0\}, 0)/\mu_m \supset (C_\lambda, P_\lambda) = (y_1\text{-axis}, 0)/\mu_m$ ($\lambda \in \mathbb{C}$). We note

(9.6.3) *Remark.* (y) is an ℓ -coordinate system for $X_\lambda \supset C$ at $P_\lambda = P$, whence (a_1, \dots, a_4) remain the same for P_λ .

Then we can apply (1b.3) as in step (L) in §1b, and following step (LG) in §1b, let $X_\lambda \supset C_\lambda$ be the twisted extension of $(X_\lambda, P_\lambda) \supset (C_\lambda, P_\lambda)$ by $u = (u_1, u_2)$. We note that $X_\lambda \supset C_\lambda (= C)$ may be identified with $X \supset C$ outside a small nbd of P_λ :

$$X_{\lambda,\text{out}} = X - X \left(\begin{matrix} [0, \beta] \times \gamma \\ u < \delta \end{matrix} \right) \supset C_{\lambda,\text{out}} = C \cap X_{\lambda,\text{out}},$$

where $\beta, \gamma, \delta \in \mathbb{C}$ are as in (1b.8.1). Then by step (G) in §1b, we have nearby extremal nbds $X_\lambda^\circ \supset C_\lambda = C$, where X_λ° is a nbd of C_λ in X_λ . Though we have not yet explicitly constructed $X_\lambda^\circ \supset C_\lambda$, we can say the following.

(9.6.4) **Lemma.** *For an arbitrary small enough λ , $X_\lambda^\circ \supset C_\lambda$ is a locally primitive extremal nbd with exactly two singular points P_λ and P'_λ , which are of type (IA) and have indices m and m' . Let I_λ be the sheaf of defining ideals of C_λ in X_λ , then the coherent sheaves $\text{gr}^1(\mathcal{O}, I_\lambda)$ on C_λ form a flat family with parameter λ , and \mathcal{L} deforms as submodules $\mathcal{L}_\lambda (\simeq \mathcal{O})$ of $\text{gr}^1(\mathcal{O}, I_\lambda)$ such that*

$$\mathcal{L}_\lambda|_{C_{\lambda,\text{out}}} = \mathcal{L}|_{C_{0,\text{out}}} \quad \text{and} \quad \mathcal{L}_\lambda \otimes \mathcal{O}_{C_\lambda, P_\lambda} = \mathcal{O}_{C_\lambda, P_\lambda} u_1.$$

In particular we have $\ell(P'_\lambda) = \ell(P')$ and $q(P'_\lambda) = q(P')$.

Proof. By smoothness of (C^\sharp, P^\sharp) , we have local primitiveness of $X_\lambda \supset C_\lambda$ in a nbd of P_λ (1b.3.1), whence local primitiveness of $X_\lambda \supset C_\lambda$ follows by the definition of twisted extension. It is clear that $X_\lambda^\circ \supset C_\lambda$ has two singular points P_λ and P'_λ with indices $m, m' (\geq 3)$, whence X_λ° has no other singular points on C_λ by (6.2). They are of type (IA) by (9.6.3) and triviality of deformation near P' . Thus $\text{deg gr}^1(\mathcal{O}, I_\lambda) = -1$ by (9.5), and $\text{gr}^1(\mathcal{O}, I_\lambda)$'s form a flat family by (1b.8.3(i)). The assertions on \mathcal{L} follow from (1b.8.3(ii)). We see that $\ell(P'_\lambda) = \ell(P')$ by $X_{\lambda,\text{out}} = X_{0,\text{out}}$, and that $q(P'_\lambda) = q(P')$ by $\mathcal{L}_\lambda = \mathcal{L}$ after the previous identification. \square

(9.7) **Proposition-Definition.** *In each of the following five cases (a)–(e), there exists a deformation $X_\lambda \supset C_\lambda$ as described above such that $\ell(P_\lambda)$ and $q(P_\lambda)$ satisfy the conditions (cf. (9.6.3)).*

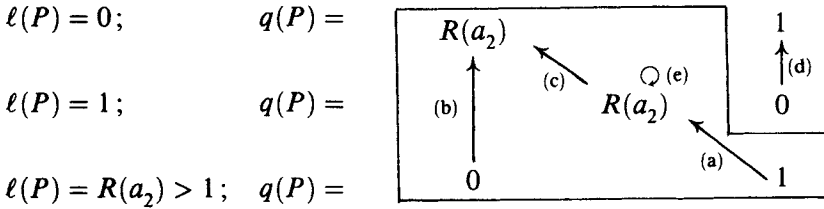
- (a) $\ell(P) = R(a_2) > 1, q(P) = 1 \Rightarrow \ell(P_\lambda) = 1, q(P_\lambda) = R(a_2) > 1,$
- (b) $\ell(P) = R(a_2) > 1, q(P) = 0 \Rightarrow \ell(P_\lambda) = 0, q(P_\lambda) = R(a_2) > 1,$
- (c) $\ell(P) = 1, q(P) = R(a_2) > 1 \Rightarrow \ell(P_\lambda) = 0, q(P_\lambda) = R(a_2) > 1,$

(d) $\ell(P) = 1, q(P) = 0 \Rightarrow \ell(P_\lambda) = 0, q(P_\lambda) = 1,$
 and if $a_2 = m - 1,$ then

(e) $\ell(P) = q(P) = 1 \Rightarrow \ell(P_\lambda) = q(P_\lambda) = 1$ and P_λ has axial multiplicity 2.

We call all these processes (or $X_\lambda^\circ \supset C_\lambda$) the LG-deformations of $X \supset C$ at P .

(9.7.1) Remark. We put a table of possible values of $\ell(P), q(P)$ with the changes (a)–(e) described (cf. (9.5.1) and (9.6.3)).



The enclosed cases correspond to $\ell(P) + q(P) > 1,$ and will be disproved in (9.8).

Proof. (a) (resp. (b)) We can choose (y) in (9.6) such that

$$\phi \equiv y_1^{\ell(P)} y_2 (y_2, y_3, y_4)^2$$

and $y_1 y_3$ (resp. y_4) generates \mathcal{L} at P . Then we set $u = (y_1 y_3, y_4)$ (resp. $(y_4, y_1 y_3)$) and choose a general invariant $\psi \equiv y_1 y_3$ (resp. y_4) mod $(y_2, y_3, y_4)^2$ such that $\text{g.c.d}\{\phi, \psi\} = 1$ and $\text{Sing}\{\phi = \psi = 0\} \subset y_1$ -axis. Thus (9.6.1) is satisfied. Since $y_1 y_3$ (resp. y_4) appears in $\phi_\lambda, (1b.5(ii'''))$ (resp. $(1b.5(ii'''))$) is satisfied and (9.6.2) is satisfied. Let $0 < \lambda \ll 1$. Since $\phi_\lambda = 0$ on $X_\lambda,$ we see

$$(9.7.2) \quad -y_1^{\ell(P)} y_2 / \lambda \equiv y_1 y_3 \quad (\text{resp. } y_4) \quad \text{mod } I_\lambda^2.$$

Thus $y_3 + y_1^{\ell(P)-1} y_2 / \lambda \in I_\lambda^{(2)}$ (resp. $y_4 + y_1^{\ell(P)} y_2 / \lambda \in I_\lambda^{(2)}$), and $\ell(P_\lambda) = 1$ (resp. 0) (cf. (2.16(i))). Hence $\{y_2, y_3\}$ (resp. $\{y_2, y_3\}$) is an ℓ -free ℓ -basis of $\text{gr}^1(\mathcal{O}, I_\lambda)$ at P_λ (cf. (9.5.1)). Using (9.6.4), we see that y_2 is an ℓ -free ℓ -basis of \mathcal{L}_λ at P_λ by (9.7.2), whence $q(P_\lambda) = R(a_2)$.

(c) (resp. (d), (e)). Let $b = R(a_2)$. We can choose (y) in (9.6) such that $\phi \equiv y_1 y_3 (y_2, y_3, y_4)^2$ and $y_1^b y_2$ (resp. $y_4, y_1^b y_2$) generates \mathcal{L} at P . Then we set $u = (y_1^b y_2, y_4)$ (resp. $(y_4, y_1^b y_2), (y_1^b y_2, y_4)$) and choose a general invariant $\psi \equiv y_4$ (resp. y_4, y_4^2) mod $(y_2, y_3, y_4)^3$ such that $\text{g.c.d}\{\phi, \psi\} = 1$ and $\text{Sing}\{\phi = \psi = 0\} \subset y_1$ -axis. Thus (9.6.1) is satisfied. Since $y_1 y_3$ appears in $\phi, (1b.5(ii'''))$ is satisfied and (9.6.2) is satisfied. Let $0 < \lambda \ll 1$. As in the cases (a) and (b), we also see

$$(9.7.3) \quad -y_1 y_3 / \lambda \equiv y_4 \quad (\text{resp. } y_4, 0) \quad \text{mod } I^{\lambda^2}.$$

It is thus similarly seen that $\ell(P_\lambda) = 0$ (resp. $0, 1$), $\{y_2, y_3\}$ (resp. $\{y_2, y_3\}, \{y_2, y_4\}$) is an ℓ -free ℓ -basis of $\text{gr}_C^1 \mathcal{O}$ at P , and $q(P_\lambda) = R(a_2)$ (resp. $1, R(a_2)$). In case (e), we note that ϕ_λ restricted to the fixed axis (y_4 -axis) has a double zero, whence P_λ has axial multiplicity 2 (1a.5). \square

(9.8) **Theorem.** *We have $\ell(P) = q(P) = 1$ and $\ell(P') + q(P') = 1$.*

Proof. By symmetry, it is enough to disprove $\ell(P) + q(P) > 1$. Since it is enough to disprove a nearby extremal nbd of $X \supset C$, we will replace $X \supset C$ with its LG-deformation (9.7) at P or P' for several times. We treat three cases.

(9.8.1) *Case $\ell(P) = 0$ and $q(P) > 1$.* By replacing $X \supset C$ with its LG-deformation at P' (9.7.1) for several times, we get a locally primitive $X \supset C$ with exactly two points P and P' with indices $m, m' (\geq 3)$ (9.6.4) such that $\ell(P') = 0$. Since LG-deformation at P' does not change “nbd” of P (9.6.4), we have $\ell(P) = 0$ and $q(P) > 1$. By definition of $\ell(P)$ and $\ell(P')$, P and P' are ordinary. Hence this contradicts (9.3) and we are done.

(9.8.2) *Case $R(a_2) > 1$.* By replacing $X \supset C$ with its LG-deformation at P (9.7.1) for several times, we get $X \supset C$ with $\ell(P) = 0$ and $q(P) > 1$ as in (9.8.1), whence we are done by (9.8.1).

(9.8.3) *Case $R(a_2) = 1$.* Similarly to the previous case, we may assume that $\ell(P) = q(P) = 1$, P has axial multiplicity 2, and that P' is ordinary, i.e. $\ell(P') = 0$. We have $q(P') = 1$ by (9.8.1).

(9.8.3.1) Since $\text{gr}_C^1 \mathcal{O} = \mathcal{L} \oplus \mathcal{O}(-1)$ (9.5), we have an ℓ -splitting $\text{gr}_C^1 \mathcal{O} = \mathcal{L} \tilde{\oplus} \mathcal{M}$ for some submodule $\mathcal{M} \simeq \mathcal{O}(-1)$ (9.1.7(iii)), which is hence an ℓ -invertible

\mathcal{O}_C -module (9.1.1). We have $ql_C(\mathcal{L}) = P^\sharp + P^\flat$ and $ql_C(\mathcal{M}) = -1 + (m' - a'_2)P^\flat$ (9.5.1) because $ql \deg(\mathcal{M}, P) = R(a_4) = 0$ by $\ell(P) = q(P) = R(a_2) = R(a_3) = 1$ and $ql \deg(\mathcal{M}, P') = R'(a'_2)$ by $\ell(P') = 0 = R'(a'_4)$ and $q(P') = 1 = R'(a'_3)$.

(9.8.3.2) Let $y = (y_1, \dots, y_4)$ be an ℓ -coordinate system and an ℓ -equation ϕ at P such that $C^\sharp = y_1$ -axis, $\phi \equiv y_1 y_3 (y_2, y_3, y_4)^2$ (2.16) and \mathcal{L} is generated by $y_1 y_2$ at P (9.1.1) (or cf. (9.4.5)). Since P has axial multiplicity 2, we see $\phi \equiv y_1 y_3 + g y_4^2 \pmod{(y_2, y_3)^2 + (y_2, y_3) y_4 + y_4^3}$, where g is an invariant unit $\in \mathbb{C}\{y\}$. Thus by replacing y_4 by $g^{\pm 1/2} y_4$, we may further assume

$$\phi \equiv y_1 y_3 + y_4^2 \pmod{(y_2, y_3)^2 + (y_2, y_3) y_4 + (y_4^3)}.$$

(9.8.3.3) Let J be a C -laminal ideal of width 2 such that $J/I^{(2)} = \mathcal{L}$. Then an argument very similar to (9.3.2) or (9.4.6) shows that J is a nested c.i. outside $\{P, P'\}$, that we have equalities $\text{Im } \alpha_J = \text{Ker } \beta_J = \mathcal{L}$ with ℓ -structure and an ℓ -isomorphism $\text{gr}^1(\mathcal{O}, J) \simeq \mathcal{M}$, and that J is $(1, 2)$ -monomializable

at ordinary point P' . We see by (9.8.3.2) that (y_4, y_2, y_3) is a $(1, 2, 2)$ -monomializing ℓ -basis of the second kind for $I \supset J$ at P . Let \mathcal{D} be the ℓ -invertible \mathcal{O}_C -module defined in (8.11.1(iii)) from (y_4, y_2, y_3) , then we have a normalized expression $ql_C \mathcal{D} = P^\sharp$. Hence we have an inequality

$$-l \deg_C \mathcal{M} \leq \frac{1}{2} l \deg_C \mathcal{L} + \frac{1}{2} l \deg_C \mathcal{D}$$

by (8.12(ii)) (with $d = 2, t = 2, s = 1, s' = 0$). This reduces to

$$\frac{a_2'}{m'} \leq \frac{1}{2} \left(\frac{1}{m} + \frac{1}{m'} \right) + \frac{1}{2 \cdot m} = \frac{1}{m} + \frac{1}{2m'} \leq \frac{1}{2}$$

by $m, m' \geq 3$, where we used $l \deg_C \mathcal{M} = -a_2'/m', l \deg_C \mathcal{L} = 1/m + 1/m'$ (9.8.3.1), and $l \deg_C \mathcal{D} = 1/m$. Thus $2a_2' \leq m'$ and we see $2a_2' < m'$ by $m' \geq 3$ and $(a_2', m') = 1$. Also by (8.12), we have ℓ -isomorphisms

$$\begin{aligned} \text{gr}^{2,0}(\mathcal{O}, J) &\simeq \mathcal{L} \quad (d = 2, n = 2, q = 1, r = 0), \\ \text{gr}^{2,1}(\mathcal{O}, J) &\simeq \mathcal{M}^{\otimes 2} \tilde{\otimes} \mathcal{D}, \end{aligned}$$

and an ℓ -exact sequence

$$0 \rightarrow \mathcal{M}^{\otimes 2} \tilde{\otimes} \mathcal{D} \rightarrow \text{gr}_C^0 J \rightarrow \mathcal{L} \rightarrow 0.$$

We note that $ql_C(\mathcal{M}^{\otimes 2} \tilde{\otimes} \mathcal{D}) = 2ql_C(\mathcal{M}) + ql_C(\mathcal{D}) = -1 + P^\sharp + (m' - 2a_2')P^b$ is a normalized expression, and that the sequence is ℓ -split as

$$\text{gr}_C^0 J = (\mathcal{M}^{\otimes 2} \tilde{\otimes} \mathcal{D}) \tilde{\oplus} \mathcal{L}_J$$

for some $\mathcal{L}_J \subset \text{gr}_C^0 J$ ℓ -isomorphic to \mathcal{L} (9.1.8(ii)) by $ql_C(\mathcal{L}) = P^\sharp + P^b$ and $m' > 2a_2'$.

(9.8.3.4) Let K be a C -primary ideal such that $J \supset K \supset F^1 J$ and $K/F^1 J = \mathcal{L}_J$. Then K is a C -laminal ideal of width 3 which is a nested c.i. outside $\{P, P'\}$, and we have ℓ -isomorphisms $\text{Ker } \beta_K = \text{Ker } \beta_J = \mathcal{L}$ and $\text{gr}^1(\mathcal{O}, K) = \text{gr}^1(\mathcal{O}, J) \simeq \mathcal{M}$ by (8.14). Then K is $(1, 3)$ -monomializable at P' (8.15.1) and the $(1, 2, 2)$ -monomializing ℓ -basis (y_4, y_2, y_3) for $I \supset J$ at P lifts to a $(1, 3, 2)$ -monomializing ℓ -basis (denoted by the same) (y_4, y_2, y_3) (by abuse of language) of the second kind for $I \supset K$ at P with the same attached semi-invariant by (8.16). Thus the invertible sheaf \mathcal{D} for K is the same as the one in (9.8.3.3). Then we have an inequality by (8.12(ii))

$$-l \deg_C \mathcal{M} \leq \frac{1}{3} l \deg_C \mathcal{L} + \frac{1}{2} l \deg_C \mathcal{D},$$

which reduces to

$$(9.8.3.5) \quad \frac{1}{m} < \frac{a_2'}{m'} \leq \frac{1}{3} \left(\frac{1}{m} + \frac{1}{m'} \right) + \frac{1}{2 \cdot m} = \frac{5}{6 \cdot m} + \frac{1}{3m'},$$

where the first inequality follows from $w_p(0) = 1/m < 1 - w_{p'}(0) = a_2'/m'$ (2.3.3) and (4.9(i)). Then we treat two cases.

(9.8.3.6) *Case* $m = 3$. By (9.8.3.5), we have $1/3 < 5/18 + 1/3m'$, whence $m' = 3, 4, 5$. If $m' = 3$, then $a'_2 = 2$ by (9.8.3.5), which contradicts $2a'_2 < m'$ proved in (9.8.3.3). We see that (9.8.3.5) reduces to $4/3 < a'_2 \leq 5 \cdot 4/(6 \cdot 3) + 1/3 = 13/9$ if $m' = 4$, and to $5/3 < a'_2 \leq 5 \cdot 5/(6 \cdot 3) + 1/3 = 31/18$ if $m' = 5$. These are both impossible, and the case $m = 3$ cannot occur.

(9.8.3.7) *Case* $m \geq 4$. Since

$$\frac{5}{6 \cdot m} + \frac{1}{3m'} \leq \frac{5}{6 \cdot 4} + \frac{1}{3 \cdot 3} = \frac{23}{72} < \frac{1}{3},$$

we see that $3a'_2 < m'$ by (9.8.3.5). Then we continue as in (9.8.3.3). By (8.12(ii)), we have ℓ -isomorphisms

$$\begin{aligned} \text{gr}^{3,0}(\mathcal{O}, K) &\simeq \mathcal{L} \quad (d = 3, n = 3, q = 1, r = 0), \\ \text{gr}^{3,1}(\mathcal{O}, K) &\simeq \mathcal{M}^{\otimes 3} \tilde{\otimes} \mathcal{D}, \end{aligned}$$

and an ℓ -exact sequence

$$0 \rightarrow \mathcal{M}^{\otimes 3} \tilde{\otimes} \mathcal{D} \rightarrow \text{gr}_C^0 K \rightarrow \mathcal{L} \rightarrow 0.$$

Since $ql_C(\mathcal{M}^{\otimes 3} \tilde{\otimes} \mathcal{D}) = 3ql_C(\mathcal{M}) + ql_C(\mathcal{D}) = -1 + P^\sharp + (m' - 3a'_2)P^b$ is a normalized expression, the sequence is also ℓ -split as $\text{gr}_C^0 K = \mathcal{M}^{\otimes 3} \tilde{\otimes} \mathcal{D} \tilde{\oplus} \mathcal{L}_K$ for some submodule \mathcal{L}_K ℓ -isomorphic to \mathcal{L} (9.1.8). Then as in (9.8.3.4), we can define a C -laminal ideal H of width 4 such that $K \supset H \supset F^1 K$ and $H/F^1 K = \mathcal{L}_K$. We see that H is a nested c.i. outside $\{P, P'\}$; H is $(1, 4)$ -monomializable at P' (8.15); (y_4, y_2, y_3) lifts to a $(1, 4, 2)$ -monomializing ℓ -basis of the second kind for $I \supset H$ at P with the same attached semi-invariant; we have ℓ -isomorphisms $\text{gr}^1(\mathcal{O}, H) \simeq \mathcal{M}$ and $\text{Ker } \beta_H \simeq \mathcal{L}$ and the same \mathcal{D} . Thus we have

$$\frac{1}{m} < \frac{a'_2}{m'} \leq \frac{1}{4} \ell \deg_C \mathcal{L} + \frac{1}{2} \ell \deg_C \mathcal{D} = \frac{1}{4} \left(\frac{1}{m} + \frac{1}{m'} \right) + \frac{1}{2 \cdot m} = \frac{3}{4 \cdot m} + \frac{1}{4m'}$$

just as (9.8.3.5). Then by $1/m < 3/(4m) + 1/(4m')$, we see $1/m < 1/m'$, whence $a'_2/m' \leq 3/(4m) + 1/(4m') < 1/m'$. Thus $a'_2 = 0$, which is a contradiction. Thus we must have $\ell(P) + q(P) = 1$ and (9.8) is proved. \square

(9.9) Let the notation and assumptions be as in (9.1). Then $\text{gr}_C^1 \mathcal{O} = \mathcal{L} \oplus \mathcal{O}(-1)$, and by (9.1.1) there is a submodule $\mathcal{M} \simeq \mathcal{O}(-1)$ such that $\text{gr}_C^1 \mathcal{O} = \mathcal{L} \tilde{\oplus} \mathcal{M}$. Then by (9.8), $ql \deg(\mathcal{M}, P) = R(a_2)$ and $ql \deg(\mathcal{M}, P') = R'(a'_2)$ (9.5.1). In particular $ql_C(\mathcal{M}) = -1 + (m - a_2)P^\sharp + (m' - a'_2)P^b$. Thus by (2.10) and $\text{gr}_C^0 \omega \simeq \mathcal{O}(-1)$, we have

(9.9.1) **Proposition.** *There is an ℓ -isomorphism $\mathcal{M} \simeq \text{gr}_C^0 \omega$.*

Indeed $\mathcal{M} \xrightarrow{\sim} \text{gr}_C^0 \omega$ is an ℓ -isomorphism because

$$ql \deg(\mathcal{M}, Q) = ql \deg(\text{gr}_C^0 \omega, Q) \quad \text{for } Q = P, P'$$

(cf. comment after $(\tilde{\oplus}.2)$ in (8.8.4)).

(9.9.2) We note that ω_X^* has a structure of an ℓ -invertible \mathcal{O}_X -module $\omega_X^{\tilde{\otimes}(-1)}$. Hence $\text{gr}_C^0(\omega_X^*)$ is ℓ -isomorphic to

$$\mathcal{O}_C \tilde{\otimes} \omega_X^{\tilde{\otimes}(-1)} \simeq (\mathcal{O}_C \tilde{\otimes} \omega_X^*)^{\tilde{\otimes}(-1)} \simeq (\text{gr}_C^0 \omega)^{\tilde{\otimes}(-1)}$$

by (8.8.6(i)) and $(\tilde{\otimes}\tilde{\otimes}.3)$ in (8.8.4), whence we have a normalized expression $ql_C(\text{gr}_C^0 \omega_X^*) = -ql_C(\text{gr}_C^0 \omega) = -ql_C(\mathcal{M}) = -1 + a_2 P^\# + a_2' P^b$. We also have an ℓ -isomorphism

$$\text{gr}_C^1(\omega_X^*) \simeq (\text{gr}_C^1 \mathcal{O}) \tilde{\otimes} \omega_X^* \simeq \mathcal{L} \tilde{\otimes} \mathcal{M}^{\tilde{\otimes}(-1)} \tilde{\oplus} \mathcal{O}_C,$$

by (8.8.6(i)) and (8.8.4). Then

(9.9.3) **Lemma.** *We have an isomorphism*

$$H^0(\omega_X^*/F^2 \omega_X^*) \simeq H^0(F^1 \omega_X^*/F^2 \omega_X^*) \simeq H^0(\mathcal{L} \tilde{\otimes} \mathcal{M}^{\tilde{\otimes}(-1)}) \oplus H^0(\mathcal{O}_C),$$

and let

$$h: H^0(\omega_X^*) \rightarrow H^0(\omega_X^*/F^2 \omega_X^*) \rightarrow H^0(\mathcal{O}_C) \simeq \mathbb{C}$$

be the projection to the factor $H^0(\mathcal{O}_C)$. Then every global section $s \in H^0(\omega_X^*)$ such that $h(s) \neq 0$ defines a zero locus $(s)_0$ which is a normal surface with only rational double points. (To be precise, $(s)_0$ is smooth outside $\{P, P'\}$, has A_{mw-1} -type (resp. $A_{m'w'-1}$ -type) singularity at P (resp. P'), where w (resp. w') is the axial multiplicity of (X, P) (resp. (X, P'))).

Proof. The first assertion follows from $H^0(\text{gr}_C^0(\omega_X^*)) \simeq H^0(\mathcal{O}_C(-1)) = 0$ (9.9.2). Let $s \in H^0(\omega_X^*)$ be such that $h(s) \neq 0$. Then $s \in H^0(F^1 \omega_X^*)$ by $H^0(\text{gr}_C^0 \omega_X^*) = 0$.

If $Q \neq P, P'$, then the image of $s(Q)$ in $(\text{gr}_C^1 \omega_X^*) \otimes \mathbb{C}(Q) = \omega_X^* \otimes I_C/I_C^2 \otimes \mathbb{C}(Q)$ is nonzero because $h(s)$ generates the subspace $\mathcal{O}_C \otimes \mathbb{C}(Q) \simeq \mathbb{C}$. Hence $(s)_0$ is smooth at Q .

By symmetry we need to consider $(s)_0$ only at $P^\#$. By (9.5.1) and $\ell(P) = 0, 1$, we can take an ℓ -coordinate system y_1, \dots, y_4 such that $C^\# = y_1$ -axis, $\text{wt } y_i \equiv a_i \pmod{m}$ ($i \in [1, 4]$), y_2 is an ℓ -free ℓ -basis of \mathcal{M} at P , y_3 (resp. y_4) is an ℓ -free ℓ -basis of \mathcal{L} at P , and $\phi \equiv y_4$ (resp. $y_1 y_3$) mod $(y_2, y_3, y_4)^2$ if $\ell(P) = 0$ (resp. 1). Let θ be an ℓ -free ℓ -basis of ω_X^* at P . By the assumption $h(s) \neq 0$, $\mathcal{O}_{C^\#}$ -component of s_p in the decomposition at $P^\#$

$$\text{gr}_C^1(\omega_X^*)^\# = (\mathcal{L} \tilde{\otimes} \mathcal{M}^{\tilde{\otimes}(-1)} \tilde{\otimes} \text{gr}_C^0 \omega)^\# \oplus \mathcal{O}_{C^\#} = \mathcal{O}_{C^\#}(y_3, y_4)\theta \oplus \mathcal{O}_{C^\#}y_2\theta$$

generates $\mathcal{O}_{C^\#}y_2\theta$. Then $s = f\theta$ for some semi-invariant $f \in (y_2, y_3, y_4)\mathbb{C}\{y\}$ such that $\text{wt } f \equiv a_2 \equiv \text{wt } y_2 \pmod{m}$ and $\partial f/\partial y_2(0) \neq 0$. Thus there exist an invariant unit $u \in \mathbb{C}\{y\}$ and a semi-invariant $g \in (y_3, y_4)\mathbb{C}\{y_1, y_3, y_4\}$ such that $\text{wt } g \equiv a_2 \pmod{m}$ and $f = u \cdot (y_2 - g)$ (Weierstrass Preparation Theorem).

The zero locus $(s)_0^\sharp$ of s in the canonical cover (X^\sharp, P^\sharp) is thus isomorphic to a hypersurface in $(y_1 y_3 y_4$ -space, 0) with defining equation $\psi(y_1, y_3, y_4) = \phi(y_1, g, y_3, y_4) \in \mathbb{C}\{y_1, y_3, y_4\}$. By the congruence relation on ϕ , we have $\psi \equiv y_4$ (resp. $y_1 y_3$) mod $(y_3, y_4)^2$.

We will see that $(s)_0^\sharp$ is a A_{w-1} -type singularity. Indeed if $\psi \equiv y_4$ then the axial multiplicity w is 1 and $(s)_0^\sharp$ is smooth. We consider the case $\psi \equiv y_1 y_3$. Since $\partial\psi/\partial y_1 \equiv y_3 \pmod{(y_3, y_4)^2}$ has $\text{wt} \equiv a_3 \not\equiv 0 \pmod{m}$ and $\partial\psi/\partial y_3 \equiv y_1 \pmod{(y_3, y_4)}$ has $\text{wt} \equiv a_1 \not\equiv 0 \pmod{m}$, we see $\partial\psi/\partial y_1 \equiv y_3 \pmod{(y_3^2, y_3 y_4, y_1 y_4^2)}$ and $\partial\psi/\partial y_3 \equiv y_1 \pmod{(y_3, y_1 y_4)}$ because $\text{wt } y_4 \equiv 0 \pmod{m}$. Hence $\partial\psi/\partial y_1$ and $\partial\psi/\partial y_3$ generate the ideal (y_1, y_3) in a nbd of origin by Nakayama's lemma, and they define y_4 -axis. On the other hand, by $\text{wt } g \equiv a_2 \pmod{m}$, we see $g(0, 0, y_4) = 0$ and $\psi(0, 0, y_4) = \phi(0, 0, 0, y_4) = c y_4^w + \dots$, where w is the axial multiplicity of (X, P) (1a.5). Thus $(s)_0^\sharp$ is an isolated singularity, and it is easy to see $(s)_0^\sharp$ is μ_m -isomorphic to a A_{w-1} -singularity $y_1 y_3 = y_4^w$ (by analytic approximation).

Hence $((s)_0, P) = (s)_0^\sharp/\mu_m$ is isomorphic to a A_{mw-1} -type singularity: $z_1 z_3 = z_4^{mw}$, where $z_1 = y_1^m$, $z_3 = y_3^m$, $z_4 = y_4$. \square

(9.9.4) **Theorem.** *The homomorphism*

$$h: H^0(\omega_X^*) \rightarrow H^0(\mathcal{O}_C) = \mathbb{C}$$

defined in (9.9.3) is a surjection.

We will prove the theorem in several steps.

(9.9.5) Let J_2 be the C -laminal ideal of width 2 such that $J_2/I^{(2)} = \mathcal{M}$. Since $J_2, I^{(2)}, \mathcal{M}$ are saturated subquotients of \mathcal{O}_X , the surjection $J_2 \rightarrow \mathcal{M}$ induces a surjection of canonical liftings at P (and also at P') by (8.8.1). Thus the homomorphism α_J for $J = J_2$ (8.2.2) is an ℓ -surjection

$$\alpha_2: \text{gr}_C^0 J_2 \rightarrow \mathcal{M}.$$

We will inductively define C -laminal ideals J_n of width n .

(9.9.6) **Lemma-Definition.** *Let J_n be a C -laminal ideal of width $n \geq 2$ such that the homomorphism α_J for $J = J_n$ (8.2.2) is an ℓ -surjection*

$$\alpha_n: \text{gr}_C^0 J_n \rightarrow \mathcal{M}.$$

Then the ℓ -exact sequence (8.8.2(v))

$$0 \rightarrow \text{Ker } \alpha_n \rightarrow \text{gr}_C^0 J_n \rightarrow \mathcal{M} \rightarrow 0$$

is ℓ -split, that is $\text{gr}_C^0 J_n = (\text{Ker } \alpha_n) \tilde{\oplus} \mathcal{M}_n$ for some submodule \mathcal{M}_n of $\text{gr}_C^0 J_n$ such that the induced map $\mathcal{M}_n \rightarrow \mathcal{M}$ is an ℓ -isomorphism. Let J_{n+1} be the C -primary ideal such that $J_n \supset J_{n+1} \supset F^1 J_n$ and $J_{n+1}/F^1 J_n = \mathcal{M}_n$. Then the

natural ℓ -surjections $\text{gr}_C^0 J_{n+1} \twoheadrightarrow \mathcal{M}_n \subset \text{gr}_C^0 J_n$ (cf. (8.8.1)) and $\mathcal{M}_n \xrightarrow{\sim} \mathcal{M}$ induce an ℓ -surjection

$$\alpha_{n+1}: \text{gr}_C^0 J_{n+1} \rightarrow \mathcal{M}.$$

In particular, J_{n+1} is a C -laminal ideal, hence of width $n + 1$.

Proof. We only need to prove ℓ -splitness by induction on n . By inductive construction of J_n , we have $\mathcal{M} = \text{Ker } \beta_n$ and $\mathcal{L} = \text{gr}^1(\mathcal{O}, J_n)$ (8.2), where $\beta_n = \beta_J$ for $J = J_n$ (8.2.2). Hence by $E(J_n)$ (8.2.2) and (8.4), we have an injection $\mathcal{O}_C \simeq \mathcal{L}^{\otimes d} \rightarrow \text{Ker } \alpha_n$. Thus $\text{deg}_C \text{Ker } \alpha_n > \text{deg } \mathcal{M}$ and the sequence is ℓ -split (9.1.8(ii)). \square

(9.9.6.1) Let us fix a number $e \in \mathbb{N}$ which is divisible by axial multiplicities of (X, P) and (X, P') . Let $C(n) \subset X$ be the closed subscheme defined by $F^{ne}(\mathcal{O}, J_e)$. Then we have

(9.9.7) **Key Lemma.** For an arbitrary $n \geq 2$, $F^{ne}(\mathcal{O}, J_e)$ is equal to the symbolic n th power $J_e^{(n)}$, $J_{ne}/F^{ne}(\mathcal{O}, J_e)$ is an ℓ -invertible $\mathcal{O}_{C(n-1)}$ -module, and we have an ℓ -isomorphism

$$\theta(n): F^{ne}(\omega_X^*, J_{ne})/F^{ne}(\omega_X^*, J_e) \xrightarrow{\sim} \omega_X^* \tilde{\otimes} (J_{ne}/F^{ne}(\mathcal{O}, J_e)) \xrightarrow{\sim} \mathcal{O}_{C(n-1)}$$

and inclusion $F^{nemm'}(\omega_X^*, J_e) \subset J_e^n \cdot \omega_X^*$.

Proof. Since ω_X^* is an ℓ -invertible \mathcal{O}_X -module, we have natural ℓ -isomorphisms by (8.8.6(i))

$$\omega_X^* \tilde{\otimes} J_{ne} \simeq F^{ne}(\omega_X^*, J_{ne}) \quad \text{and} \quad \omega_X^* \tilde{\otimes} F^{ne}(\mathcal{O}, J_e) \simeq F^{ne}(\omega_X^*, J_e)$$

inducing an ℓ -isomorphism

$$F^{ne}(\omega_X^*, J_{ne})/F^{ne}(\omega_X^*, J_e) \simeq \omega_X^* \tilde{\otimes} (J_{ne}/F^{ne}(\mathcal{O}, J_e))$$

by ($\tilde{\otimes}$.5) in (8.8.4), provided that $J_{ne} \supset F^{ne}(\mathcal{O}, J_e)$ which is to be proved later.

(9.9.8) Let us first assume that $\mathcal{F} = J_{ne}/F^{ne}(\mathcal{O}, J_e)$ is an ℓ -invertible $\mathcal{O}_{C(n-1)}$ -module and $F^{ne}(\mathcal{O}, J_e) = J_e^{(n)}$. Then $F^{ne}(\mathcal{O}, J_e) \subset I^{(2)}$ and α_{ne} gives the identification $J_{ne} + I^{(2)}/I^{(2)} = \mathcal{M}$, whence the identity map induces the surjection $\mathcal{F} \twoheadrightarrow \mathcal{M}$ of saturated subquotients and it is an ℓ -surjection (8.8.1). Thus by ($\tilde{\otimes}$.0) and ($\tilde{\otimes}$.4) in (8.8.4), we have an ℓ -surjection

$$\sigma: \omega_X^* \tilde{\otimes} \mathcal{F} \tilde{\otimes} \mathcal{O}_C \rightarrow (\omega_X^* \tilde{\otimes} \mathcal{M} \simeq) \mathcal{O}_C,$$

where the last ℓ -isomorphism is due to (9.9.1). Then it is an ℓ -isomorphism (8.8.2(v)) by $\text{Ker } \sigma = 0$. Hence the ℓ -invertible $\mathcal{O}_{C(n-1)}$ -module $\omega_X^* \tilde{\otimes} \mathcal{F}$ has an ℓ -free ℓ -basis with $\text{wt} \equiv 0 \pmod m$ (resp. m'), hence ℓ -trivial at P (resp. P'). Then we see that $\omega_X^* \tilde{\otimes} \mathcal{F}$ is an invertible $\mathcal{O}_{C(n-1)}$ -module such that

$$(\omega_X^* \tilde{\otimes} \mathcal{F}) \otimes \mathcal{O}_C \simeq ((\omega_X^* \tilde{\otimes} \mathcal{F}) \tilde{\otimes} \mathcal{O}_C \simeq) \mathcal{O}_C$$

by (8.8) in (8.8.4). Since $H^1(\mathcal{O}_{C(n-1)}) = 0$, we have an isomorphism $\theta(n): \omega_X^* \tilde{\otimes} \mathcal{F} \xrightarrow{\sim} \mathcal{O}_{C(n-1)}$ of invertible $\mathcal{O}_{C(n-1)}$ -modules. Since $\omega_X^* \tilde{\otimes} \mathcal{F}$ and $\mathcal{O}_{C(n-1)}$ are ℓ -trivial at P and P' , $\theta(n)$ is an ℓ -isomorphism.

(9.9.9) It remains to show that $F^{ne}(\mathcal{O}, J_e) = J_e^{(n)}$, $J_{ne}/F^{ne}(\mathcal{O}, J_e)$ is an ℓ -invertible $\mathcal{O}_{C(n-1)}$ -module, and the last inclusion of (9.9.7) at each point $Q \in C$. Let $Q \in C - \{P, P'\}$. By the surjectivity of α_{ne} , let $\{u_1, u_2\}$ be a basis of $\text{gr}_C^1 = I/I^2$ at P such that $\mathcal{O}_{C,Q}u_1 = \mathcal{M}_Q$ and $u_1 \in J_{ne}$. Then by (8.3(i)), we see that $J_e = (u_1, u_2^e)$, $J_{ne} = (u_1, u_2^{ne})$, and $F^{ne}(\mathcal{O}, J_e) = J_e^n$ near Q . In particular, we have the global equality $F^{ne}(\mathcal{O}, J_e) = J_e^{(n)}$ since both are C -primary. Furthermore, by $u_2^{ne} \in F^{ne}(\mathcal{O}, J_e)$, we see that $J_{ne}/F^{ne}(\mathcal{O}, J_e) \simeq \mathcal{O}/\{(u_1, u_2^e)^n: u_1\} \simeq \mathcal{O}/J_e^{n-1}$. Thus the sheaf is invertible and the inclusion at Q is obvious by $F^{ne}(\mathcal{O}, J_e) = J_e^n$.

(9.9.10) By symmetry, it is enough to consider at P . Let $y = (y_1, \dots, y_4)$ and ϕ be an ℓ -coordinate system and an ℓ -equation at P such that $C^\sharp = y_1$ -axis, $\text{wt } y_i \equiv a_i \pmod{m}$ ($i \in [1, 4]$), $\phi \equiv y_4 \pmod{(y_2, y_3, y_4)^2}$ if $\ell(P) = 0$ (resp. 1), $y_2 \in J_{ne}^\sharp$, where J_{ne}^\sharp is the canonical lifting of J_{ne} at P . We note that we used ℓ -surjection α_{ne} for the last condition. We treat two cases.

(9.9.11) *Case $\ell(P) = 0$.* Again by (8.3(i)), (y_3, y_2) is a $(1, e)$ -monomializing (resp. $(1, ne)$ -monomializing) ℓ -basis for $I \supset J_e$ (resp. $I \supset J_{ne}$) at P . Thus as in (9.9.9), the canonical liftings are calculated as $J_e^\sharp = (y_2, y_3^e)$, $J_{ne}^\sharp = (y_2, y_3^{ne})$, and $F^{ne}(\mathcal{O}, J_e)^\sharp = (J_e^\sharp)^n$. Hence $J_{ne}^\sharp/F^{ne}(\mathcal{O}, J_e)^\sharp \simeq \mathcal{O}/J_3^{\sharp n-1}$ is similarly checked. By $F^{ne}(\mathcal{O}, J_e)^\sharp = J_e^{\sharp n}$, the inclusion follows from the easy

(9.9.11.1) **Sublemma.** *If $J^\sharp, K^\sharp \subset \mathcal{O}_{X^\sharp, P^\sharp}$ are ideals generated by semi-invariants and $k \in \mathbb{Z}_m$, then*

$$(J^{\sharp m} K^\sharp)_{\{k\}} \subset (J_{\{0\}}^\sharp) \cdot K_{\{k\}}^\sharp \quad (\text{cf. (2.5)}).$$

Indeed for arbitrary semi-invariants $u_1, \dots, u_m \in J^\sharp$ and $v \in K^\sharp$ such that $\text{wt } u_1 + \dots + \text{wt } u_m + \text{wt } v \equiv k \pmod{m}$, there exist $a, b \in [0, m]$ such that $a < b$ and $\sum_{i \leq a} \text{wt } u_i \equiv \sum_{i \leq b} \text{wt } u_i \pmod{m}$, whence $\sum_{a < i \leq b} \text{wt } u_i \equiv 0 \pmod{m}$. Then (9.9.11.1) follows from

$$u_1 \cdots u_m v = \left(\prod_{a < i \leq b} u_i \right) \cdot \left(\prod_{i \leq a} u_i \cdot \prod_{b < i} u_i \cdot v \right) \in J_{\{0\}}^\sharp \cdot K_{\{k\}}^\sharp.$$

(9.9.12) *Case $\ell(P) = 1$.* By $\ell(P) > 0$, X^\sharp is singular and we see that the axial multiplicity w satisfies $w > 1$. Then we may rewrite $\phi \equiv y_1 y_3 \pmod{(y_2, y_3, y_4)^2}$ as

$$(g_1 y_1 + g_3) y_3 = y_2 g_2 + y_4^w,$$

Hence the proof of (0.4.5) is completed and thus the Flip Theorems (0.4.1) and (0.2.5) are proved and the results mentioned in §0 follow.

10. A SLIGHT GENERALIZATION OF (0.4.5) AND COMMENTS ON (1a.7)

In this section, we state a slight generalization (10.2) of (0.4.5) and then make comments on (1a.7).

The proof of (10.2) is basically the same as that of (0.4.5). Therefore, we will only indicate changes to be made together with simple explanations.

(10.1) Let k be an algebraically closed field of characteristic 0. Let C be a connected reduced projective curve over k and let $X \supset C$ be a 3-dimensional normal formal scheme along C which has only terminal singularities. We consider the following condition:

$$(*) \quad -K_X \text{ is ample on } C \text{ and } H^1(X, \mathcal{O}_X(L)) = 0 \text{ for all Weil divisors } L \text{ on } X \text{ such that } nL \sim 0 \text{ or } nK_X \text{ for some } n > 0.$$

We note that $(*)$ is satisfied by the formal schemes associated to extremal nbds by (1.2.1) and (1.12). The following is the slight generalization of (0.4.5).

(10.2) **Proposition.** *Let $X \supset C$ be as in (10.1) satisfying the condition $(*)$ and such that C is irreducible. Then one of the following on the linear system $| -aK_X |$ ($a = 1$, or 2) holds.*

- (i) $| -K_X |$ has a member D with only rational double points, or
- (ii) $| -2K_X |$ has a member D so that the double cover Z of X with branch locus D has only canonical singularities.

For instance, this applies to a divisorial contraction of a 3-fold contracting an irreducible divisor to a curve.

(10.3) Except for the obvious changes due to the fact that $X \supset C$ is a formal scheme along C rather than the germ of an analytic space along C , most arguments use only the condition $(*)$. The only nontrivial changes are the following.

(10.4) **Changes in §1.** Let $X \supset C$ be as in (10.1) with the condition $(*)$. Since $H^1(X, \mathcal{O}_X) = 0$ $(*)$, the description of $\text{Pic } X$ (1.3) and $\text{Cl}^{\text{sc}} X$ (1.9) for irreducible C are also proved by the same argument. By $(*)$, the splitting cover $X' \supset C'$ (1.12) satisfies the condition:

$$(**) \quad -K_{X'} \text{ is ample on } C' \text{ and } H^1(X', \mathcal{O}_{X'}) = H^1(X', \omega_{X'}) = 0.$$

Since (1.13)–(1.16) need only the property $(**)$, the results corresponding to (1.13)–(1.16) hold, which are the results needed in later sections.

(10.5) **Changes in §1b.** A “nearby extremal nbd” for a parameter λ should be replaced by the generic deformation $X_\lambda \supset C_\lambda$ over $\overline{k((\lambda))}$ which is to be defined in a similar way. It is easy to see that $X_\lambda \supset C_\lambda$ satisfies the condition $(*)$.

(10.6) **Changes to be made in §§6 and 7.** The only place in this paper which uses the property that the total contraction $X \rightarrow Y$ is an isomorphism outside C is (6.2(i)) which asserts that X has index > 1 . Therefore, for our formal scheme $X \supset C \simeq \mathbb{P}^1$ with $(*)$, (6.2(i)) should be changed to “ X has at most three singular points in C ” and thus (6.7) should have an extra case (6.7.0) where X is Gorenstein.

Hence §7 should treat the extra case (6.7.0), which is very easy: $|-K_X|$ has a smooth member by (1.3) since $\mathcal{O}_C(-K_X) \simeq \mathcal{O}(-1)$.

(10.7) It was after this paper had been written that the author learned that [KSB] contained the complete classification of 3-fold terminal singularities [KSB, Theorem 6.5] and settled

(10.8) **Theorem** (Kollár, Shepherd-Barron). *Small deformations of 3-fold terminal singularities are terminal.*

(10.9) If we use it, the conditions (1b.3(ii)) and hence (1b.5) became unnecessary, and therefore the arguments for (1b.3), (4.7), (4.12), and (9.6) became simpler but not much. Therefore the revision at this point does not seem worth the trouble, and we decided not to revise the paper.

APPENDIX A. SUMMARY OF LOCAL CLASSIFICATION OF $X \supset C \ni P$

In this appendix, we will summarize the classification of the local structure of an extremal nbd $X \supset C \simeq \mathbb{P}^1$ at a singular point P .

(A.1) Let $X \supset C \simeq \mathbb{P}^1$ be an extremal nbd and $P \in C$ a singular point of X of index m . Let $\pi^\sharp: (X^\sharp, P^\sharp) \rightarrow (X, P)$ be the canonical cover of the germ, and let $C^\sharp = \pi^{\sharp-1}(C)_{\text{red}} \subset X^\sharp$. We note that (X^\sharp, P^\sharp) is at most an isolated cDV singularity (hence embeddable in $(\mathbb{C}^4, 0)$) with μ_m -action which is free on $X^\sharp - P^\sharp$ and that π^\sharp is the μ_m -quotient morphism.

There exists a set $(y) = (y_1, y_2, y_3, y_4)$ of μ_m -semi-invariant convergent power series giving a μ_m -equivariant embedding of (X^\sharp, P^\sharp) into $(\mathbb{C}^4, 0)$.

(A.2) **Notation.** Given a generating character $\chi \in \text{Hom}(\mu_m, \mathbb{C}^*) \simeq \mathbb{Z}_m$, we write $\text{wt } f \equiv a$ for a μ_m -semi-invariant f if $g(f) = \chi(g)^a f$ for all $g \in \mu_m$; $\text{wt}(f_1, \dots, f_r) \equiv (a_1, \dots, a_r)$ for μ_m -semi-invariant f_1, \dots, f_r if $\text{wt } f_i \equiv a_i$ for all i .

(A.3) **Summary.** By choosing the above (y) and χ properly, we have the following case-by-case description of $\mathbb{C}^4 \supset X^\sharp \supset C^\sharp$, in which g_i 's are μ_m -semi-invariant convergent power series in y with the specified weights such that X^\sharp has at most an isolated cDV point at the origin. Each result is in (4.2)–(4.4) unless otherwise mentioned.

(IA) There exist positive integers a_1, a_2, α, β such that $(a_1 a_2, m) = 1, \alpha a_1 + \beta a_2 = m, \text{wt}(y) \equiv (a_1, a_2, -a_1, 0),$

$$C^\sharp: y_1^{a_2} - y_2^{a_1} = y_3 = y_4 = 0,$$

$$X^\sharp: g_2 \cdot (y_1^{a_2} - y_2^{a_1}) + g_3 \cdot y_3 + g_4 \cdot y_4 = 0,$$

where $\text{wt}(g_2, g_3, g_4) \equiv (-a_1 a_2, a_1, 0).$

(IB) This does not occur by (6.3).

(IC) $\text{wt}(y) \equiv (2, m - 2, 0, 1)$ and m is an odd integer $\geq 5,$

$$C^\sharp: y_1^{m-2} - y_2^2 = y_3 = y_4 = 0,$$

$$X^\sharp: g_2 \cdot (y_1^{m-2} - y_2^2) + y_3 + g_4 \cdot y_4 = 0,$$

where $\text{wt}(g_2, g_4) \equiv (4, -1).$

In this case, (y_1, y_2, y_4) induces an embedding $(X^\sharp, 0) \simeq (\mathbb{C}^3, 0)$ in which C^\sharp is given by $y_1^{m-2} - y_2^2 = y_4 = 0.$ The assertion on $\text{wt}(y)$ is a combination of (4.9.(i)), (5.5(i)) and (6.5).

(IIA) $\text{wt}(y) \equiv (1, 1, 3, 2), m = 4,$

$$C^\sharp: y_1 - y_2 = y_3 = y_4 = 0,$$

$$X^\sharp: g_2 \cdot (y_1 - y_2) + g_3 \cdot y_3 + g_4 \cdot y_4 = 0,$$

where $\text{wt}(g_2, g_3, g_4) \equiv (1, 3, 0), g_2, g_3 \notin (y)^2, g_4(0) = 0.$

(IIB) $\text{wt}(y) \equiv (3, 2, 1, 1), m = 4,$

$$C^\sharp: y_1^2 - y_2^3 = y_3 = y_4 = 0,$$

$$X^\sharp: y_1^2 - y_2^3 + g_3 \cdot y_3 + g_4 \cdot y_4 = 0,$$

where $\text{wt}(g_3, g_4) \equiv (1, 1), g_3 \cdot y_3 + g_4 \cdot y_4 \notin (y)^3.$

(III) $m = 1,$

$$C^\sharp: y_1 - y_2 = y_3 = y_4 = 0,$$

$$X^\sharp: y_1 \cdot (y_1 - y_2) + g_3 \cdot y_3 + g_4 \cdot y_4 = 0,$$

where $g_3(0) = g_4(0) = 0.$

This will be proved at the end of (A.3).

(IA[∨]) There exist positive integers $a_1, a_2, w_1, w_2, \alpha, \beta, s, \bar{m}$ such that $s, \bar{m} > 1, s\bar{m} = m, a_2 \leq \bar{m}/2, \alpha a_1 + \beta a_2 = \bar{m}, \alpha w_1 + \beta w_2 \equiv 0 (m), w_i \equiv a_i (\bar{m})$ for $i = 1, 2, (w_1 w_2, m) = 1, \text{wt}(y) \equiv (w_1, w_2, -w_1, 0),$

$$C^\sharp: y_1^{a_2 s} - y_2^{a_1 s} = y_3 = y_4 = 0,$$

$$X^\sharp: g_2 \cdot (y_1^{a_2 s} - y_2^{a_1 s}) + g_3 \cdot y_3 + g_4 \cdot y_4 = 0,$$

where $\text{wt}(g_2, g_3, g_4) \equiv (-s a_1 a_2, a_1, 0);$ and $\partial g_3 / \partial y_1(0) \neq 0$ or $g_4 = 1.$

(IC[∨]) This does not occur by (6.1).

(II[∨]) $\text{wt}(y) \equiv (1, 3, 3, 2), m = 4,$

$$C^\sharp: y_1^2 - y_2^2 = y_3 = y_4 = 0,$$

$$X^\sharp: y_1^2 - y_2^2 + g_3 \cdot y_3 + g_4 \cdot y_4 = 0,$$

where $\text{wt}(g_3, g_4) \equiv (3, 0)$ and $g_4(0) = 0$.

We note that the μ_m -action on $X^\sharp - \{0\}$ is free in all the above cases by the smoothness of $X^\sharp - \{0\}$.

Proof for case (III). The assertion on the equation of X^\sharp is equivalent to $i_p(1) = 1$ (2.16(ii)). If $i_p(1) > 1$, then by a separating L -deformation (4.7) one obtains an extremal nbd $X' \supset C'$ with two ordinary double points Q, R on C' . Since $w_Q^*(1) = w_R^*(1) = 1$ (4.9(ii)), X' has another singular point on C' by $\sum_{x \in C'} w_x^*(1) \leq 1$ (2.3.3). It contradicts (6.2(ii)). \square

(A.4) *Remarks.* (i) For the simplicity of the formulation, (y) above is different from (x) which is used in (4.2). Our y_i is x_i in (4.2) for $i = 1, 2$ and it is F_i in (4.3) for $i = 3, 4$,

(ii) we note that $(X^\sharp, 0)$ is smooth or a cA type point (1a.2), except in the case (IA),

(iii) when $X \supset C$ is as in (10.2) and not an extremal nbd, we still have the same local results (cf. §10, especially (10.6)), and

(iv) we note that a_1, a_2 in (IA[∨]) need not be prime to m , as in an example of $X \supset C \ni P$ for which $a_1 = 3, a_2 = 2, w_1 = 3 + 41 \cdot 2 = 85, w_2 = 2 + 41 \cdot 3 = 125, \alpha = 13, \beta = 1, s = 6, \bar{m} = 41, m = 41 \cdot 6 = 246$.

APPENDIX B. POSSIBLE SINGULARITIES ON AN EXTREMAL NBD

In this appendix, we make a list of possible singularities of an extremal nbd as a summary.

(B.1) **Summary.** Let $X \supset C \simeq \mathbb{P}^1$ be an extremal nbd. The following is the list of all the possible singularities of X on C .

Case (1) (treated in (6.7.1) and (7.3.1)):

- (a) a type (IA) point P and at most one type (III) point Q ,
- (b) a type (IIA) point P and at most one type (III) point Q ,
- (c) a type (IA[∨]) point P and at most one type (III) point Q ,
- (d) a type (II[∨]) point P (cf. (6.1(iii))).

In (1), a general member $D \in |-K_X|$ satisfies $D \cap C = \{P\}$ (7.3) and has only rational double points as singularities.

Case (2) (treated in (6.7.2) and (7.3.2)):

- (e) a type (IC) point P , or
- (f) a type (IIB) point P .

Case (3) (treated in (6.7.3) and (7.3.3)):

(g) two type (IA) points P, Q and at most one type (III) point R such that index $Q = 2$ and P is a type cA point of index ≥ 3 and size $= 1$ (cf. (7.3)).

In (2) and (3), a general member $E \in |-2K_X|$ is such that $E \cap C = \{P\}$ and such that the double cover of X , branched along E and at Q , has only canonical singularities.

Case (4) (treated in (6.7.4) and in §9):

(h) two type (IA) points P, Q with indices ≥ 3 .

In (4), a general member $D \in |-K_X|$ contains C , is smooth outside of $\{P, Q\}$, and has rational double points of type A at P and Q (9.10).

(B.2) *Remark.* When $X \supset C$ is as in (10.2) and not an extremal nbd, the results are the same except that we have an extra case:

Case (5) (this is not an extremal nbd by (6.2(i))):

(i) at most one type (III) point P .

In this case $(-K_X \cdot C) = 1$, $\mathcal{O}(-K_X)$ is generated by global sections, and $|-K_X|$ has a smooth member (10.6).

This is the Gorenstein case as mentioned at (10.6) and the same argument as in the proof of (III) of (A.3) shows that X has at most one type (III) singular point on C .

LIST OF NOTATION AND TERMINOLOGY

Numbers on the right show the paragraphs where the notations/symbols are defined/explained.

[#]	#	(1.12), (1a.1), (1b.1), (2.4), (9.1)
[b]	b	(9.1)
[{]	$(1, d)$ -monomializable	(8.10)
	$(1, d)$ -monomializing ℓ -basis	(8.10)
	$(1, d, b)$ -monomializable at P	(8.11)
	$(1, d, b)$ -monomializing ℓ -basis of the first kind	(8.11)
	$(1, d, b)$ -monomializing ℓ -basis of the second kind	(8.11)
	$(C^\#, P^\#)$	(2.4), (9.1)
	(C^b, P^b)	(9.1)
	(C^\dagger, P^\dagger)	(2.5)
	(I), (IA), (IB), (IC)	(4.2)
	(II), (IIA), (IIB)	(4.2)
	(III)	(4.2)
	$(I^\vee), (IA^\vee), (IC^\vee)$	(4.2)
	(II^\vee)	(4.2)
[0]	0-sequence	Conventions
[A]	a_i	(2.6), (4.2), (9.1)

	a'_i	(9.1)
	almost ordinary	(4.5)
	attached semi-invariant	(8.11)
	axial multiplicity	(1a.5)
[B]	$B(Y, P)$	(8.8.5)
[C]	$C^\sharp(i)$	(2.5)
	C^\sharp -ord	(2.5)
	C^\sharp -ow	(2.5)
	C^\sharp -ow*	(2.8)
	C^\sharp -wt	(2.5)
	$C(n)$	(9.9.6.1)
	C -laminal	(8.2)
	$\text{Cl}^{\text{sc}} X$	(1.5)
	cA	(1a.2)
	canonical lifting (of a homomorphism)	(8.8.2)
	canonical lifting (of a sequence)	(8.8.2)
	canonical lifting (of a sheaf)	(8.8.1)
	canonical singularity	(1.0)
	cD	(1a.2)
	cDV point	(1a.2)
	cE	(1a.2)
	$\text{cl}(W, P)$	(1.7)
	$\mathcal{E}_{\text{all}}(u), \mathcal{E}_{\text{in}}(u), \mathcal{E}_{\text{out}}(u), \mathcal{E}_{\text{mid}}(u)$	(1b.8.1)
[D]	$\text{deg}(\text{deg}: QL(Y) \rightarrow \mathbb{Q})$	(8.9.1)
	$\widetilde{\text{det}}$ (e.g. $\widetilde{\text{det}} \mathcal{M}$)	(8.8.4), (8.9)
	\mathcal{D}_P	(8.11.1)
[E]	$E(J)$	(8.2.2)
	e	(9.9.6.1)
	$\text{ex}_4(P(b, c))$	(2.13.1)
	extremal nbd	(1.1)
[F]	F_2, F_3, F_4	(4.3)
	$F^n(\mathcal{S}, J)$	(8.2)
	$F^n \mathcal{S}$	(2.2)
	first kind	(8.11)
	fixed axis	(1a.5)

- [G] $\text{gr}^\bullet \mathcal{O} (= \bigoplus_{n \geq 0} \text{gr}_C^n \mathcal{O})$ (2.2)
- $\text{gr}^\bullet(\mathcal{S}, J)$ (8.2.1)
- $\text{gr}^n(\mathcal{S}, J)$ (8.2)
- $\text{gr}^{n,i}(\mathcal{S}, J)$ (8.6)
- $\text{gr}_C^\bullet = \bigoplus_{n \geq 0} \text{gr}_C^n$ (2.2)
- gr_C^n (2.2)
- [H] $h: H^0(\omega_X^*) \rightarrow H^0(\mathcal{O}_C) = \mathbb{C}$ (9.9.3)
- [I] I^\sharp, I^\flat (9.1)
- $i_P(n)$ (2.2.1)
- imprimitive (1.7)
- index (1.0)
- induced ℓ – structure (8.8.1)
- [J] J_n (9.9.6)
- J -filtration (8.2.2)
- [K] $\mathcal{H}_n, \mathcal{H}'_n, \mathcal{H}_n^\circ$ (8.11)
- [L] L' -deformation (4.12.2)
- L -deformation (4.7)
- L -smoothing (4.7)
- LG -deformation (9.7)
- laminal (8.2)
- $l \text{ deg}$ (e.g. $l \text{ deg}_Y \mathcal{M}$) (8.9.1)
- lif (8.8.1)
- locally a nested c.i. (8.4)
- locally ℓ -free (8.9)
- locally primitive (1.7)
- locally primitive at P (1.7)
- locally $q\ell$ -free (8.9)
- \mathcal{L} (9.2)
- ℓ (e.g. $\ell(P)$) (9.4.7)
- ℓ -basis (8.8.3)
- ℓ -character (2.5)
- ℓ -coherent (8.9)
- ℓ -coordinate (2.5)
- ℓ -coordinate system (1a.5)

	ℓ -equation	(1a.5)
	ℓ -exact	(8.8.2), (8.9)
	ℓ -free ℓ -basis at P	(8.8.3)
	ℓ -free \mathcal{O}_Y -module at P	(8.8.3)
	ℓ -homomorphism	(8.8.2), (8.9)
	ℓ -injection	(8.8.2), (8.9)
	ℓ -invertible	(8.8.3), (8.9)
	ℓ -isomorphism	(8.8.2), (8.9)
	ℓ -split	(8.8.2), (8.9)
	ℓ -splitting	(8.8.2), (8.9)
	ℓ -structure at P	(8.8)
	ℓ -surjection	(8.8.2), (8.9)
	ℓ -trivial	(8.8.3)
[M]	minimal	(2.6)
	monomializable	(8.10), (8.11)
	monomializing	(8.10), (8.11)
	\mathcal{M} The paragraph preceding (9.3.2), (9.4.5), (9.8.3.1), (9.9)	
[N]	nearby extremal nbd	(1b.1)
	nested c.i.	(8.4)
	nested complete intersection	(8.4)
	normalized expression	(8.9.1)
	normalized ℓ -coordinates	(2.6)
	\mathbb{N}	Conventions
[O]	ord	(2.5), (9.1)
	ord'	(9.1)
	ord(C^\sharp)	(2.5)
	ordered (ℓ -basis)	(8.10), (8.11)
	ordinary	(4.5)
	ow	(2.5), (9.1)
	ow _{i}	(4.1)
	ow'	(9.1)
	ow(C^\sharp)	(2.5)
	ow*	(2.8)
[P]	$P(b, c)$	(2.13.1)
	primitive point	(1.7)

	pure of rank	(8.5)
	pure width	(8.2)
[Q]	QL (e.g. $QL(Y)$)	(8.9.1)
	q (e.g. $q(P)$)	(9.5.1)
	ql (e.g. $ql_Y(\mathcal{M})$)	(8.9.1)
	$ql \deg(\mathcal{M}, P)$	(8.8.5)
	ql -free	(8.8.3)
	ql -invertible	(8.8.3), (8.9)
	quo	(8.8.1)
	\mathbb{Q}	Conventions
[R]	R	(2.8), (9.1)
	R'	(9.1)
	rational singularity	(1.0)
	reduced	(3.4.3), (3.5.1)
	res	(8.8.2)
[S]	$S(d)$	(5.2)
	$SSQ(\mathcal{S}), SSQ(\mathcal{S}, \mu_m)$	(8.8)
	\tilde{S} (e.g. $\tilde{S}^n(\mathcal{M})$)	(8.8.4), (8.9)
	saturated submodule	(2.2)
	saturated chain	(8.2)
	saturated subquotient	(8.8)
	saturation of submodule	(8.2)
	saturation of chain	(8.2)
	second kind	(8.11)
	semi-Cartier divisor class group	(1.5)
	separating L -deformation	(4.7)
	size, siz_p	(4.1.1)
	splitting cover	(1.12.1)
	splitting degree	(1.7)
	subindex	(1.7)
	subquotient	(8.8)
	\mathfrak{S}_n (symmetric group of degree n)	(2.15.1)
[T]	TL ($TL: QL(Y) \rightarrow \mathbb{Z}$)	(8.9.1)
	t	(2.6), (9.1)
	t'	(9.1)

	terminal singularity	(1.0)
	total contraction	(1.1)
	twisted extension	(1b.8.1)
[U]	U	(2.8), (9.1)
	U'	(9.1)
	u_1, u'_1	(9.2)
	u_2, u'_2	The paragraph preceding (9.3.2)
	$\mathcal{U} \begin{pmatrix} I \times \delta \times \delta \\ u \end{pmatrix}$	(1b.7)
[W]	w, w'	(9.9.3)
	$w_P(n)$	(2.2.1)
	$w_P^*(n) = \binom{n+1}{2} i_P(1) - w_P(n)$	(2.2.1)
	width (of an element), $\text{width}_{j,s}$	(8.2.2)
	width (of a laminal ideal)	(8.2)
	wt	(2.5), (9.1)
	wt'	(9.1)
[X]	$X_{\lambda, \text{in}}, X_{\lambda, \text{mid}}, X_{\lambda, \text{out}}$	(1b.8.1)
	$X \begin{pmatrix} I \times \gamma \\ \nu < \delta \end{pmatrix}$	(1b.8.1)
	$X[D, d, \phi]$	(1.11)
	$x = (x_1, \dots, x_4)$	(2.6), (9.1)
	$x' = (x'_1, \dots, x'_4)$	(9.1)
	$\mathcal{X}_{\text{all}}(u), \mathcal{X}_{\text{in}}(u), \mathcal{X}_{\text{mid}}(u), \mathcal{X}_{\text{out}}(u)$	(1b.8.1)
	$\mathcal{X} \begin{pmatrix} I \times \gamma \times \delta \\ u < \delta \end{pmatrix}$	(1b.7)
[Z]	$\mathbb{Z}_+, \mathbb{Z}_{++}, \mathbb{Z}_m$	Conventions
[[]]	[] (round up)	Conventions
[[]]	[] (Gaussian symbol)	Conventions
	$[\cdot, \cdot, \cdot]$	(2.11)
	$[n]$ (e.g. $M[n]$ for $n \in \mathbb{Z}_m$)	(2.5), (8.8.4)
	Rem : $M[n]$ is defined in two places in equivalent ways.	
[{ }]	$\{n\}$ (e.g. $M_{\{n\}}$)	(2.5)
[α]	α_j	(8.2.2)
	α_n	(2.2.1)
	$\alpha_n : \text{gr}_C^0 J_n \rightarrow \mathcal{M}$	(9.9.6)
	Rem : The above two α_n 's have nothing to do with each other.	

[β]	β_J	(8.2.2)
	β_n	(2.2.1)
[γ]	$\gamma^{n,i}(\mathcal{S}, J)$	(8.6)
	$\gamma^n(\mathcal{S}, J)$	(8.2.2)
	$\tilde{\gamma}^{n,i}(\mathcal{S}, J)$	(8.10), (8.11.1), (8.12)
	$\tilde{\gamma}^n(\mathcal{S}, J)$	(8.10), (8.12)
	γ_C^n	(2.2)
[Δ]	Δ_J	(8.2.2)
[δ]	$\delta_{i,j}$ (Kronecker's δ)	Conventions
	$\delta_{J^\sharp, P^\sharp}, \delta_{J^\natural}$	(8.11)
[θ]	$\theta(n)$	(9.9.7)
[π]	$\pi^\sharp: (X^\sharp, P^\sharp) \rightarrow (X, P), \pi^\flat: (X^\flat, P^\flat) \rightarrow (X, P')$	(9.1)
	$\pi[D, d, \phi]$	(1.11)
[σ]	$\sigma: J^\sharp \rightarrow J^\dagger / J^{\dagger(2)}$	(2.11)
[τ]	$\tau: \mathcal{O}_{X^\sharp, P^\sharp} \rightarrow \mathcal{O}_{C^\dagger, P^\dagger}$	(2.5)
[Φ]	$\Phi^i \text{gr}^n(\mathcal{S}, J)$	(8.6)
[μ]	μ_m	Conventions
	μ_d -cover associated to $[D, \phi]$	(1.11)
[\oplus]	$\tilde{\oplus}$ (e.g. $\mathcal{L} \tilde{\oplus} \mathcal{M}$)	(8.8.2), (8.8.4), (8.9)
[\otimes]	$\tilde{\otimes}, \tilde{\otimes}$ (e.g. $\mathcal{L} \tilde{\otimes} \mathcal{M}, \mathcal{L}^{\tilde{\otimes} n}$)	(8.8.4), (8.9)
[\parallel]	$\parallel \parallel$	(2.13.1), (3.4.3)

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