Flipping Cubical Meshes

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Abstract. We define and examine flip operations for quadrilateral and hexahedral meshes, similar to the flipping transformations previously used in triangular and tetrahedral mesh generation.

Keywords: quadrilateral meshing, hexahedral meshing, flip graph connectivity, mesh improvement.

1 Introduction

In this paper we propose flipping transformations for cubical meshes, such as quad and hex meshes. These flipping transformations are precise analogues of the well-known flipping transformations for simplicial meshes. We did not invent these transformations. They have already appeared—although they were not explicitly enumerated—in the mathematical polytope literature [1, 25]. We have not found any previous references in the meshing literature. The closest work seems to be the quad refinement operations of Canann, Muthukrishnan, and Phillips [4] and the hex reconnection primitives of Knupp and Mitchell [12].

As in the simplicial case, quad and hex flipping could be useful for mesh generation, improvement, and refinement/derefinement. We also use flipping to study existence questions for hexahedral meshes. Up until this present paper, the theoretical results on quad-to-hex extension have been positive, suggesting that (under simple necessary conditions) it should always be possible. Mitchell [17] and Thurston [22] independently showed that any quad surface mesh with an even number of quadrilaterals, and topologically equivalent to a sphere, can be extended to a *topological* hex volume mesh, which allows elements to have warped, nonplanar sides. (An even number of surface quadrilaterals is a necessary condition, because hex elements each have six sides, and each interior face will account for two hex sides.) Eppstein [6] gave another proof of this result and improved the complexity bound from $\mathcal{O}(n^2)$ to $\mathcal{O}(n)$ elements. Each of the two proofs generalize to certain more topologically complicated inputs as well. However, it has not been clear under what circumstances these topological meshes can be realized using flat-faced cuboids. In this context we discuss a simple 10-element boundary complex, the *bicuboid*, which we have been unable to mesh, and use flipping to show that many possible topological meshes for this complex are unrealizable.

This is primarily a theoretical paper—realizability is not of great practical importance since the isoparametric trilinear elements typically used with unstructured hexahedral meshes allow the cells to have curved boundaries. However, the existence or nonexistence of flat-faced meshes may shed light on the question of how many curved elements are needed to achieve satisfactory mesh quality. In addition, our study of flip graph connectivity relates to mesh improvement algorithms: by characterizing the connected components of the quadrilateral flip graph for simply-connected regions (Theorem 5) we show that any local improvement operation that remeshes bounded regions of a quadrilateral mesh can be simulated by a sequence of simpler flip and parity-change operations.

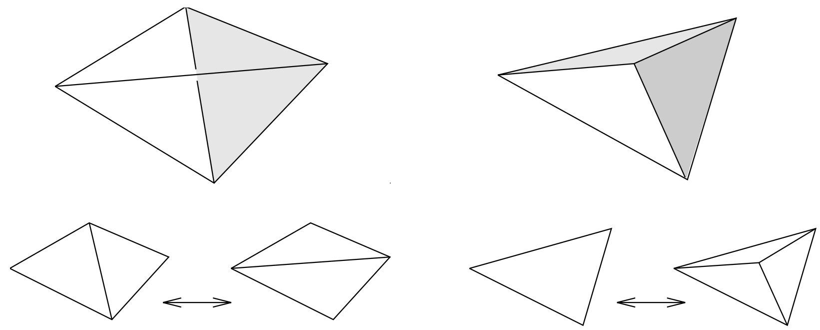

Fig. 1. The 2-2 and 1-3 flips for triangular meshes correspond to switching upper and lower facets of a tetrahedron.

2 Definitions

A *polyhedron* is a set of closed planar polygonal faces in $\mathbb{R}^3$, that intersect only at shared vertices and edges, and such that each edge is shared by exactly two faces and each vertex is shared by exactly one cycle of faces.[1] A polyhedron divides $\mathbb{R}^3$ into a bounded *interior* and an unbounded *exterior*, each of which may contain more than one connected component.

A *cuboid* in $\mathbb{R}^d$ is a convex polytope, combinatorially equivalent to the unit cube $[0, 1]^d$. *Combinatorial equivalence* means that the faces (vertices, edges, and so forth) of the cuboid are in one-to-one correspondence with the faces of the unit cube, and that this correspondence preserves intersections. In $\mathbb{R}^2$ a cuboid is simply a convex quadrilateral, called a *quad* for short. In $\mathbb{R}^3$ a cuboid is a convex polyhedron called a *hex*.

A *quad surface mesh* is a polyhedron whose faces are all quads. A *hex mesh* is a quad surface mesh, along with interior quads that subdivide the polyhedron interior into hexes. The intersection of any pair of hexes is either empty, a single vertex, a single edge, or a single quad. Quad surface meshes and hex meshes are both examples of *cubical complexes* [2, 11, 15], the analogue of simplicial complexes in which each k-dimensional face is a k-dimensional cuboid.

We also consider *topological hex meshes*: cubical complexes homeomorphic to the input domain, with a boundary mesh combinatorially equivalent to that of the input. In this context, we call the quad and hex meshes defined earlier *geometric meshes* to emphasize the geometric requirements on their cell shapes. Topological meshes can be *realized* by specifying locations in the domain for each internal vertex of the complex; the realization is a hex mesh if each cell's eight vertex locations have a convex hull that is a cuboid, and no two of these cuboids have intersecting interiors. A realization is *self-intersecting* if each four vertices of a quadrilateral in the mesh have locations that form a convex quadrilateral in the domain, without respect for whether these quadrilaterals form the boundaries of disjoint cuboids; a self-intersecting mesh can be viewed as a generalized geometric mesh in which some cells are allowed to be inverted.

3 Flipping

Consider the well-known *flip* transformation for a triangular mesh, which switches one diagonal for the other in a convex quadrilateral Q formed by two triangular elements sharing a side. We can think of this 2-2 flip (two elements in and two elements out) as exchanging the upper and lower facets of a three-dimensional simplex P that projects to Q, as shown on the left in Figure 1. This view leads us to consider the refinement transformation that subdivides a triangle into three smaller ones by adding an interior vertex as a 1-3 flip (Figure 1 right side). The reverse of the 1-3 flip is a 3-1 flip, which derefines the triangulation by removing a vertex.

[1] This definition of polyhedron allows multiple connected components (two cubes can be one polyhedron!), as well as voids and tunnels. It disallows non-manifold boundaries, such as those used to model objects made from two different materials.

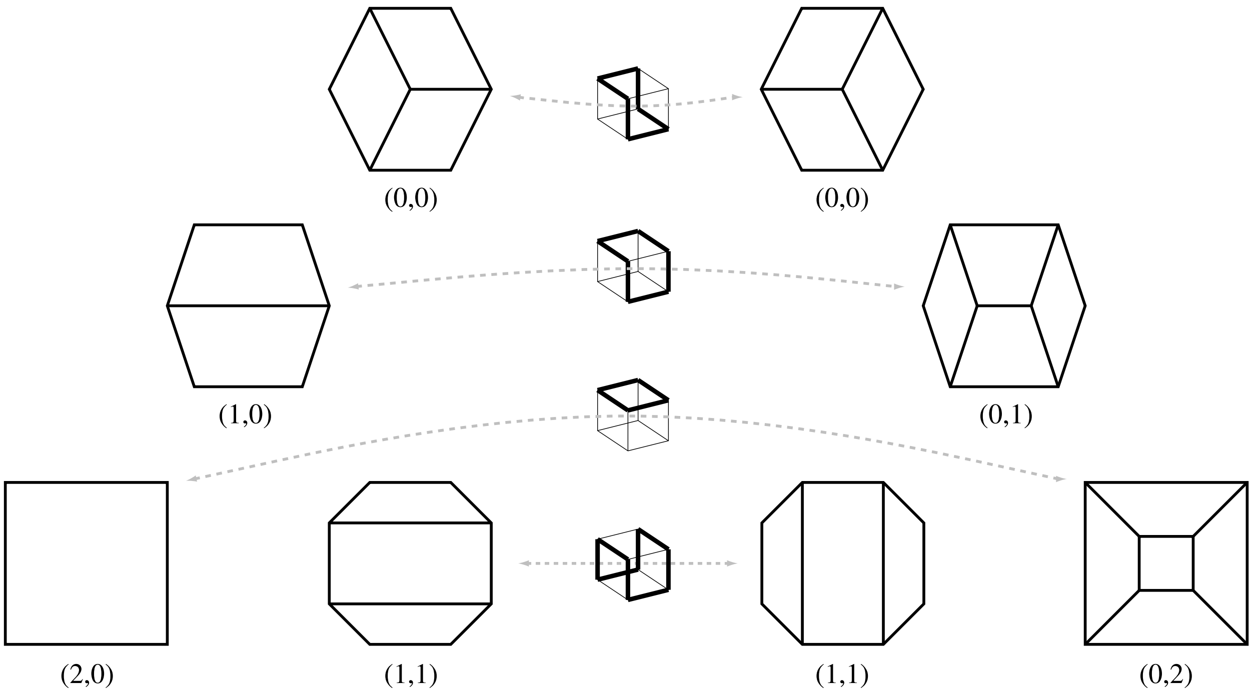

Fig. 2. Flips for quad meshes correspond to switching upper and lower facets of a three-dimensional cuboid. The bold cuboid edges mark the division between upper and lower facets.

The almost-as-well-known flips for tetrahedral meshes can be defined analogously by the exchange of the upper and lower facets of a four-dimensional simplex. The flips include the 2-3 flip, which exchanges two tetrahedra sharing a face for three sharing an edge, and the 1-4 flip which subdivides a tetrahedron by adding an interior vertex, along with their reverse flips.

This view of flipping extends to quad and hex meshes. The flips for quad meshes are induced by the combinatorially distinct exchanges of upper and lower facets of a three-dimensional cuboid, as shown in Figure 2, and the flips for hexahedral meshes are induced by the combinatorially distinct exchanges of upper and lower facets of a four-dimensional hypercube, as shown in Figure 3.

The facets of a d-dimensional cube or hypercube can be grouped into d opposite pairs. In order to discover all possible flips, we describe any set of facets of a cube or hypercube by how it relates to these pairs. To any set S of cube or hypercube facets, we assign a pair of integers (X,Y), where X denotes the number of pairs of facets eXcluded from S, and Y denotes the number of pairs of facets Yncluded in S. The remaining $d-X-Y$ pairs of facets have one of their two facets included in S, and the other facet excluded from S. The symmetries of the hypercube include all permutations of opposite pairs of facets, and all swaps of the two facets in each pair, so any two sets S described by the same pair (X,Y) are combinatorially equivalent.

The meshes in Figures 2 and 3 are labeled by these pairs of numbers. If a set S is represented by a pair (X,Y), the complementary set is represented by pair (Y,X). So, a flip can be described as the replacement of a set of mesh cells combinatorially equivalent to the set (X,Y) by a different set of cells combinatorially equivalent to (Y,X). In order to find all possible flips, it remains only to determine which pairs (X,Y) form topological disks.

Lemma 1. *Let S be a set of facets of a d-hypercube described by pair (X,Y). Then S forms a topological disk if and only if $X+Y<d$.*

Proof. If $X + Y < d$, there is some facet $f \in S$ such that the opposite facet is not contained in S; then f is adjacent to every facet in S. We can retract all facets to f by an affine transformation that shrinks the coordinate axis perpendicular to f; therefore S has a retraction to a $(d-1)$-dimensional disk and is itself a disk.

On the other hand, suppose $X + Y = d$. We show by induction on X that S has a retraction to a Y-sphere, and therefore cannot be a disk. More specifically, we can retract S in a facet-preserving way to the boundary of a Y-hypercube. As a base case, if $X = 0$ and $Y = d$, S is the whole boundary of the d-hypercube. Otherwise, S is formed by removing a pair of facets from a set S^+ that can be retracted to a $(Y+1)$-hypercube C^+. Removing the final two facets from S^+ corresponds, in its retraction, to removing two opposite facets from the boundary of C^+, and the remaining facets of C^+ can be contracted to a Y-hypercube by again using an affine map that contracts the coordinate axis perpendicular to the removed facets. □

We cannot allow sets S that do not form disks to count as flips, because exchanging such a set for its complement would change the topology of the domain. The possible flips in a d-dimensional cubical mesh correspond to subsets of a $(d+1)$-hypercube, and are therefore given by the pairs (X,Y)–(Y,X) where $X + Y \leq d$.

Corollary 1. *The d-dimensional cubical meshes have $(d+1)(d+2)/2$ combinatorially distinct flippable submeshes (X,Y) which can be grouped into $\lfloor (d+2)^2/4 \rfloor$ flip pairs (X,Y)–(Y,X).*

The possible quadrilateral mesh flips are therefore $(2,0)$–$(0,2)$ (changing one quadrilateral for five), $(1,0)$–$(0,1)$ (changing two quadrilaterals for four), and two changes of three for three quads: $(0,0)$–$(0,0)$ and $(1,1)$–$(1,1)$. along with their reverses. We can picture the $(2,0)$ mesh as corresponding to the unique lowest face of a cuboid, and the flipped $(0,2)$ mesh as the unique highest cuboid face together with the four neighboring side faces. The $(1,0)$–$(0,1)$ flip can similarly be pictured with unique lowest and highest edges, and the $(0,0)$–$(0,0)$ flip can be pictured with unique lowest and highest vertices. The remaining $(1,1)$–$(1,1)$ flip transforms the top, front, and back faces of a cube into the bottom, left, and right faces.

Analogously, the flips for hex meshes are induced by all possible exchanges of upper and lower facets of a four-dimensional cuboid, as shown in Figure 3. The $(3,0)$–$(0,3)$ flip converts one hex to seven, and can be pictured with unique lowest and highest facets of a hypercuboid in $\mathbb{R}^4$. The $(2,0)$–$(0,2)$ converts two hexes to six, and can be pictured with unique highest and lowest two-dimensional faces of a hypercuboid. There are two different flips from three to five hexes: $(1,0)$–$(0,1)$ and $(2,1)$–$(1,2)$. In the $(2,1)$ three-hex mesh, the hexes are connected in a path. This mesh can be pictured most symmetrically with a unique lowest hypercuboid facet bordered by two other mutually disjoint almost-lowest facets, and the complementary $(1,2)$ mesh can be pictured with a unique highest facet (a sort of hex tunnel in Figure 3) surrounded by four other high facets. The other three-hex mesh in our set of flips, $(1,0)$, has three hexes connected in a cycle, and can be pictured with a unique lowest hypercuboid edge surrounded by three facets; its complementary five-hex mesh, $(0,1)$, has a unique highest edge, whose endpoints do not appear on the lower convex hull, surrounded by three facets. Finally, there are two different flips involving sets of four hexes. The $(1,1)$ flip corresponds to a lower-convex-hull facet connectivity of a clique minus an edge, and can be pictured with a unique lowest two-dimensional hypercuboid face; the $(0,0)$ flip has four mutually adjacent hexes, and can be pictured with a unique lowest hypercuboid vertex.

The regions of $\mathbb{R}^3$ occupied by the cells of a flip are all combinatorially inequivalent: a cuboid for the $(3,0)$–$(0,3)$ flip, a bicuboid for the $(2,0)$–$(0,2)$ flip, a twisted rhombic dodecahedron for the $(1,0)$–$(0,1)$ flip, a tricuboid for the $(2,1)$–$(1,2)$ flip, a fourteen-faced shape resembling a square orthobicupola for the $(1,1)$–$(1,1)$ flip, and a rhombic dodecahedron for the $(0,0)$–$(0,0)$ flip. These shapes can be determined from the two-dimensional quadrilateral flips: if one forms a polyhedron from the convex hull of two parallel copies of an (X,Y) quadrilateral mesh, the result is the region of an $(X,Y+1)$–$(Y+1,X)$ flip (for instance, the $(0,2)$ bicuboid is formed by the convex hull of two parallel two-quadrilateral $(0,1)$ meshes), and if one

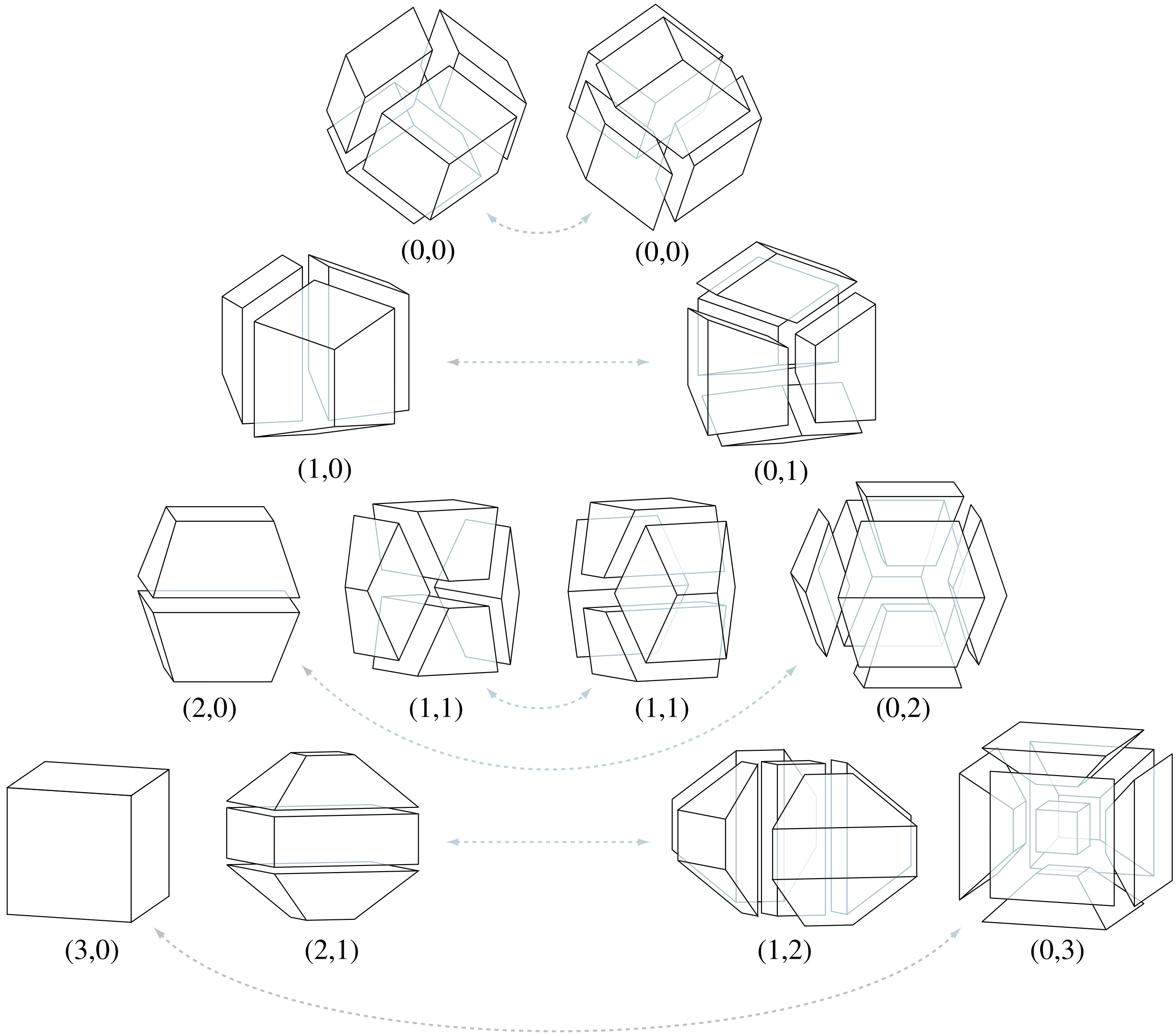

Fig. 3. The flips for hex meshes correspond to switching upper and lower facets of a four-dimensional cube.

instead forms the convex hull of an (X,Y) mesh and a parallel (Y,X) mesh, the resulting three-dimensional shape is the region for an (X,Y)–(Y,X) flip.

Three of the hexahedral flips are listed by Knupp and Mitchell [12]: the $(3,0)$–$(0,3)$ flip is given as an example of their "pillowing" transformation, the $(2,0)$–$(0,2)$ flip is given as an example of their "inflating hex ring" transformation, and the $(0,0)$–$(0,0)$ flip is called the "rotate three hexes primitive" despite its symmetric action on four hexes. The $(3,0)$–$(0,3)$ and $(2,0)$–$(0,2)$ flips are also used by Maréchal [14] to reduce the complexity of some case analysis in his conformal mesh refinement algorithm.

4 Flippability

We say that a set of cells in a mesh is *flippable* if it can be replaced by one of the flips described above. For topological meshes, a set of cells is flippable whenever it has the appropriate connectivity pattern, but for geometric meshes additional conditions will likely be required. For instance, in triangular meshes, a pair of triangles is flippable if and only if it forms a convex quadrilateral.

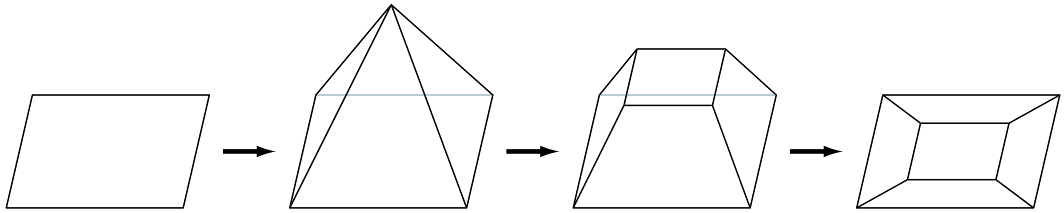

Fig. 4. The $(2,0)$–$(0,2)$ quad flip and $(3,0)$–$(0,3)$ hex flip can be performed geometrically by projecting the truncation of a higher dimensional pyramid.

We say that a hex mesh is *generic* if no three vertices are collinear and no two quadrilaterals are coplanar. We say that a type of flip is *automatically flippable* if, whenever the configuration of cells on one side of the flip occurs in a generic geometric or self-intersecting mesh, the flip can be performed resulting in another generic (possibly self-intersecting) mesh. That is, for these types of flips, flippability of geometric meshes may involve convexity constraints disallowing inverted elements, but does not involve additional constraints on the flatness of cell boundaries. In triangular, quadrilateral, or tetrahedral meshes, all flips are automatically flippable, because all cell boundaries are always flat. The requirement that each quadrilateral face be flat leads to more interesting theory in the hexahedral case.

Theorem 1. *The* $(3,0)$–$(0,3)$*,* $(0,3)$–$(3,0)$*,* $(2,0)$–$(0,2)$*,* $(0,2)$–$(2,0)$*, and* $(0,0)$–$(0,0)$ *hex mesh flips are automatically flippable.*

Proof. The result is trivial for the $(0,3)$–$(3,0)$ flip, since it just involves removal of some faces from the $(0,3)$ mesh. We can flip the $(3,0)$ mesh by viewing the initial cuboid as the base of a pyramid in four dimensions, forming a four-dimensional cuboid by truncating the pyramid's apex by a hyperplane, and replacing the base by the orthogonal projection of the truncated pyramid's remaining faces (Figure 4). There are enough degrees of freedom in this construction to preserve genericity.

To understand the $(2,0)$–$(0,2)$ flips, let's analyze the conditions under which the six-cuboid $(0,2)$ mesh is possible. This mesh (shown in the upper right corner of Figure 3) breaks a *bicuboid* into a *ring* of four cuboids connected to the bicuboid sides, and a *core* of two cuboids connected to the top and bottom.

Let's first look at the ring. The planes P_i ($i = 0,1,2,3$) where its cuboids connect to each other are determined by triples of points on the bicuboid. Each such plane contains four ring vertices, three of which are already determined. Fixing the position of the fourth vertex v_i on plane P_i causes the planarity constraints of two inner quadrilaterals to combine to fix the position of v_{i+1} on plane P_{i+1}, etc: $v_{i+1} = f_i(v_i)$. In order for the ring to be formed correctly, these functions must return back to the starting position: $v_0 = f_3(f_2(f_1(f_0(v_0))))$. But each f_i is an affine transformation, and we already know three noncollinear points of P_0 that are returned to themselves by the composition $f_3(f_2(f_1(f_0(x))))$: the three original bicuboid vertices. So the composition must be the identity map, and *any* point of P_0 is mapped by the functions f_i to four points determining a geometrically embedded ring.

The only remaining question is, when are these four points coplanar? We say that a point x in P_0 is *plane-generating* when its quadruple $x, f_0(x), f_1(f_0(x)), f_2(f_1(f_0(x)))$ forms a planar quadrilateral. Then the plane-generating points form a linear subspace of P_0: if you have any two such points p and q, and a third point r on line pq, then each point in the quadruple of r is just the same linear combination of the images of p and q, so the quadruple of r lies on a plane that is the same linear combination of the planes for p and q. Note also that we already know two points in P_0 that determine a coplanar quadrilateral: the vertices of the top and bottom faces of the bicuboid. So if there were a third affinely independent point determining a coplanar quadrilateral then *all* points of P_0 would determine a coplanar quadrilateral. So the existence of a flat middle face for the $(2,0)$ mesh determines the existence of a flat middle face for the $(0,2)$ hex and vice

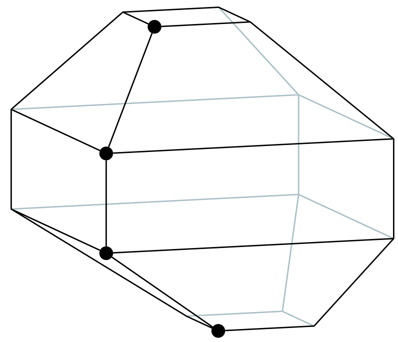

Fig. 5. Unflippable $(2,1)$ configuration.

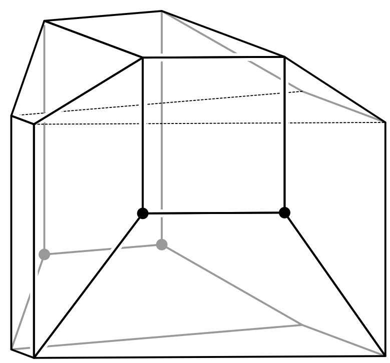

Fig. 6. Unflippable $(1,1)$ configuration.

versa. Since there are two degrees of freedom for placing the points of the middle face, we can do so in such a way that the mesh remains generic.

Finally, consider the $(0,0)$–$(0,0)$ flip. The hexes of a $(0,0)$ mesh form a rhombic dodecahedron. Color the degree-4 vertices of the dodecahedron gray, and color the degree-3 vertices alternately white and black, so that the cuboids in the given $(0,0)$ mesh have one white vertex each. To show that it can be subdivided into black cuboids, lift the vertices of the mesh into $\mathbb{R}^4$ by assigning w-coordinates as follows. Place the interior point a (the apex) on the plane $w = 1$ and the gray vertices $g_1, g_2, g_3, g_4, g_5, g_6$ on the plane $w = 0$. Each quadruple $ag_ig_jg_k$ determines a hyperplane, and these hyperplanes determine the w-coordinates of the black and white vertices. The lifted cuboid mesh is the upper surface of a hypercuboid. The single remaining hypercuboid vertex z (the zenith) is located at the intersection of four hyperplanes, each determined by a black point and its three gray neighbors. Projecting the lower facets of the resulting hypercuboid back to $\mathbb{R}^3$ gives us the flipped cuboid mesh. In particular, the projection of z lies in the interior of the original rhombic dodecahedron. $\square$

The same argument shows that, for geometric hex meshes, regardless of genericity, the $(3,0)$–$(0,3)$, $(0,3)$–$(3,0)$, $(2,0)$–$(0,2)$, $(0,2)$–$(2,0)$, and $(0,0)$–$(0,0)$ flips are possible if and only if the set of cuboids to be flipped forms a strictly convex subset of the domain.

However, not all flips are automatically flippable:

Theorem 2. *The $(1,1)$–$(1,1)$, $(2,1)$–$(1,2)$ and $(1,2)$–$(2,1)$ flips are not automatically flippable for geometric or self-intersecting meshes. The $(1,0)$–$(0,1)$ flip is not automatically flippable for geometric meshes.*

Proof. One can form a set of vertices in the pattern of a $(2,1)$ mesh by placing four axis-aligned squares in four parallel planes: two large ones above each other in the two inner planes, and two small ones inside

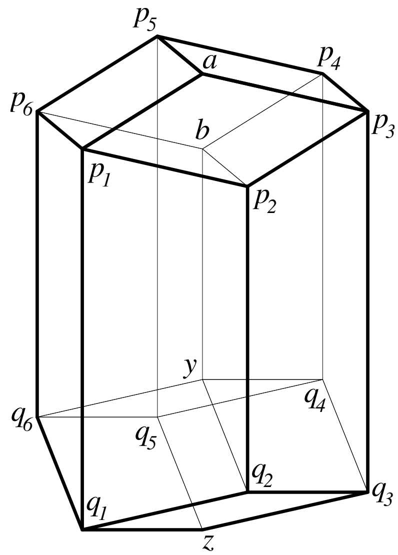

Fig. 7. Vertex labels of $(0,1)$ configuration for proof of $(1,0)$ unflippability.

the projections of the large squares in the remaining planes (Figure 5). With this placement, the three-hex $(2,1)$ mesh has all faces flat. However, the small squares can be translated arbitrarily, causing the diagonal interior faces of the five-hex $(1,2)$ mesh (such as the face formed by the four marked vertices in the figure) to be warped, so that the $(2,1)$ mesh is not flippable.

Similarly, by centering these four squares on the same axis, but then shifting a pair of corresponding vertices in the two central squares along the edges connecting these two vertices to their counterparts in the outer two squares, one can create an unflippable $(1,2)$ mesh.

Next, choose parameters $r > 0$ and $0 < \theta < \pi/4$, and let $p(i,j)$ denote the point $(r\cos i\theta, r\sin i\theta, j) \in \mathbb{R}^3$. Then the eight points $p(\pm 1,0)$, $p(\pm 2,1)$, $p(\pm 3,1)$, $p(\pm 4,0)$ form the vertices of a cuboid in which all facets are trapezoids (shown as the top eight vertices of Figure 6). If we make another copy of this set of eight vertices, translated downwards, and connect each bottom face of the cuboid to each bottom face of the copy, we get a $(1,1)$ mesh. However, if the copies of $p(-2,1)$ and $p(-3,1)$ (the black marked points in the figure) are shifted less far than the copies of $p(2,1)$ and $p(3,1)$ (the grey marked points), then these four points will form a skew quadrilateral, preventing this mesh from being flipped.

Finally, consider the $(1,0)$ mesh, and label the 14 vertices of the corresponding $(0,1)$ mesh as illustrated in Figure 7. Suppose also that the segments p_iq_i, az, and by are all parallel. We have the following equations for the two interior vertices b and y:

$$b = p_6p_1p_2 \cap p_2p_3p_4 \cap p_4p_5p_6$$
$$y = q_6q_1q_2 \cap q_2q_3q_4 \cap q_4q_5q_6$$

Here, each triple denotes a plane through the three points. Now consider shrinking the mesh by simultaneously contracting the parallel edges p_iq_i and az. As long as these edges have non-zero length, the $(0,1)$ mesh is still valid. But if we make them short enough, by is inverted, so the top and bottom cubes in the $(1,0)$ mesh intersect. Thus, the $(0,1)$–$(1,0)$ flip is not automatically flippable. □

Open Problem 1 *Is the $(1,0)$ flip automatically flippable for self-intersecting meshes? Is the $(0,1)$ flip automatically flippable?*

Our results on automatic flippability are related to scene analysis questions studied by Whiteley and Sugihara (see [23]). In its most basic form, scene analysis asks whether a given planar drawing of vertices

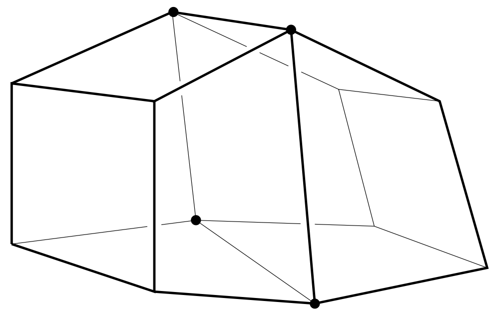

Fig. 8. The bicuboid.

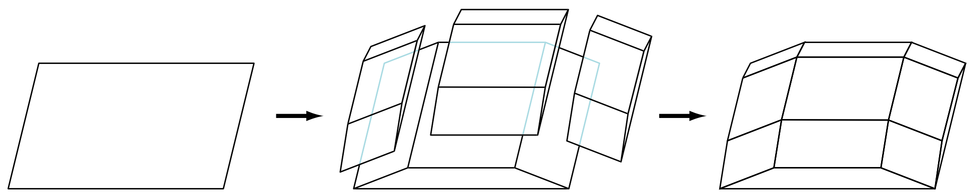

Fig. 9. 3-by-3 subdivision of boundary can be simulated by covering quads with four cuboids.

connected by faces is the projection of flat faces in $\mathbb{R}^3$. Here we are interested in whether a given set of hexes is the projection of a cuboid in $\mathbb{R}^4$, however our three-dimensional scenes are not generic (due to the flatness constraints) so Whiteley's formula for the number of degrees of freedom does not seem to apply. It would be interesting to apply the matroid techniques used in scene analysis and rigidity theory to hex meshing.

5 Bicuboid

In this section, we discuss a simple but important solid, called a *bicuboid*. A bicuboid has the same topology as two hex elements sharing a face, as shown in Figure 8. We call a bicuboid *warped* if its central four vertices, shown dotted in Figure 8, are not coplanar. The bicuboid is important because, as we now show, the question of whether topologically meshable domains have polyhedral hex meshes can be reduced to this case, improving a reduction to a more complicated but finite set of cases from our previous work [6].

Theorem 3. *If every bicuboid has a geometric hex mesh, then any quad surface mesh, with an even number of quad elements and topologically equivalent to a sphere, can be extended to a geometric hex volume mesh.*

Proof. We start by finding a topological hex volume mesh as in [6, 17, 22]. In such a mesh, each element is a topological cuboid, meaning a cuboid with curved edges and faces. We then subdivide all topological cuboids into $m \times m \times m$ grids of small topological cuboids that are "geometrically close" to true cuboids. For example, if we require each face of a small topological cuboid to fit within a thin slab defined by parallel planes, we could pick a sufficiently large m in order to make this possible.

Each small topological cuboid is then subdivided into seven pieces as in a $(3,0)$–$(0,3)$ flip: a true cuboid in the center, and six "almost-cuboidal" warped hexes, each with five planar faces and one nonplanar face. The almost-cuboidal hexes of the subdivided mesh match up in pairs to form bicuboids. Meshing each bicuboid and combining the results gives a mesh for the overall domain.

The only remaining problem is that the $m \times m \times m$ subdivision also subdivides the boundary of the domain, which is not allowed. To get around this problem, we let m be a power of three, and simulate the 3×3 subdivision of a boundary quadrilateral by placing four flat cuboids along it, as shown in Figure 9, so that the cuboids from adjacent boundary quadrilaterals meet face-to-face. The number of degrees of freedom

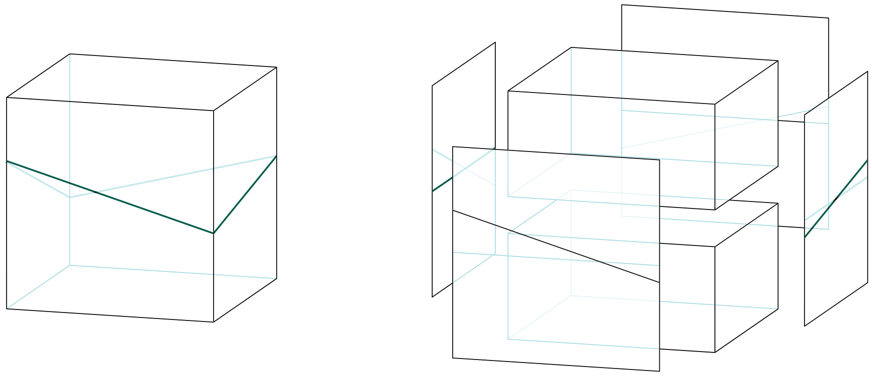

Fig. 10. Non-generic warped bicuboid (left) with a self-intersecting mesh (right).

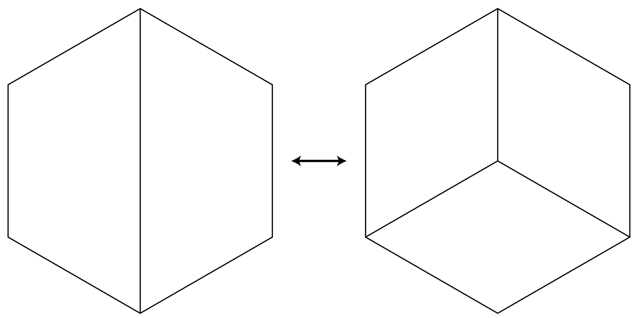

Fig. 11. Parity-changing transformation of a quadrilateral mesh.

for the new cuboid vertices exceeds the number of coplanarity constraints, allowing this construction to be performed geometrically. □

Theorem 3 shows that a bicuboid is the crucial test case for the question of whether every geometric quad surface mesh can be extended to a geometric hex mesh. Theorem 1, however, leads us to suspect that the answer is no, since it shows that a warped bicuboid cannot be meshed by any generic mesh that can be reached from the two-hex mesh by automatically flippable flips—reversing those flips would give a geometric two-hex mesh, which does not exist.

Open Problem 2 *Does every warped bicuboid have a geometric hex mesh? Does every warped cuboid have a generic self-intersecting hex mesh?*

The question is not open for non-generic self-intersecting meshes: a warped cuboid formed by placing the four vertices of a warped quadrilateral around the equator of a unit cube has a non-generic self-intersecting six-hex mesh in which the inner quadrilateral's vertices also lie on the surface of the cube (Figure 10). Less degenerately, if one has a warped bicuboid such that the convex hull of its top and bottom quadrilaterals forms a cuboid, then the toroidal difference between the bicuboid and this central cuboid can be partitioned into four triangular prisms; if we then slice the central cuboid in two by a plane, this plane divides a face of each prism in two, forming degenerate cuboids each with a single 180° dihedral.

6 Parity

Each of the flip transformations preserves the parity (odd or even) of the number of hex elements in the mesh. In two dimensions, any meshable domain has meshes of both parities, as can easily be seen by performing

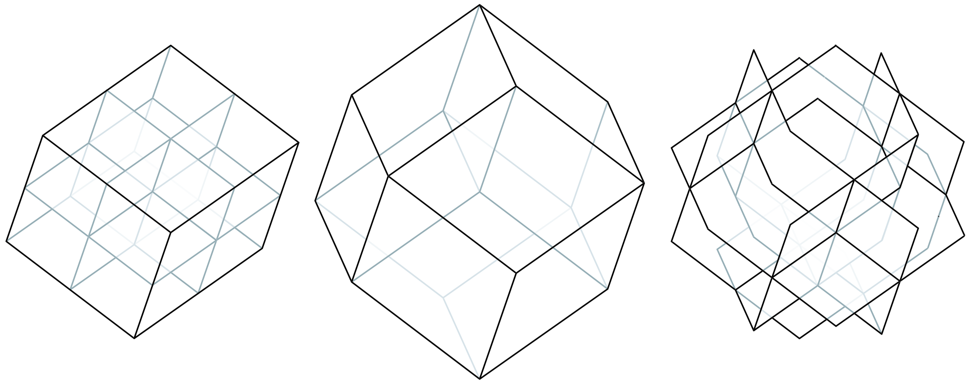

Fig. 12. Cuboid divided into octants by three dual surfaces (left), and the set of dual surfaces (right) for a more complicated four-hex mesh (center).

a two-to-three quad replacement (not a flip), as shown in Figure 11, so not all quad meshes are obtainable by flipping from a given starting mesh. A similar result holds for hex meshes, but the construction is more complicated:

Theorem 4. *If a given quad surface mesh can be extended to a topological hex mesh, it has topological hex meshes with both even and odd numbers of hexes.*

Proof. Suppose we have some hex mesh, and we wish to find an alternative mesh with the opposite parity. Following Mitchell [17] and Thurston [22], we form three quadrilateral surface patches within each hex, medial between each of the three pairs of opposite faces; these patches join up to form an arrangement of surfaces, with one triple intersection per cuboid (Figure 12). The boundaries of these surfaces form a graph on the domain boundary that is the dual of the original quad surface mesh.

Conversely, if we have an arrangement of surfaces that meets the domain boundary in the dual of the quad mesh, where at most three surfaces meet in any point and we have no pinch points or other topological anomalies, we can form a graph with one vertex per triple intersection and with edges connecting the vertices along the curves where two surfaces meet. As long as this graph is connected and has no self-loops or multiple adjacencies, we can form a valid topological hex mesh whose cuboids and quadrilaterals correspond to the vertices and edges of this graph. If the graph does have self-loops or multiple adjacencies, Mitchell [17] shows that they can be removed by the "pillowing" ($(3,0)$–$(0,3)$ flip) operation.

In order to change the parity of the number of hexes in the mesh, then, we can introduce a new surface in such a way that the number of triple intersections increases by an odd number. A suitable candidate is Boy's surface [3, 7], a projective plane embedded in $\mathbb{R}^3$ in such a way that there is one triple intersection point, connected to itself by double-intersection curves in a pattern of three self-loops. We place a copy of Boy's surface within a simply-connected region of the dual arrangement, in such a way that the augmented arrangement remains connected and continues to avoid multiple adjacencies. Then each of the three loops of Boy's surface will cross the other arrangement surfaces an even number of times, and the double intersection curves of those other surfaces will cross the new Boy's surface an even number of times, so that the only change in parity will be from the single triple intersection of Boy's surface with itself. The resulting graph may have some self-loops or multiple adjacencies, but we can remove these by pillowing without further changes of parity. The hex mesh formed by this graph will therefore have the opposite parity from the mesh we started with. □

Open Problem 3 *Are there domains with geometric hex meshes of both parities?*

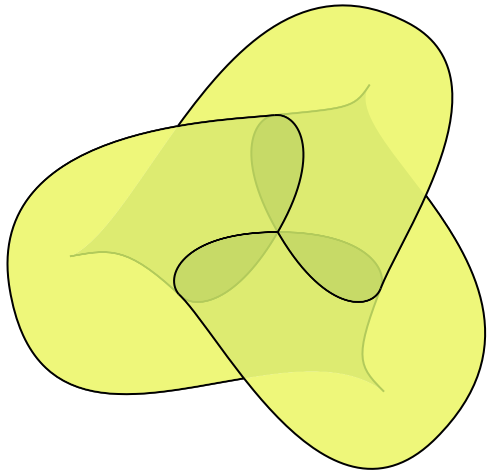

Fig. 13. Boy's surface.

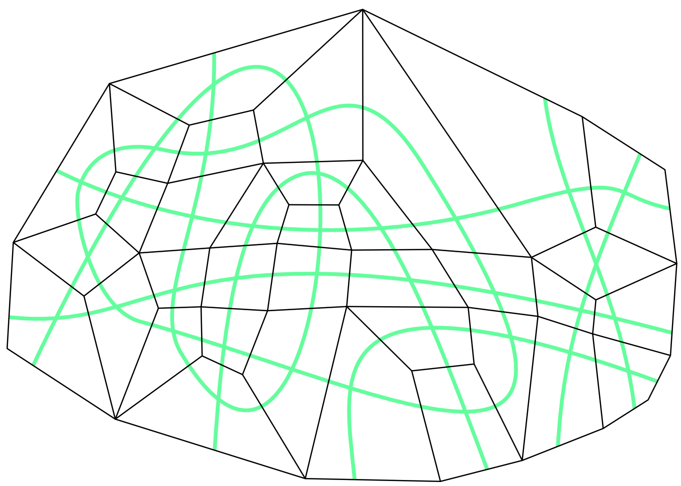

Fig. 14. Curve arrangement dual to a quadrilateral mesh.

For quadrilateral meshes, one can allow parity-changing operations by adding Figure 11 to the repertoire of flips.

Open Problem 4 *Is there a simpler parity-changing operation for hexahedral meshes? What is the minimum possible number of hexahedra involved in such an operation?*

For non-simply-connected domains, parity changing may be easier: for instance, a torus with a hexagonal cross section and an odd number of segments may be filled with either two or three hexes per segment, as in the planar parity-changing operation. For instance, with three segments, this leads to an even mesh with six hexes and an odd mesh with nine. However such a torus cannot be part of a mesh of a simply-connected domain because the complement of the torus would have a contractible odd cycle.

7 Bubble-wrapping and Shelling

In order to prove our results in the next section on flip graph connectivity, we need a technical strengthening of the Mitchell-Thurston result on topological mesh existence: every ball-shaped domain with an even number of quadrilateral faces has a *shellable* hexahedral mesh, where a *shelling* of a complex [25] is an ordering

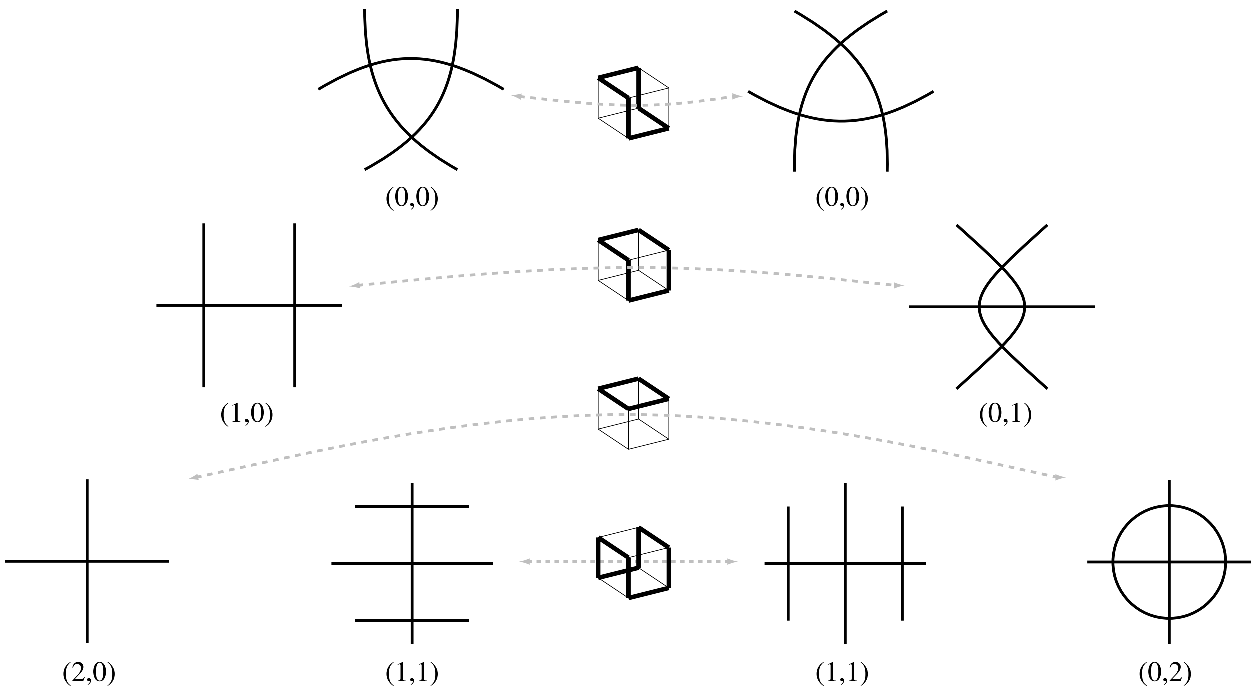

Fig. 15. Actions of quadrilateral flips on dual curves.

of its cells such that the union of the cells in any suffix of the ordering forms a simply-connected set, and such that each cell meets the union of the set of cells coming after it in the ordering in a simply-connected subset. In this section we prove a simpler version of this strengthening: define a *pseudo-shelling* to be an ordering of the cells such that the union of the cells in any suffix of the ordering is simply connected, without the requirement on how each cell meets this union. That is, we want to remove elements one by one from our mesh in such a way that the remaining elements always form a ball-shaped domain. Then we prove here that every simply-connected domain with an even number of quadrilateral faces has a pseudo-shelling.

To do this, we take a dual view similar to that in Theorem 4 and in [18]. For any topological quadrilateral mesh on a topological-sphere surface, define a curve arrangement by connecting each opposite pair of edge midpoints in each quadrilateral by a segment of a curve (Figure 14). This is a *simple* arrangement, meaning that at most two curves intersect at any point, and whenever two curves intersect they cross each other at a single point. Because the surface has no boundary, the curves have no endpoints; each curve crossing corresponds to a mesh element. Each element removal in a pseudo-shelling corresponds to a flip in the boundary quadrilateral mesh, and we consider these flips in terms of their effect on the dual curve arrangement (Figure 15). By finding a sequence of flips that can be used to transform any such arrangement into the arrangement dual to a single cube's boundary, we will show that a pseudo-shelling exists.

Lemma 2. *A simple curve arrangement is dual to a quadrilateral mesh if and only if the graph formed by its crossing vertices and edges is 3-connected.*

Proof. Any simple curve arrangement can be dualized by forming a quadrilateral for each crossing, however the resulting collection of cells must meet face-to-face and vertex-to-vertex to be a cell complex. It is not hard to see that, if the graph has one or two vertices that separate it into nontrivial components, then it must have a degeneracy that prevents it from being a cell complex; for instance, a single articulation point corresponds to a quadrilateral that is adjacent to itself along a pair of edges. Conversely, if an arrangement forms a 3-connected planar graph, then by Steinitz' theorem [25] it can be represented as the vertices and edges of a convex polyhedron, and the polar polyhedron's facets form the desired dual mesh. □

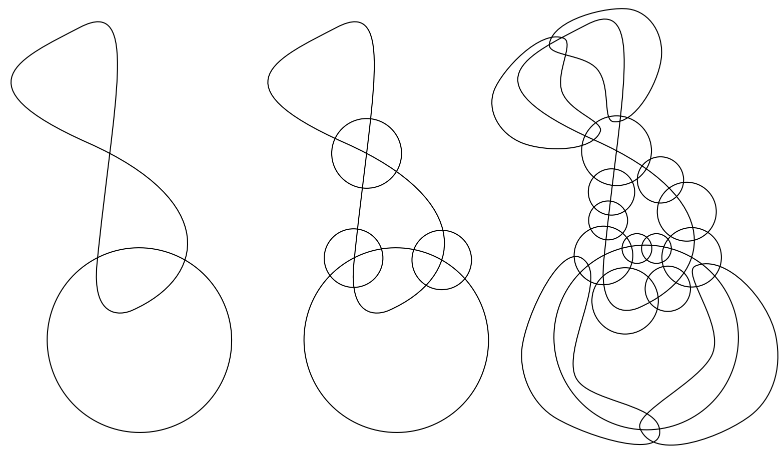

Fig. 16. Bubble-wrapping a curve arrangement

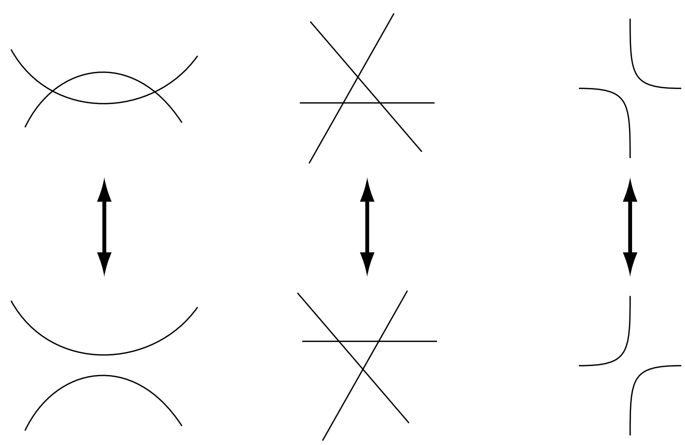

Fig. 17. Three operations on arrangement: pulling apart or pushing together (left); inverting a triangle (middle), and switching (right).

Since not all arrangements are 3-connected, it is convenient to work with a modification of the mesh that is guaranteed to be 3-connected. For a given arrangement A, we define the *bubble-wrapped arrangement* $B(A)$ by adding curves to A as follows: first, form a small circular curve around each vertex of A. Second, repeat this step around each new crossing point formed by the new circles and the edges of A. Each edge has two circles added in this second step, and these circles should be drawn so that they cross each other, and so that the edge passes through their intersection. Figure 16 depicts these bubble-wrapping steps.

Lemma 3. *If A is a connected arrangement, $B(A)$ is 3-connected.*

Lemma 4. *If A is the dual to a quadrilateral mesh, $B(A)$ can be formed by flips from A.*

Proof. We create all the circles used in the bubble-wrapping process by $(2,0)$–$(0,2)$ flips, and cross the two circles on each edge by a $(1,0)$–$(0,1)$ flip. It is not hard to see that each step in this process preserves 3-connectivity of the arrangement. $\square$

We now define three operations that modify the topology of a curve arrangement in simpler ways than the modifications performed by the flip duals; in some ways these operations are similar to the well-known Reidemeister moves for knots in $\mathbb{R}^3$, or the Whitney moves for curves in $\mathbb{R}^2$ [16, 24]. If the arrangement

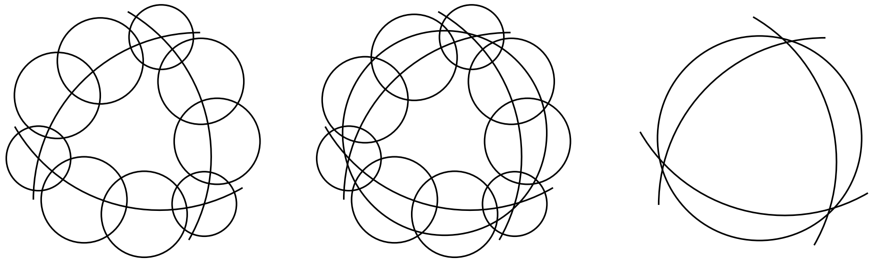

Fig. 18. Steps in inversion of a triangle in a bubble-wrapped arrangement.

has a pair of crossings that are connected by two edges, without other curves between them, we can *pull apart* the crossings and form an arrangement with two fewer vertices; conversely we can *push together* two nearby curves to add two vertices to an arrangement (Figure 17, left). If the arrangement has a face formed by a triangle of three crossings and three curves, we can *invert* the triangle by passing one curve across the crossing of the other two (Figure 17, middle). Finally, if the arrangement contains two nearby connected and non-crossing segments of curves, we can *switch* these segments by removing them from the arrangement and reconnecting the curves by two other non-crossing segments (Figure 17, right).

Lemma 5. *Suppose two connected arrangements A and A' are related to each other by one of the three operations described above. Then $B(A)$ can be transformed into $B(A')$ by a sequence of flips.*

Proof. We begin by describing the simplest case, inversion of a triangle. If A were not bubble-wrapped, we could perform this operation by a single $(0,0)$–$(0,0)$ flip, however the difficulty is in rearranging the bubble-wrap curves to allow this flip without violating 3-connectivity. Suppose A contains a triangle that can be inverted; its bubble-wrapped version $B(A)$ is depicted in Figure 18 (left). We perform a $(2,0)$–$(0,2)$ flip at one of the crossings of $B(A)$ in this triangle, adding a new curve to the arrangement, and perform a sequence of $(1,0)$–$(0,1)$ and $(0,0)$–$(0,0)$ flips to extend the region interior to this new curve until it covers the whole triangle without extending outside the bubble-wrapping of the triangle (Figure 18, middle). Next, we remove the nine curves forming the bubble-wrapping around the triangle, by using $(0,0)$–$(0,0)$ and $(0,1)$–$(1,0)$ flips to reduce the size of the region each one contains until it can be removed by a $(0,2)$–$(2,0)$ flip (Figure 18, right). At this point the triangle is exposed in the interior of the newly added curve and can be inverted by a $(0,0)$–$(0,0)$ flip, after which we reverse the above steps to restore the bubble-wrapping around the triangle. It is not difficult to verify that each step in this process can be performed while preserving 3-connectivity of the overall arrangement.

Similarly, pulling apart and pushing together can be performed by modifying the bubble-wrapping to remove the eight curves protecting the crossings and edges in the pulled-apart region, and replace them by a single curve surrounding an adjacent face of the arrangement, in a way that allows the change to A to be made by a single $(1,0)$–$(0,1)$ or $(0,1)$–$(1,0)$ flip. Switching can be performed by adding curves around the face of the arrangement between the switched edges, so that one curve crosses both switched edges and another curve passes between them, in a way that allows the change to A to be made by a single $(1,1)$–$(1,1)$ flip. The details are more complicated than inversion, and we omit them. $\square$

Lemma 6. *Any simply-connected domain, with a boundary consisting of an even number of quadrilaterals, has a mesh with a pseudo-shelling.*

Proof. As stated at the start of the section, we describe the pseudo-shelling in terms of the effect each removal of a hexahedral element causes to the curve arrangement A dual to the domain boundary. By Lemma 4,

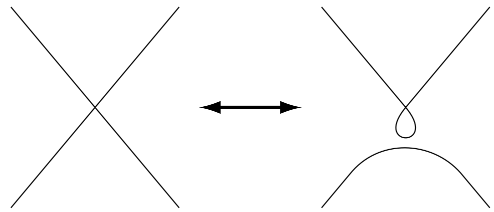

Fig. 19. Switching near a crossing.

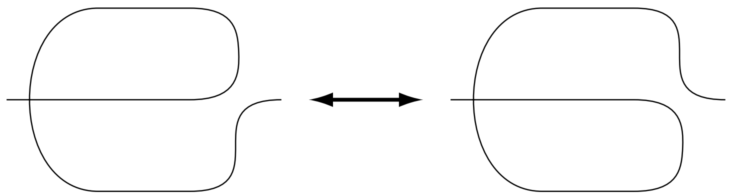

Fig. 20. Switching near a loop.

we can form an initial sequence of flips to bubble-wrap A. Next, we apply Lemma 5 to perform a sequence of switching operations near each crossing of A. Each such operation creates a loop connecting the crossing to itself (Figure 19), and reduces the number of crossings without such loops; it is always possible to choose a way to perform the switching operation that preserves the connectivity of the overall arrangement.

After this step, the transformed arrangement must consist of a single simple closed curve, decorated with a sequence of loops. Since A initially had an even number of crossings, and each flip preserves parity, there are an even number of loops along the curve. Note that a switching operation near the crossing of a loop can result in a new loop on the other side of the simple closed curve (Figure 20). We perform additional switching operations near the loops' crossings if necessary so that the loops can be grouped into adjacent pairs, with one member of each pair interior to the simple closed curve and one member exterior. Each pair can be removed by a sequence of a push, triangle inversion, and two pull operations (Figure 21); this sequence of operations is known to topologists as the *Whitney trick*. After repeated Whitney tricks, the arrangement is transformed into a simple closed curve protected by bubbles; we remove bubbles until only two are left, forming the dual arrangement of a cube, the desired final state of a pseudo-shelling. □

The existence of a mesh with a shelling, rather than just a pseudo-shelling, can be deduced via the methods used to prove Theorem 5 below.

8 Connectivity

The results of Section 6 naturally raise the question whether parity is the only obstacle to reachability by flips, or whether there might be some more subtle property of a mesh that prevents it being formed by flipping from some other mesh. Phrased another way, can every mesh transformation that replaces some bounded submesh by another be simulated by a sequence of flips? Figure 22 depicts a difficult example, closely related to a hexahedral meshing problem posed by Schneiders [20]. We leave finding a flip sequence for this example as a puzzle for the reader.

For any domain with boundary mesh, and a type of mesh to use for that domain, define the *flip graph* [19] to be a graph with (infinitely many) vertices corresponding to possible meshes of the domain, and an edge connecting two vertices whenever the corresponding two meshes can be transformed into each other by a single flip. In this framework, the question above can be phrased as asking for a description of the connected components of the flip graph.

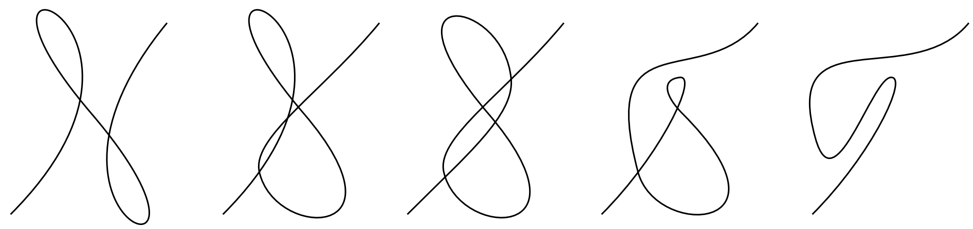

Fig. 21. Removing two adjacent loops by a push, invert, and two pull operations (the Whitney trick).

Theorem 5. *The flip graph for topological quad meshes of any simply-connected domain has exactly two connected components.*

Proof. Due to parity, there must be at least two components. It remains to show that any two meshes M_1 and M_2 of the same parity can be flipped into each other. Consider a three-dimensional ball, the upper hemisphere of which is partitioned into mesh M_1 and the lower hemisphere of which is partitioned into mesh M_2, the two meshes meeting along the equator of the ball at corresponding boundary edges. Because M_1 and M_2 have the same parity, the ball has an even number of quadrilateral faces and so has a topological hex mesh H.

The possible existence of pairs of quadrilaterals meeting in more than one edge (along the ball's equator) does not lead to difficulties with degenerate meshes, as we can remove these degeneracies by performing $(2,0)$–$(0,2)$ and $(0,2)$–$(2,0)$ flips in M_1 and M_2 respectively.

We now wish to remove the cuboids from H one by one, reducing the initial ball to successively smaller shapes. At each step in the removal process, we will maintain a mesh M describing describing the upper boundary of the remaining shape; initially $M = M_1$. As long as each removed cuboid touches the upper boundary of M in a simply connected set of faces, the change to M caused by its removal is a flip. If we can remove all the cuboids in this way, we will have flipped M_1 into M_2.

The requirement of simply-connected incidence with M for each removed cuboid is very similar to the definition of a *shelling* of a complex, as described in the previous section; it differs from a shelling only in the requirement that cuboids be removed from the upper boundary. As shown in the previous section, we can at least choose H to be pseudo-shellable. We then partition the pseudo-shelling sequence into *layers* of independent sets of hexahedral elements, such that no two elements in the same layer share a facet. Let B_i denote the ball formed by the union of layer i and all successive layers. The boundary of each B_i is a quadrilateral mesh; we thicken each quadrilateral of this mesh into a hexahedron, forming another layer of hexahedra separating layer i from layer $i-1$. In terms of the dual surface arrangement this corresponds to adding a topological-sphere surface surrounding B_i. After performing this thickening process, we have a mesh with alternating nested layers of cells: half of the layers consist of cells connecting two nested and combinatorially equivalent quad meshes, while the other half of the layers consist of cells not sharing any facets with each other.

We order the cells of this mesh as follows: start by choosing some cell adjacent to M_1, then some cell in the next layer adjacent to the starting cell, and continue "drilling down" by removing at most one cell from each layer until reaching the center of the set of concentric spheres. Then, remove the layers from the inside out, removing all cells from a given layer before starting the next outer layer. The existence of an appropriate ordering within each layer follows from the existence of shellings for all planar complexes. $\square$

As a consequence, the set of flips together with the parity-changing operation depicted in Figure 11 form a complete set of local mesh transformations, in that they are able to simulate any other transformation operation. In fact, since the parity-changing operation can simulate many of the possible flips (Figure 23), it and the $(2,0)$–$(0,2)$ flip together form a complete set.

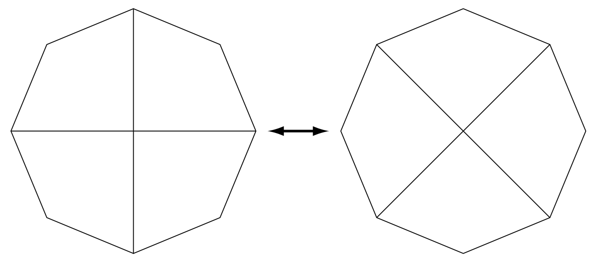

Fig. 22. Transformation of one four-quad mesh into another. Can it be simulated by a sequence of flips?

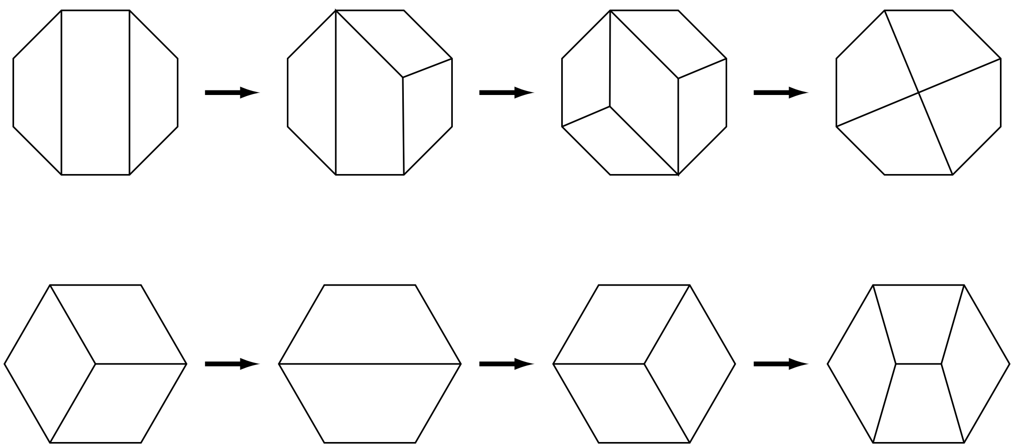

Fig. 23. Simulation of flips by parity-change operation: half of $(1,1)$–$(1,1)$ flip (top), $(0,0)$–$(0,0)$ flip (bottom, first and third positions), and $(1,0)$–$(0,1)$ flip (bottom, second and fourth positions).

Combining the proof method with our previous linear-complexity hex meshing techniques [6] shows that any two equal-parity quad meshes M_1 and M_2 of a simply-connected region can be flipped one to the other (as topological meshes) in a number of steps linear in the total number of cells in the two meshes.

Conversely to the techniques of this proof, any sequence of flips from one quad mesh to another can be seen as forming a hex mesh of a three dimensional region bounded above and below by the two meshes. The assumption of simple connectivity is necessary for the result, as can be seen from Figure 24: the region bounded by the two meshes has odd cycles on its boundary (triangles formed by one diagonal edge of the four-quad mesh and two edges of the eight-quad mesh) that can be contracted to points in the interior of the region; Mitchell [17] has shown that such contractible odd cycles make hex meshing an impossibility. Therefore there is no way to flip one mesh to the other, despite their having equal parity.

Open Problem 5 *Can we characterize the flip graphs for geometric quad meshes of non-simply-connected regions? Is the number of components of the flip graph determined by the number of topological holes in the region?*

For the analogous problem of geometric triangle mesh flip graph connectivity, a single connected component is known to exist due to the ability to flip to the Delaunay triangulation, and the ability to add or remove vertices by changing the weights of a regular triangulation. However, higher dimensional flip graphs can be disconnected [19]. There has also been some work on finding efficient sequences of flips [8, 9, 21] although finding the shortest such sequence even for triangulations of convex polygons remains a major open problem.

Open Problem 6 *Can the flip graph for topological or geometric hex meshes of a simply connected region have more than two components?*

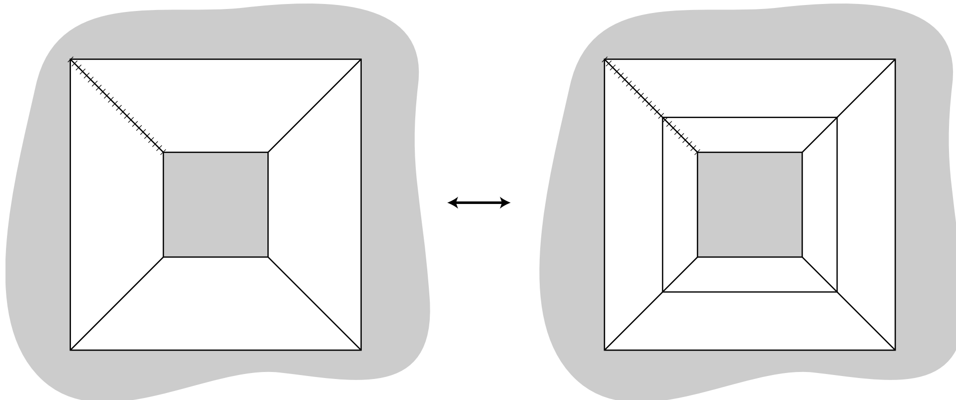

Fig. 24. Transformation of a non-simply-connected quad mesh that cannot be achieved by flips. The three edges indicated by hash marks form a contractible triangle in the region bounded above and below by the left and right meshes.

Answering this question seems to require a four dimensional generalization of Mitchell and Thurston's result. Even the tetrahedral mesh version of this problem seems difficult: topological tetrahedron mesh flip graphs are connected [13], and flipping can be used to form Delaunay triangulations [5, 10], but this does not seem to imply geometric tetrahedral flip graph connectivity since the flipping-based Delaunay construction algorithms do not allow the initial triangulation or sequence of flips to be chosen arbitrarily.

9 Conclusions

In this paper, we have introduced (or at least popularized) flipping moves for quad and hex meshes. We have also considered the conditions under which flips are realizable, tested the connectivity of the flip graph, and used flipping to show some evidence that not all quad surface meshes can be extended to hex volume meshes.

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