Flipping Edges on Triangulations

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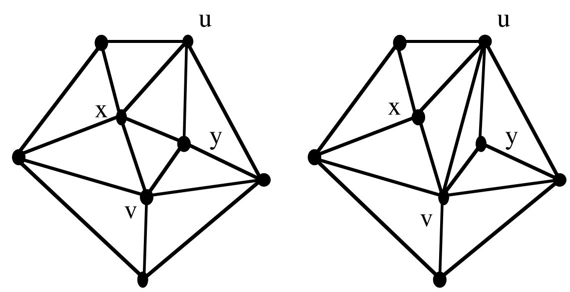
Abstract

In this paper we study the problem of flipping edges in triangulations of polygons and point sets. We prove that if a polygon Q_n has k reflex vertices, then any triangulation of Q_n can be transformed to another triangulation of Q_n with at most $O(n+k^2)$ flips. We produce examples of polygons with two triangulations T and T' such that to transform T to T' requires $O(n^2)$ flips. These results are then extended to triangulations of point sets. We also show that any triangulation of an n point set always has $\frac{n-4}{2}$ edges that can be flipped.

1. Introduction

Let $P_n = \{v_1, \ldots, v_n\}$ be a collection of points on the plane. A *triangulation of* P_n is a partitioning of the convex hull $Conv(P_n)$ of P_n into a set of triangles $T = \{t_1, \ldots, t_m\}$ with disjoint interiors in such a way that the vertices of each triangle t_i of T are points of P_n. The elements of P_n will be called the vertices of T and the edges of the triangles $t_1, \ldots, t_m$ of T will be called the edges of T. The degree $d(v_i)$ of a vertex v_i of T is the number of edges of T that have v_i as an endpoint. We say that an edge e of T can be flipped if e is contained in the boundary of two triangles t_i and t_j of T and $C = t_i \cup t_j$ is a convex quadrilateral. By flipping e we mean the operation of removing e from T and replacing it by the other diagonal of C. See Figure 1.

Given a collection of points $P_n = \{v_1, \ldots, v_n\}$ we define the graph $G_T(P_n)$, the graph of triangulations of P_n, to be the graph such that the vertices of $G_T(P_n)$ are the triangulations of P_n, two triangulations being adjacent if one can be obtained from the other by flipping an edge.



Two triangulations of a point set. The second one is obtained from the first by flipping edge xy.

Figure 1

Given two triangulations T' and T'' of P_n, we say that they are at distance k if there is a sequence of triangulations $T_0 = T', \ldots, T_k = T''$ such that T_{i+1} can be obtained from T_i by flipping an edge of it, $i = 0, \ldots, k-1$. This is equivalent to saying that if we consider T' and T'' as vertices of $G_T(P_n)$ their distance in it is k. We will also say that T' can be *transformed* into T'' by flipping k edges.

Triangulations of polygons with or without holes, the flipping of edges in them and their corresponding graphs of triangulations are defined in an analogous way. Throughout this paper, P_n will be used to denote point sets and Q_n will always denote polygons. The vertices of Q_n will always be assumed to be labeled $v_1, \ldots, v_n$ in the clockwise direction.

Triangulations of point sets and polygons on the plane have been studied intensely in the literature both because of their intrinsic beauty and for their use in many problems, such as image processing [22], mesh generation for finite element methods [2, 9, 23, 29], scattered data interpolation [15, 18] and many others such as computer graphics, solid modeling and geographical information systems [1, 3, 4, 17, 19, 20, 21, 25, 27, 28]. In this paper we study triangulations of point sets, polygons and polygons with holes on the plane.

It is well known that if a polygon Q_n is convex, then the diameter of $G_T(Q_n)$ is at most $2(n-3)$. Graphs of triangulations of convex polygons have been studied in [8, 24]. If Q_n is a convex polygon on n vertices, $G_T(Q_n)$ is isomorphic to the *rotation graph RG(n-2)*. The vertex set of *RG(n-2)* is the set of all binary trees with *n-2* vertices, [24].

It is also known that the graph of triangulations of a simple polygon Q_n with n vertices is connected [3, 6, 11, 12, 13, 17] and that its diameter is at most $O(n^2)$[8]. Some further result on the graph of triangulations of convex polygons have been obtained in [8].

In Section 2 we give a new and simple proof that the graph of triangulations of a polygon, *with or without holes*, is connected. Next we show that there are polygons with $2n$ vertices such that the diameter of their graph of triangulations is $O(n^2)$. We would like to remark here that our proofs do not use Delauney flips at all. A similar result to ours, concerning triangulations of point sets appears in [6], however, the flips used there are only use Delauney flips. In fact, from the results of our paper, we conclude that Delauney flips or triangulations are not an essential tool in the study of triangulations; they may even hinder their study! We then develop two algorithms that transform any triangulation T of Q_n into any other triangulation $T^©$. The number of flips required by our first algorithm is at most the number of edges of the *visibility graph* of Q_n. Our second algorithm uses at most $cn+k^2$ flips where k is the number of *reflex* vertices of Q_n.

In Section 3 we study triangulations of point sets on the plane. Our main result in that section is to prove that any triangulation of a point set P_n of n points on the plane contains at least $\frac{(n-4)}{2}$ edges that can be flipped. Our bound is tight. We would like to remark here for those readers familiar with *regular* triangulations that our results are for arbitrary triangulations of point sets, not for regular triangulations. We recall that regular triangulations are known to have at least $n-2$ flips; moreover some of the flips allowed for regular triangulations are not allowed in our case.

2. Triangulations of Polygons

We start this section by giving a simple proof that the graph of triangulations $G_T(Q_n)$ of a simple polygon Q_n is connected and that the diameter of $G_T(Q_n)$ is at most the number of edges of the visibility graph of Q_n.

Let T be a triangulation of a polygon Q_n, and v_i and v_j be vertices of Q_n such that the line segment v_iv_j connecting them is not an edge of T. We say that v_iv_j can be inserted in T by flipping $k-1$ edges if there is a sequence of triangulations $T_1=T,\ldots,T_k$ such that v_iv_j is an edge of T_k and T_{i+1} can be obtained from T_i by flipping an edge of it, $i=1,\ldots,k-1$. We say that a vertex v_i of Q_n is exposed if it lies in the convex hull of Q_n. Consider the two vertices v_{i-1} and v_{i+1} of Q_n adjacent to v_i. The shortest polygonal chain joining v_{i-1} to v_{i+1} totally contained in Q_n will be denoted by $P_{i-1,i+1}$

The *visibility graph* of Q_n is the graph with vertex set $\{v_1,\ldots,v_n\}$. Two vertices v_i and v_j of Q_n

are adjacent in the visibility graph of Q_n if the line segment joining them is contained in Q_n. We now prove:

Lemma 2.1: Let Q_n be a simple polygon, v_i an exposed vertex of Q_n and T a triangulation of Q_n. Then it is always possible to insert all the edges of $P_{i-1,i+1}$ into T using exactly as many flips as the number of edges of T, not in $P_{i-1,i+1}$, that intersect $P_{i-1,i+1}$.

Proof: Suppose that at least one edge e of $P_{i-1,i+1}$ is not in T. Consider the polygon P_e formed by the union of all triangles of T intersected by e and the chain of vertices of P_e joining the endpoints of e. At least one of these vertices, say w, is a convex vertex of P_e, and thus the edge joining v_i to w can be flipped decreasing the number of edges of T that intersect e by one. Our result follows (see Figure 2).

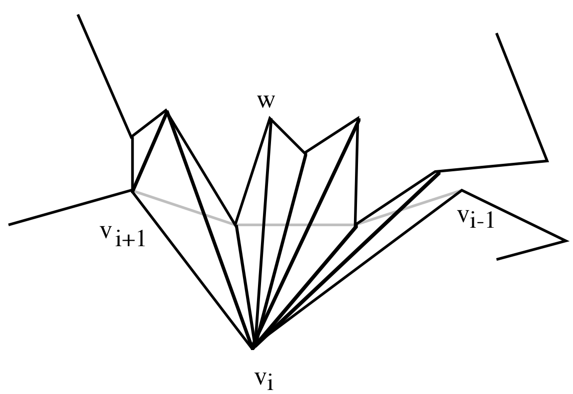


Figure 2

We can now prove:

Theorem 2.1: The graph of triangulations $G_T(Q_n)$ of a simple polygon is connected. Moreover, the diameter of $G_T(Q_n)$ is at most the number of edges of the visibility graph of Q_n.

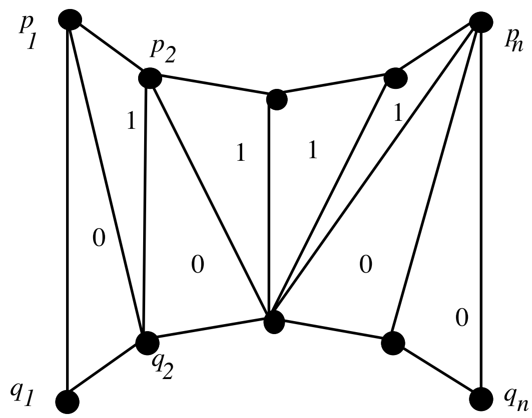
Proof: Let v_i be an exposed vertex of Q_n, and T_1 and T_2 two triangulations of Q_n. By Lemma 2.1 we can insert in each of T_1 and T_2 all the edges of $P_{i-1,i+1}$ to obtain two new triangulations T_1' and T_2' of Q_n. Delete from Q_n the subpolygon bounded by the vertices of $P_{i-1,i+1}$ and v_i. This will result in a collection of simple polygons with disjoint interiors. Each of these polygons has two triangulations induced by T_1' and T_2' respectively and fewer vertices than Q_n. Our result now follows by induction on the number of vertices of Q_n. Our argument actually gives a diameter of twice the number of edges of the visibility graph of Q_n. A simple modification to it will give the claimed bound; the details are left to the reader.

To prove the second part of our result, we simply notice that each edge of the visibility graph of Q_n incident to v_i may be used twice; the first time while inserting $P_{i-1,i+1}$ into T_1 and the second time when we insert T_2 into $P_{i-1,i+1}$. Once we delete v_i from Q_n these edges are not used again, and our result follows.

The bound on the diameter of $G_T(Q_n)$ given in Theorem 2.1 can, in general, be bad. For example, when Q_n is a convex polygon, the visibility graph of Q_n has $O(n^2)$ edges, while the diameter of $G_T(Q_n)$ is at most $2(n-2)$. On the positive side, if the visibility graph of Q_n has few edges, Theorem 2.1 gives us an efficient method to transform one triangulation into another one. Notice that if the visibility graph of Q_n has few edges, it has many reflex vertices. Thus the question of studying the tradeoffs in the diameter of $G_T(Q_n)$ and the number of reflex vertices of Q_n becomes relevant. We address this question now.

We start by producing a polygon Q_n with $2n$ vertices such that the diameter of $G_T(Q_n)$ is exactly $(n-1)^2$.

Consider the polygon with $2n$ vertices $Q_{2n} = \{p_1, \ldots, p_n, q_1, \ldots, q_n\}$ such that $\{p_1, \ldots, p_n\}$ lie on a convex curve, $\{q_1, \ldots, q_n\}$ lie on a concave curve and the line joining p_i to p_j, $1 \le i < j \le n$ leaves all the elements of $\{q_1,\ldots,q_n\}$ below it, and all the elements of $\{p_1,\ldots,p_n\}$ lie above any line joining q_i to q_j, $1 \le i < j \le n$; see Figure 3.



Q_{10}

Figure 3

We now show that there are two triangulations of Q_{2n} such that to transform one into the other requires exactly $(n-1)^2$ flips. This will prove our result.

Consider any triangulation T of Q_{2n}. We assign a code to T as follows:

Each triangle t_i of T has either two vertices in $\{p_1,\ldots,p_n\}$ or two vertices in $\{q_1,\ldots,q_n\}$. In the first case, assign a 1 to t_i; in the second case, t_i is assigned a 0. See Figure 3.

If we read the numbers assigned to the triangles of T from left to right, we obtain an ordered sequence of 0's and 1's; this sequence is the code assigned to our triangulation.

The triangulation of Q_{10} presented in Figure 3 receives the code 01011100. It is clear that each triangulation of Q_{2n} is thus assigned a sequence containing $n-1$ 0's and $n-1$ 1's. Clearly, each sequence of $n-1$ 0's and $n-1$ 1's also defines a unique triangulation of Q_{2n}, and thus we have a one-to-one correspondence between the set of triangulations of Q_{2n} and the set of binary sequences containing $n-1$ 1's and $n-1$ 0's. Flippings of triangulations can be easily identified within this encoding. An internal edge of a triangulation T can be flipped if the triangles of T containing it have been assigned a 1 and a 0. Moreover, a flip of T corresponds to a transposition in the code of T of a 0 with a 1!

Consider the triangulations T_1 and T_2 of Q_{2n} that receive the encodings 11...100...0 and 00...011...1. It is now clear that to transform T_1 to T_2 we need $(n-1)^2$ flips. We have just obtained:

Theorem 2.2: The diameter of $G_T(Q_{2n})$ is exactly $(n-1)^2$.

We close this section by proving that if Q_n is a polygon with k reflex vertices, then the diameter of $G_T(Q_n)$ is $\Omega(n+k^2)$, i.e. the diameter of the graph of triangulations of a polygon depends heavily on the number of its reflex vertices; the number of convex vertices of Q_n hardly matters at all! We now prove:

Theorem 2.3: Let Q_n be a simple polygon with k reflex vertices. Then the diameter of $G_T(P_n)$ is at most $O(n+k^2)$.

Several lemmas, definitions and observations will be needed before we can prove Theorem 2.2.

Two vertices v_i and v_j of a polygon Q_n are called *c-connected* if they are visible and the vertices $v_{i+1},\ldots,v_{j-1}$ of Q_n are all convex, addition taken mod n. If in addition, v_i and v_j are reflex vertices of Q_n, we call v_i and v_j *consecutive reflex* vertices of Q_n.

Let v_iv_j be the line segment joining vertices v_i and v_j. If v_iv_j is such that, for each edge e of T intersecting v_iv_j, the end vertex of e below v_iv_j is a convex vertex of Q_n, or for each edge e of T

intersecting $v_i v_j$ the end vertex of e above $v_i v_j$ is a convex vertex of Q_n, we call $v_i v_j$ a *proper diagonal* of T.

The following lemma will prove useful to us:

Lemma 2.1: Let $v_i v_j$ be a proper diagonal of a triangulation T of a polygon Q_n. Then if $v_i v_j$ is intersected by t edges of T, $v_i v_j$ can be inserted in T using at most $2t$ flips.

Proof: Let $v_i v_j$ be a proper diagonal of T. Assume without loss of generality that for each edge e of T intersecting $v_i v_j$, the end vertex of e below $v_i v_j$ is a convex vertex of Q_n. See Figure 4.

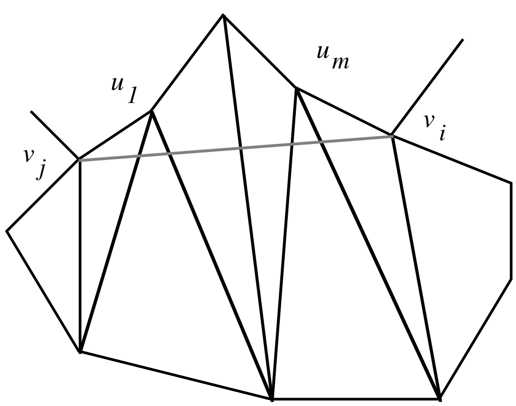


Figure 4

Let $Q_{i,j}$ be the subpolygon of Q_n obtained by joining all the triangles of T intersected by $v_i v_j$ and consider the triangulation T' of $Q_{i,j}$ induced by T in $Q_{i,j}$. Suppose that $v_i v_j$ is intersected by t edges of T', $t \geq 1$. We now show that $v_i v_j$ can be inserted in T' by flipping at most $2t$ edges. To show this, it is enough to show that by flipping at most two edges of T' we can obtain a new triangulation of $Q_{i,j}$ in which $v_i v_j$ is intersected by $t-1$ edges. Let $u_1, \ldots, u_m$ be the vertices of $Q_{i,j}$ between v_j and v_i in the clockwise direction. At least one of these vertices, say u_l, is a convex vertex of $Q_{i,j}$; otherwise, v_i and v_j would not be visible in Q_n. If in T' u_l is adjacent to exactly one element in the chain $v_{i+1}, \ldots, v_{j-1}$, then the edge connecting them in T' can be flipped, reducing by one the number of edges that intersect $v_i v_j$. If u_l is adjacent to at least 3 vertices of $Q_{i,j}$ in $v_{i+1}, \ldots, v_{j-1}$, say v_{s-1}, v_s, v_{s+1}, then we can flip the edge $u_l v_s$ inserting $v_{s-1} v_{s+1}$ and our result follows. Suppose then that u_l is adjacent to exactly two vertices, say v_s and v_{s+1} in $v_{i+1}, \ldots, v_{j-1}$. See Figure 5.

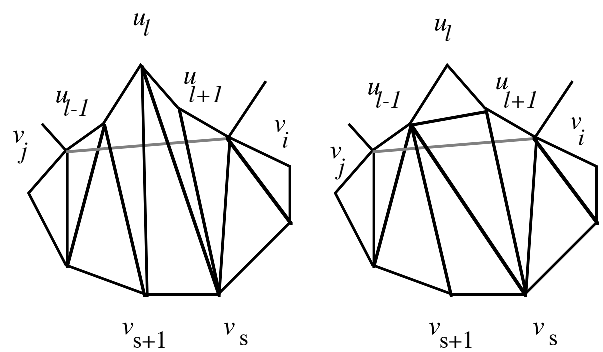


Figure 5

Notice that since u_l is convex, we can flip $u_l v_{s+1}$. Next flip $u_l v_s$, and the number of edges intersecting $v_i v_j$ has gone down by one! Our result now follows.

A polygon Q_n is called a spiral polygon if the vertices of Q_n can be labeled $v_1, \ldots, v_s, v_{s+1}, \ldots, v_n$ such that $v_1, \ldots, v_s$ are reflex vertices of Q_n and $v_{s+1}, \ldots, v_n$ are convex vertices of Q_n. We now prove:

Lemma 2.2: Let Q_n be a spiral polygon. Then the diameter of $G_T(Q_n)$ is at most $2n-6$.

Proof: We define a special triangulation T_0 of Q_n as follows: First join $p'_0 = v_{n-1}$ to all the vertices of Q_n visible from it. Let p_1 and p'_1 be the last reflex and convex vertices visible from p_{n-1} respectively. See Figure 6. Join p_1 and p'_1 and iterate our construction until we obtain a triangulation of Q_n. See Figure 6. We now claim that any triangulation of Q_n is at distance at most $n-3$ from T_0. Let T be any triangulation of Q_n. If v_{n-1} is adjacent in T to all the vertices visible from it, our result follows by induction. Otherwise, it is not difficult to see that T

contains an edge that can be flipped, increasing the degree of v_{n-1} by one. Once v_{n-1} is connected to all the vertices of Q_n visible from it, the edge p_1p_1' must be present in the current triangulation of Q_n. Since each flip adds one diagonal of T_0 and T_0 has $n-3$ diagonals, our result now follows.

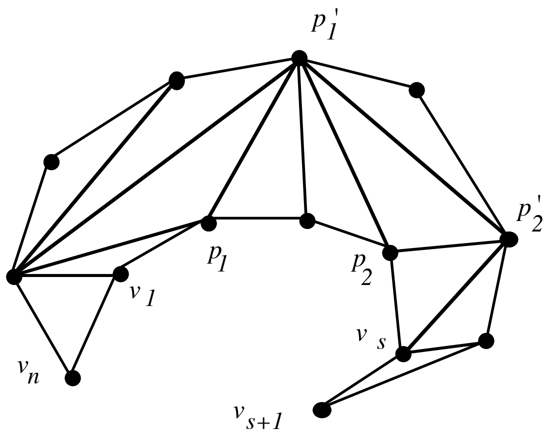


Figure 6

Suppose next that Q_n has k reflex vertices labeled $v_{i_1},\dots,v_{i_k}$ such that $i_1<\dots<i_k$. For each $j=1,\dots,k$ let R_j be the shortest polygonal chain contained in Q_n joining v_{i_j} to $v_{i_{j+1}}$, addition taken mod k. Finally let $R=R_1\cup\dots\cup R_k$. See Figure 7.

The following lemma, which is easy to prove, is given without proof:

Lemma 2.3: Any edge joining two vertices of Q_n intersects at most two edges of R. Moreover if e is an edge of R and T is any triangulation of Q_n either e is an edge of T or e is a proper diagonal of T.

We now prove the last lemma we need to prove Theorem 2.2, namely:

Lemma 2.4: Let T be any triangulation of Q_n. Then all the edges of R can be inserted in T using at most $4(n-3)$ flips.

Proof: Let T be any triangulation of Q_n, and w be any edge of T. Then by Lemma 2.4, w intersects at most two edges of R. Since T has $n-3$ edges, the number of intersections between the edges of T and those of R is at most $2(n-3)$. However since all the edges of R are proper edges of T, each of these intersections can be removed by flipping at most two edges. Thus flipping at most $4(n-3)$ edges, we insert in T all the edges of R.

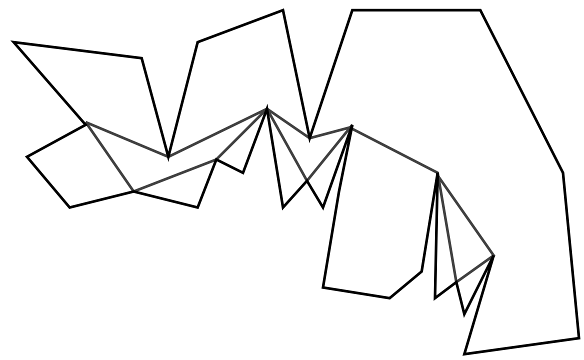


Figure 7

We can now finish the proof of Theorem 2.2.

Proof of Theorem 2.2. Let T and T' be any two triangulation of Q_n. By Lemma 2.4, by flipping at most $4(n-3)$ edges, we can transform each of them into triangulations T_1 and T_1' respectively of Q_n such that each of them contains all the edges of R.

Notice that the edges of R induce a partition of Q_n into a set of polygons of either one of these two types:

a) At most k convex or spiral polygons $Q^1,\dots,Q^m$, $m\le k$ bounded by edges of Q_n and edges of R

b) A set of polygons $R_1,\dots,R_s$ bounded by the edges of R such that the total number of edges of these polygons is at most k.

Notice that the total number of edges bounding $Q^1,\dots,Q^m$ is at most $n+k$. Both of T_1 and T_1' induce triangulations of $Q^1,\dots,Q^m$ which may be different. Since each $Q^1,\dots,Q^m$ is a spiral or a convex polygon, by Lemma 2.2 the triangulations induced by

T_1 in $Q^1,\ldots,Q^m$ can be transformed into those induced by T_1' in $Q^1,\ldots,Q^m$ using at most $2((n+k)-3)$ flips. Since the total number of edges bounding all the polygons in $R_1,\ldots,R_s$ is at most k, then by Theorem 2.1 or [8] the triangulations induced in them by T_1 and T_1' can be transformed into each other with at most $O(k^2)$ flips. Our result now follows.

3. Triangulations of Point Sets

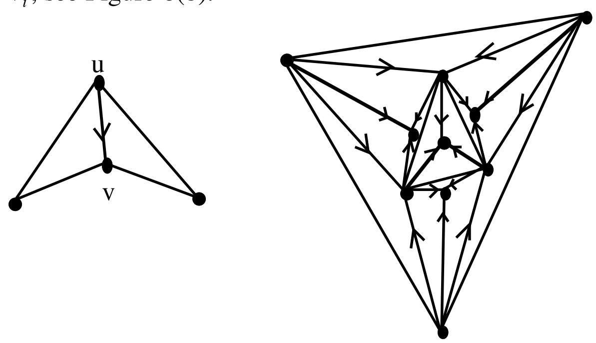
In this section we study triangulations of point sets on the plane. Our main goal is to answer the following question: Given a triangulation T of a collection $P_n = \{v_1,\ldots,v_n\}$ of n points on the plane, how many edges of T can be flipped? We show:

Theorem 3.1: Any triangulation of a collection P_n of n points on the plane contains at least $\frac{(n-4)}{2}$ diagonals that can be flipped. The bound is tight.

Some definitions will be needed before we can prove Theorem 3.1. Let T be a triangulation of P_n. Let us divide the set of edges of T into two subsets, $F(T)$, consisting of all the edges of T that can be flipped, and $NF(T)$, which contains those edges of T that are not flippable. Clearly all the edges of T contained in the boundary of $Conv(P_n)$ are not flippable. We orient the edges of $NF(T)$ as follows according to the following rules:

R1) If e is an edge of the convex hull of P_n, orient it in the clockwise direction around the boundary of the boundary of the convex hull $Conv(P_n)$ of P_n.

R2) If e is not in $Conv(P_n)$ let $C = t_i \cup t_j$ be the quadrilateral formed by the union of the two triangles t_i and t_j of T containing e in their common boundary. See Figure 8(a). Since C is not convex, it follows that one of the end vertices of e, say v_i, is a reflex vertex of C while the other end vertex of e, say v_j, is a convex vertex of C. Orient e from v_j to v_i; see Figure 8(b).



(a) (b)

Figure 8.

Let v_i be any vertex of T. We now define $d^-(v_i)$ to be the number of edges v_iv_j of T that cannot be flipped and that are oriented from v_j to v_i. Notice that $d(v_i)$ is the total number of edges of T incident with v_i, whereas $d^-(v_i)$ involves only edges of T that cannot be flipped. We now prove:

Lemma 3.2: Let v_i be any vertex of T. Then $d^-(v_i) \le 3$. Moreover if $d(v_i) \ge 4$ in T then $d^-(v_i)$ is at most 2.

Proof: It is clear that if v_i is in $Conv(P_n)$ then $d^-(v_i) = 1$. Suppose then that v_i is in the interior of $Conv(P_n)$. Two cases arise:

a) $d(v_i) = 3$ in T. In this case, it is easy to verify that all the edges of T incident with v_i are non-flippable and are oriented towards v_i. It follows that $d^-(v_i) = 3$.

b) $d(v_i) > 3$ in T. In this case it is trivial to verify that no more than two edges of T can be oriented towards v_i.

We are now ready to prove Theorem 3.1.

Proof of Theorem 3.1: Let P_n be a point set on the plane, T a triangulation of P_n and let S be the set of elements of P_n with degree 3 in T that are not in the convex hull of P_n. We now prove that T contains at least $\frac{(n-4)}{2}$ edges that can be flipped.

By adding a point w in the exterior of $Conv(P_n)$ and joining it to all the vertices of $Conv(P_n)$, we obtain a proper triangulation of the plane with $n+1$ points which by Euler's Theorem contains $3n-3$ edges. Let us classify the edges adjacent to w as non-flippable edges and orient them from w to their other vertex in $Conv(P_n)$. Next orient all non-flippable edges of T according to R1) and R2). Notice that with these orientations, $d^-(v_i) = 2$ for all the elements of P_n of in $Conv(P_n)$.

Remove from T all the elements of S. Notice that we will remove exactly $3|S|$ edges of T which are not flippable. Furthermore, notice that what remains is still a triangulation T' of $P_n - S + \{w\}$, which by Euler's formula contains exactly $2(|P_n - S| + 1) - 4 = 2(n - |S|) + 2$ triangles. Moreover, any elements v_i of $P_n - S + \{w\}$ that are not on the convex hull of P_n have degree at least 4 in T, and thus by Lemma 3.2 have $d^-(v_i) \leq 2$ in T.

Let Q be the set of vertices of $P_n - S + \{w\}$ that have $d^-(v_i) = 2$. Then by Lemma 3.2, we can associate to each element v_i of Q in the interior of $Conv(P_n)$ a different triangle $t(v_i)$ of T' which is also a triangle in T. See Figure 9.

To each vertex v_i of T'in the convex hull of P_n we can also associate a different 'triangle' of T' among those having w as one of their vertices. That is, to each vertex of T', except w and the vertices of T with $d^-(v_i) < 2$, we can associate a different triangle of T' that contains no element of S. Let m be the number of vertices of T that are on the boundary of $Conv(P_n)$ or have $d^-(v_i) = 2$. Since T' has $2(n - |S|) + 2$ triangles, it follows that $|S| \leq 2(n - |S|) + 2 - m$. It is easy to verify that the number of edges of T that can be flipped is minimized when all of the vertices v_i of $P_n - S$ not in $Conv(P_n)$ have $d^-(v_i) = 2$.

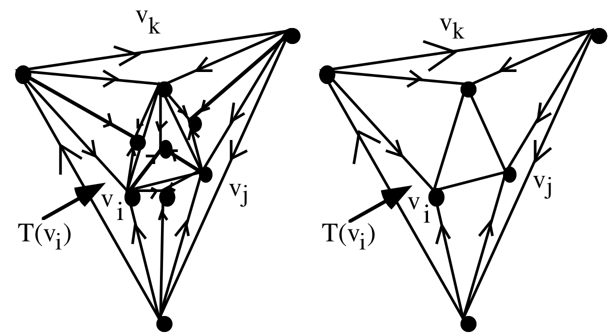


Figure 9

In this case, since we can associate to each element of $P_n - S$ a different empty triangle of T', we can easily verify that $|S| = n - |S| - 2$, that is,

(1) $\quad n = 2|S| + 2$.

Since T' contains $3(|P_n - S| + 1) - 6 = 3|P_n - S| - 3$ edges and each vertex of $P_n - S$ has $d^-(v_i) = 2$, the number of flippable edges of T (i.e. those edges of that are not oriented in T') is exactly :

(2) $\quad k = (3(n - |S|) - 3) - 2(n - |S|) = n - |S| - 3$.

Using (1) and (2) we get $k = \frac{n-4}{2}$ which concludes the first part of our proof.

We now show that our bound is tight. We give two different examples. Our first example is obtained as follows: Take any collection of m points that are the vertices of a convex polygon P_m on the plane, together with any triangulation of it. Next add to the interior of each triangle of this triangulation an extra vertex adjacent to the three vertices of each triangle. If the convex polygon has m vertices, our final point set has $2m - 3$ points, and the only edges that can be flipped are the $m - 3$ edges used to triangulate P_m. Trivially if $n = 2m - 2$, $m - 3 = \frac{n-4}{2}$.

More interesting examples with $3n+(n-2)=4n-2$ points in which exactly $\frac{(4n-2)-4}{2}=2n-3$ edges can be flipped will now be presented.

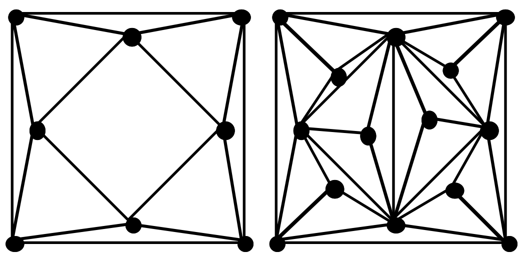


Figure 10

Consider a regular polygon R_n with n vertices. For every edge e of R_n place a point p_e in the interior of R_n on the perpendicular through the mid point of e and at distance ε from it. Join the end vertices of e to p_e and using all the new points construct a second regular polygon $R_n^{\copyright}$ with n vertices contained in R_n. Triangulate the interior of $R_n^{\copyright}$ and add a point to the interior of each triangle t of this triangulation of $R_n^{\copyright}$ adjacent to all the vertices of t. Now in the middle of each edge of $R_n^{\copyright}$, add a new vertex at distance $\delta < \varepsilon$ and join it to the vertices of the triangle containing it. This construction is illustrated for a square in Figure 10. It is not hard o see that the only edges of the triangulation we just defined that can be flipped are the edges of $R_n^{\copyright}$ plus the edges of the triangulation of $R_n^{\copyright}$. These are exactly $2n-3$ edges and since this construction will yield exactly $4n-2$ points our result follows.

We conclude this section by showing that some of our results for polygons presented in Section 2 can be easily generalized to point sets. We prove first:

Theorem 3.2: There are collections P_{2n} of $2n$ points on the plane such that the diameter of $G(P_{2n})$ is greater than $(n-1)^2$.

Proof: Let P_{2n} be the set of vertices of the polygon Q_{2n} presented in Section 2. Notice that any triangulation of P_{2n} will necessarily include the edges of Q_{2n}. Our result now follows by extending the triangulations of Q_{2n} at distance $(n-1)^2$ to triangulations of $Conv(P_{2n})$.

4. Polygons with holes

To finish this paper, we notice that the proof of Theorem 2.1 can be easily modified to show that the graph of triangulations of point sets and polygons with holes is connected. Theorem 2.3 can also be easily modified to work for polygons with holes. To avoid being repetitive, we leave the details of these proofs to the reader. Thus we have:

Theorem 4.1: The graph of triangulations of a point sets or polygons with holes on the plane is connected.

Theorem 4.2: Let Q_n be a simple polygon with k reflex vertices and admitting holes. Then the diameter of $G_T(P_n)$ is at most $O(n+k^2)$.

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