FLOCKING AND SYNCHRONIZATION OF PARTICLE MODELS

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Abstract. In this note, we present a multi-dimensional flocking model rigorously derived from a vector oscillatory chain model and study the connection between the Cucker-Smale flocking model and the Kuramoto synchronization model appearing in the statistical mechanics of nonlinear oscillators. We provide an alternative direct approach for frequency synchronization to the Kuramoto model as an application of the flocking estimate for the Cucker-Smale model.

1. Introduction. The purpose of this paper is to unify some popular notions in the *flocking* and *synchronization* of dynamical systems. In particular, we first consider a theme already addressed by Ha and Slemrod [9]: the relation of two, usually separately studied, flocking models, i.e. the Cucker-Smale system [4, 5, 12] and the oscillator chain system [6] pioneered by Erdmann et al. Both systems are known to exhibit *flocking* behavior, but until the paper [9] the two models had not been linked. In the original [9] paper only a simple scalar version was presented, so we use this paper to present the general vector case where the flock propagates in d-dimensional Euclidean space. Of

course, the value of such an approach is that it both unifies the theories and provides a rigorous proof of the flocking phenomena for both Cucker-Smale and the oscillator chain. As a second application of the *flocking* estimates, we show that the well-known Kuramoto synchronization model [10] also falls within our theory and hence a rigorous "flocking" theorem is immediately derivable.

The paper is divided into two sections after this Introduction. Section 2 is devoted to the oscillator chain and the application of Tikhonov's theorem to recover the Cucker-Smale system. *Flocking* is then proved for the Cucker-Smale system. Section 3 recalls Kuramoto's synchronization model [10] and again derives a rigorous proof of *flocking*.

2. From mechanical model to flocking model. In this section, we present a new particle model which reveals the flocking phenomena and can be rigorously derived from a singularly perturbed mechanical system. This is a multi-dimensional analogue of the attempt given in [9].

2.1. *Formal derivation.* Consider a simple mechanical model of mobile agents with a steep attracting force and a friction coefficient ($\varepsilon > 0$) moving in physical space $\mathbb{R}^d$:

$$\frac{dx_i}{dt} = v_i, \tag{2.1a}$$

$$\varepsilon \frac{dv_i}{dt} = \frac{\lambda}{N} \sum_{j=1}^{N} \nabla_x \varphi(x_j - x_i) - v_i, \tag{2.1b}$$

where λ is the nonnegative coupling constant, and the mobile agents are represented by point particles with the same mass ε, which will set the time scale for the flocking to occur. Damping is needed as a self-organizing mechanism to remove energy from the configuration, a role which may be played by bird wings in flight. The force potential φ is assumed to be isotropic and only dependent on the distance between two mobile agents: with abuse of notation, we set

$$\varphi(x) := \varphi(|x|).$$

Then the equation (2.1b) can be written as

$$\varepsilon \frac{dv_i}{dt} = \frac{\lambda}{N} \sum_{j=1}^{N} \varphi'(|x_j - x_i|) \frac{x_j - x_i}{|x_j - x_i|} - v_i. \tag{2.2}$$

We now take a time derivative of (2.2) to find the singularly perturbed second-order ODE system:

$$\varepsilon \frac{d^2 v_i}{dt^2} = \frac{\lambda}{N} \sum_{j=1}^{N} \left[\frac{\varphi'(r_{ij})}{r_{ij}} (v_j - v_i) + \left(\frac{\varphi''(r_{ij})}{r_{ij}^2} - \frac{\varphi'(r_{ij})}{r_{ij}^3} \right) ((x_j - x_i) \cdot (v_j - v_i))(x_j - x_i) \right] - \frac{dv_i}{dt}, \tag{2.3}$$

where $r_{ij} = |x_j - x_i|$. In the formal zero mass limit $\varepsilon \to 0$, the combined systems (2.1a) and (2.3) become

$$\frac{dx_i}{dt} = v_i,$$

$$\frac{dv_i}{dt} = \frac{\lambda}{N} \sum_{j=1}^{N} \left[\frac{\varphi'(r_{ij})}{r_{ij}} (v_j - v_i) + \left(\frac{\varphi''(r_{ij})}{r_{ij}^2} - \frac{\varphi'(r_{ij})}{r_{ij}^3} \right) ((x_j - x_i) \cdot (v_j - v_i))(x_j - x_i) \right]$$

$$:= \frac{\lambda}{N} \sum_{j=1}^{N} \left[\psi_1(r_{ij})(v_j - v_i) + \psi_2(r_{ij})((x_j - x_i) \cdot (v_j - v_i))(x_j - x_i) \right].$$
$$(2.4)$$

Note that for a simple choice of $\varphi(x) = \frac{|x|^2}{2}$ (which corresponds to the potential in the harmonic oscillator problem with a unit spring constant), the coefficients in the second equation of (2.4) become

$$\psi_1 = 1, \qquad \psi_2 = 0;$$

hence we can recover the Cucker-Smale system with all-to-all communication rate. The new component in the system (2.4),

$$\psi_2(r_{ij})((x_j - x_i) \cdot (v_j - v_i))(x_j - x_i),$$

seems to be related with the rotational motion of the flocking group. On the other hand, in the one-dimensional case $d = 1$, the two terms in the R.H.S. of the second equation of (2.4) can be combined to result in the corresponding term in the Cucker-Smale system, i.e.,

$$\frac{\lambda}{N} \sum_{j=1}^{N} \left[\psi_1(r_{ij})(v_j - v_i) + \psi_2(r_{ij})((x_j - x_i) \cdot (v_j - v_i))(x_j - x_i) \right]$$

$$= \frac{\lambda}{N} \sum_{j=1}^{N} \left(\psi_1(r_{ij}) + \psi_2(r_{ij})r_{ij}^2 \right)(v_j - v_i).$$

This is why we do not see this new component in Ha and Slemrod's one-dimensional analysis in [9].

We next look for an admissible class of potentials φ which guarantees the nonnegativity of the coefficients ψ_1 and ψ_2, i.e.,

$$\psi_1(r) = \frac{\varphi'(r)}{r} \geq 0, \qquad \psi_2(r) = \frac{1}{r} \left(\frac{\varphi'(r)}{r} \right)' \geq 0. \qquad (2.5)$$

By the direct observation of (2.5), we classify a potential φ:

$$\varphi(r) := \int_0^r s\psi_1(s)ds,$$

where ψ_1 satisfies

$$(i) \ \psi_1 > 0, \quad \psi_1' > 0,$$
$$(ii) \ \lim_{r \to 0^+} \psi_1(r) = R_1 > 0. \qquad (2.6)$$

The second condition (ii) in (2.6) is imposed for the strict dissipativity of the total energy (see Lemma 2.1 and Theorem 2.3). Note that once ψ_1 is determined in the above process, then the communication rate ψ_2 is determined by the relation

$$\psi_2(r) = \frac{1}{r}\psi_1'(r).$$

On the other hand, the admissible conditions (2.6) can be expressed in terms of the potential φ:

$$\varphi' > 0, \qquad \varphi'' \geq \frac{\varphi'}{r}, \qquad \frac{\varphi'}{r} \to R_1 \quad \text{as } r \to 0^+. \tag{2.7}$$

It is easy to see that the conditions (2.6) and (2.7) are equivalent to each other. Below, we present two admissible cases:

EXAMPLES. (i) Consider

$$\psi_1(r) = e^{-\frac{1}{1+r}}.$$

Then it is easy to see that this special choice satisfies the condition (2.6).

(ii) Consider

$$\varphi(r) = r^2 + r^4.$$

Again, this potential satisfies (2.7).

2.2. *Rigorous justification.* In this section, we provide a rigorous justification for the passage from the singularly perturbed system (2.1a) and (2.3) to the system (2.4) as a direct application of Tikhonov's theorem.

We first rewrite the particle system (2.1a) and (2.3) as a first-order system:

$$\frac{dx_i}{dt} = v_i, \quad 1 \leq i \leq N, \tag{2.8a}$$

$$\frac{dv_i}{dt} = z_i, \tag{2.8b}$$

$$\varepsilon\frac{dz_i}{dt} = \frac{\lambda}{N} \sum_{j=1}^{N} \Big(\psi_1(r_{ij})(v_j - v_i) + \psi_2(r_{ij})((x_j - x_i) \cdot (v_j - v_i))(x_j - x_i) \Big) - z_i. \tag{2.8c}$$

For the reader's convenience, we recall the classical theorem on the singular perturbation limit due to A. N. Tikhonov.

Consider the slow–fast dynamical system:

$$\begin{aligned}
\frac{dy_i}{dt} &= f_i(y, z, t), \quad i = 1, \ldots, n, \\
\mu_j \frac{dz_j}{dt} &= F_j(y, z, t), \quad j = 1, \ldots, m,
\end{aligned} \tag{2.9}$$

where $y = (y_1, \ldots, y_n), z = (z_1, \ldots, z_m)$, and μ_j are small positive parameters depending on a parameter μ in such a way that

$$\lim_{\mu \to 0} \mu_j(\mu) = 0, \quad \lim_{\mu \to 0} \frac{\mu_{j+1}}{\mu_j} = 0 \quad \text{or} \quad 1.$$

THEOREM 2.1 (Tikhonov [13]). Suppose the following conditions hold.

(1) The degenerate systems obtained by setting all $\mu_j = 0$,

$$\frac{dy_i}{dt} = f_i(y, z, t), \qquad F_j(y, z, t) = 0,$$

have continuous solutions.

(2) The roots $z_j = \psi_j(y, t)$ of $F_j(y, z, t) = 0$ have continuous first partial derivatives and are stable; i.e., the expression $\sum_j (z_j - \psi_j(y, t)) F_j(y, z, t)$ is negative in a suitable deleted neighborhood $\mathcal{N}$ of the roots.

Then as $\mu \to 0$, the solutions of (2.9) tend to the corresponding solutions of the degenerate system with the initial data $(y_i^0, z_j^0, t^0) \in \mathcal{N}$, and this convergence is uniform in a closed interval $[t_1, T_*]$ for any $t_1 > t^0$ and a constant $T_* > t_1$.

In our case, F_j is the right-hand side of (2.8c). The roots of $F_j = 0$ are of course the sum on the right-hand side of (2.8c), and hence the second assertion (2) in Tikhonov's theorem above becomes

$$\sum_{i=1}^N \left(z_i - \frac{\lambda}{N} \sum_{j=1}^N \Big(\psi_1(r)(v_j - v_i) + \psi_2(r)((x_j - x_i) \cdot (v_j - v_i))(x_j - x_i) \Big) \right)$$
$$\cdot \left(\frac{\lambda}{N} \sum_{j=1}^N \Big(\psi_1(r)(v_j - v_i) + \psi_2(r)((x_j - x_i) \cdot (v_j - v_i))(x_j - x_i) \Big) - z_i \right) < 0,$$

which is trivially true in the whole state space, excluding the roots themselves.

THEOREM 2.2. Let $(x_i^\varepsilon(t), v_i^\varepsilon(t), z_i^\varepsilon(t))$ be the solution to the singularly perturbed system (2.8a) - (2.8c) with initial data $(x_i(0), v_i(0), z_i(0))$. Then for any small positive time t_1 and a given $T_* > t_1$, the solutions $(x_i^\varepsilon(t), v_i^\varepsilon(t), z_i^\varepsilon(t))$ converge to $(x_i^0(t), v_i^0(t), z_i^0(t))$ uniformly in $t \in [t_1, T_*]$, where $(x_i^0(t), v_i^0(t), z_i^0(t))$ satisfy the unperturbed system:

$$\frac{dx_i^0}{dt} = v_i^0, \quad t \geq 0, \quad 1 \leq i \leq N,$$
$$\frac{dv_i^0}{dt} = z_i^0,$$
$$z_i^0 = \frac{\lambda}{N} \sum_{j=1}^N \Big(\psi_1(r^0)(v_j^0 - v_i^0) + \psi_2(r^0)[(x_j^0 - x_i^0) \cdot (v_j^0 - v_i^0)](x_j^0 - x_i^0) \Big).$$

Proof. The proof follows from a direct application of Theorem 2.1. A more modern approach which generalizes Tikhonov's original idea as well as the method of averaging is in Artstein and Vigodner [2], and we also refer to [14] for reference. $\square$

2.3. *Flocking estimate.* In this part, we study the flocking estimate for the formally derived new system from the mechanical particle system:

$$\frac{dx_i}{dt} = v_i, \tag{2.11a}$$

$$\frac{dv_i}{dt} = \frac{\lambda}{N} \sum_{j=1}^{N} \psi_1(|x_j - x_i|)(v_j - v_i) \tag{2.11b}$$

$$+ \frac{\lambda}{N} \sum_{j=1}^{N} \psi_2(|x_j - x_i|)((x_j - x_i) \cdot (v_j - v_i))(x_j - x_i).$$

LEMMA 2.1. Let (x_i, v_i) be the solutions to the system (2.11a)–(2.11b). Then the total momentum and energy satisfy

$$(i) \ \frac{d}{dt} \sum_{i=1}^{N} v_i = 0,$$

$$(ii) \ \frac{d}{dt} \sum_{i=1}^{N} |v_i|^2 = -\frac{\lambda}{N} \sum_{1 \leq i,j \leq N} \psi_1(|x_j - x_i|)|v_j - v_i|^2$$

$$- \frac{\lambda}{N} \sum_{1 \leq i,j \leq N} \psi_2(|x_j - x_i|)|(x_j - x_i) \cdot (v_j - v_i)|^2.$$

Proof. (i) The conservation of momentum is due to the fact that the right-hand side of (2.11b) is skew-symmetric under the transformation $i \leftrightarrow j$.

(ii) For the energy dissipation estimate, we take an inner product with (2.11b) and $2v_i$, and then sum it over i to find the desired result. $\square$

REMARK 2.1. Compared to the dissipation estimate for the Cucker-Smale flocking system [4, 5, 7, 8], we have an extra dissipation term

$$\frac{\lambda}{N} \sum_{1 \leq i,j \leq N} \psi_2(|x_j - x_i|)|(x_j - x_i) \cdot (v_j - v_i)|^2,$$

which enhances the flocking.

For the flocking estimate, we first introduce averaged quantities and fluctuations around them:

$$\langle x \rangle := \frac{1}{N} \sum_{i=1}^{N} x_i, \qquad \langle v \rangle := \frac{1}{N} \sum_{i=1}^{N} v_i,$$

and

$$\bar{x}_i := x_i - \langle x \rangle, \qquad \bar{v}_i := v_i - \langle v \rangle.$$

Then it easy to see that the dynamics for the averaged quantities and fluctuations are completely decoupled so that

$$\langle v(t) \rangle = \langle v(0) \rangle, \qquad \langle x(t) \rangle = \langle x(0) \rangle + t \langle v(0) \rangle,$$

and

$$\frac{d\bar{x}_i}{dt} = \bar{v}_i, \tag{2.12a}$$

$$\frac{d\bar{v}_i}{dt} = \frac{\lambda}{N} \sum_{j=1}^{N} \psi_1(|\bar{x}_j - \bar{x}_i|)(\bar{v}_j, \bar{v}_i) \tag{2.12b}$$

$$+ \frac{\lambda}{N} \sum_{j=1}^{N} \psi_2(|\bar{x}_j - \bar{x}_i|)((\bar{x}_j - \bar{x}_i) \cdot (\bar{v}_j - \bar{v}_i))(\bar{x}_j - \bar{x}_i).$$

For the flocking estimate, it suffices to get the dissipation estimate for the system (2.12a)–(2.12b).

THEOREM 2.3. *The limit system (2.11a)–(2.11b) admits flocking in the time-asymptotic limit. More precisely, for any solution (x_i, v_i) with a finite initial energy for perturbation, we have*

$$(i) \ \Big(\sum_{i=1}^{N} |\bar{v}_i(t)|^2 \Big) \le e^{-2\lambda R_1 t} \Big(\sum_{i=1}^{N} |\bar{v}_i(0)|^2 \Big), \quad t > 0,$$

$$(ii) \ \max_{1 \le i \le N} |\bar{x}_i(t)| \le \max_{1 \le i \le N} |\bar{x}_{i0}| + \frac{1}{\lambda R_1} \Big(\sum_{i=1}^{N} |\bar{v}_i(0)|^2 \Big)^{\frac{1}{2}},$$

where R_1 is a positive constant appearing in (2.6).

Proof. (i) As in Lemma 2.1, we have

$$\frac{d}{dt} \sum_{i=1}^{N} |\bar{v}_i|^2 = -\frac{\lambda}{N} \sum_{1 \le i,j \le N} \psi_1(|\bar{x}_j - \bar{x}_i|)|\bar{v}_j - \bar{v}_i|^2$$

$$-\frac{\lambda}{N} \sum_{1 \le i,j \le N} \psi_2(|\bar{x}_j - \bar{x}_i|)|(\bar{x}_j - \bar{x}_i) \cdot (\bar{v}_j - \bar{v}_i)|^2$$

$$\le -\frac{\lambda}{N} \sum_{1 \le i,j \le N} \psi_1(|\bar{x}_j - \bar{x}_i|)|\bar{v}_j - \bar{v}_i|^2$$

$$\le -2\lambda R_1 \sum_{i=1}^{N} |\bar{v}_i|^2.$$

We now integrate the above Gronwall's inequality to get the desired result.

(ii) It follows from the result in (i) that

$$|\bar{v}_i(t)| \le e^{-\lambda R_1 t} \Big(\sum_{i=1}^{N} |\bar{v}_i(0)|^2 \Big)^{\frac{1}{2}}.$$

We now use the above estimate to find

$$|\bar{x}_i(t) - \bar{x}_{i0}| \le \int_0^t |\bar{v}_i(s)| ds \le \frac{1}{\lambda R_1} \Big(\sum_{i=1}^{N} |\bar{v}_i(0)|^2 \Big)^{\frac{1}{2}} (1 - e^{-\lambda R_1 t}).$$

This yields the desired result. $\qquad \square$

REMARK 2.2. For the flocking estimates for the particle and kinetic Cucker-Smale model, we refer to [3, 4, 5, 7, 8, 12].

As a corollary of Theorem 2.2 and Theorem 2.3, we obtain estimates for relative velocities and the flocking group's spatial diameter for the singularly perturbed system (2.8a)-(2.8c).

COROLLARY 2.1. Let $\varepsilon > 0$ be sufficiently small. Then for any small positive time t_1 and a given $T_* > t_1$, there exists a positive constant C such that the perturbed solutions $(x_i^\varepsilon, v_i^\varepsilon)$ to (2.8a)–(2.8c) satisfy

$$
\begin{aligned}
&(i) \ |v_i^\varepsilon(t) - v_j^\varepsilon(t)| < 2\varepsilon + Ce^{-\lambda R_1 t}, \qquad t \in [t_1, T_*], \\
&(ii) \ |x_i^\varepsilon(t) - x_j^\varepsilon(t)| \leq 2\varepsilon + C.
\end{aligned}
$$

Here $(x_i^0(0), v_i^0(0))$ is the initial datum corresponding to the unperturbed solution $(x_i^0(t), v_i^0(t))$, and the constant C depends only on the initial datum of the unperturbed solution.

Proof. It follows from Theorem 2.2 that $\exists \ t_1, T_* > 0$ such that

$$
\max_{1 \leq i \leq N} |v_i^\varepsilon(t) - v_i^0(t)| < \varepsilon, \qquad \max_{1 \leq i \leq N} |x_i^\varepsilon(t) - x_i^0(t)| < \varepsilon, \quad t \in [t_1, T_*].
$$

On the other hand, we use Theorem 2.3 to estimate

$$
\max_{1 \leq i,j \leq N} |v_i^0(t) - v_j^0(t)| \leq \Big(2 \sum_{i=1}^N |\bar{v}_i^0(t)|^2 \Big)^{\frac{1}{2}} \leq \Big(2 \sum_{i=1}^N |\bar{v}_i^0(0)|^2 \Big)^{\frac{1}{2}} e^{-\lambda R_1 t}, \qquad t \in [0, \infty).
$$

We simply apply the triangle inequalities:

$$
\begin{aligned}
|v_i^\varepsilon(t) - v_j^\varepsilon(t)| \ &\leq \ |v_i^\varepsilon(t) - v_i^0(t)| + |v_j^\varepsilon(t) - v_j^0(t)| + |v_i^0(t) - v_j^0(t)|, \qquad t \in [t_1, T_*] \\
&\leq \ 2\varepsilon + \Big(2 \sum_{i=1}^N |\bar{v}_i^0(0)|^2 \Big)^{\frac{1}{2}} e^{-\lambda R_1 t}.
\end{aligned}
$$

We use Theorem 2.3 to find

$$
\begin{aligned}
|x_i^0(t) - x_j^0(t)| \ &= \ |\bar{x}_i^0(t) - \bar{x}_j^0(t)| \\
&\leq \ |\bar{x}_i^0(0) - \bar{x}_j^0(0)| + \int_0^t |\bar{v}_i(s) - \bar{v}_j(s)| ds \\
&\leq \ |\bar{x}_i^0(0) - \bar{x}_j^0(0)| + \frac{1}{\lambda R_1} \Big(2 \sum_{i=1}^N |\bar{v}_i^0(0)|^2 \Big)^{\frac{1}{2}}.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
|x_i^\varepsilon(t) - x_j^\varepsilon(t)| \ &\leq \ |x_i^\varepsilon(t) - x_i^0(t)| + |x_j^\varepsilon(t) - x_j^0(t)| + |x_i^0(t) - x_j^0(t)|, \qquad t \in [t_1, T_*] \\
&\leq \ 2\varepsilon + |\bar{x}_i^0(0) - \bar{x}_j^0(0)| + \frac{1}{\lambda R_1} \Big(2 \sum_{i=1}^N |\bar{v}_i^0(0)|^2 \Big)^{\frac{1}{2}}.
\end{aligned}
$$

$\square$

3. From the Kuramoto model to the Cucker-Smale model. In this section, we present a rather interesting bridge between the Kuramoto phase model and the Cucker-Smale flocking model.

Consider the Kuramoto phase model describing the dynamics of nonlinear oscillators $\{x_k = e^{i\theta_k}\}$ moving on the circle with randomly distributed natural frequency Ω_k and sinusoidal coupling between oscillators:

$$\frac{d\theta_k}{dt} = \Omega_k + \frac{\lambda}{N} \sum_{h=1}^{N} \sin(\theta_h - \theta_k), \quad t > 0, \quad k = 1, \ldots, N,$$

subject to the initial phase

$$\theta_k(0) = \theta_{k0}, \qquad k = 1, \ldots, N.$$

The Kuramoto model has been extensively studied in the last thirty years after Kuramoto's seminal work [10] and is still a popular subject for the modeling of synchronization phenomena arising from statistical physics (see a review article [1]). In particular, Kuramoto [10] observed that for the model in the $N \to \infty$ limit, there is a continuous dynamical phase transition at a critical value of the coupling strength λ_c. In particular, for $\lambda > \lambda_c$, the system shows the phase and frequency synchronization. We now differentiate the Kuramoto model to derive the dynamics of the frequency $\omega_i = \dot{\theta}_i$:

$$\frac{d\omega_i}{dt} = \frac{\lambda}{N} \sum_{j=1}^{N} \cos(\theta_j - \theta_i)(\omega_j - \omega_i), \quad t > 0, \quad i = 1, \ldots, N.$$

We take the initial frequency of oscillators to be the natural frequency of oscillators, i.e.,

$$\omega_i(0) = \Omega_i, \quad i = 1, \ldots, N.$$

3.1. *Dynamics of standard deviations.* In this part, we present the dynamics of standard deviations for physical variables. We introduce averaged natural frequency, phase and frequency:

$$\langle \Omega \rangle := \frac{1}{N} \sum_{i=1}^{N} \Omega_i, \quad \langle \theta \rangle := \frac{1}{N} \sum_{i=1}^{N} \theta_i, \quad \langle \omega \rangle := \frac{1}{N} \sum_{i=1}^{N} \omega_i.$$

Note that $\langle \Omega \rangle$ is a stationary random variable, and $\langle \theta \rangle, \langle \omega \rangle$ satisfy

$$\langle \theta(t) \rangle = \langle \theta_0 \rangle + t \langle \Omega \rangle, \qquad \langle \omega(t) \rangle = \langle \Omega \rangle.$$

We next introduce fluctuations of the state variables around the averaged quantities:

$$\bar{\Omega}_i := \Omega_i - \langle \Omega \rangle, \quad \bar{\theta}_i := \theta_i - \langle \theta \rangle, \quad \bar{\omega}_i = \omega_i - \langle \omega \rangle.$$

Then the fluctuations $\bar{\theta}_i$ and $\bar{\omega}_i$ satisfy

$$\frac{d\bar{\theta}_i}{dt} = \bar{\Omega}_i + \frac{\lambda}{N} \sum_{j=1}^{N} \sin(\bar{\theta}_j - \bar{\theta}_i), \tag{3.1a}$$

$$\frac{d\bar{\omega}_i}{dt} = \frac{\lambda}{N} \sum_{j=1}^{N} \cos(\bar{\theta}_j - \bar{\theta}_i)(\bar{\omega}_j - \bar{\omega}_i). \tag{3.1b}$$

Note that the equation (3.1b) takes the form of the Cucker-Smale model in the regime $\cos(\theta_j - \theta_i) > 0$. At this point, we denote with σ_θ, σ_ω and σ_Ω the standard deviations of θ_i, ω_i and Ω_i respectively:

$$\sigma_\theta^2 := \frac{1}{N} \sum_{i=1}^{N} |\bar{\theta}_i|^2, \qquad \sigma_\omega^2 := \frac{1}{N} \sum_{i=1}^{N} |\bar{\omega}_i|^2, \quad \sigma_\Omega^2 := \frac{1}{N} \sum_{i=1}^{N} |\bar{\Omega}_i|^2.$$

We study the dynamics of these quantities in the following lemma.

LEMMA 3.1. Let $(\bar{\theta}_i, \bar{\omega}_i)$ be smooth solutions to the system (3.1a) and (3.1b). Then the σ_θ and σ_ω standard deviations satisfy

$$(i) \ \frac{d}{dt} \sum_{i=1}^{N} |\bar{\theta}_i|^2 = 2 \sum_i \bar{\theta}_i \bar{\Omega}_i - \frac{\lambda}{N} \sum_{1 \le i,j \le N} (\bar{\theta}_j - \bar{\theta}_i) \sin(\bar{\theta}_j - \bar{\theta}_i),$$

$$(ii) \ \frac{d}{dt} \sum_{i=1}^{N} |\bar{\omega}_i|^2 = -\frac{\lambda}{N} \sum_{1 \le i,j \le N} \cos(\bar{\theta}_j - \bar{\theta}_i) |\bar{\omega}_j - \bar{\omega}_i|^2.$$

Proof. We only consider the second identity; the first identity can be treated similarly. We multiply the equation (3.1b) by $2\bar{\omega}_i$, sum over i and use the symmetry trick $i \leftrightarrow j$ to find

$$\begin{aligned}
\frac{d}{dt} \sum_{i=1}^{N} |\bar{\omega}_i|^2 &= \frac{2\lambda}{N} \sum_{i,j} \cos(|\bar{\theta}_j - \bar{\theta}_i|) \bar{\omega}_i \cdot (\bar{\omega}_j - \bar{\omega}_i) \\
&= -\frac{2\lambda}{N} \sum_{i,j} \cos(|\bar{\theta}_j - \bar{\theta}_i|) \bar{\omega}_j \cdot (\bar{\omega}_j - \bar{\omega}_i) \\
&= -\frac{\lambda}{N} \sum_{i,j} \cos(|\bar{\theta}_j - \bar{\theta}_i|) |\bar{\omega}_j - \bar{\omega}_i|^2.
\end{aligned}$$

$\square$

We next proceed with a dissipation estimate for the system (3.1a) and (3.1b). To this aim, we estimate the dynamics of the standard deviations in the following proposition.

PROPOSITION 3.1. For given $T > 0$, let $(\bar{\theta}_i, \bar{\omega}_i) \in C^1([0, T))$ be solutions to the system (3.1a) and (3.1b) satisfying the following stability condition:

$$\sup_{0 \le t < T} \sigma_\theta(t) < \frac{\pi}{4\sqrt{N}}. \tag{3.2}$$

Then the standard deviations σ_θ and σ_ω satisfy the system of differential inequalities:

$$(i) \ \left| \frac{d\sigma_\theta}{dt} \right| \le \sigma_\omega,$$

$$(ii) \ \frac{d\sigma_\omega}{dt} \le -\lambda \cos(2\sqrt{N}\sigma_\theta)\sigma_\omega.$$

Proof. (i) We use the definitions of variances σ_θ^2, σ_ω^2 and $\frac{d\bar{\theta}_i}{dt} = \bar{\omega}_i$ to find

$$\left| \frac{d\sigma_\theta^2}{dt} \right| = \frac{1}{N} \left| \frac{d}{dt} \sum_{i=1}^{N} (\bar{\theta}_i)^2 \right| \le \frac{2}{N} \sum_{i=1}^{N} |\bar{\theta}_i||\bar{\omega}_i| \le \frac{2}{N} \left(\sum_{i=1}^{N} |\bar{\theta}_i|^2 \right)^{\frac{1}{2}} \left(\sum_{i=1}^{N} |\bar{\omega}_i|^2 \right)^{\frac{1}{2}}, \tag{3.3}$$

where we used Cauchy-Schwarz's inequality.

On the other hand, note that

$$\left|\frac{d\sigma_\theta^2}{dt}\right| = 2\sigma_\theta\left|\frac{d\sigma_\theta}{dt}\right|, \quad \sum_{i=1}^N |\bar{\theta}_i|^2 = N\sigma_\theta^2, \quad \sum_{i=1}^N |\bar{\omega}_i|^2 = N\sigma_\omega^2.$$

Thus in (3.3) we have

$$2\sigma_\theta\left|\frac{d\sigma_\theta}{dt}\right| \leq 2\sigma_\theta\sigma_\omega,$$

and we divide the above inequality by σ_θ to find the desired result.

(ii) We use $|\bar{\theta}_j - \bar{\theta}_i| \leq 2\sqrt{N}\sigma_\theta \leq \frac{\pi}{2}$ and the nonincreasing property of $\cos(\theta)$ in the interval $[0, \frac{\pi}{2}]$ to find

$$\cos(\bar{\theta}_j - \bar{\theta}_i) \geq \cos(2\sqrt{N}\sigma_\theta).$$

On the other hand, note that

$$\begin{aligned}
2\sigma_\omega\frac{d\sigma_\omega}{dt} &= \frac{d\sigma_\omega^2}{dt} = \frac{1}{N}\frac{d}{dt}\sum_{i=1}^N |\bar{\omega}_i|^2 = -\frac{\lambda}{N^2}\sum_{1\leq i,j\leq N}\cos(\bar{\theta}_j - \bar{\theta}_i)|\bar{\omega}_j - \bar{\omega}_i|^2 \\
&\leq -2\lambda\cos(2\sqrt{N}\sigma_\theta)\sigma_\omega^2,
\end{aligned}$$

where we used

$$\sum_{1\leq i,j\leq N} |\bar{\omega}_i - \bar{\omega}_j|^2 = 2N\sum_{i=1}^N |\bar{\omega}_i|^2 = 2N^2\sigma_\omega^2.$$

$\square$

3.2. *Frequency synchronization.* In this part, we study the dissipation estimate for standard deviations, which will give the existence of a frequency locked state:

$$\omega_i = \omega_j, \qquad i \neq j.$$

The Lyapunov approach employed in this section was introduced in [7] in the context of the Cucker-Smale system with general communication rates. Consider the following system of differential inequalities:

$$\left|\frac{d\sigma_\theta}{dt}\right| \leq \sigma_\omega, \qquad \frac{d\sigma_\omega}{dt} \leq -\lambda\cos(2\sqrt{N}\sigma_\theta)\sigma_\omega \tag{3.4}$$

and define Lyapunov functionals

$$\mathcal{H}^\pm[\sigma_\theta(t), \sigma_\omega(t)] := \sigma_\omega(t) \pm \lambda\int_{\sigma_\theta(0)}^{\sigma_\theta(t)}\cos(2\sqrt{N}s)ds.$$

PROPOSITION 3.2 (Dissipation estimate). *For given $T > 0$, let $(\sigma_\theta, \sigma_\omega) \in C^1([0,T))$ be solutions to the system (3.1a) and (3.1b) satisfying the stability condition (3.2). Then the following stability estimate holds. For $t \geq 0$,*

$$(i)\ \mathcal{H}^\pm[\sigma_\theta(t), \sigma_\omega(t)] \leq \mathcal{H}^\pm[\sigma_\theta(0), \sigma_\omega(0)],$$

$$(ii)\ \sigma_\omega(t) + \lambda\left|\int_{\sigma_\theta(0)}^{\sigma_\theta(t)}\cos(2\sqrt{N}s)ds\right| \leq \sigma_\omega(0).$$

Proof. (i) We use Proposition 3.1, that is, relations (3.4), to show that

$$\frac{d}{dt}\mathcal{H}^{\pm}[t] = \frac{d}{dt}\sigma_\omega(t) \pm \lambda\cos(2\sqrt{N}\sigma_\theta(t))\frac{d}{dt}\sigma_\theta(t)$$

$$\leq -\lambda\cos(2\sqrt{N}\sigma_\theta)\sigma_\omega \pm \lambda\cos(2\sqrt{N}\sigma_\theta(t))\frac{d}{dt}\sigma_\theta(t)$$

$$= \lambda\cos(2\sqrt{N}\sigma_\theta)\Big(-\sigma_\omega \pm \frac{d}{dt}\sigma_\theta(t)\Big)$$

$$\leq 0.$$

(ii) It follows from the result in (i) that we have

$$\sigma_\omega(t) + \lambda\int_{\sigma_\theta(0)}^{\sigma_\theta(t)}\cos(2\sqrt{N}s)ds \leq \sigma_\omega(0),$$

$$\sigma_\omega(t) + \lambda\int_{\sigma_\theta(t)}^{\sigma_\theta(0)}\cos(2\sqrt{N}s)ds \leq \sigma_\omega(0),$$

which concludes the proof. $\qquad\square$

Finally we can prove our main result.

THEOREM 3.1 (Frequency synchronization). Suppose the standard deviations of the initial configurations $\sigma_\theta(0)$ and $\sigma_\omega(0)$ satisfy

$$\sigma_\theta(0) < \frac{\pi}{16\sqrt{N}}, \qquad \sigma_\omega(0) < \frac{\lambda\pi}{16\sqrt{2N}}.$$

Then we have

$$(i) \quad \sigma_\omega(t) \leq \sigma_\omega(0)e^{-\frac{\lambda t}{\sqrt{2}}}, \quad t \geq 0,$$

$$(ii) \lim_{t\to\infty}|\omega_i(t) - \omega_j(t)| = 0, \qquad 1 \leq i, j \leq N.$$

Proof. The proof for (ii) directly follows from the estimate in (i):

$$|\omega_i(t) - \omega_j(t)| \leq 2\sqrt{N}\sigma_\omega(t) \leq 2\sqrt{N}\sigma_\omega(0)e^{-\frac{\lambda t}{\sqrt{2}}} \to 0, \qquad \text{as } t \to \infty.$$

We now consider the estimate (i). Since $\sigma_\theta(0) < \frac{\pi}{16\sqrt{N}}$ and σ_θ is a continuous function of t, there exists a maximal time $T_* \in (0, \infty]$ such that

$$\sup_{0\leq t\leq T_*}\sigma_\theta(t) \leq \frac{\pi}{8\sqrt{N}}. \tag{3.5}$$

Step A (Decay estimate in the time zone $[0, T_*)$): In this time zone, the condition (3.5) and relation (3.4) through Proposition 3.1 imply

$$\sigma_\omega(t) \leq \sigma_\omega(0)e^{-\frac{\lambda t}{\sqrt{2}}}.$$

On the other hand, the first inequality in (3.4) yields

$$|\sigma_\theta(t) - \sigma_\theta(0)| \leq \int_0^t \Big|\frac{d\sigma_\theta(s)}{ds}\Big|ds \leq \int_0^t \sigma_\omega(s)ds \leq \frac{\sqrt{2}}{\lambda}\sigma_\omega(0) < \frac{\pi}{16\sqrt{N}}.$$

Hence we have

$$\sigma_\theta(t) < \sigma_\theta(0) + \frac{\pi}{16\sqrt{N}} < \frac{\pi}{8\sqrt{N}}, \quad t \in [0, T_*].$$

Step B ($\mathbf{T_* = \infty}$): Suppose the maximal time T_* is finite, i.e., $T_* < \infty$. Then by the estimate given in Step A, we have

$$\sigma_\theta(T_*) < \frac{\pi}{8\sqrt{N}}, \quad \sigma_\omega(T_*) \le \sigma_\omega(0) < \frac{\lambda\pi}{16\sqrt{2N}}.$$

Hence by the same argument given in Step A, we can find a $\delta > 0$ such that

$$\sup_{0 \le t \le T_*+\delta} \sigma_\theta(t) \le \frac{\pi}{8\sqrt{N}}.$$

This contradicts the maximality of T_*. Hence $T_* = \infty$ and the proof is complete.

$\square$

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