Technical Notes and Correspondence

Flocking in Fixed and Switching Networks

Herbert G. Tanner, Ali Jadbabaie, and George J. Pappas

Abstract—This note analyzes the stability properties of a group of mobile agents that align their velocity vectors, and stabilize their inter-agent distances, using decentralized, nearest-neighbor interaction rules, exchanging information over networks that change arbitrarily (no dwell time between consecutive switches). These changes introduce discontinuities in the agent control laws. To accommodate for arbitrary switching in the topology of the network of agent interactions we employ nonsmooth analysis. The main result is that regardless of switching, convergence to a common velocity vector and stabilization of inter-agent distances is still guaranteed as long as the network remains connected at all times.

Index Terms—Algebraic graph theory, cooperative control, multiagent systems, nonsmooth systems.

I. INTRODUCTION

In this note, we interpret Reynolds' flocking model [3] as a mechanism for achieving velocity synchronization and regulation of relative distances within a group of agents, and derive decentralized controllers which provably give rise to such a phenomenon, even when information exchange between the agents can change *arbitrarily* fast. Since flocking is defined in many different ways in literature [4]–[7], the emphasis in this note is not on reproducing flocking, but rather on providing a decentralized coordination method in the case where the rate of change of the network, over which agent information is disseminated, affords no bounds.

We make a distinction between the *sensing* and the *communication* network. These two networks need not necessarily coincide, a fact that further motivates a nonsmooth approach to cooperative control design and analysis. Under the assumption of connected (but arbitrarily switching) communication network topology, we construct local control laws, composed of artificial potential field [8], [9], and neighbor velocity difference terms, that allow a group of mobile agents with double integrator dynamics to align their velocities, move with a common speed and achieve desired interagent distances while avoiding collisions with each other. We establish the stability properties of the interconnected closed loop system using nonsmooth control analysis and algebraic graph theory.

A. Related Work

The mechanism triggering formation clustering without centralized coordination in groups of autonomous moving creatures such as flocks

of birds, schools of fish, crowds of people [10], [11] has also been investigated in ecology and theoretical biology, in the context of animal aggregation and social cohesion in animal groups (see for example [12], [13]). A computer model mimicking animal aggregation was proposed by [3]. At the same time, several researchers in the area of statistical physics and complexity theory have addressed flocking and schooling behavior in the context of non-equilibrium phenomena in many-degrees-of-freedom dynamical systems and self organization in systems of self-propelled particles [4], [14], [15]. Similar problems have become a major thrust in systems and control theory, in the context of cooperative control, distributed control of multiple vehicles and formation control. Within the space limitations of a technical note, no literature review can be anywhere close to being complete, but the interested reader is referred to [7] and [16]–[27]. The main goal in the work cited above is to develop a decentralized control strategy so that a global objective, such as a tight formation with desired inter-vehicle distances, is achieved.

In related work on time-varying interconnections [28], a node has to be connected to all other nodes over all time. If, on the other hand, dwell time is assumed between switching instances, as in [17], the stability analysis can be based on recent results for switched nonlinear systems [29] as sketched in Remark IV-B. (The analysis in [22] involves only velocity synchronisation and is performed in discrete time.) The main contribution of this note is in providing a stability result for the case where the topology of agent interconnections changes in a completely arbitrary manner, and without dwell time between switching instants.

II. PROBLEM FORMULATION

Consider a group of N mobile agents moving on the plane, with dynamics expressed by double integrators

$$\dot{r}_i = v_i \tag{1a}$$

$$\dot{v}_i = u_i = u_i = \alpha_i + a_i, \qquad i = 1, \dots, N \tag{1b}$$

where $r_i = (x_i, y_i)^T$ is the position of agent i, $v_i = (\dot{x}_i, \dot{y}_i)^T$ its velocity, and $u_i = (u_x, u_y)^T$ its acceleration inputs. Let the relative position vector between agents i and j be denoted $r_{ij} = r_i - r_j$. Agent i is steered via its acceleration input u_i which consists of two components, α_i and a_i. Component α_i in (1) aims at aligning the velocity vectors of all the agents. Component a_i is a vector in the direction of the negated gradient of an artificial potential function, V_i, and is used for collision avoidance and cohesion in the group. Let R be the sensing radius of agent i. Agents beyond this range are assumed not to affect a_i.

A collision is assumed to have occurred when the coordinates of two agents coincide. The problem is to design the control input (1) so that if connectivity is maintained in the group, agent velocities are synchronized, collisions are avoided, and pair-wise distances between agents that sense each other are stabilized to steady state values within a given range.

III. PRELIMINARY DEFINITIONS AND THE CASE OF FIXED COMMUNICATION TOPOLOGY

For the sake of completeness, let us first consider the case where the communication network is time-invariant. We represent the communication network by means of a graph, which determines how velocity information propagates in the group.
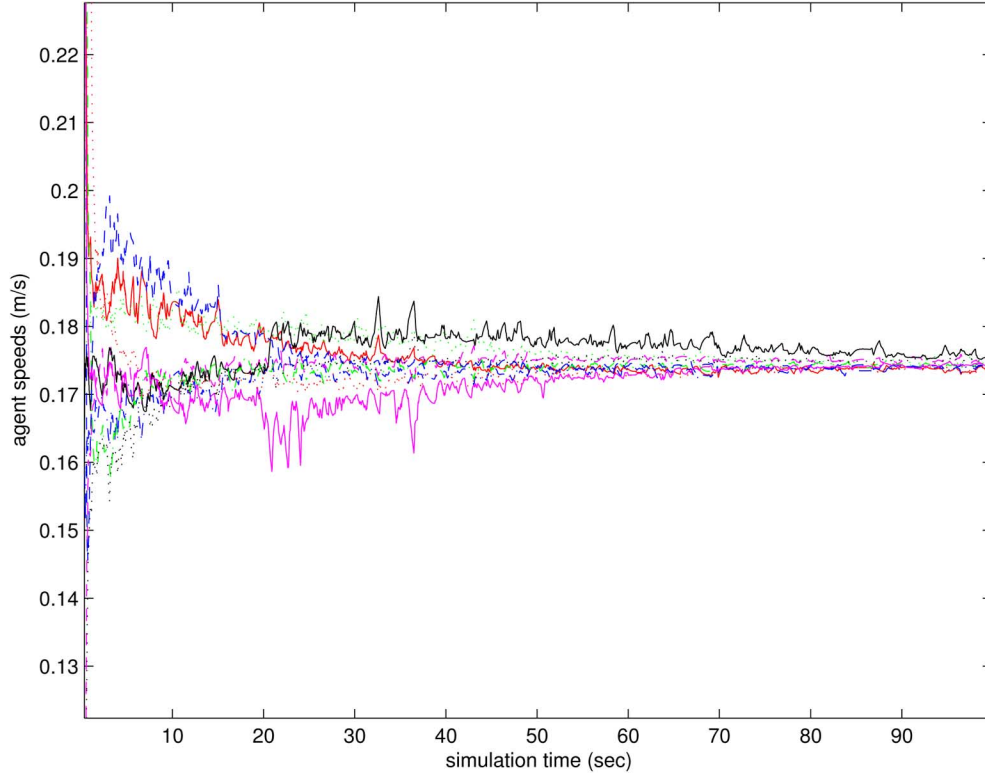
Fig. 1. Convergence of agent speeds. The agents communicate their velocities over a network that changes randomly at every simulation step, with a given probability. Network topology switches introduce discontinuities.

Definition 1 (Velocity Graph): The velocity graph, $\mathcal{G}_c = \{\mathcal{V}, \mathcal{E}_c\}$, is an undirected graph consisting of

- a set of vertices (nodes), $\mathcal{V} = \{1, \ldots, N\} \subset \mathbb{N}$, indexed by the agents in the group;
- a set of edges, $\mathcal{E}_c = \{(i,j) \in \mathcal{V} \times \mathcal{V} \mid i \sim j\}$, ($\sim$ denotes adjacency) containing unordered pairs of nodes that represent communication links.

The velocity graph neighbors of agent i are assumed to belong to a set $\mathcal{N}_c(i) \triangleq \{j \mid (i,j) \in \mathcal{E}_c\} \subseteq \mathcal{V} \setminus \{i\}$.

Agents within distances smaller than R are interacting through artificial potential "forces." Each such interaction is associated with a link in the *sensing* network of the group, which, being position dependent, is represented by the position graph defined as follows.

Definition 2 (Position Graph): The position graph, $\mathcal{G}_s = \{\mathcal{V}, \mathcal{E}_s\}$, is an undirected graph consisting of

- a set of vertices (nodes), $\mathcal{V} = \{1, \ldots, N\} \subset \mathbb{N}$, indexed by the agents in the group.
- a set of edges, $\mathcal{E}_s = \{(i,j) \in \mathcal{V} \times \mathcal{V} \mid \|r_i - r_j\| \leq R\}$, containing unordered pairs of nodes that represent sensing links.

Similarly, position graph neighbors of agent i define a set $\mathcal{N}_s(i) \triangleq \{j \mid (i,j) \in \mathcal{E}_s\} \subseteq \mathcal{V} \setminus \{i\}$. Contrary to the velocity graph, the position graph is time-varying, depending on the agents' relative positions.

Consider a function V_{ij} that depends on the distance between position neighbors:

Definition 3 (Potential Function): Potential V_{ij} is a differentiable, nonnegative, function of the distance $\|r_{ij}\|$ between agents i and j, such that

1) $V_{ij}(\|r_{ij}\|) \to +\infty$ as $\|r_{ij}\| \to 0$.
2) V_{ij} attains its unique minimum when agents i and j are located at a desired distance.
3) $(d/d\|r_{ij}\|)V_{ij} = 0$, if $\|r_{ij}\| > R$.

Definition 3 ensures that minimization of the inter-agent potential functions implies cohesion and separation in the group. By defining

V_{ij} according to Definition 3 we attempt to regulate distances between agents in $\mathcal{G}_s$ within the range $(0, R)$.

The total potential of agent i is:

$$V_i = \sum_{j \in \mathcal{N}_{s(i)}} V_{ij}(\|r_{ij}\|) \tag{2}$$

and the control input for agent i is defined as

$$u_i = \underbrace{- \sum_{j \in \mathcal{N}_c(i)} (v_i - v_j)}_{\alpha_i} \underbrace{- \sum_i^N \nabla_{r_i} V_i}_{a_i}. \tag{3}$$

Let us define the dynamical system derived from (1) by stacking the position and velocity vectors. This system has $(\bar{r}, v)$ as its state, where $\bar{r} = (B_{\mathcal{K}_N} \otimes I)r$ is the stack vector of all relative positions between agents, r is the stack vector of agent positions, v is the stack vector of all agent velocities, $\otimes$ denotes the Kronecker matrix product, $B_{\mathcal{K}_N}$ is the oriented incidence matrix of the complete graph with N vertices, $\mathcal{K}_N$ (for an arbitrary orientation), and I is the identity matrix of appropriate dimension. This dynamics is expressed as

$$\dot{\bar{r}} = (B_{\mathcal{K}_N} \otimes I_2)v \tag{4a}$$
$$\dot{v} = u \tag{4b}$$

where u is the stack vector of all agent inputs, defined in (1). The convergence properties of (4) can be analyzed using standard invariance arguments [1], [17], [21]. Due to space limitations, in this note we focus on the case of switching communication topology, which is discussed in Section IV.
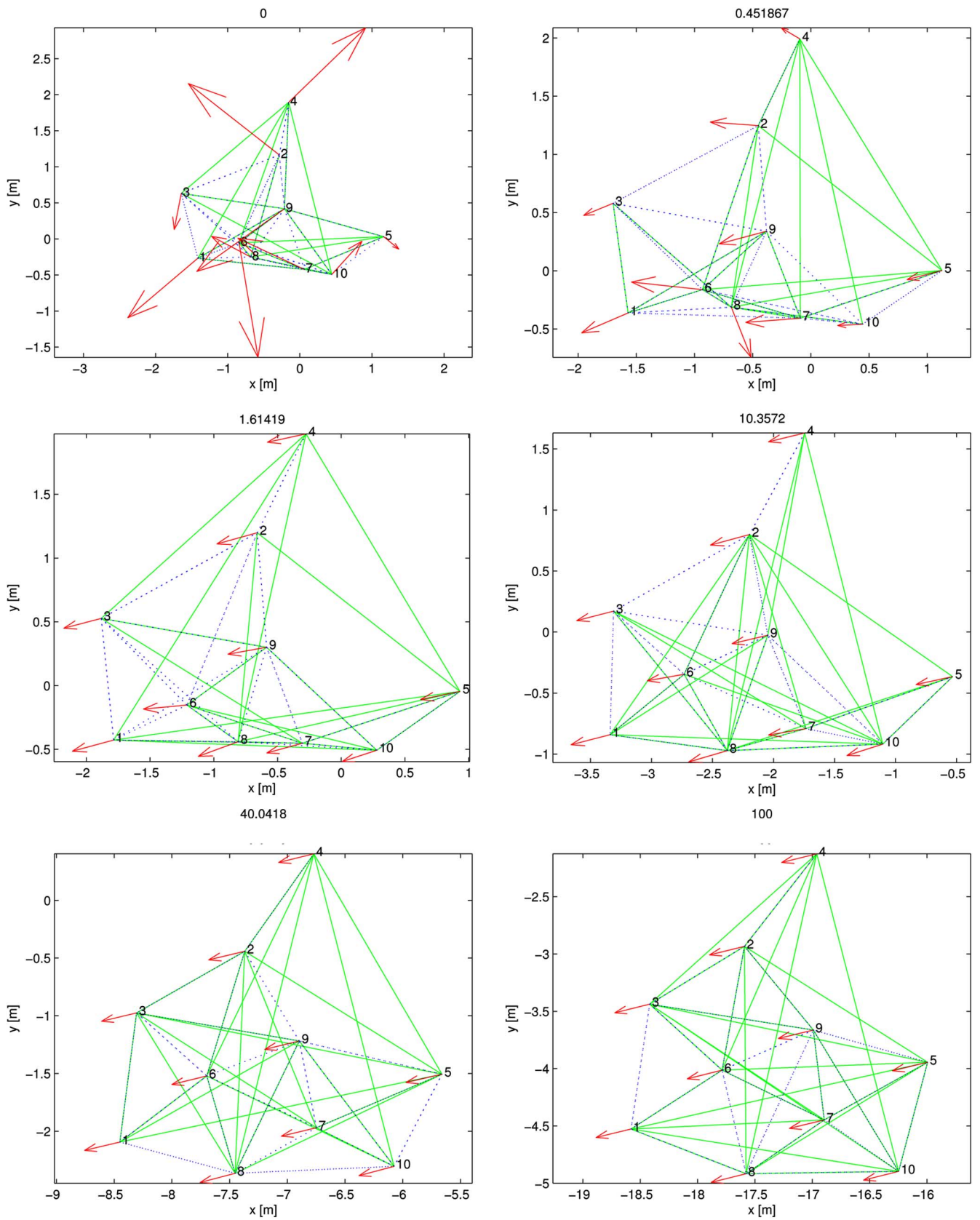
Fig. 2. Successive simulation time snapshots of flocking with dynamic interconnection topology. (Top left) Initial condition. (Bottom right) Position after 100 simulation seconds. The time stamp of each snapshot is shown on top of the corresponding figure.

IV. COORDINATION WITH SWITCHING COMMUNICATION TOPOLOGY

A. Switching Without Dwell Time

In this section, we assume that the topology of the communication network can switch arbitrarily fast. In this case, the velocity graph of Definition 1 is time-varying. Since α_i in (3) now depends on the time-varying $\mathcal{N}_c(i)$, topology switches will introduce discontinuities to the right hand side of (3). The stability of the discontinuous dynamics is analyzed using differential inclusions [30] and nonsmooth analysis [31]. Since the control signal u is switching, (4) is expressed in terms of differential inclusions

$$\dot{r} = (B_{\mathcal{K}_N} \otimes I_2)v \qquad (5a)$$

$$\dot{v} \in^{a.e} K[u] \qquad (5b)$$

where $B_{\mathcal{K}_N}$ is the incidence matrix of the complete graph with N vertices, $\mathcal{K}_N$, $K[\cdot]$ is a differential inclusion [32], and $a.e$ stands for "almost everywhere." We do not make any assumption on the uniqueness of the solutions of (5).

Theorem 1 (Flocking in Networks With Arbitrary Switching): Consider a system of N mobile agents with dynamics (5), each steered by control law (3). Let both the position and velocity graphs be time-varying, but always connected. Then all pairwise velocity differences converge asymptotically to zero, collisions between the agents are avoided, and the system approaches a local extremum of agent potentials (2).

Proof: Consider the Lyapunov-like function

$$W(\bar{r}, v) = \frac{1}{2} \sum_{i=1}^{N} \left(V_i + v_i^T v_i \right). \qquad (6)$$

The position graph is time-varying, but the associated topology changes do not introduce discontinuities, since the potential function is differentiable at the transition point. Since the position graph is assumed to be always connected, by definition there is a path (in the position graph) from every vertex to every other vertex. The graph's diameter, therefore, cannot be larger than $N - 1$. This implies that the largest distance between any two agents in the graph, (by the triangle inequality) is smaller than $(N-1)R$. As a result, $\sum_{(i,j) \in \mathcal{V} \times \mathcal{V}} \|r_{ij}\| \leq (N(N-1)^2 R)/2$. Thus, $\bar{r}$ always evolves in a closed and bounded set. Similarly, the level sets of W define compact sets in the space of agent velocities: $W \leq c \Rightarrow \sum_i v_i^2 \leq c \Rightarrow \|v_i\|^2 \leq c$. Consequently, the set $\{\bar{r}, v\}$ such that $W \leq c$, for $c > 0$ is closed by continuity. Boundedness follows from connectivity: From $W \leq c$ we have that $V_{ij} \leq c$. Connectivity ensures that a path connecting nodes i and j has length at most $N - 1$. Thus $\|r_{ij}\| \leq V_{ij}^{-1}\left(c(N-1)\right)$. Similarly, $v_i^T v_i \leq c$ yielding $\|v_i\| \leq \sqrt{c}$. Therefore, the set

$$\Omega = \left\{ (\bar{r}, v) \mid \sqrt{\|\bar{r}\|^2 + \|v\|^2} \leq \sqrt{c} + \frac{N(N-1)^2 R}{2} \right\} \qquad (7)$$

is compact. The invariant properties of Ω will be established in the sequel once W is shown to be non-increasing. Function W is differentiable, but its derivative along the system's trajectories is not a quantity that can be evaluated at the switching instants, for we do not know the value of $\dot{v}$. We can only ensure that $\dot{v} \in^{a.e.} K[u]$. The right-hand side of (5) can be expanded as follows:

$$\dot{r} = (B_{\mathcal{K}_N} \otimes I_2)v$$

$$\dot{v} \in^{a.e} K[-(L_c \otimes I_2)v] - \begin{pmatrix} \nabla_{r_1} V_1 \\ \vdots \\ \nabla_{r_N} V_n \end{pmatrix}.$$

Let ϕ_v be an arbitrary element of $K[-(L_c \otimes I_2)v]$. The *generalized* derivative of W, along a vector ϕ belonging in the set given by the right-hand side of (5), is expressed as

$$W^\circ(\bar{r}, v; \phi) = \frac{1}{2} \sum_{i=1}^{N} \dot{V}_i + v^T \phi_v - \sum_{i=1}^{N} v_i^T \nabla_{r_i} V_i.$$

Based on the fact that $\nabla_{r_{ij}} V_{ij} = \nabla_{r_i} V_{ij} = -\nabla_{r_j} V_{ij}$, we have

$$\frac{1}{2} \sum_{i=1}^{N} \dot{V}_i = \frac{1}{2} 2 \sum_{i=1}^{N} \dot{r}_i^T \sum_{(i,j) \in \mathcal{E}_s} \nabla_{r_{ij}} V_{ij} = \sum_{i=1}^{N} \dot{r}_i^T \nabla_{r_i} V_i. \qquad (8)$$

Thus $W^\circ(\bar{r}, v; \phi)$, using (8), becomes

$$W^\circ(\bar{r}, v; \phi) = \sum_{i=1}^{N} v_i^T \nabla_{r_i} V_i + v^T \phi_v - \sum_{i=1}^{N} v_i^T \nabla_{r_i} V_i = v^T \phi_v. \qquad (9)$$

The invariance principle in [33] examines the worst case for the rate of change of W, which evaluates to $m(r, v) = \max_{\phi_v \in K[-(L_c \otimes I_2)v]} \left\{ v^T \phi_v \right\}$. Theorem 1 of [32] enables us to write: $v^T K[-(L_c \otimes I_2)v] = K\left[-v^T(L_c \otimes I_2)v\right]$. From the definition of the differential inclusion, it follows that $m(r, v) = \max \overline{\mathrm{co}} \left\{-v^T(L_c \otimes I_2)v\right\}$. For a *connected* velocity graph $\mathcal{G}_c$, L_c is positive semi-definite and therefore all quadratic forms of the type $-v^T(L_c \otimes I_2)v$ are nonpositive, regardless of the topology of the graph. Convex hulls of nonpositive numbers are nonpositive intervals, and thus $m(r, v)$ cannot be positive. The largest value it can have is zero. Rewriting v^T as: $v^T = (v_{1x}, v_{1y}, v_{2x}, v_{2y}, \ldots, v_{Nx}, v_{Ny})$, we have that $-v^T(L \otimes I_2)v = v_x^T L_c v_x + v_y^T L_c v_y$, which implies that $m(r, v) = 0$ iff $v_x = c_x \mathbf{1}_N$ and $v_y = c_y \mathbf{1}_N$, where $c_x, c_y \in \mathbb{R}$. Applying the invariance principle of [33] to the system described by the (set valued) vector field $(\dot{r}, \dot{v})$, it follows that for initial conditions in Ω, the Filippov solutions of the system converge to a subset of $\{v \mid v_x, v_y \in \mathrm{span}\{\mathbf{1}\}\}$. If v_x and v_y are aligned with $\mathbf{1}$, then, for any two agents i and j, $\dot{r}_{ij} = v_i - v_j = 0$. In $\{v \mid v_x, v_y \in \mathrm{span}\{\mathbf{1}\}\}$, the acceleration dynamics reduces to

$$\dot{v} = (B_s \otimes I_2) \left[\ldots \quad (\nabla_{r_{ij}} v_{ij}) \quad \ldots \right]^T \qquad (10)$$

which implies that both $\dot{v}_x$ and $\dot{v}_y$ belong to the range of the oriented incidence matrix B_s of the position graph $\mathcal{G}_s$ (for an arbitrary orientation). For a connected velocity graph, $\mathrm{range}(B_c) = \mathrm{span}\{\mathbf{1}\}^\perp$ and, therefore

$$\dot{v}_x, \dot{v}_y \in \mathrm{span}\{\mathbf{1}\} \cap \mathrm{span}\{\mathbf{1}\}^\perp \equiv \{0\}. \qquad (11)$$

Thus, the right-hand side of (10) is zero at steady state, implying that V_i is *locally* minimized. Configurations corresponding to such local minima may not be isolated; however, (11) ensures that the system is stable there, so $d/(dt\|r_{ij}\|) = 0, \forall i, j \in \mathcal{E}$. However, if V_{ij} happens to be locally convex in the $(0, R)$ range, then V_i will have a unique extremum and inter-agent distances between agents connected in $\mathcal{G}_s$ are stabilized to their desired points. Collision avoidance is ensured by the definition of V_{ij} and the fact that W is decreasing. ∎

Maintaining connectivity in the group while the network topology is switching based on the distance between the agents is a major issue. In the present analysis, this assumption is instrumental in showing the stability of the flocking motion of the group. The nonsmooth invariance theorem of Ryan [33], does not require Ω to be compact, however the compactness and invariance of Ω implies the necessary *precompactness* of the solutions. If connectivity is lost, one cannot guarantee that $r_{ij} \in \Omega$ and thus stability may not be guaranteed.

B. Switching With Dwell Time

Although, a detailed stability analysis of this case is beyond the scope of this note, we wish to highlight an alternative methodology should the stronger condition that requires a dwell time between switches of the velocity graph is made, in addition to network connectivity. The approach described here is different from the one followed in [34], where all control signals are continuous, and in [17], where single integrator dynamics with no potential agent interaction is considered.

Here, the stability analysis can be based on recent results for switched nonlinear systems [29]. System (5)–(3) can be thought of as a switched nonlinear system $\dot{x} = f_\sigma(x)$, $\sigma : [0, \infty) \to \mathcal{P}$, where $\mathcal{P}$ is a finite index set. The dwell time assumption implies that there are always intervals of some length $\tau > 0$ between the consecutive discontinuities of the switching signal σ. For each $p \in \mathcal{P}$, i) the right-hand side of (5)–(3) (denoted here f_p) is locally Lipschitz, ii) W_p (the Lyapunov function (6) when dynamics p is activated) is positive definite and radially unbounded, and iii) W_p is continuous (thus $W_p(t_i) \leq W_p(t_j)$ whenever $t_j < t_i$ and $\sigma(t_i) = \sigma(t_j) = p$), and (iv) $-\nabla_{(\bar{r}, v)} W_p f_p$ is positive semi-definite. These conditions are sufficient to ensure that the "auxiliary output" $(B_\mathcal{Q} \otimes I_2) v_\mathcal{Q} \to 0$ [29], where $\mathcal{Q}$ is the union of time intervals when $\sigma = p$, and $B_\mathcal{Q}$, $v_\mathcal{Q}$ denote the incidence matrix and velocity vector during $t \in \mathcal{Q}$, respectively. Note in [29] that the dwell time assumption is instrumental in constructing $\mathcal{Q}$, on which $-\nabla_{(\bar{r}, v)} W_p f_p$ is integrable. Since the auxiliary output convergences for an arbitrary p among the finite set $\mathcal{P}$, we will eventually have $(B \otimes I_2) v \to 0$.

V. NUMERICAL SIMULATIONS

A group of ten mobile agents with dynamics (1) is initialized with random initial (x, y) positions in a rectangular area of 6.25 m^2 centered at the origin. Velocities were also randomly selected with magnitudes in the $(0,1)$ m/s range, and with arbitrary directions. Randomly generated adjacency matrices defined connected position and velocity graphs. Each call to the dynamic equation matlab function that implements(1)–(3), by the numerical integration function (ode45) can initiate a random switch to a completely different connected communication graph. Such switching happens with a given probability, but it is *not* otherwise restricted (for instance, in terms of dwell time). Fig. 2 describes the evolution of a group of ten agents, where the velocity graph topology is switching in the aforementioned manner. We depict the velocity graph edges in solid (green) line segments and the position graph edges in dotted (blue) segments. Each snapshot shows a different velocity graph, although the topology could have undergone several changes between these two time instants. Fig. 1 gives the time history of agent velocities. Convergence is fast, probably because with the network neighbors changing, an agent can have access to the velocities of a large set of its groupmates, rather than a restricted set of constant neighbors. Frequent topology switchings produce transients, but stability and overall convergence trend is evident.

VI. CONCLUDING REMARKS

We show that the multiagent behavior induced by our control law is robust to arbitrary changes in the sensing and communication networks, as long as these remain connected during the motion. We prove that agent potential functions are locally minimized and velocity vectors converge asymptotically to a common vector, by exploiting the algebraic connectivity of the underlying sensing and communication graphs.

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Uniformly Stabilizing Sets of Switching Sequences for Switched Linear Systems

Ji-Woong Lee and Geir E. Dullerud

Abstract—A set of switching sequences of a discrete-time switched linear system is said to be uniformly stabilizing if the system is uniformly exponentially stable over all switching sequences that belong to the set. Based on the recent result that a switched linear system is uniformly stable if and only if an increasing countable union of linear matrix inequality conditions is satisfied, we characterize all uniformly stabilizing sets of switching sequences. A discrete-time hybrid automaton of the automobile engine is considered to illustrate the result.

Index Terms—Discrete linear inclusions, hybrid automata, linear matrix inequalities (LMIs), uniform exponential stability.

I. INTRODUCTION

The switched system is an abstraction of complex hybrid automata where continuous state variables and discrete states coexist and depend on each other [1]–[3]. The analysis and synthesis of a switched system is often relatively a lot easier than that of the original hybrid dynamical system that the switched system abstracts, since the continuous state does not affect the transitions among discrete state values in the switched system abstraction. The vector of continuous state variables is called the *state*, and the discrete state values are called the *modes*; transition sequences between modes are called *switching sequences*. There are three benchmark problems for switched systems [4]: stabilization under arbitrary switching sequences, stabilization under a switching path constraint, and construction of stabilizing switching sequences.

For discrete-time switched linear systems, control-oriented complete solutions to the first two problems concerning (uniform) stabilization were presented and also extended to disturbance attenuation in [5], [6].

In this note, we solve the third benchmark problem and characterize all uniformly stabilizing sets of switching sequences for the discrete-time switched linear system. Recent results in [5], [6] include that the switched linear system, possibly with a switching path constraint, is uniformly stable (and contractive) if and only if the union of an increasing family of linear matrix inequality conditions holds. They draw on the operator-theoretic analysis of linear time-varying systems [7], [8], but exploit the fact that each switching sequence gives rise to a linear time-varying system whose coefficients vary within a finite set. Based on these results, this note extends the existing results on the existence and construction of stabilizing switching sequences [9], [10], and presents a complete characterization of uniformly stabilizing sets of switching sequences for the first time. The result is then applied to the automobile engine control problem described in [10].

Notation: If $\mathbf{X}, \mathbf{Y} \in \mathbb{R}^{n \times n}$ are symmetric and $\mathbf{Y} - \mathbf{X}$ is positive definite, we write $\mathbf{X} < \mathbf{Y}$. The identity matrix is denoted by $\mathbf{I}$ with n understood. For $x \in \mathbb{R}^n$, denoted by $\|x\|$ is the Euclidean vector norm $\|x\| = \sqrt{x^{\mathrm{T}} x}$ of x.

II. ANALYSIS

Given positive integers n and N, let

$$\mathcal{A} = \{\mathbf{A}_1, \ldots, \mathbf{A}_N\} \tag{1}$$

with each $\mathbf{A}_i \in \mathbb{R}^{n \times n}$. Let $\boldsymbol{\Omega}$ be the set of every infinite sequence in $\{1, \ldots, N\}$; each element of $\boldsymbol{\Omega}$ shall be called a *switching sequence*. A discrete-time linear time-varying system whose coefficient jumps within the finite set $\mathcal{A}$ has the state-space representation

$$x(t + 1) = \mathbf{A}_{\theta(t)} x(t) \tag{2}$$

for some $\boldsymbol{\theta} = (\theta(0), \theta(1), \ldots) \in \boldsymbol{\Omega}$, and for all $t = 0, 1, \ldots$. If $\theta(t) = i$, then the system (2) is said to be in *mode* i at time t. In general, if $\boldsymbol{\Theta}$ is a nonempty subset of $\boldsymbol{\Omega}$, then the pair $(\mathcal{A}, \boldsymbol{\Theta})$ is identified with the family of systems (2) over all $\boldsymbol{\theta} \in \boldsymbol{\Theta}$, and called a discrete-time *switched linear system*. In particular, the pair $(\mathcal{A}, \boldsymbol{\Omega})$ is called a *discrete linear inclusion*.

Definition 1: Let $\mathcal{A}$ be as in (1); let $\boldsymbol{\Theta}$ be a nonempty subset of $\boldsymbol{\Omega}$. The switched linear system $(\mathcal{A}, \boldsymbol{\Theta})$ is said to be *asymptotically stable* if

$$\lim_{t \to \infty} \|x(t)\| = 0$$

for all $x(0) \in \mathbb{R}^n$ and for all $\boldsymbol{\theta} \in \boldsymbol{\Theta}$. If, for each $\boldsymbol{\theta} \in \boldsymbol{\Theta}$, there exist $c_{\boldsymbol{\theta}} \geq 1$ and $\lambda_{\boldsymbol{\theta}} \in (0, 1)$ such that

$$\|x(t)\| \leq c_{\boldsymbol{\theta}} \lambda_{\boldsymbol{\theta}}^{t - t_0} \|x(t_0)\|$$

whenever $t \geq t_0 \geq 0$ and for all $x(t_0) \in \mathbb{R}^n$, then the system $(\mathcal{A}, \boldsymbol{\Theta})$ is said to be *pointwise uniformly stable*. If there exist $c \geq 1$ and $\lambda \in (0, 1)$ such that

$$\|x(t)\| \leq c \lambda^{t - t_0} \|x(t_0)\|$$