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Flops and Poisson Deformations of Symplectic Varieties

Dedicated to Professor Heisuke Hironaka on his 77-th birthday

By

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§1. Introduction

In [Na] we have dealt with a deformation of a *projective* symplectic variety. This paper, on the contrary, deals with a deformation of a *local* symplectic variety. More exactly, we mean by a local symplectic variety, a normal variety X satisfying

1. there is a birational projective morphism from X to an affine normal variety Y,

2. there is an everywhere non-degenerate d-closed 2-form ω on the regular part U of X such that, for any resolution $\pi : \tilde{X} \to X$ with $\pi^{-1}(U) \cong U$, ω extends to a regular 2-form on $\tilde{X}$.

In the remainder, we call such a variety a *convex* symplectic variety. A convex symplectic variety has been studied in [K-V], [Ka 1] and [G-K]. One of main difficulties we meet is the fact that tangent objects $\mathbf{T}^1_X$ and $\mathbf{T}^1_Y$ are *not* finite dimensional, since Y may possibly have non-isolated singularities; hence the usual deformation theory does not work well. Instead, in [K-V], [G-K], they introduced a Poisson scheme and studied a *Poisson deformation* of it. A Poisson deformation is the deformation of the pair of a scheme itself and a Poisson structure on it. When X is a convex symplectic variety, X admits a natural Poisson structure induced from a symplectic 2-form ω; hence one can consider

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its Poisson deformations. Then they are controlled by the *Poisson cohomology.* The Poisson cohomology has been extensively studied by Fresse [Fr 1], [Fr 2]. In some good cases, it can be described by well-known topological data (Corollary 10). The first application of the Poisson deformation theory is the following two results:

Corollary 25. *Let Y be an affine symplectic variety with a good $\mathbf{C}^*$-action and assume that the Poisson structure of Y is positively weighted. Let*

$$X \xrightarrow{f} Y \xleftarrow{f'} X'$$

be a diagram such that,

1. f (resp. f') is a crepant, birational, projective morphism.
2. X (resp. X') has only terminal singularities.
3. X (resp. X') is $\mathbf{Q}$-factorial.

Then both X and X' have locally trivial deformations to an affine variety Y_t obtained as a Poisson deformation of Y. In particular, X and X' have the same kind of singularities.

A typical situation of Corollary 25 is a symplectic flop. At this moment, we need the "good $\mathbf{C}^*$ condition" to make sure the existence of an algebraization of certain formal Poisson deformation. For the exact definition of a good $\mathbf{C}^*$-action, see Appendix. But even if Y does not have such an action, one can prove:

Corollary 31. *Let Y be an affine symplectic variety. Let*

$$X \xrightarrow{f} Y \xleftarrow{f'} X'$$

be a diagram such that,

1. f (resp. f') is a crepant, birational, projective morphism.
2. X (resp. X') has only terminal singularities.
3. X (resp. X') is $\mathbf{Q}$-factorial.

If X is smooth, then X' is smooth.

The proofs of Corollaries 25 and 31 are essentially based on [Ka 1], where he proved that the smoothness is preserved in a symplectic flop under certain assumptions. Corollaries 25 and 31 are local versions of Corollary 1 of [Na]. More general facts can be found in Corollary 30.

The following is the second application:

Corollary 28. *Let Y be an affine symplectic variety with a good $\mathbf{C}^*$-action. Assume that the Poisson structure of Y is positively weighted, and Y has only terminal singularities. Let $f : X \to Y$ be a crepant, birational, projective morphism such that X has only terminal singularities and such that X is $\mathbf{Q}$-factorial. Then the following are equivalent.*

(a) *X is non-singular.*

(b) *Y is smoothable by a Poisson deformation.*

In the proof of Corollary 28, we observe that the pro-representable hulls (= formal Kuranishi spaces) of the Poisson deformations of X and Y are isomorphic. Here we just use the assumption that Y has only terminal singularities. Thus, any formal Poisson deformation of Y is obtained from that of X by the contraction map; this makes it possible for us to obtain (a) from (b). But, what we really want, is just that the formal Kuranishi space for X *dominates* that for Y. The author believes that this would be true if Y does not have terminal singularities. So our final goal would be the following conjecture:

Conjecture[1]. *Let Y be an affine symplectic variety with a good $\mathbf{C}^*$-action. Assume that the Poisson structure of Y is positively weighted. Then the following are equivalent.*

(1) *Y has a crepant projective resolution.*

(2) *Y has a smoothing by a Poisson deformation.*

The contents of this paper are as follows. In §2 we introduce the Poisson cohomology of a Poisson algebra according to Fresse [Fr 1], [Fr 2]. In Propositions 5, we shall prove that a Poisson deformation of a Poisson algebra is controlled by the Poisson cohomology. In particular, when the Poisson algebra is smooth, the Poisson cohomology is computed by the Lichnerowicz-Poisson complex. Since the Lichnerowicz-Poisson complex is defined also for a smooth Poisson scheme, one can define the Poisson cohomology for a smooth Poisson scheme. In §3, we restrict ourselves to the Poisson structures attached to a convex symplectic variety X. When X is smooth, the Poisson cohomology can be identified with the truncated De Rham cohomology (Proposition 9). When X

[1]After submitting this paper, the author showed in [Na 6] that the conjecture is true if the minimal model conjecture holds.

has only terminal singularities, its Poisson deformations are the same as those of the regular locus U of X. Thus the Poisson deformations of X are controlled by the truncated De Rham cohomology of U. Theorem 14 and Corollary 15 assert that, the Poisson deformation functor of a convex symplectic variety with terminal singularities, has a pro-representable hull and it is unobstructed. These are more or less already known. But we reproduce them here so that they fit our aim and our context. (see also [G-K], Appendix). Kaledin's twistor deformation is also easily generalized to our singular case; but this generalization is very useful in the proof of Corollary 25. At the end of this section we shall prove the following two key results:

Theorem 17. *Let X be a convex symplectic variety with terminal singularities. Let $(X, \{\ ,\ \})$ be the Poisson structure induced by the symplectic form on the regular part. Assume that X^{an} is $\mathbf{Q}$-factorial. Then any Poisson deformation of $(X, \{\ ,\ \})$ is locally trivial as a flat deformation* (*after forgetting Poisson structure*).

Theorem 19. *Let X be a convex symplectic variety with terminal singularities. Let L be a* (*not necessarily ample*) *line bundle on X. Then the twistor deformation $\{X_n\}_{n\geq 1}$ of X associated with L is locally trivial as a flat deformation.*

§4 deals with a convex symplectic variety with a good $\mathbf{C}^*$-action. The main results of this section are Corollary 25 and Corollary 28 explained above. These are actually corollaries to Theorem 19 and Theorem 17 respectively. In §5 we consider the general case where Y does not have a good $\mathbf{C}^*$-action. Corollary 30 is a similar statement to Corollary 25 in the general case; but for the lack of algebraizations, it is not clear, at this moment, how the singularities of X' are related with those of X. Finally, we shall prove Corollary 31 explained above. In §6 one can find a concrete example of a Poisson deformation (Example 32). Example 33 is an example of a singular symplectic flop. The final section is Appendix, where some well-known results on good $\mathbf{C}^*$-actions are proved. The main result of Appendix is Corollary A.10. For a non-compact variety, the analytic category and the algebraic category are usually quite different. However, Corollary A.10 asserts that when we have a good $\mathbf{C}^*$-action, both categories are well-matched.

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§2. Poisson Deformations

(i) **Harrison cohomology** (cf. [Ge-Sc]): Let S be a commutative $\mathbf{C}$-algebra and let A be a commutative S-algebra. Let S_n be the n-th symmetric group. Then S_n acts from the left hand side on the n-tuple tensor product $A \otimes_S ... \otimes_S A$ as

$$\pi(a_1 \otimes ... \otimes a_n) := a_{\pi^{-1}(1)} \otimes ... \otimes a_{\pi^{-1}(n)},$$

where $\pi \in S_n$. This action extends naturally to an action of the group algebra $\mathbf{C}[S_n]$ on $A \otimes_S ... \otimes_S A$. For $0 < r < n$, an element $\pi \in S_n$ is called a *pure shuffle* of type $(r, n-r)$ if $\pi(1) < ... < \pi(r)$ and $\pi(r+1) < ... < \pi(n)$. Define an element $s_{r,n-r} \in \mathbf{C}[S_n]$ by

$$s_{r,n-r} := \Sigma \mathrm{sgn}(\pi)\pi,$$

where the sum runs through all pure shuffles of type $(r, n-r)$. Let N be the S-submodule of $A \otimes_S A ... \otimes_S A$ generated by all elements

$$\{s_{r,n-r}(a_1 \otimes ... \otimes a_n)\}_{0<r<n,\, a_i \in A}.$$

Define $\mathrm{ch}_n(A/S) := (A \otimes_S A ... \otimes_S A)/N$. Let M be an A-module. Then the *Harrison chain* $\{\mathrm{ch}_{\cdot}(A/S; M)\}$ is defined as follows:

1. $\mathrm{ch}_n(A/S; M) := \mathrm{ch}_n(A/S) \otimes_S M$
2. the boundary map $\partial_n : \mathrm{ch}_n(A/S; M) \to \mathrm{ch}_{n-1}(A/S; M)$ is defined by $\partial_n(a_1 \otimes ... \otimes a_n \otimes m) :=$

$$a_1 \otimes ... \otimes a_n m + \Sigma_{1 \le i \le n-1} (-1)^{n-i} a_1 \otimes ... \otimes a_i a_{i+1} \otimes ... \otimes a_n \otimes m$$
$$+ (-1)^n a_2 \otimes ... \otimes a_n \otimes a_1 m.$$

We define the n-th *Harrison homology* $\mathrm{Har}_n(A/S; M)$ just as the n-th homology of $\mathrm{ch}_{\cdot}(A/S; M)$.

The *Harrison cochain* $\{\mathrm{ch}^{\cdot}(A/S; M)\}$ is defined as follows

1. $\mathrm{ch}^n(A/S; M) := \mathrm{Hom}_S(\mathrm{ch}_n(A/S), M)$
2. the coboundary map $d^n : \mathrm{ch}^n(A/S; M) \to \mathrm{ch}^{n+1}(A/S; M)$ is defined by $(d^n f)(a_1 \otimes ... \otimes a_{n+1}) := (-1)^{n+1} a_1 f(a_2 \otimes ... \otimes a_{n+1})$

$$+\Sigma_{1 \le i \le n} (-1)^{n+1-i} f(a_1 \otimes ... \otimes a_i a_{i+1} \otimes ... \otimes a_{n+1}) + f(a_1 \otimes ... \otimes a_n) a_{n+1}.$$

We define the n-th *Harrison cohomology* $\mathrm{Har}^n(A/S; M)$ as the n-th cohomology of $\mathrm{ch}^{\cdot}(A/S; M)$.

Example 1. Assume that $S = \mathbf{C}$. Then $\mathrm{ch}_2(A/\mathbf{C}; A) = \mathrm{Sym}^2_{\mathbf{C}}(A) \otimes_{\mathbf{C}} A$ and $\mathrm{ch}_1(A/\mathbf{C}; A) = A \otimes_{\mathbf{C}} A$. The boundary map ∂_2 is defined as

$$\partial_2([a_1 \otimes a_2] \otimes a) := a_1 \otimes a_2 a - a_1 a_2 \otimes a + a_2 \otimes a_1 a.$$

We see that $\mathrm{im}(\partial_2)$ is a right A-submodule of $A \otimes_{\mathbf{C}} A$. Let $I \subset A \otimes_{\mathbf{C}} A$ be the ideal generated by all elements of the form $a \otimes b - b \otimes a$ with $a, b \in A$. Then we have a homomorphism of right A-modules $I \to (A \otimes_{\mathbf{C}} A)/\mathrm{im}(\partial_2)$. One can check that its kernel coincides with I^2. Hence we have

$$\Omega^1_{A/\mathbf{C}} \cong \mathrm{Har}_1(A/\mathbf{C}; A).$$

In fact, the Harrison chain $\mathrm{ch}_{\cdot}(A/\mathbf{C}; A)$ is quasi-isomorphic to the cotangent complex $L^{\cdot}_{A/\mathbf{C}}$ for a $\mathbf{C}$-algebra A (cf. [Q]).

Let A and S be the same as above. We put $S[\epsilon] := S \otimes_{\mathbf{C}} \mathbf{C}[\epsilon]$, where $\epsilon^2 = 0$. Let us consider the set of all $S[\epsilon]$-algebra structures of the $S[\epsilon]$-module $A \otimes_S S[\epsilon]$ such that they induce the original S-algebra A if we take the tensor product of $A \otimes_S S[\epsilon]$ and S over $S[\epsilon]$. We say that two elements of this set are *equivalent* if and only if there is an isomorphism of $S[\epsilon]$-algebras between them which induces the identity map of A over S. We denote by $D(A/S, S[\epsilon])$ the set of such equivalence classes. Fix an $S[\epsilon]$-algebra structure $(A \otimes_S S[\epsilon], *)$. Here $*$ just means the corresponding ring structure. Then we define $\mathrm{Aut}(*, S)$ to be the set of all $S[\epsilon]$-algebra automorphisms of $(A \otimes_S S[\epsilon], *)$ which induces the identity map of A over S.

Proposition 2. *Assume that A is a free S module.*

(1) *There is a one-to-one correspondence between* $\mathrm{Har}^2(A/S; A)$ *and* $D(A/S, S[\epsilon])$.

(2) *There is a one-to-one correspondence between* $\mathrm{Har}^1(A/S; A)$ *and* $\mathrm{Aut}(*, S)$.

Proof. We shall only give a proof to (1). The proof of (2) is left to the readers. Denote by $*$ a ring structure on $A \otimes_S S[\epsilon] = A \oplus A\epsilon$. For $a, b \in A$, write

$$a * b = ab + \epsilon\phi(a, b)$$

with some $\phi : A \times A \to A$. The multiplication of an element of $S[\epsilon]$ and an element of $A \otimes_S S[\epsilon]$ should coincide with the action of $S[\epsilon]$ as the $S[\epsilon]$-module;

hence $a * \epsilon = a\epsilon$ and $a\epsilon * \epsilon = 0$. Then

$$a * (b\epsilon) = a * (b * \epsilon) = (a * b) * \epsilon$$

$$= \{ab + \phi(a,b)\epsilon\} * \epsilon = ab\epsilon + \phi(a,b) * (\epsilon * \epsilon) = ab\epsilon.$$

Similarly, we have $(a\epsilon) * (b\epsilon) = 0$. Therefore, $*$ is determined completely by ϕ. By the commutativity of $*$, $\phi \in \mathrm{Hom}_S(\mathrm{Sym}^2_S(A), A)$. By the associativity: $(a * b) * c = a * (b * c)$, we get

$$\phi(ab, c) + c\phi(a, b) = \phi(a, bc) + a\phi(b, c).$$

This condition is equivalent to that $\phi \in \mathrm{Ker}(d^2)$, where d^2 is the 2-nd coboundary map of the Harrison cochain. Next let us observe when two ring structures $*$ and $*'$ are equivalent. As above, we write $a * b = ab + \epsilon\phi(a,b)$ and $a *' b = ab + \epsilon\phi'(a,b)$. Assume that a map $\psi : A \oplus A\epsilon \to A \oplus A\epsilon$ gives an equivalence. Then, for $a \in A$, write $\psi(a) = a + f(a)\epsilon$ with some $f : A \to A$. One can show that $\psi(a\epsilon) = a\epsilon$. Since $\psi(a) *' \psi(b) = \psi(a * b)$, we see that

$$\phi'(a,b) - \phi(a,b) = f(ab) - af(b) - bf(a).$$

This implies that $\phi' - \phi \in \mathrm{im}(d^1)$.

Remark 3. Assume that S is an Artinian ring and A is flat over S. Then A is a free S-module and for any flat extension A' of A over $S[\epsilon]$, $A' \cong A \otimes_S S[\epsilon]$ as an $S[\epsilon]$-module.

(ii) **Poisson cohomology** (cf. [Fr1], [Fr2]): Let A and S be the same as (i). Assume that A is a free S-module. Let us consider the graded free S-module $\mathrm{ch.}(A/S) := \oplus_{0<m}\mathrm{ch}_m(A/S)$ and take its super-symmetric algebra $\mathcal{S}(\mathrm{ch.}(A/S))$. By definition, $\mathcal{S}(\mathrm{ch.}(A/S))$ is the quotient of the tensor algebra $T(\mathrm{ch.}(A/S)) := \oplus_{0\le n}(\mathrm{ch.}(A/S))^{\otimes n}$ by the two-sided ideal M generated by the elements of the form: $a \otimes b - (-1)^{pq} b \otimes a$, where $a \in \mathrm{ch}_p(A/S)$ and $b \in \mathrm{ch}_q(A/S)$. We denote by $\bar{\mathcal{S}}(\mathrm{ch.}(A/S))$ the truncation of the degree 0 part. In other words,

$$\bar{\mathcal{S}}(\mathrm{ch.}(A/S)) := \oplus_{0<n}(\mathrm{ch.}(A/S))^{\otimes n}/M.$$

Now let us consider the graded A-module

$$\bar{\mathcal{S}}(\mathrm{ch.}(A/S)) \otimes_S A := \mathrm{ch.}(A/S) \otimes_S A \oplus (\mathcal{S}^2(\mathrm{ch.}(A/S)) \otimes_S A) \oplus ...$$

The Harrison boundary maps ∂ on $\mathrm{ch.}(A/S) \otimes_S A$ naturally extends to those on $\mathcal{S}^n(\mathrm{ch.}(A/S)) \otimes_S A$. In fact, for $a_i \in \mathrm{ch}_{p_i}(A/S) \otimes_S A$, $1 \le i \le n$, denote by

$a_1 \cdots a_n \in \mathcal{S}^n_A(\mathrm{ch.}(A/S) \otimes_S A)$ their super-symmetric product. We then define ∂ inductively as

$$\partial(a_1...a_n) := \partial(a_1)a_2...a_n + (-1)^{p_1} a_1 \cdot \partial(a_2...a_n).$$

In this way, each $\mathcal{S}^n_A(\mathrm{ch.}(A/S) \otimes_S A) = \mathcal{S}^n(\mathrm{ch.}(A/S)) \otimes_S A$ becomes a chain complex. By taking the dual,

$$\mathrm{Hom}_A(\mathcal{S}^n(\mathrm{ch.}(A/S)) \otimes_S A, A)$$

$$= \mathrm{Hom}_S(\mathcal{S}^n(\mathrm{ch.}(A/S)), A)$$

becomes a cochain complex:

$$(1)\qquad \begin{array}{cccc} \uparrow d & & & \\ \mathrm{Hom}(\mathrm{ch}_3, A) & ... & ... & \\ \uparrow d & \uparrow d & \uparrow d & \\ \mathrm{Hom}(\mathrm{ch}_2, A) & \mathrm{Hom}(\mathrm{ch}_2 \otimes \mathrm{ch}_1, A) & ... & \\ \uparrow d & \uparrow d & \uparrow d & \\ \mathrm{Hom}(\mathrm{ch}_1, A) & \mathrm{Hom}(\wedge^2 \mathrm{ch}_1, A) & \mathrm{Hom}(\wedge^3 \mathrm{ch}_1, A) & ... \end{array}$$

Here we abbreviate $\mathrm{ch}_i(A/S)$ by ch_i and $\mathrm{Hom}_S(...)$ by $\mathrm{Hom}(...)$. We want to make the diagram above into a double complex when A is a Poisson S-algebra.

Definition. A Poisson S-algebra A is a commutative S-algebra with an S-linear map

$$\{\ ,\ \} : \wedge^2_S A \to A$$

such that

1. $\{a, \{b, c\}\} + \{b, \{c, a\}\} + \{c, \{a, b\}\} = 0$
2. $\{a, bc\} = \{a, b\}c + \{a, c\}b.$

We assume now that A is a Poisson S-module such that A is a free S-module. We put $\bar{T}_S(A) := \oplus_{0<n}(A)^{\otimes n}$. We shall introduce an S-bilinear bracket product

$$[\ ,\ ] : \bar{T}_S(A) \times \bar{T}_S(A) \to \bar{T}_S(A)$$

in the following manner. Take two elements from $\bar{T}_S(A)$: $f = f_1 \otimes ... \otimes f_p$ and $g = g_1 \otimes ... \otimes g_q$. Here each f_i and each g_i are elements of A. Let $\pi \in S_{p+q}$ be a pure shuffle of type (p,q). For the convention, we put $f_{i+p} := g_i$. Then the shuffle product is defined as

$$f \cdot g := \Sigma \mathrm{sgn}(\pi) f_{\pi^{-1}(1)} \otimes ... \otimes f_{\pi^{-1}(p+q)},$$

where the sum runs through all pure shuffle of type (p,q). For each term of the sum (which is indexed by π), let I_π be the set of all i such that $\pi^{-1}(i) \leq p$ and $\pi^{-1}(i+1) \geq p+1$ (which implies that $f_{\pi^{-1}(i+1)} = g_{\pi^{-1}(i+1)-p}$). Then we define $[f,g]$ as

$$\Sigma \mathrm{sgn}(\pi)(\Sigma_{i \in I_\pi}(-1)^{i+1} f_{\pi^{-1}(1)} \otimes ... \otimes \{f_{\pi^{-1}(i)}, f_{\pi^{-1}(i+1)}\} \otimes ... \otimes f_{\pi^{-1}(p+q)}).$$

The bracket $[\ ,\ ]$ induces that on $\mathrm{ch.}(A/S)$ by the quotient map $\bar{T}_S(A) \to \mathrm{ch.}(A/S)$. By abuse of notation, we denote by $[\ ,\ ]$ the induced bracket. We are now in a position to define coboundary maps

$$\delta : \mathrm{Hom}_S(\mathcal{S}^{s-1}(\mathrm{ch.}(A/S)), A) \to \mathrm{Hom}_S(\mathcal{S}^s(\mathrm{ch.}(A/S)), A)$$

so that $\mathrm{Hom}_S(\bar{\mathcal{S}}(\mathrm{ch.}(A/S)), A)$ is made into a double complex together with d already defined. We take an element of the form $x_1 \cdots x_s$ from $\mathcal{S}^s(\mathrm{ch.}(A/S))$ with each x_i being a homogeneous element of $\mathrm{ch.}(A/S)$.

For $f \in \mathrm{Hom}_S(\bar{\mathcal{S}}^{s-1}(\mathrm{ch.}(A/S)), A)$, we define

$$\delta(f)(x_1 ... x_s) := \sum_{1 \leq i \leq s} (-1)^{\sigma(i)} \overline{[x_i, f(x_1 \cdots \breve{x}_i \cdots x_s)]}$$

$$- \sum_{i<j} (-1)^{\tau(i,j)} f([x_i, x_j] \cdots \breve{x}_i \cdots \breve{x}_j \cdots x_s).$$

Here $\overline{[\ ,\ ]}$ is the composite of $[\ ,\ ]$ and the truncation map $\mathrm{ch.}(A/S) \to \mathrm{ch}_1(A/S)(= A)$. Moreover,

$$\sigma(i) := \deg(x_i) \cdot (\deg(x_1) + ... + \deg(x_{i-1}))$$

and

$$\tau(i,j) := \deg(x_i)(\deg x_1 + ... + \deg x_{i-1})$$

$$+ \deg(x_j)(\deg(x_1) + ... + \breve{\deg(x_i)} + ... + \deg(x_{j-1})).$$

We now obtain a double complex $(\mathrm{Hom}_S(\bar{\mathcal{S}}(\mathrm{ch}.(A/S)), A), d, \delta)$. The n-th *Poisson cohomology* $\mathrm{HP}^n(A/S)$ for a Poisson S-algebra A is the n-th cohomology of the total complex (by $d + \delta$) of this double complex.

$$
\begin{tikzcd}
{} & & \\
\mathrm{Hom}(\mathrm{ch}_3, A) \arrow[u, "d"] \arrow[r, "\delta"] & \cdots & \cdots \\
\mathrm{Hom}(\mathrm{ch}_2, A) \arrow[u, "d"] \arrow[r, "\delta"] & \mathrm{Hom}(\mathrm{ch}_2 \otimes \mathrm{ch}_1, A) \arrow[u, "d"] \arrow[r, "\delta"] & \cdots \arrow[u, "d"] \\
\mathrm{Hom}(\mathrm{ch}_1, A) \arrow[u, "d"] \arrow[r, "\delta"] & \mathrm{Hom}(\wedge^2 \mathrm{ch}_1, A) \arrow[u, "d"] \arrow[r, "\delta"] & \mathrm{Hom}(\wedge^3 \mathrm{ch}_1, A) \arrow[u, "d"] & \cdots
\end{tikzcd} \tag{2}
$$

Example 4. We shall calculate δ explicitly in a few cases. As in the diagram above, we abbreviate Hom_S by Hom, and $\mathrm{ch}_i(A/S)$ by ch_i.

(i) Assume that $f \in \mathrm{Hom}(\mathrm{ch}_1, A)$.

$$\delta(f)(a \wedge b) = \{a, f(b)\} + \{f(a), b\} - f(\{a, b\}).$$

(ii) Assume that $\varphi \in \mathrm{Hom}(\mathrm{ch}_2, A)$. For $(a, b) \in \mathrm{Sym}^2_S(A)(= \mathrm{ch}_2)$, and for $c \in A(= \mathrm{ch}_1)$,

$$\begin{aligned} \delta(\varphi)((a, b) \cdot c) &= [(a, b), \varphi(c)] + [c, \varphi(a, b)] - \varphi([(a, b), c]) \\ &= \{c, \varphi(a, b)\} - \varphi(\{c, b\}, a) - \varphi(\{c, a\}, b). \end{aligned}$$

(iii) Assume that $\psi \in \mathrm{Hom}(\wedge^2 \mathrm{ch}_1, A)$.

$$\begin{aligned} \delta(\psi)(a \wedge b \wedge c) &= \{a, \psi(b, c)\} + \{b, \psi(c, a)\} + \{c, \psi(a, b)\} \\ &\quad + \psi(a, \{b, c\}) + \psi(b, \{c, a\}) + \psi(c, \{a, b\}). \end{aligned}$$

Let A be a Poisson S-algebra such that A is a free S-module. We put $S[\epsilon] := S \otimes_{\mathbf{C}} \mathbf{C}[\epsilon]$, where $\epsilon^2 = 0$. Let us consider the set of all Poisson $S[\epsilon]$-algebra structures on the $S[\epsilon]$-module $A \otimes_S S[\epsilon]$ such that they induce the original Poisson S-algebra A if we take the tensor product of $A \otimes_S S[\epsilon]$ and

S over $S[\epsilon]$. We say that two elements of this set are equivalent if and only if there is an isomorphism of Poisson $S[\epsilon]$-algebras between them which induces the identity map of A over S. We denote by $PD(A/S, S[\epsilon])$ the set of such equivalence classes. Fix a Poisson $S[\epsilon]$-algebra structure $(A \otimes_S S[\epsilon], *, \{\ ,\ \})$. Then we define $\mathrm{Aut}(*, \{\ ,\ \}, S)$ to be the set of all automorphisms of Poisson $S[\epsilon]$-algebras of $(A \otimes_S S[\epsilon], *, \{\ ,\ \})$ which induces the identity map of A over S.

Proposition 5. (1) *There is a one-to-one correspondence between* $HP^2(A/S)$ *and* $PD(A/S, S[\epsilon])$.

(2) *There is a one-to-one correspondence between* $HP^1(A/S)$ *and* $\mathrm{Aut}(*, \{\ ,\ \}, S)$.

Proof. (1): As explained in Proposition 2, giving an $S[\epsilon]$-algebra structure $*$ on $A \oplus A\epsilon$ is equivalent to giving $\varphi \in \mathrm{Hom}_S(\mathrm{Sym}^2_S(A), A)$ with $d(\varphi) = 0$ such that $a * b = ab + \epsilon\varphi(a, b)$. Assume that $\{\ ,\ \}_\epsilon$ is a Poisson bracket on $(A \oplus A\epsilon, *)$ which is an extension of the original Poisson bracket $\{\ ,\ \}$ on A. We put

$$\{a, b\}_\epsilon = \{a, b\} + \psi(a, b)\epsilon.$$

Since $\{a, b\epsilon\}_\epsilon = \{a, b * \epsilon\}_\epsilon = \{a, b\}\epsilon$ and $\{a\epsilon, b\epsilon\} = 0$, the Poisson structure $\{\ ,\ \}_\epsilon$ is completely determined by ψ. By the skew-commutativity of $\{\ ,\ \}$, $\psi \in \mathrm{Hom}_S(\wedge^2_S A, A)$. The equality

$$\{a, b * c\}_\epsilon = \{a, b\}_\epsilon * c + \{a, c\}_\epsilon * b$$

is equivalent to the equality

$$(\star):\quad \psi(a, bc) - c\psi(a, b) - b\psi(a, c)$$

$$= \varphi(\{a, b\}, c) + \varphi(\{a, c\}, b) - \{a, \varphi(b, c)\}.$$

The equality

$$\{a, \{b, c\}_\epsilon\}_\epsilon + \{b, \{c, a\}_\epsilon\}_\epsilon + \{c, \{a, b\}_\epsilon\}_\epsilon = 0$$

is equivalent to the equality

$$(\star\star): \psi(a, \{b, c\}) + \psi(b, \{c, a\}) + \psi(c, \{a, b\})$$

$$+\{a, \psi(b, c)\} + \{b, \psi(c, a)\} + \{c, \psi(a, b)\} = 0.$$

We claim that the equality $(\star)$ means $\delta(\varphi) + d(\psi) = 0$ in the diagram:

$$\mathrm{Hom}(\mathrm{Sym}^2(A), A) \xrightarrow{\delta} \mathrm{Hom}(\mathrm{Sym}^2(A) \otimes A, A) \xleftarrow{d} \mathrm{Hom}(\wedge^2 A, A).$$

By Example 4, (ii), we have shown that

$$\delta(\varphi)((a,b)\cdot c) = \{c, \varphi(a,b)\} - \varphi(\{c,b\},a) - \varphi(\{c,a\},b).$$

On the other hand, for the Harrison boundary map

$$\partial : \mathrm{Sym}^2(A)\otimes_S A\otimes_S A \to \wedge^2 A\otimes_S A,$$

we have

$$\partial((a,b)\otimes c\otimes 1) = b(a\wedge c) - ab\wedge c + a(b\wedge c).$$

Since d is defined as the dual map of ∂, we see that

$$d\psi((a,b)\cdot c) = \psi(c,ab) - a\psi(c,b) - b\psi(c,a).$$

As a consequence, we get

$$(\delta\varphi + d\psi)((a,b)\cdot c) = \psi(c,ab) - a\psi(c,b) - b\psi(c,a)$$

$$+\{c,\varphi(a,b)\} - \varphi(\{c,b\},a) - \varphi(\{c,a\},b).$$

By changing a and c each other, we conclude that $\delta(\varphi) + d(\psi) = 0$.

By the equality $(\star\star)$ and Example 4, (iii), we see that $(\star\star)$ means $\delta(\psi) = 0$ for the map $\delta : \mathrm{Hom}(\wedge^2 A, A) \to \mathrm{Hom}(\wedge^3 A, A)$. Next, let us observe when two Poisson structures (φ,ψ) and (φ',ψ') (on $A\oplus A\epsilon$) are equivalent. Assume that, for $f\in \mathrm{Hom}_S(A,A)$,

$$\chi_f : A\oplus A\epsilon \to A\oplus A\epsilon$$

gives such an equivalence between both Poisson structures, where $\chi_f(a) = a + f(a)\epsilon$, $\chi_f(a\epsilon) = a\epsilon$ for $a\in A$. Since χ_f gives an equivalence of $S[\epsilon]$-algebras,

$$(\varphi' - \varphi)(a,b) = f(ab) - af(b) - bf(a) = -d(f)(a,b)$$

by Proposition 2. The map χ_f must be compatible with two Poisson structure:

$$\{\chi_f(a), \chi_f(b)\}'_\epsilon = \chi_f(\{a,b\}_\epsilon).$$

The left hand side equals

$$\{a,b\} + [\phi'(a,b) + \{a, f(b)\} + \{f(a), b\}]\epsilon.$$

The right hand side equals

$$\{a,b\} + [f(\{a,b\}) + \psi(a,b)]\epsilon.$$

Thus, we have

$$(\psi' - \psi)(a,b) = -\delta(f)(a,b),$$

and the proof of (1) is now complete. We omit the proof of (2). □

We next consider the case where A is formally smooth over S. We put $\Theta_{A/S} := \mathrm{Hom}_A(\Omega^1_{A/S}, A)$. We make $\oplus_{i>0} \wedge^i_A \Theta_{A/S}$ into a complex by defining the coboundary map

$$\delta : \wedge^i \Theta_{A/S} \to \wedge^{i+1}\Theta_{A/S}$$

as

$$\delta f(da_1 \wedge ... \wedge da_{i+1}) := \sum_j (-1)^{j+1}\{a_j, f(da_1 \wedge ... \wedge \breve{da_j} \wedge ... \wedge da_{i+1})\}$$

$$- \sum_{j<k} (-1)^{j+k+1} f(d\{a_j, a_k\}) \wedge da_1 \wedge ... \wedge \breve{da_j} \wedge ... \wedge \breve{da_k} \wedge ... \wedge da_{i+1}),$$

for $f \in \wedge^i\Theta_{A/S} = \mathrm{Hom}_A(\Omega^i_{A/S}, A)$. This complex is called the *Lichnerowicz-Poisson* complex. One can connect this complex with our Poisson cochain complex $\mathcal{C}^{\cdot}(A/S)$. In fact, there is a map $\mathrm{ch}_1 \otimes_S A \to \Omega^1_{A/S}$ (cf. Example 1). This map induces, for each i, $\wedge^i \mathrm{ch}_1(A/S) \otimes_S A \to \Omega^i_{A/S}$. By taking the dual, we get

$$\wedge^i \Theta_{A/S} \to \mathrm{Hom}_A(\wedge^i \mathrm{ch}_1(A/S) \otimes_S A, A) = \mathrm{Hom}(\wedge^i \mathrm{ch}_i(A/S), A).$$

By these maps, we have a map of complexes

$$\wedge^{\cdot}\Theta_{A/S} \to \mathcal{C}^{\cdot}(A/S).$$

Proposition 6. *For a Poisson S-algebra A, assume that A is formally smooth over S and that A is a free S-module. Then $(\wedge^{\cdot}\Theta_{A/S}, \delta) \to (\mathcal{C}^{\cdot}(A/S), d + \delta)$ is a quasi-isomorphism.*

For the proof of Proposition 6, see Fresse [Fr 1], Proposition 1.4.9.

Definition. Let $T := \mathrm{Spec}(S)$ and X a T-scheme. Then $(X, \{\ ,\ \})$ is a Poisson scheme over T if $\{\ ,\ \}$ is an $\mathcal{O}_T$-linear map:

$$\{\ ,\ \} : \wedge^2_{\mathcal{O}_T} \mathcal{O}_X \to \mathcal{O}_X$$

such that, for $a, b, c \in \mathcal{O}_X$,

1. $\{a, \{b, c\}\} + \{b, \{c, a\}\} + \{c, \{a, b\}\} = 0$

2. $\{a, bc\} = \{a, b\}c + \{a, c\}b$.

We assume that X is a *smooth* Poisson scheme over T, where $T = \mathrm{Spec}(S)$ with a local Artinian $\mathbf{C}$-algebra S with $S/m_S = \mathbf{C}$. Then the Lichnerowicz-Poisson complex can be globalized [2] to the complex on X

$$\mathcal{LC}^{\cdot}(X/T) := (\wedge^{\cdot}\Theta_{X/T}, \delta).$$

We define the i-th Poisson cohomology as

$$\mathrm{HP}^i(X/T) := \mathbf{H}^i(X, \mathcal{LC}^{\cdot}(X/T)).$$

Remark 7. When $X = \mathrm{Spec}(A)$, $\mathrm{HP}^i(X/T) = \mathrm{HP}^i(A/S)$. In fact, there is a spectral sequence induced from the stupid filtration:

$$E_1^{p,q} := H^q(X, \mathcal{LC}^p(X/T)) => \mathrm{HP}^i(X/T).$$

Since each $\mathcal{LC}^p(X/T)$ is quasi-coherent on the affine scheme X, $H^q(X, \mathcal{LC}^p) = 0$ for $q > 0$. Therefore, this spectral sequence degenerate at E_2-terms and we have

$$\mathrm{HP}^i(X/T) = H^i(\Gamma(X, \mathcal{LC}^{\cdot})),$$

where the right hand side is nothing but $HP^i(A/S)$ by Proposition 6.

One can generalize Proposition 5 to smooth Poisson schemes. Let S be an Artinian $\mathbf{C}$-algebra and put $T := \mathrm{Spec}(S)$. Let X be a Poisson T-scheme which is smooth over T. We put $T[\epsilon] := \mathrm{Spec}S[\epsilon]$ with $\epsilon^2 = 0$. A Poisson deformation $\mathcal{X}$ of X over $T[\epsilon]$ is a Poisson $T[\epsilon]$-algebra such that $\mathcal{X}$ is flat over $T[\epsilon]$ and there is a Poisson isomorphism $\mathcal{X} \times_{T[\epsilon]} T \cong X$ over T. Two Poisson deformations $\mathcal{X}$ and $\mathcal{X}'$ are equivalent if there is an isomorphism $\mathcal{X} \cong \mathcal{X}'$ as Poisson $T[\epsilon]$-schemes such that it induces the identity map of X over T. Denote by $PD(X/T, T[\epsilon])$ the set of equivalence classes of Poisson deformations of X over $T[\epsilon]$. For a Poisson deformation $\mathcal{X}$ of X over $T[\epsilon]$, we denote by $Aut(\mathcal{X}, T)$ the set of all automorphisms of $\mathcal{X}$ as a Poisson $T[\epsilon]$-scheme such that they induce the identity map of X over T.

Proposition 8. (1) *There is a one-to-one correspondence between* $\mathrm{HP}^2(X/T)$ *and* $\mathrm{PD}(X/T, T[\epsilon])$.

(2) *For a Poisson deformation $\mathcal{X}$ of X over $T[\epsilon]$, there is a one-to-one correspondence between* $\mathrm{HP}^1(X/T)$ *and* $Aut(\mathcal{X}, T)$.

[2] The definition of the Poisson cochain complex is subtle because the sheafication of each component of the Harrison complex is not quasi-coherent (cf. [G-K]).

Proof. We only prove (1). For an affine open covering $\mathcal{U} := \{U_i\}_{i \in I}$ of X, construct a double complex $\Gamma(\mathcal{LC}^{\cdot}(\mathcal{U}, X/T))$ as follows:

$$
(3)\qquad
\begin{array}{ccccc}
\delta\Big\uparrow & & \delta\Big\uparrow & & \\
\prod_{i_0} \Gamma(\mathcal{LC}^2(U_{i_0}/T)) & \longrightarrow & \prod_{i_0,i_1} \Gamma(\mathcal{LC}^2(U_{i_0 i_1}/T)) & \longrightarrow & \ldots \\
\delta\Big\uparrow & & \delta\Big\uparrow & & \\
\prod_{i_0} \Gamma(\mathcal{LC}^1(U_{i_0}/T)) & \longrightarrow & \prod_{i_0,i_1} \Gamma(\mathcal{LC}^1(U_{i_0,i_1}/T)) & \longrightarrow & \ldots
\end{array}
$$

Here the horizontal maps are Čech coboundary maps. Since each $\mathcal{LC}^p$ is quasi-coherent, one can calculate the Poisson cohomology by the total complex associated with this double complex:

$$\mathrm{HP}^i(X/T) = H^i(\Gamma(\mathcal{LC}^{\cdot}(\mathcal{U}, X/T))).$$

An element $\zeta \in \mathrm{HP}^2(X/T)$ corresponds to a 2-cocycle

$$(\prod \zeta_{i_0}, \prod \zeta_{i_0,i_1}) \in \prod_{i_0} \Gamma(\mathcal{LC}^2(U_{i_0}/T)) \oplus \prod_{i_0,i_1} \Gamma(\mathcal{LC}^1(U_{i_0 i_1}/T)).$$

By Proposition 5, (1), ζ_{i_0} determines a Poisson deformation $\mathcal{U}_{i_0}$ of U_{i_0} over $T[\epsilon]$. Moreover, $\zeta_{i_0 i_1}$ determines a Poisson isomorphism $\mathcal{U}_{i_0}|_{U_{i_0 i_1}} \cong \mathcal{U}_{i_1}|_{U_{i_0 i_1}}$. One can construct a Poisson deformation of $\mathcal{X}$ of X by patching together $\{\mathcal{U}_{i_0}\}$. Conversely, a Poisson deformation $\mathcal{X}$ is obtained by patching together local Poisson deformations $\mathcal{U}_i$ of U_i for an affine open covering $\{U_i\}_{i \in I}$ of X. Each $\mathcal{U}_i$ determines $\zeta_i \in \Gamma(\mathcal{LC}^2(U_i/T))$, and each Poisson isomorphism $\mathcal{U}_i|_{U_{ij}} \cong \mathcal{U}_j|_{U_{ij}}$ determines $\zeta_{ij} \in \Gamma(\mathcal{LC}^1(U_{ij}))$. Then

$$(\prod \zeta_i, \prod \zeta_{ij}) \in \prod_i \Gamma(\mathcal{LC}^2(U_i/T)) \oplus \prod_{i,j} \Gamma(\mathcal{LC}^1(U_{ij}/T))$$

is a 2-cocycle: hence gives an element of 2-nd Čech cohomology. □

§3. Symplectic Varieties

Assume that X_0 is a non-singular variety over $\mathbf{C}$ of dimension $2d$. Then X_0 is called a *symplectic manifold* if there is a 2-form $\omega_0 \in \Gamma(X_0, \Omega^2_{X_0})$ such that $d\omega_0 = 0$ and $\wedge^d \omega_0$ is a nowhere-vanishing section of $\Omega^{2d}_{X_0}$. The 2-form ω_0 is called a *symplectic form*, and it gives an identification $\Omega^1_{X_0} \cong \Theta_{X_0}$. For a local section f of $\mathcal{O}_{X_0}$, the 1-form df corresponds to a local vector field H_f by this identification. We say that H_f is the *Hamiltonian vector field* for f. If we put $\{f, g\} := \omega(H_f, H_g)$, then X_0 becomes a Poisson scheme over Spec($\mathbf{C}$). Now let us consider a Poisson deformation X of X_0 over $T := \mathrm{Spec}(S)$ with a local Artinian $\mathbf{C}$-algebra S with $S/m_S = \mathbf{C}$. The Poisson bracket $\{\ ,\ \}$ on X can be written as $\{f, g\} = \Theta(df \wedge dg)$ for a relative bi-vector (Poisson bi-vector) $\Theta \in \Gamma(X, \wedge^2 \Theta_{X/T})$. The restriction of Θ to the central fiber X is nothing but the Poisson bi-vector for the original Poisson structure, which is non-degenerate because it is defined via the symplectic form ω_0. Hence Θ is also a non-degenerate relative bi-vector. It gives an identification of $\Theta_{X/T}$ with $\Omega^1_{X/T}$. Hence $\Theta \in \Gamma(X, \wedge^2\Theta_{X/T})$ defines an element $\omega \in \Gamma(X, \Omega^2_{X/T})$ that restricts to ω_0 on X_0. One can define the Hamiltonian vector field $H_f \in \Theta_{X/T}$ for $f \in \mathcal{O}_X$.

Proposition 9. *Assume that X is a Poisson deformation of a symplectic manifold X_0 over an Artinian base T. Then $\mathcal{LC}^{\cdot}(X/T)$ is quasi-isomorphic to the truncated De Rham complex $(\Omega^{\geq 1}_{X/T}, d)$.*

Proof. By the symplectic form ω, we have an identification $\phi : \Theta_{X/T} \cong \Omega^1_{X/T}$; hence, for each $i \geq 1$, we get $\wedge^i \Theta_{X/T} \cong \Omega^i_{X/T}$, which we denote also by ϕ (by abuse of notation). We shall prove that $\phi \circ \delta(f) = d\phi(f)$ for $f \in \wedge^i \Theta_{X/T}$. In order to do that, it suffices to check this for the f of the form: $f = \alpha f_1 \wedge ... \wedge f_i$ with $\alpha \in \mathcal{O}_X$, f_1, ..., $f_i \in \Theta_{X/T}$. It is enough to check that

$$d\phi(f)(H_{a_1} \wedge ... \wedge H_{a_{i+1}}) = \delta f(da_1 \wedge ... \wedge da_{i+1}).$$

We shall calculate the left hand side. In the following, for simplicity, we will not write the $\pm$ signature exactly as $(-1)^{\cdots}$, but only write $\pm$ because it does not cause any confusion. We have

$$(L.H.S.) = d(\alpha\omega(f_1, \cdot) \wedge ... \wedge \omega(f_i, \cdot))(H_{a_1} \wedge ... \wedge H_{a_{i+1}})$$

$$= \sum_{1 \leq j \leq i+1} (-1)^{j+1}\Big(\sum_{\{l_1, ..., l_i\} = \{1, ..., \check{j}, ..., i+1\}} \pm H_{a_j}(\alpha\omega(f_1, H_{a_{l_1}}) \cdots \omega(f_i, H_{a_{l_i}}))$$

$$+ \sum_{1 \leq j < k \leq i+1} (-1)^{j+k}\Big(\sum_{\{l_1, ..., \check{l}, ..., l_i\} = \{1, ..., \check{j}, ..., \check{k}, ..., i+1\}} \pm \alpha\omega(f_1, H_{a_{l_1}}) \times ...$$

$$... \times \omega(f_l, [H_{a_j}, H_{a_k}]) \times ... \times \omega(f_i, H_{a_{l_i}}))$$

$$= \sum_{1 \leq j \leq i+1} (-1)^{j+1}\Big(\sum \pm H_{a_j}(\alpha f_1(da_{l_1}) \cdots f_i(da_{l_i})))$$

$$+ \sum_{1 \le j < k \le i+1} (-1)^{j+k} (\sum \pm \alpha f_1(da_{l_1}) \cdots f_l(d\{a_j, a_k\}) \cdots f_i(da_{l_i}))$$

$$= \sum_{1 \le j \le i+1} (-1)^{j+1} H_{a_j}(\alpha f(da_1 \wedge ... \wedge \check{da_j} \wedge ... \wedge da_{i+1}))$$

$$+ \sum_{1 \le j < k \le i+1} (-1)^{j+k} \alpha f(d\{a_j, a_k\} \wedge da_1 \wedge ... \wedge \check{da_j} \wedge ... \wedge \check{da_k} \wedge ... \wedge da_{i+1})$$

$$= \sum_{1 \le j \le i+1} (-1)^{j+1} \{a_j, \alpha f(da_1 \wedge ... \wedge \check{da_j} \wedge ... \wedge da_{i+1})\}$$

$$+ \sum_{1 \le j < k \le i+1} (-1)^{j+k} \alpha f(d\{a_j, a_k\} \wedge ... \wedge \check{da_j} \wedge ... \wedge \check{da_k} \wedge ... \wedge da_{i+1})$$

$= (R.H.S.).$ □

Corollary 10. *Assume that X is a Poisson deformation of a symplectic manifold X_0 over an Artinian base T. If $H^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = 0$, then $\mathrm{HP}^2(X/T) = H^2((X_0)^{an}, S)$, where $(X_0)^{an}$ is a complex analytic space associated with X_0 and S is the constant sheaf with value in S.*

Proof. By the distinguished triangle

$$\Omega^{\ge 1}_{X/T} \to \Omega^{\cdot}_{X/T} \to \mathcal{O}_X \xrightarrow{[1]} \Omega^{\ge 1}_{X/T}[1]$$

we have an exact sequence

$$\to \mathrm{HP}^i(X/T) \to \mathbf{H}^i(\Omega^{\cdot}_{X/T}) \to H^i(\mathcal{O}_X) \to .$$

Here $\mathbf{H}^i(X, \Omega^{\cdot}_{X/T}) \cong H^i((X_0)^{an}, S)$; from this we obtain the result. We prove this by an induction of $\mathrm{length}_{\mathbf{C}}(S)$. We take $t \in S$ such that $t \cdot m_S = 0$. For the exact sequence

$$0 \to \mathbf{C} \xrightarrow{t} S \to \bar{S} \to 0,$$

define $\bar{X} := X \times_T \bar{T}$, where $\bar{T} := \mathrm{Spec}(\bar{S})$. Then we obtain a commutative diagrams of exact sequences:

(4)

$$\begin{array}{ccccccc}
\longrightarrow & \mathbf{H}^i(X_0, \Omega^{\cdot}_{X_0}) & \longrightarrow & \mathbf{H}^i(X, \Omega^{\cdot}_{X/T}) & \longrightarrow & \mathbf{H}^i(\bar{X}, \Omega^{\cdot}_{\bar{X}/\bar{T}}) \\
& \cong\downarrow & & \downarrow & & \cong\downarrow \\
\longrightarrow & \mathbf{H}^i((X_0)^{an}, \Omega^{\cdot}_{(X_0)^{an}}) & \longrightarrow & \mathbf{H}^i(X^{an}, \Omega^{\cdot}_{X^{an}/T}) & \longrightarrow & \mathbf{H}^i(\bar{X}^{an}, \Omega^{\cdot}_{\bar{X}^{an}/\bar{T}})
\end{array}$$

By a theorem of Grothendieck [G], the first vertical maps are isomorphisms and the third vertical maps are isomorphisms by the induction. Hence the middle vertical maps are also isomorphisms. By the Poincare lemma (cf. [De]), we know that $\mathbf{H}^i(X^{an}, \Omega^{\cdot}_{X^{an}/T}) \cong H^i((X_0)^{an}, S)$. □

Example 11. When $f : X \to T$ is a proper smooth morphism of $\mathbf{C}$-schemes, by GAGA, we have

$$\mathbf{R}^i f_* \Omega^{\cdot}_{X/T} \otimes_{\mathcal{O}_T} \mathcal{O}_{T^{an}} \cong R^i (f^{an})_* \mathbf{C} \otimes_{\mathbf{C}} \mathcal{O}_{T^{an}}$$

without the Artinian condition for T. But when f is not proper, the structure of $\mathbf{R}^i f_* \Omega^{\cdot}_{X/T}$ is complicated. For example, Put $X := \mathbf{C}^2 \setminus \{xy = 1\}$, where x and y are standard coordinates of $\mathbf{C}^2$. Let $f : X \to T := \mathbf{C}$ be the map defined by $(x, y) \to x$. Set $\hat{T} := \mathrm{Spec}\mathbf{C}[[x]]$ and $T_n := \mathrm{Spec}\mathbf{C}[x]/(x^{n+1})$. Define $\hat{X} := X \times_T \hat{T}$ and define $\hat{f}$ to be the natural map from $\hat{X} \to \hat{T}$. Finally put $X_n := X \times_T T_n$. Then

1. $\mathbf{R}^1 f_* \Omega^{\cdot}_{X/T}$ is a quasi-coherent sheaf on T, and $\mathbf{R}^1 f_* \Omega^{\cdot}_{X/T}|_{T \setminus \{0\}}$ is an invertible sheaf.
2. proj.lim $H^1(X_n, \Omega^{\cdot}_{X_n/T_n}) = 0$.

Definition. Let X_0 be a normal variety of dimension $2d$ over $\mathbf{C}$ and let U_0 be its regular part. Then X_0 is a symplectic variety if U_0 admits a 2-form ω_0 such that

1. $d\omega_0 = 0$,
2. $\wedge^d \omega_0$ is a nowhere-vanishing section of $\wedge^d \Omega^1_{U_0}$,
3. for any resolution $\pi : Y_0 \to X_0$ of X_0 with $\pi^{-1}(U_0) \cong U_0$, ω_0 extends to a (regular) 2-form on Y_0.

If X_0 is a symplectic variety, then U_0 becomes a Poisson scheme. Since $\mathcal{O}_{X_0} = (j_0)_* \mathcal{O}_{U_0}$, the Poisson bracket $\{\ ,\ \}$ on U_0 uniquely extends to that on X_0. Thus X_0 is a Poisson scheme. By definition, its Poisson bi-vector Θ_0 is non-degenerate over U_0. The Θ_0 identifies Θ_{U_0} with $\Omega^1_{U_0}$; by this identification, $\Theta_0|_{U_0}$ corresponds to ω_0. A symplectic variety X_0 has rational Gorenstein singularities; in other words, X has canonical singularities of index 1. When X_0 has only terminal singularities, $\mathrm{Codim}(\Sigma_0 \subset X_0) \geq 4$ for $\Sigma_0 := \mathrm{Sing}(X_0)$.

Definition. Let X_0 be a symplectic variety. Then X_0 is *convex* if there is a birational projective morphism from X_0 to an affine normal variety Y_0. In this case, Y_0 is isomorphic to $\mathrm{Spec}\Gamma(X_0, \mathcal{O}_{X_0})$.

Lemma 12. *Let X_n be a Poisson deformation of a convex symplectic variety X_0 over $T_n := \mathrm{Spec}(S_n)$ with $S_n := \mathbf{C}[t]/(t^{n+1})$. We define $U_n \subset X_n$ to be locus where $X_n \to S_n$ is smooth. Assume that X_0 has only terminal singularities. Then $\mathrm{HP}^2(U_n/T_n) \cong H^2((U_0)^{an}, S_n)$, where S_n is the constant sheaf over $(U_0)^{an}$ with value in S_n.*

Proof. Since X_0 has terminal singularities, X_0 is Cohen-Macaulay and $\mathrm{Codim}(\Sigma_0 \subset X_0) \geq 4$. Similarly, X_n is Cohen-Macaulay and $\mathrm{Codim}(\Sigma_n \subset X_n) \geq 4$ for $\Sigma_n := \mathrm{Sing}(X_n \to T_n)$. The affine normal variety Y_0 has symplectic singularities; hence Y_0 has rational singularities. This implies that $H^i(X_0, \mathcal{O}_{X_0}) = 0$ for $i > 0$. Since X_0 is Cohen-Macaulay and $\mathrm{Codim}(\Sigma_0 \subset X_0) \geq 4$, we see that $H^1(U_0, \mathcal{O}_{U_0}) = H^2(U_0, \mathcal{O}_{U_0}) = 0$ by the depth argument. By using the exact sequences

$$0 \to \mathcal{O}_{U_0} \xrightarrow{t^k} \mathcal{O}_{U_k} \to \mathcal{O}_{U_{k-1}} \to 0$$

inductively, we conclude that $H^1(\mathcal{O}_{U_n}) = H^2(\mathcal{O}_{U_n}) = 0$. Then, by Corollary 10, we have $\mathrm{HP}^2(U_n/T_n) \cong H^2((U_0)^{an}, S_n)$. □

Let X_n be the same as Lemma 12. Put $T_n[\epsilon] := \mathrm{Spec}(S_n[\epsilon])$ with $\epsilon^2 = 0$. As in Proposition 8, we define $\mathrm{PD}(X_n/T_n, T_n[\epsilon])$ to be the set of equivalence classes of the Poisson deformations of X_n over $T_n[\epsilon]$. Let $\mathcal{X}_n$ be a Poisson deformation of X_n over $T_n[\epsilon]$. Then we denote by $\mathrm{Aut}(\mathcal{X}_n, T_n)$ the set of all automorphisms of $\mathcal{X}_n$ as a Poisson $T_n[\epsilon]$-scheme such that they induce the identity map of X_n over T_n. Then we have:

Proposition 13.

(1) *There is a one-to-one correspondence between* $\mathrm{HP}^2(U_n/T_n)$ *and* $\mathrm{PD}(X_n/T_n, T_n[\epsilon])$.

(2) *There is a one-to-one correspondence between* $\mathrm{HP}^1(U_n/T_n)$ *and* $\mathrm{Aut}(\mathcal{X}_n.T_n)$.

Proof. Assume that $\mathcal{U}_n$ is a Poisson deformation of U_n over $T_n[\epsilon]$. Since $\mathrm{Codim}(\Sigma_n \subset X_n) \geq 3$ and X_n is Cohen-Macaulay, by [K-M, 12.5.6],

$$\mathrm{Ext}^1(\Omega^1_{X_n/T_n}, \mathcal{O}_{X_n}) \cong \mathrm{Ext}^1(\Omega^1_{U_n/T_n}, \mathcal{O}_{U_n}).$$

This implies that, over $T_n[\epsilon]$, $\mathcal{U}_n$ extends uniquely to an $\mathcal{X}_n$ so that it gives a flat deformation of X_n. Let us denote by $j : \mathcal{U}_n \to \mathcal{X}_n$ the inclusion map. Then, by the depth argument, we see that $\mathcal{O}_{\mathcal{X}_n} = j_*\mathcal{O}_{\mathcal{U}_n}$. Therefore, the Poisson structure on $\mathcal{U}_n$ also extends uniquely to that on $\mathcal{X}_n$. Now Proposition

8 implies (1). As for (2), let $\mathcal{U}_n$ be the locus of $\mathcal{X}_n$ where $\mathcal{X}_n \to T_n[\epsilon]$ is smooth. Then, we see that

$$\mathrm{Aut}(\mathcal{U}_n, T_n) = \mathrm{Aut}(\mathcal{X}_n, T_n),$$

which implies (2) again by Proposition 8. $\square$

Let X be a convex symplectic variety with terminal singularities. We regard X as a Poisson scheme by the natural Poisson structure $\{\ ,\ \}$ induced by the symplectic form on the regular locus $U := (X)_{\mathrm{reg}}$. For a local Artinian $\mathbf{C}$-algebra S with $S/m_S = \mathbf{C}$, we define $\mathrm{PD}(S)$ to be the set of equivalence classes of the pairs of Poisson deformations $\mathcal{X}$ of X over $\mathrm{Spec}(S)$ and Poisson isomorphisms $\phi : \mathcal{X} \times_{\mathrm{Spec}(S)} \mathrm{Spec}(\mathbf{C}) \cong X$. Here $(\mathcal{X}, \phi)$ and $(\mathcal{X}', \phi')$ are equivalent if there is a Poisson isomorphism $\varphi : \mathcal{X} \cong \mathcal{X}'$ over $\mathrm{Spec}(S)$ which induces the identity map of X over $\mathrm{Spec}(\mathbf{C})$ via ϕ and ϕ'. We define the *Poisson deformation functor*:

$$\mathrm{PD}_{(X,\{\ ,\ \})} : (\mathrm{Art})_{\mathbf{C}} \to (\mathrm{Set})$$

by $\mathrm{PD}(S)$ for $S \in (\mathrm{Art})_{\mathbf{C}}$.

Theorem 14. *Let $(X, \{\ ,\ \})$ be a Poisson scheme associated with a convex symplectic variety with terminal singularities. Then $\mathrm{PD}_{(X,\{\ ,\ \})}$ has a pro-representable hull in the sense of Schlessinger. Moreover* PD *is pro-representable.*

Proof. We have to check Schlessinger's conditions [Sch] for the existence of a hull. By Proposition 13, $\mathrm{PD}(\mathbf{C}[\epsilon]) = H^2(U^{an}, \mathbf{C}) < \infty$. Other conditions are checked in a similar way as the case of usual deformations. For the last statement, we have to prove the following. Let $\mathcal{X}$ be a Poisson deformation of X over an Artinian base T, and let $\bar{\mathcal{X}}$ be its restriction over a closed subscheme $\bar{T}$ of T. Then, any Poisson automorphism of $\bar{\mathcal{X}}$ over $\bar{T}$ inducing the identity map on X, extends to a Poisson automorphism of $\mathcal{X}$ over T. Let R be the pro-representable hull of PD and put $R_n := R/(m_R)^{n+1}$. Take a formal versal Poisson deformation $\{\mathcal{X}_n\}$ over $\{R_n\}$. Note that, if we are given an Artinian local R-algebra S with residue field $\mathbf{C}$, then we get a Poisson deformation X_S of X over Spec(S). We then define $\mathrm{Aut}(S)$ to be the set of all Poisson automorphisms of X_S over $\mathrm{Spec}(S)$ which induce the identity map of X. Let

$$\mathrm{Aut} : (\mathrm{Art})_R \to (\mathrm{Set})$$

be the covariant functor defined in this manner. We want to prove that $\mathrm{Aut}(S) \to \mathrm{Aut}(\bar{S})$ is surjective for any surjection $S \to \bar{S}$. It is enough to check this only for a small extension $S \to \bar{S}$, that is, the kernel I of $S \to \bar{S}$ is generated by an element a such that $am_S = 0$. For each small extension $S \to \bar{S}$, one can define the *obstruction map*

$$\mathrm{ob} : \mathrm{Aut}(\bar{S}) \to a \cdot \mathrm{HP}^2(U)$$

in such a way that any element $\phi \in \mathrm{Aut}(\bar{S})$ can be lifted to an element of $\mathrm{Aut}(S)$ if and only if $\mathrm{ob}(\phi) = 0$. The obstruction map is constructed as follows. For $\phi \in \mathrm{Aut}(\bar{S})$, we have two Poisson extensions $X_{\bar{S}} \to X_S$ and $X_{\bar{S}} \xrightarrow{\phi} X_{\bar{S}} \to X_S$. This gives an element of $a \cdot \mathrm{HP}^2(U)$ (cf. Proposition 13 [3]). Obviously, if this element is zero, then these two extensions are equivalent and ϕ extends to a Poisson automorphism of X_S.

Case 1 ($S = S_{n+1}$ and $\bar{S} := S_n$): We put $S_n := \mathrm{C}[t]/(t^{n+1})$. We shall prove that $\mathrm{Aut}(S_{n+1}) \to \mathrm{Aut}(S_n)$ is surjective. Taking Proposition 13, (2) into consideration, we say that X has T^0-lifting property if, for any Poisson deformation X_n of X over $T_n := \mathrm{Spec}(S_n)$ and its restriction X_{n-1} over $T_{n-1} := \mathrm{Spec}(S_{n-1})$, the natural map $\mathrm{HP}^1(U_n/T_n) \to \mathrm{HP}^1(U_{n-1}/T_{n-1})$ is surjective.

Claim. *X has T^0-lifting property.*

Proof. Note that X_n is Cohen-Macaulay. Let U_n be the locus of X_n where $X_n \to T_n$ is smooth. We put

$$K_n := \mathrm{Coker}[H^0(U^{an}, S_n) \to H^0(U_n, \mathcal{O}_{U_n})].$$

By the proof of Corollary 10, there is an exact sequence

$$0 \to K_n \to \mathrm{HP}^1(U_n/T_n) \to H^1(U^{an}, S_n) \to 0.$$

Since $H^1(U, \mathcal{O}_U) = 0$, the restriction map $H^0(U_n, \mathcal{O}_{U_n}) \to H^0(U_{n-1}, \mathcal{O}_{U_{n-1}})$ is surjective. Hence the map $K_n \to K_{n-1}$ is surjective. On the other hand, $H^1(U^{an}, S_n) \to H^1(U^{an}, S_{n-1})$ is also surjective; hence the result follows. $\square$

Note that $t \to t + \epsilon$ induces the commutative diagram of exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (t^{n+1}) & \longrightarrow & S_{n+1} & \longrightarrow & S_n & \longrightarrow & 0 \\ & & \cong\downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & (t^n\epsilon) & \longrightarrow & S_n[\epsilon] & \longrightarrow & S_{n-1}[\epsilon] \times_{S_{n-1}} S_n & \longrightarrow & 0 \end{array} \tag{5}$$

[3]Exactly, one can prove the following. Let $T := \mathrm{Spec}(S)$ with a local Artinian $\mathbf{C}$-algebra S with $S/m_s = \mathbf{C}$. Let $X \to T$ be a Poisson deformation of a convex symplectic variety X_0 with only terminal singularities. Assume that T is a closed subscheme of T' defined by the ideal sheaf $I = (a)$ such that $a \cdot m_{S'} = 0$. Denote by $\mathrm{PD}(X/T, T')$ the set of equivalence classes of Poisson deformations of X over T'. If $\mathrm{PD}(X/T, T') \neq \emptyset$, then $\mathrm{HP}^2(U_0) \cong \mathrm{PD}(X/T, T')$.

Applying Aut to this diagram, we obtain

$$
\begin{array}{ccccc}
\mathrm{Aut}(S_{n+1}) & \longrightarrow & \mathrm{Aut}(S_n) & \xrightarrow{\mathrm{ob}} & t^{n+1}\cdot \mathrm{HP}^2(U) \\
\downarrow & & \downarrow & & \cong\downarrow \\
\mathrm{Aut}(S_n[\epsilon]) & \longrightarrow & \mathrm{Aut}(S_{n-1}[\epsilon]\times_{S_{n-1}} S_n) & \xrightarrow{\mathrm{ob}} & t^n\epsilon\cdot \mathrm{HP}^2(U)
\end{array} \tag{6}
$$

The T^0-lifting property implies that the map $\mathrm{Aut}(S_n[\epsilon]) \to \mathrm{Aut}(S_{n-1}[\epsilon]\times_{S_{n-1}} S_n)$ is surjective. Hence, by the commutative diagram, we see that $\mathrm{Aut}(S_{n+1}) \to \mathrm{Aut}(S_n)$ is surjective.

Case 2 (general case): For any small extension $S \to \bar{S}$, one can find the following commutative diagram for some n:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & aS & \longrightarrow & S & \longrightarrow & \bar{S} & \longrightarrow & 0 \\
 & & \cong\downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (t^{n+1}) & \longrightarrow & S_{n+1} & \longrightarrow & S_n & \longrightarrow & 0
\end{array} \tag{7}
$$

Applying Aut to this diagram, we get:

$$
\begin{array}{ccccc}
\mathrm{Aut}(S) & \longrightarrow & \mathrm{Aut}(\bar{S}) & \xrightarrow{\mathrm{ob}} & a\cdot \mathrm{HP}^2(U) \\
\downarrow & & \downarrow & & \cong\downarrow \\
\mathrm{Aut}(S_{n+1}) & \longrightarrow & \mathrm{Aut}(S_n) & \xrightarrow{\mathrm{ob}} & t^{n+1}\cdot \mathrm{HP}^2(U)
\end{array} \tag{8}
$$

By Case 1, we already know that $\mathrm{Aut}(S_{n+1}) \to \mathrm{Aut}(S_n)$ is surjective. By the commutative diagram we see that $\mathrm{Aut}(S) \to \mathrm{Aut}(\bar{S})$ is surjective. □

Corollary 15. *Let $(X, \{\ ,\ \})$ be the same as Theorem 14. Then*

(1) *X has T^1-lifting property.* (cf. [Kaw, Na 5])

(2) $\mathrm{PD}_{(X,\{\ ,\ \})}$ *is unobstructed.*

Proof. (1): We put $S_n := \mathbf{C}[t]/(t^{n+1})$ and $T_n := \mathrm{Spec}(S_n)$. Let X_n be a Poisson deformation of X over T_n and let X_{n-1} be its restriction over T_{n-1}. By Proposition 13,(1), we have to prove that $\mathrm{HP}^2(U_n/T_n) \to \mathrm{HP}^2(U_{n-1}/T_{n-1})$ is surjective. By Lemma 12, $\mathrm{HP}^2(U_n/T_n) \cong H^2(U^{an}, S_n)$. Since $H^2(U^{an}, S_n) = H^2(U^{an}, \mathbf{C}) \otimes_{\mathbf{C}} S_n$, we conclude that this map is surjective.

(2): By Theorem 14, PD has a pro-representable hull R. Denote by $h_R : (\mathrm{Art})_{\mathbf{C}} \to (\mathrm{Set})$ the covariant functor defined by $h_R(S) := \mathrm{Hom}_{\mathrm{local}\ \mathbf{C}-\mathrm{alg.}}(R, S)$. Since PD is pro-representable by Theorem 14, $h_R = \mathrm{PD}$. We write R as

$\mathbf{C}[[x_1, ..., x_r]]/J$ with $r := \dim_{\mathbf{C}} m_R/(m_R)^2$. Let S and S_0 be the objects of $(\mathrm{Art})_{\mathbf{C}}$ such that $S_0 = S/I$ with an ideal I such that $Im_S = 0$. Then we have an exact sequence (cf. [Gr, (1.7)])

$$h_R(S) \to h_R(S_0) \xrightarrow{ob} (J/m_R J)^* \otimes_{\mathbf{C}} I.$$

By sending t to $t+\epsilon$, we have the commutative diagram of exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (t^{n+1}) & \longrightarrow & S_{n+1} & \longrightarrow & S_n & \longrightarrow & 0 \\ & & \cong\downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & (t^n\epsilon) & \longrightarrow & S_n[\epsilon] & \longrightarrow & S_{n-1}[\epsilon] \times_{S_{n-1}} S_n & \longrightarrow & 0 \end{array} \tag{9}$$

Applying h_R to this diagram, we obtain

$$\begin{array}{ccccc} h_R(S_{n+1}) & \longrightarrow & h_R(S_n) & \xrightarrow{\mathrm{ob}} & t^{n+1} \otimes (J/m_R J)^* \\ \downarrow & & \downarrow & & \cong\downarrow \\ h_R(S_n[\epsilon]) & \longrightarrow & h_R(S_{n-1}[\epsilon] \times_{S_{n-1}} S_n) & \xrightarrow{\mathrm{ob}} & t^n\epsilon \otimes (J/m_R J)^* \end{array} \tag{10}$$

By (1), we see that $h_R(S_n[\epsilon]) \to h_R(S_{n-1}[\epsilon] \times_{S_{n-1}} S_n)$ is surjective. Then, by the commutative diagram, we conclude that $h_R(S_{n+1}) \to h_R(S_n)$ is surjective. $\square$

Twistor deformations (cf. [Ka 1]): Let X be a convex symplectic variety with terminal singularities. We put $U := X_{reg}$. Let $\{\ ,\ \}$ be the natural Poisson structure on X defined by the symplectic form ω on U. Fix a line bundle L on X^{an}. Define a class $[L]$ of L as the image of L by the map

$$H^1(U^{an}, \mathcal{O}^*_{U^{an}}) \to \mathbf{H}^2(U^{an}, \Omega^{\cdot}_{U^{an}}) \cong H^2(U^{an}, \mathbf{C}).$$

We put $S_n := \mathbf{C}[t]/(t^{n+1})$ and $T_n := \mathrm{Spec}(S_n)$. By Proposition 13, (1), the element $[L] \in H^2(U^{an}, \mathbf{C})$ determines a Poisson deformation X_1 of X over T_1. We shall construct Poisson deformations X_n over T_n inductively. Assume that we already have a Poisson deformation X_n over T_n. Define X_{n-1} to be the restriction of X_n over T_{n-1}. Since $H^1(X^{an}, \mathcal{O}_{X^{an}}) = H^2(X^{an}, \mathcal{O}_{X^{an}}) = 0$, L extends uniquely to a line bundle L_n on $(X_n)^{an}$. Denote by L_{n-1} the restriction of L_n to $(X_{n-1})^{an}$. Consider the map $S_n \to S_{n-1}[\epsilon]$ defined by $t \to t + \epsilon$. This map induces

$$\mathrm{PD}(S_n) \to \mathrm{PD}(S_{n-1}[\epsilon]).$$

The class $[L_{n-1}] \in H^2(U^{an}, S_{n-1})$ determines a Poisson deformation $(X_{n-1})'$ of X_{n-1} over $T_{n-1}[\epsilon]$. Assume that X_n satisfies the condition

$(*)_n$: $[X_n] \in \mathrm{PD}(S_n)$ is sent to $[(X_{n-1})'] \in \mathrm{PD}(S_{n-1}[\epsilon])$.

Note that X_1 actually has this property. We shall construct X_{n+1} in such a way that X_{n+1} satisfies $(*)_{n+1}$. Look at the commutative diagram:

$$\begin{array}{ccc} \mathrm{PD}(S_{n+1}) & \longrightarrow & \mathrm{PD}(S_n) \\ \downarrow & & \downarrow \\ \mathrm{PD}(S_n[\epsilon]) & \longrightarrow & \mathrm{PD}(S_{n-1}[\epsilon] \times_{S_{n-1}} S_n) \end{array} \tag{11}$$

Note that we have an element

$$[X_n \leftarrow X_{n-1} \rightarrow (X_{n-1})')] \in \mathrm{PD}(S_{n-1}[\epsilon] \times_{S_{n-1}} S_n).$$

Identifying $\mathrm{HP}^2(U_n/T_n)$ with $H^2(U^{an}, S_n)$, $[L_n]$ is sent to $[L_{n-1}]$ by the map

$$\mathrm{HP}^2(U_n/T_n) \rightarrow \mathrm{HP}^2(U_{n-1}/T_{n-1}).$$

Now, by Proposition 13,(1), we get a lifting $[(X_n)'] \in \mathrm{PD}(S_n[\epsilon])$ of

$$[X_n \leftarrow X_{n-1} \rightarrow (X_{n-1})')] \in \mathrm{PD}(S_{n-1}[\epsilon] \times_{S_{n-1}} S_n)$$

corresponding to $[L_n]$. By the standard argument used in T^1-lifting principle (cf. proof of Corollary 15, (2)), one can find a Poisson deformation X_{n+1} such that $[X_{n+1}] \in \mathrm{PD}(S_{n+1})$ is sent to $[(X_n)'] \in \mathrm{PD}(S_n[\epsilon])$ in the diagram above. Moreover, since PD is pro-representable, such $[X_{n+1}]$ is unique. By the construction, X_{n+1} satisfies $(*)_{n+1}$. This construction do not need the sequence of line bundles L_n on $(X_n)^{an}$; we only need the sequence of line bundles on $(U_n)^{an}$. For example, if we are given a line bundle L^0 on U^{an}. Then, since $H^i(U^{an}, \mathcal{O}_{U^{an}}) = 0$ for $i = 1, 2$, we have a unique extension $L^0_n \in \mathrm{Pic}((U_n)^{an})$. By using this, one can construct a formal deformation of X.

Definition. (1) When $L \in \mathrm{Pic}(X^{an})$, we call the formal deformation $\{X_n\}_{n \geq 1}$ the *twistor deformation* of X associated with L.

(2) More generally, for $L^0 \in \mathrm{Pic}(U^{an})$, we call, the formal deformation $\{X_n\}_{n \geq 1}$ similarly constructed, the *quasi-twistor deformation* of X associated with L^0. When L^0 extends to a line bundle L on X^{an}, the corresponding quasi-twistor deformation coincides with the twistor deformation associated with L.

We next define the *Kodaira-Spencer class* of the formal deformation $\{X_n\}$. As before, we denote by U_n the locus of X_n where $f_n : X_n \rightarrow T_n$ is smooth. We put $f^0_n := f_n|_{U_n}$. The extension class $\theta_n \in H^1(U, \Theta_{U_{n-1}/T_{n-1}})$ of the exact sequence

$$0 \rightarrow (f^0_n)^* \Omega^1_{T_n/\mathbf{C}} \rightarrow \Omega^1_{U_n/\mathbf{C}} \rightarrow \Omega^1_{U_n/T_n} \rightarrow 0$$

is the Kodaira Spencer class for $f_n : X_n \to T_n$. Here note that $\Omega^1_{T_n} \cong \mathcal{O}_{T_{n-1}} dt$.

Lemma 16. *Let $\{X_n\}$ be the twistor deformation of X associated with $L \in \mathrm{Pic}(X^{an})$. Write $L_n \in \mathrm{Pic}(X_n^{an})$ for the extension of L to X_n. Let $\omega_n \in \Gamma(U_n, \Omega^2_{U_n/T_n})$ be the symplectic form defined by the Poisson T_n-scheme X_n. Then*

$$\imath(\theta_{n+1})(\omega_n) = [L_n] \in H^1(U, \Omega^1_{U_n/T_n}),$$

where the left hand side is the interior product.

Proof. We use the same notation in the definition of a twistor deformation. By the commutative diagram

$$\begin{array}{ccc} (X_n)' & \longrightarrow & X_{n+1} \\ \downarrow & & \downarrow \\ T_n[\epsilon] & \longrightarrow & T_{n+1} \end{array} \tag{12}$$

we get the commutative diagram of exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_{U_n} d\epsilon & \longrightarrow & \Omega^1_{(U_n)'/T_n}|_{U_n} & \longrightarrow & \Omega^1_{U_n/T_n} & \longrightarrow & 0 \\ & & \cong\uparrow & & \cong\uparrow & & \cong\uparrow & & \\ 0 & \longrightarrow & \mathcal{O}_{U_n} dt & \longrightarrow & \Omega^1_{U_{n+1}/\mathbf{C}}|_{U_n} & \longrightarrow & \Omega^1_{U_n/T_n} & \longrightarrow & 0 \end{array} \tag{13}$$

The second exact sequence is the Kodaira-Spencer's sequence where the first term is $(f^0)^*\Omega^1_{T_{n+1}/\mathbf{C}}$ and the third term is $\Omega^1_{U_{n+1}/T_{n+1}}|_{U_n}$. Let $\eta \in H^1(U, \Theta_{U_n/T_n})$ be the extension class of the first exact sequence. By the definition of $(X_n)'$, we have $i(\eta)(\omega_n) = [L_n]$. On the other hand, the extension class of the second exact sequence is θ_{n+1}. Hence $\eta = \theta_{n+1}$. □

Let $\{X_n\}$ be the twistor deformation of X associated with $L \in \mathrm{Pic}(X)$. For each n, we put $Y_n := \mathrm{Spec}\Gamma(X_n, \mathcal{O}_{X_n})$. Y_n is an affine scheme over T_n. Since $H^1(X, \mathcal{O}_X) = 0$, $\Gamma(X_n, \mathcal{O}_{X_n}) \to \Gamma(X_{n-1}, \mathcal{O}_{X_{n-1}})$ is surjective. Define

$$Y_\infty := \mathrm{Spec}(\varprojlim \Gamma(X_n, \mathcal{O}_{X_n})).$$

Note that Y_∞ is an affine variety over $T_\infty := \mathrm{Spec}\mathbf{C}[[t]]$. Fix an ample line bundle A on X. Since $H^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = 0$, A extends uniquely to ample line bundles A_n on X_n. Then, by [EGA III, Théorème (5.4.5)], there is an *algebraization* X_∞ of $\{X_n\}$ such that X_∞ is a projective scheme over Y_∞ and $X_\infty \times_{Y_\infty} Y_n \cong X_n$ for all n. By [ibid, Theoreme 5.4.1], the algebraization X_∞ is unique. We denote by g_∞ the projective morphism $X_\infty \to Y_\infty$.

Theorem 17. *Let X be a convex symplectic variety with terminal singularities. Let $(X, \{\ ,\ \})$ be the Poisson structure induced by the symplectic form on the regular part. Assume that X^{an} is $\mathbf{Q}$-factorial* [4]*. Then any Poisson deformation of $(X, \{\ ,\ \})$ is locally trivial as a flat deformation (after forgetting Poisson structure).*

Proof. Define a subfunctor

$$\mathrm{PD}_{lt} : (\mathrm{Art})_{\mathbf{C}} \to (\mathrm{Set})$$

of PD by setting $\mathrm{PD}_{lt}(S)$ to be the set of equivalence classes of Poisson deformations of $(X, \{\ ,\ \})$ over $\mathrm{Spec}(S)$ which are *locally trivial* as usual flat deformations. One can check that PD_{lt} has a pro-representable hull. Let $X_n \to T_n$ be an object of $\mathrm{PD}_{lt}(S_n)$, where $S_n := \mathrm{C}[t]/(t^{n+1})$ and $T_n := \mathrm{Spec}(S_n)$. Write $T^1_{X_n/T_n}$ for $\underline{\mathrm{Hom}}(\Omega^1_{X_n/T_n}, \mathcal{O}_{X_n})$. By Proposition 13, we have a natural map

$$\mathrm{HP}^2(U_n/T_n) \to \mathrm{Ext}^1(\Omega^1_{X_n/T_n}, \mathcal{O}_{X_n}).$$

Define $T(X_n/T_n)$ to be the kernel of the composite

$$\mathrm{HP}^2(U_n/T_n) \to \mathrm{Ext}^1(\Omega^1_{X_n/T_n}, \mathcal{O}_{X_n}) \to H^0(X_n, T^1_{X_n/T_n}).$$

Let $\mathrm{PD}_{lt}(X_n/T_n; T_n[\epsilon])$ be the set of equivalence classes of Poisson deformations of X_n over $T_n[\epsilon]$ which are locally trivial as usual deformations. Here $T_n[\epsilon] := \mathrm{Spec}(S_n[\epsilon])$ and $S_n[\epsilon] = \mathbf{C}[t, \epsilon]/(t^{n+1}, \epsilon^2)$. Two Poisson deformations of X_n over $T_n[\epsilon]$ are equivalent if there is a Poisson $T_n[\epsilon]$-isomorphisms between them which induces the identity of X_n. Then there is a one-to-one correspondence between $T(X_n/T_n)$ and $\mathrm{PD}_{lt}(X_n/T_n; T_n[\epsilon])$.

Lemma 18. $T(X_n/T_n) = \mathrm{HP}^2(U_n/T_n)$.

Proof. Since $H^0(X, T^1_{X_n/T_n}) \subset H^0(X^{an}, T^1_{X_n^{an}/T_n})$, it suffices to prove that $\mathrm{HP}^2(U_n/T_n) \to H^0(X^{an}, T^1_{X_n^{an}/T_n})$ is the zero map. In order to do this, for $p \in \Sigma(= \mathrm{Sing}(X))$, take a Stein open neighborhood $X_n^{an}(p)$ of $p \in X_n$, and put $U_n^{an}(p) := X^{an}(p) \cap U_n^{an}$. We have to prove that $H^2(U^{an}, S_n) \to H^2(U_n^{an}(p), S_n)$ is the zero map. In fact, on one hand,

$$\mathrm{HP}^2(U_n/T_n) \cong H^2(U^{an}, S_n)$$

[4] Since X is convex, there is a projective birational morphism f from X to an affine variety Y. Take a reflexive sheaf F on X^{an} of rank 1. The direct image $f^{an}_* F^*$ of the dual sheaf F^* is a coherent sheaf on the Stein variety Y^{an}. Hence $f^{an}_* F^*$ has a non-zero global section; in other words, there is an injection $\mathcal{O}_{X^{an}} \to F^*$. By taking its dual, F is embedded in $\mathcal{O}_{X^{an}}$. Thus, $F = \mathcal{O}(-D)$ for an analytic effective divisor D. So, for any reflexive sheaf F of rank 1, the double dual sheaf $(F^{\otimes m})^{**}$ becomes an invertible sheaf for some m.

by Lemma 12. On the other hand, $H^0(X_n^{an}(p), T^1_{X_n^{an}/T_n}) \cong H^1(U_n^{an}(p), \Theta_{U_n^{an}(p)})$ (cf. the proof of [Na, Lemma 1]). By the symplectic form $\omega_n \in \Gamma(U_n, \Omega^2_{X_n/T_n})$, $\Theta_{U_n^{an}(p)}$ is identified with $\Omega^1_{U_n^{an}(p)}$. Hence

$$H^0(X_n^{an}(p), T^1_{X_n^{an}/T_n}) \cong H^1(U_n^{an}(p), \Omega^1_{U_n^{an}(p)}).$$

By these identifications, the map $\mathrm{HP}^2(U_n/T_n) \to H^0(X_n^{an}(p), T^1_{X_n^{an}/T_n})$ coincides with the composite

$$H^2(U^{an}, S_n) \to H^2(U^{an}(p), S_n) \to H^1(U^{an}(p), \Omega^1_{U_n^{an}(p)}),$$

where the second map is induced by the spectral sequence

$$E_1^{p,q} := H^q(U_n^{an}(p), \Omega^p_{U_n^{an}(p)}) => H^{p+q}(U^{an}, S_n)$$

(for details, see the proof of [Na, Lemma 1]). Let us consider the commutative diagram

$$\begin{array}{ccccc} \mathrm{Pic}(X^{an}) \otimes_{\mathbf{Z}} S_n & \longrightarrow & \mathrm{Pic}(U^{an}) \otimes_{\mathbf{Z}} S_n & \xrightarrow{\cong} & H^2(U^{an}, S_n) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Pic}(X^{an}(p)) \otimes_{\mathbf{Z}} S_n & \longrightarrow & \mathrm{Pic}(U^{an}(p)) \otimes_{\mathbf{Z}} S_n & \xrightarrow{\cong} & H^2(U^{an}(p), S_n) \end{array} \tag{14}$$

Here the second map on the first row is an isomorphism because $H^1(U^{an}, \mathcal{O}_{U^{an}}) = H^2(U^{an}, \mathcal{O}_{U^{an}}) = 0$. Since $\mathrm{Codim}(\Sigma \subset X) \geq 3$, any line bundle on U^{an} extends to a coherent sheaf on X^{an}. Thus, by the $\mathbf{Q}$-factoriality of X^{an}, the first map on the first row is surjective. If we take $X^{an}(p)$ small enough, then $\mathrm{Pic}(X^{an}(p)) = 0$. Now, by the commutative diagram above, we conclude that $H^2(U^{an}, S_n) \to H^2(U^{an}(p), S_n)$ is the zero map. This completes the proof of Lemma 18. □

Let us return to the proof of Theorem 17. The functor PD has T^1-lifting property by Corollary 15. By Lemma 18, PD_{lt} also has T^1-lifting property. Let R and R_{lt} be the pro-representable hulls of PD and PD_{lt} respectively. Then these are both regular local $\mathbf{C}$-algebra. There is a surjection $R \to R_{lt}$ because PD_{lt} is a sub-functor of PD. By Lemma 18, the cotangent spaces of R and R_{lt} coincides. Hence $R \cong R_{lt}$. □

Theorem 19. *Let X be a convex symplectic variety with terminal singularities. Let L be a (not necessarily ample) line bundle on X^{an}. Then the twistor deformation $\{X_n\}$ of X associated with L is locally trivial as a flat deformation.*

Proof. Define $U_n \subset X_n$ to be the locus where $X_n \to T_n$ is smooth. We put $\Sigma := \mathrm{Sing}(X)$. For each point $p \in \Sigma$, we take a Stein open neighborhood $p \in X_n(p)$ in $(X_n)^{an}$, and put $U_n^{an}(p) := X_n(p) \cap U_n^{an}$. Let $L_n \in \mathrm{Pic}(X_n)$ be the (unique) extension of L to X_n. We shall show that $[L_n] \in H^2(U^{an}, S_n)$ is sent to zero by the map

$$H^2(U^{an}, S_n) \to H^2(U^{an}(p), S_n).$$

This is enough for us to prove that the twistor deformation $\{X_n\}$ is locally trivial. In fact, we have to show that the *local* Kodaira-Spencer class $\theta_{n+1}^{loc}(p) \in H^1(U_n^{an}(p), \Theta_{U_n^{an}(p)})$ is zero. By the same argument as Lemma 16, one can show that

$$\iota(\theta_{n+1}^{loc}(p))(\omega_n) = [L_n|_{U^{an}(p)}] \in H^1(U_n^{an}(p), \Omega^1_{U_n^{an}(p)}).$$

Now let us consider the commutative diagram induced from the Hodge spectral sequences:

$$\begin{array}{ccc} H^2(U^{an}, S_n) & \longrightarrow & H^1(U_n, \Omega^1_{U_n/T_n}) \\ \downarrow & & \downarrow \\ H^2(U^{an}(p), S_n) & \longrightarrow & H^1(U_n^{an}(p), \Omega^1_{U_n^{an}/T_n}) \end{array} \tag{15}$$

For the existence of the first horizontal map, we use Grothendieck's theorem [G] and the fact $H^i(U_n, \mathcal{O}_{U_n}) = 0$ $(i = 1, 2)$ (cf. Lemma 12). Since X_n^{an} is Cohen-Macaulay and $\mathrm{Codim}(\Sigma \subset X) \geq 4$, we have $H^i(U_n^{an}(p), \mathcal{O}_{U_n^{an}(p)}) = 0$ for $i = 1, 2$, by the depth argument. This assures the existence of the second horizontal map. The vertical map on the right-hand side is just the composite of the maps

$$H^1(U_n, \Omega^1_{U_n/T_n}) \to H^1(U_n^{an}, \Omega^1_{U_n^{an}/T_n}) \to H^1(U_n^{an}(p), \Omega^1_{U_n^{an}(p)/T_n}).$$

If $[L_n] \in H^2(U^{an}, S_n)$ is sent to zero by the map

$$H^2(U^{an}, S_n) \to H^2(U^{an}(p), S_n),$$

then, by the diagram, $[L_n|_{U^{an}(p)}] = 0$. Thus, the local Kodaira-Spencer class $\theta_{n+1}^{loc}(p)$ vanishes. Let us consider the same diagram in the proof of Theorem 17.

$$\begin{array}{ccccc} \mathrm{Pic}(X^{an}) \otimes_{\mathbf{Z}} S_n & \longrightarrow & \mathrm{Pic}(U^{an}) \otimes_{\mathbf{Z}} S_n & \xrightarrow{\cong} & H^2(U^{an}, S_n) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Pic}(X^{an}(p)) \otimes_{\mathbf{Z}} S_n & \longrightarrow & \mathrm{Pic}(U^{an}(p)) \otimes_{\mathbf{Z}} S_n & \xrightarrow{\cong} & H^2(U^{an}(p), S_n) \end{array} \tag{16}$$

Since L_n is a line bundle of $(X_n)^{an}$ and $\mathrm{Pic}((X_n)^{an}) \cong \mathrm{Pic}(X^{an})$, $[L_n] \in H^2(U^{an}, S_n)$ comes from $\mathrm{Pic}(X^{an})$. If we take $X^{an}(p)$ small enough, then $\mathrm{Pic}(X^{an}(p)) = 0$. Hence, by the commutative diagram, we see that $[L_n] \in H^2(U^{an}, S_n)$ is sent to zero by the map $H^2(U^{an}, S_n) \to H^2(U^{an}(p), S_n)$. □

§4. Symplectic Varieties with Good $\mathbf{C}^*$-actions

Let X be a convex symplectic variety with terminal singularities and, in addition, with a $\mathbf{C}^*$-action. We put $Y := \mathrm{Spec}\,\Gamma(X, \mathcal{O}_X)$. Then the natural morphism $g : X \to Y$ is a $\mathbf{C}^*$-equivariant morphism. We assume that Y has a good $\mathbf{C}^*$-action with a unique fixed point $0 \in Y$. By definition, $V := Y - \mathrm{Sing}(Y)$ admits a symplectic 2-form ω; hence it gives a Poisson structure $\{\ ,\ \}$ on Y. We assume that this Poisson structure has a positive weight $l > 0$ with respect to the $\mathbf{C}^*$-action, that is,

$$\deg\{a, b\} = \deg(a) + \deg(b) - l$$

for all homogeneous elements $a, b \in \mathcal{O}_Y$. Now let us consider the Poisson deformation functor $\mathrm{PD}_{(X,\{,\})}$ (cf. Section 3). By Theorem 14, it is pro-represented by a certain complete regular local $\mathbf{C}$-algebra $R = \lim R_n$ and a universal formal Poisson deformation $\{X_n^{univ}\}$ of X over it.

Lemma 20. *The $\mathbf{C}^*$-action on X naturally induces a $\mathbf{C}^*$-action on R and $\{X_n^{univ}\}$.*

Proof. Take an infinitesimal Poisson deformation $(X_S, \{\ ,\ \}_S; \iota)$ of X over $S = \mathrm{Spec}(A)$ with $A/m = \mathbf{C}$. By definition, $\iota : X_S \otimes_A A/m \cong X$ is an identification of the central fiber with X. Since X is a $\mathbf{C}^*$-variety, for each $\lambda \in \mathbf{C}^*$, we get an isomorphism $\phi_\lambda : X \to X$. By the assumption, $\phi_\lambda^*\{\ ,\ \} = \lambda^l\{\ ,\ \}$. Then $(X_S, \lambda^l\{\ ,\ \}_S; \phi_\lambda \circ \iota)$ gives another Poisson deformation of X over S. This operation naturally gives a $\mathbf{C}^*$-action on R and $\{X_n\}$. □

We shall investigate the $\mathbf{C}^*$-action of R. In order to do that, take $L \in H^1(U^{an}, \mathcal{O}^*_{U^{an}})$ and consider the corresponding quasi-twistor deformation of X. We define a $\mathbf{C}^*$-action on $\mathbf{C}[[t]]$ so that t has weight l. This induces a $\mathbf{C}^*$-action on each quotient ring $S_n := \mathbf{C}[t]/(t^{n+1})$. We put $T_n := \mathrm{Spec}(S_n)$.

Lemma 21. *Any quasi-twistor deformation $\{X_n\}$ of X has a $\mathbf{C}^*$-action so that $\{X_n\} \to \{T_n\}$ is $\mathbf{C}^*$-equivariant.*

Proof. Let $R \to \mathbf{C}[[t]]$ be the surjection determined by our quasi-twistor deformation. We shall prove this map is $\mathbf{C}^*$-equivariant. For $\lambda \in \mathbf{C}^*$, let

$\lambda^l : T_n \to T_n$ be the morphism induced by $t \to \lambda^l t$. We shall lift $\mathbf{C}^*$-actions of X_n inductively. More explicitly, for each $\lambda \in \mathbf{C}^*$, we shall construct an isomorphism $\phi_{\lambda,n} : X_n \to X_n$ in such a way that:

(i) the following diagram commutes

$$\begin{array}{ccc} X_n & \xrightarrow{\phi_{\lambda,n}} & X_n \\ \downarrow & & \downarrow \\ T_n & \xrightarrow{\lambda^l} & T_n \end{array} \tag{17}$$

(ii) $(\phi_{\lambda,n})^*\{\ ,\ \}_n = \lambda^l\{\ ,\ \}_n$, and

(iii) the collection $\{\phi_{\lambda,n}\}$, $\lambda \in \mathbf{C}^*$ gives a $\mathbf{C}^*$-action of X_n.

Suppose that it can be achieved. As in Lemma 20, let us fix an original identification $\iota : X_n \times_{T_n} T_0 \cong X$. Let $h_n : R \to S_n$ and $h_n^\lambda : R \to S_n$ be the maps determined by $(X_n, \{\ ,\ \}_n; \iota)$ and $(X_n, \lambda^l\{\ ,\ \}_n; \phi_\lambda \circ \iota)$ respectively. Let $\lambda \in \mathbf{C}^*$ act on R_n as $\psi_{\lambda,n} : R_n \to R_n$. By definition, $h_n \circ \psi_{\lambda,n} = h_n^\lambda$. Then the existence of $\phi_{\lambda,n}$ implies that there is a commutative diagram

$$\begin{array}{ccc} R_n & \xrightarrow{h_n} & S_n \\ {\scriptstyle \psi_{\lambda,n}}\downarrow & & \downarrow{\scriptstyle \lambda^l} \\ R_n & \xrightarrow{h_n} & S_n \end{array} \tag{18}$$

The construction of $\phi_{\lambda,n}$ goes as follows. We assume that $\phi_{\lambda,n-1}$ already exist. Let $U_{n-1} \subset X_{n-1}$ be the locus where $X_{n-1} \to T_{n-1}$ is smooth. Let $\omega_{n-1} \in \Gamma(U_{n-1}, \Omega^2_{U_{n-1}/T_{n-1}})$ be the symplectic 2-form corresponding to the Poisson structure $\{\ ,\ \}_{n-1}$. By the assumption, $X_{n-1} \to T_{n-1}$ is a $\mathbf{C}^*$-equivariant morphism. The symplectic 2-form ω_{n-1} has weight l with the induced $\mathbf{C}^*$-action on U_{n-1}. Let $L_{n-1} \in \mathrm{Pic}(U^{an}_{n-1})$ be the (unique) extension of $L \in \mathrm{Pic}(U^{an})$. By T^1-lifting principle, the extension of X_{n-1} to X_n is determined by an element $\theta_n \in \mathbf{H}^2(U_{n-1}, \wedge^{\cdot}\Theta_{U_{n-1}/T_{n-1}})$, where $\wedge^{\cdot}\Theta_{U_{n-1}/T_{n-1}}$ is the Lichnerowicz-Poisson complex defined in §2. The symplectic 2-form ω_{n-1} gives an identification (cf. §2):

$$\mathbf{H}^2(U_{n-1}, \wedge^{\cdot}\Theta_{U_{n-1}/T_{n-1}}) \cong \mathbf{H}^2(U_{n-1}, \Omega^{\geq 1}_{U_{n-1}/T_{n-1}}).$$

By definition of the twistor deformation, θ_n is sent to $[L_{n-1}]$. Since $[L_{n-1}]$ and ω_{n-1} have respectively weights 0 and l for the $\mathbf{C}^*$-action, θ_n should have weight $-l$. This is what we want. □

Since any direction in $H^2(X^{an}, \mathbf{Q})$ is realized in a suitable quasi-twistor deformation, $\mathbf{C}^*$ has only weight l on the maximal ideal m_R of R. Thus, one can write R as $\mathbf{C}[[t_1, ..., t_m]]$, where t_i are all eigen-elements with weight l. Let $\hat{\mathcal{Y}} := \mathrm{Spec} \lim \Gamma(X_n^{univ}, \mathcal{O}_{X_n^{univ}})$. The $\mathbf{C}^*$-action on $\{X_n^{univ}\}$ induces a $\mathbf{C}^*$-action on $\hat{\mathcal{Y}}$. Let $\hat{\mathcal{X}} \to \hat{\mathcal{Y}}$ be the algebraization of $\{X_n^{univ}\}$ over $\hat{\mathcal{Y}}$. Since Y and R are both positively weighted, $\hat{\mathcal{Y}}$ is also positively weighted. The $\mathbf{C}^*$-action of the formal scheme $\{X_n^{univ}\}$ induces a $\mathbf{C}^*$-action of $\hat{\mathcal{X}}$ in such a way that $\hat{\mathcal{X}} \to \hat{\mathcal{Y}}$ becomes $\mathbf{C}^*$-equivariant.

Lemma 22. *There is a projective birational morphism of algebraic varieties with $\mathbf{C}^*$-actions*

$$\mathcal{X} \to \mathcal{Y}$$

over Spec $\mathbf{C}[t_1, ..., t_m]$ *which is an algebraization of $\hat{\mathcal{X}} \to \hat{\mathcal{Y}}$. Moreover, $\mathcal{X}$ and $\mathcal{Y}$ admit natural Poisson structures over* Spec $\mathbf{C}[t_1, ..., t_m]$.

Proof. Let A be the completion of the coordinate ring of the affine scheme $\hat{\mathcal{Y}}$ at the origin. Then A becomes a complete local ring with a good $\mathbf{C}^*$-action. The $\mathbf{C}^*$-equivariant projective morphism $\hat{\mathcal{X}} \to \hat{\mathcal{Y}}$ induces a $\mathbf{C}^*$-equivariant projective morphism $\hat{\mathcal{X}}_A \to \mathrm{Spec}(A)$. By Lemma A.8, there is a $\mathbf{C}^*$-linearised ample line bundle on $\hat{\mathcal{X}}_A$. By Lemma A.2, there is a $\mathbf{C}$-algebra R of finite type such that $\hat{R} = A$. Put $\mathcal{Y} := \mathrm{Spec}(R)$. Since R is generated by eigenvectors (homogeneous elements) of $\mathbf{C}^*$-action, R contains t_i. So R is a ring over $\mathbf{C}[t_1, ..., t_m]$. By Proposition A.5, there is a $\mathbf{C}^*$-equivariant projective morphism $\mathcal{X} \to \mathcal{Y}$ which algebraizes $\hat{\mathcal{X}}_A \to \mathrm{Spec}(A)$. This automatically algebraizes $\hat{\mathcal{X}} \to \hat{\mathcal{Y}}$. The complete local ring A admits a Poisson structure over $\mathbf{C}[[t_1, ..., t_m]]$ induced by that of $\Gamma(\hat{\mathcal{Y}}, \mathcal{O}_{\hat{\mathcal{Y}}})$. This Poisson structure induces a Poisson structure of R over $\mathbf{C}[t_1, ..., t_m]$ because, if $a, b \in A$ are homogeneous, then $\{a, b\} \in A$ is again homogeneous. The corresponding relative Poisson bi-vector Θ of $\mathcal{Y}$ is non-degenerate on the smooth part. Hence it defines a relative symplectic 2-form on the smooth part of $\mathcal{Y}$. This relative symplectic 2-form is pulled back to $\mathcal{X}$ and defines a relative Poisson structure of $\mathcal{X}$. $\square$

Let us fix an algebraic line bundle L on X. Since $H^i(X, \mathcal{O}_X) = 0$ for $i = 1, 2$, there is a unique line bundle $\hat{L} \in \mathrm{Pic}(\hat{\mathcal{X}})$ extending L. Let $\hat{L}_A \in \mathrm{Pic}(\hat{\mathcal{X}}_A)$ be the pull-back of $\hat{L}$ to $\hat{\mathcal{X}}_A$. Since $\hat{L}$ is fixed by the $\mathbf{C}^*$-action of $\hat{\mathcal{X}}$, $\hat{L}_A$ is fixed by the $\mathbf{C}^*$-action of $\hat{\mathcal{X}}_A$. By Lemma A.8, for some $k > 0$, $(\hat{L}_A)^{\otimes k}$ is $\mathbf{C}^*$-linearized. By Proposition A.6, there is a $\mathbf{C}^*$-linearized line bundle on $\mathcal{X}$ extending $(\hat{L}_A)^{\otimes k}$. Thus, by replacing L by its suitable multiple, we may assume that L extends to a line bundle on $\mathcal{X}$. Let U be the regular part of X and let $[L] \in H^2(U^{an}, \mathbf{C})$ be the associated class with $L|_U$. Let us denote by M the maximal ideal of $\mathbf{C}[t_1, ..., t_m]$ and identify $(M/M^2)^*$ with $H^2(U^{an}, \mathbf{C})$.

Then $[L]$ can be written as a linear combination

$$a_1 t_1^* + a_2 t_2^* + ... + a_m t_m^*$$

with the dual base $\{t_i^*\}$ of $\{t_i\}$. Take a base change of

$$\mathcal{X} \to \mathrm{Spec}\ \mathbf{C}[t_1, ..., t_n]$$

by the map

$$\mathrm{Spec}\ \mathbf{C}[t] \to \mathrm{Spec}\ \mathbf{C}[t_1, ..., t_n]$$

with $t_i = a_i t$. Then we have a 1-parameter deformation $\mathcal{X}^L$ of X over $T :=$ Spec $\mathbf{C}[t]$. As we have shown in Lemma 21, this deformation gives an algebraization of the twistor deformation $X^L_\infty \to$ Spec $\mathbf{C}[[t]]$. We put $\mathcal{Y}^L :=$ Spec $\Gamma(\mathcal{X}^L, \mathcal{O}_{\mathcal{X}^L})$. Now let us consider the birational projective morphism

$$g_T : \mathcal{X}^L \to \mathcal{Y}^L$$

over Spec $\mathbf{C}[t]$. Let $\eta \in \mathrm{Spec}\mathbf{C}[t]$ be the generic point and let X^L_η and Y^L_η be the generic fibers. Then we get a birational projective morphism

$$g_\eta : X^L_\eta \to Y^L_\eta.$$

Proposition 23 (Kaledin). *Assume that X is smooth and L is ample. Then $g_\eta : X^L_\eta \to Y^L_\eta$ is an isomorphism.*

Proof. Denote by $T (\cong \mathrm{Spec}\mathbf{C}[t])$ the base space of our algebraized twistor deformation $\mathcal{X}^L$. Since T has a good $\mathbf{C}^*$-action, $\mathcal{X}^L$ is smooth over T. The line bundle L on X uniquely extends to a line bundle $\mathcal{L}$ on $\mathcal{X}^L$. Moreover, $\mathcal{X}^L$ is a Poisson T-scheme extending the original Poisson scheme X; thus, the symplectic 2-form ω on X extends to a relative symplectic 2-form $\omega_T \in \Gamma(\mathcal{X}^L, \Omega^2_{\mathcal{X}^L/T})$. Let $\theta_T \in H^1(\mathcal{X}^L, \Theta_{\mathcal{X}^L/T})$ be the extension class (Kodaira-Spencer class) of the exact sequence

$$0 \to (f_T)^*\Omega^1_{T/\mathbf{C}} \to \Omega^1_{\mathcal{X}^L/\mathbf{C}} \to \Omega^1_{\mathcal{X}^L/T} \to 0.$$

By Lemma 16, we see that, in $H^1(\mathcal{X}^L, \Omega^1_{\mathcal{X}^L/T})$, $i(\theta_T)(\omega_T) = [\mathcal{L}]$. We put $\omega_\eta := \omega_T|_{X^L_\eta}$, $\theta_\eta := \theta_T|_{X^L_\eta}$ and $L_\eta := \mathcal{L}|_{X_\eta}$. Then, in $H^1(X^L_\eta, \Omega^1_{X^L_\eta/k(\eta)})$, we have an equality:

$$i(\theta_\eta)(\omega_\eta) = [L_\eta].$$

Since g_η is a proper birational morphism, we only have to show that X_η does not contain a proper curve defined over $k(\eta)$. Now let $\iota : C \to X_\eta$ be a morphism from a proper regular curve C defined over $k(\eta)$ to X_η. We shall prove that $\iota(C)$ is a point. Let $\theta_C \in H^1(C, \Theta_{C/k(\eta)})$ be the Kodaira-Spencer class for $h : C \to \mathrm{Spec}k(\eta)$. In other words, θ_C is the extension class of the exact sequence

$$0 \to h^*\Omega^1_{k(\eta)/\mathbf{C}} \to \Omega^1_{C/\mathbf{C}} \to \Omega^1_{C/k(\eta)} \to 0.$$

Then, by the compatibility of Kodaira-Spencer classes, we have

$$i(\theta_C)(\iota^*\omega_\eta) = \iota^*(i(\theta_\eta)(\omega_\eta)).$$

The left hand side is zero because $\iota^*\omega_\eta = 0$. On the other hand, the right hand side is $\iota^*[L_\eta]$. Since L is ample, L_η is also ample. If $\iota(C)$ is not a point, then $\iota^*[L_\eta] \neq 0$, which is a contradiction. □

The following is a generalization of Proposition 23 to the singular case.

Proposition 24. *Assume that X has only terminal singularities and* $L \in \mathrm{Pic}(X)$.

(a) *If L is ample, then $g_\eta : X^L_\eta \to Y^L_\eta$ is an isomorphism.*

(b) *Let X^+ be another convex symplectic variety over Y with terminal singularities and assume that L becomes the proper transform of an ample line bundle L^+ on X^+. Then g_η is a small birational morphism; in other words,* $\mathrm{codimExc}(g_\eta) \geq 2$.

Proof. (i) We shall use the same notation as the proof of Proposition 23. We note that the Kodaira-Spencer class $\theta_T \in \mathrm{Ext}^1(\Omega^1_{\mathcal{X}^L/T}, \mathcal{O}_{\mathcal{X}^L})$ is contained in $H^1(\mathcal{X}^L, \Theta_{\mathcal{X}^L/T})$ because the twistor deformation is locally trivial by Theorem 19. Let $\mathcal{U} \subset \mathcal{X}^L$ be the locus where $\mathcal{X}^L \to T$ is smooth. Denote by U_η the generic fiber of $\mathcal{U} \to T$. Let $(\theta_T)^0 \in H^0(\mathcal{U}, \Theta_{\mathcal{U}/T})$ be the restriction of θ_T to $\mathcal{U}$. The relative Poisson structure on $\mathcal{X}^L$ over T gives an element $(\omega_T)^0 \in H^0(\mathcal{U}, \Omega^2_{\mathcal{U}/T})$. Note that, in general, $(\omega_T)^0$ cannot extend to a global section of $\Omega^2_{\mathcal{X}^L/T}$. Let $[\mathcal{L}]^0 \in H^1(\mathcal{U}, \Omega^1_{\mathcal{U}/T})$ be the class corresponding to a restricted line bundle $\mathcal{L}|_{\mathcal{U}}$. Then, $(\theta_T)^0$, $(\omega_T)^0$ and $[\mathcal{L}]^0$ defines respectively the classes

$$\theta^0_\eta \in H^1(U_\eta, \Theta^1_{U_\eta/k(\eta)}),$$

$$\omega^0_\eta \in H^0(U_\eta, \Omega^2_{U_\eta/k(\eta)})$$

and

$$[L_\eta]^0 \in H^1(U_\eta, \Omega^1_{U_\eta/k(\eta)}).$$

We then have

$$i(\theta^0_\eta)(\omega^0_\eta) = [L_\eta]^0.$$

(ii)(Construction of a good resolution): We shall construct a good equivariant resolution of $\mathcal{X}^L$. In order to do that, first take an equivariant resolution $\pi_0 : \tilde{X} \to X$ of X, that is, $(\pi_0)_*\Theta_{\tilde{X}} = \Theta_X$. Here $\Theta_X := \underline{\mathrm{Hom}}(\Omega^1_X, \mathcal{O}_X)$. By Theorem 19, our twistor deformation gives us a sequence of locally trivial formal deformations of X:

$$X \to X_1 \to ... \to X_n \to ...$$

We shall construct resolutions $\pi_n : \tilde{X}_n \to X_n$ inductively so that there is an affine open cover $X_n = \cup_{i\in I} U_{n,i}$ such that $(\pi_n)^{-1}(U_{n,i}) \cong (\pi_0)^{-1}(U) \times_{T_0} T_n$. Note that, if this could be done, then $(\pi_n)_*\Theta_{\tilde{X}_n/T_n} = \Theta_{X_n/T_n}$. Moreover, if we let $\tilde{\theta}_n \in H^1(\tilde{X}_{n-1}, \Theta_{\tilde{X}_{n-1}/T_{n-1}})$ be the Kodaira-Spencer class of $\tilde{X}_n \to T_n$, then $\tilde{\theta}_n$ coincides with the Kodaira-Spencer class $\theta_n \in H^1(X_{n-1}, \Theta_{X_{n-1}/T_{n-1}})$ of $X_n \to T_n$ because $\tilde{\theta}_n$ is mapped to zero by the map

$$H^1(\tilde{X}_{n-1}, \Theta_{\tilde{X}_{n-1}/T_{n-1}}) \to H^0(X_n, R^1(\pi_{n-1})_*\Theta_{\tilde{X}_{n-1}/T_{n-1}}).$$

Now assume that we are given such a resolution $\pi_n : \tilde{X}_n \to X_n$. Take the affine open cover $\{U_{n,i}\}_{i\in I}$ of X_n as above. We put $\tilde{U}_{n,i} := (\pi_n)^{-1}(U_{n,i})$. For $i, j \in I$, there is an identification $U_{n,i}|_{U_{ij}} \cong U_{n,j}|_{U_{ij}}$ determined by X_n. For each $i \in I$, let $\mathcal{U}_{n,i}$ and $\tilde{\mathcal{U}}_{n,i}$ be trivial deformations of $U_{n,i}$ and $\tilde{U}_{n,i}$ over T_{n+1} respectively. For each $i, j \in I$, take a T_{n+1}-isomorphism

$$g_{ji} : \mathcal{U}_{n,i}|_{U_{ij}} \to \mathcal{U}_{n,j}|_{U_{ij}}$$

such that $g_{ji}|_{T_n} = id$. Then

$$h_{ijk} := g_{ij} \circ g_{jk} \circ g_{ki}$$

gives an automorphism of $\mathcal{U}_{n,i}|_{U_{ijk}}$ over T_{n+1} such that $h_{ijk}|_{T_n} = id$. Since $\pi_n : \tilde{X}_n \to X_n$ is an equivariant resolution, g_{ij} extends uniquely to

$$\tilde{g}_{ij} : \tilde{\mathcal{U}}_{n,i}|_{U_{ij}} \cong \tilde{\mathcal{U}}_{n,j}|_{U_{ij}}.$$

One can consider $\{h_{ijk}\}$ as a 2-cocycle of the Čech cohomology of Θ_X; hence it gives an element $ob \in H^2(X, \Theta_X)$. But, since X_n extends to X_{n+1}, $ob = 0$. Therefore, by modifying g_{ij} to g'_{ij} suitably, one can get

$$g'_{ij} \circ g'_{jk} \circ g'_{ki} = id.$$

Then

$$\tilde{g}'_{ij} \circ \tilde{g}'_{jk} \circ \tilde{g}'_{ki} = id.$$

Now $\tilde{X}_n$ also extends to $\tilde{X}_{n+1}$ and the following diagram commutes:

$$\begin{array}{ccc} \tilde{X}_n & \longrightarrow & \tilde{X}_{n+1} \\ \downarrow & & \downarrow \\ X_n & \longrightarrow & X_{n+1} \end{array} \tag{19}$$

By Théorème (5.4.5) of [EGA III], one has an algebraization $\tilde{X}^L_\infty \to Y_\infty$ of $\{\tilde{X}_n \to Y_n\}$. Moreover, the morphism $\{\pi_n : \tilde{X}_n \to X_n\}$ induces $\pi_\infty : \tilde{X}^L_\infty \to X^L_\infty$. By the construction, the $\mathbf{C}^*$-action on X^L_∞ lifts to $\tilde{X}^L_\infty$. Then $\tilde{X}^L_\infty \to Y_\infty$ is algebraized to a $\mathbf{C}^*$-equivariant projective morphism $\tilde{\mathcal{X}}^L \to \mathcal{Y}^L$ in such a way that it factors through $\mathcal{X}^L$.

(iii) Let $\pi : \tilde{\mathcal{X}}^L \to \mathcal{X}^L$ be the equivariant resolution constructed in (ii). Let us denote by $\tilde{X}_\eta$ the generic fiber of $\tilde{\mathcal{X}}^L \to T$. This resolution gives an equivariant resolution $\pi_\eta : \tilde{X}_\eta \to X_\eta$. In particular, $(\pi_\eta)_*\Theta_{\tilde{X}/k(\eta)} = \Theta_{X_\eta/k(\eta)}$. Let $\tilde{\theta}_\eta \in H^1(\tilde{X}_\eta, \Theta_{\tilde{X}_\eta/k(\eta)})$ be the Kodaira-Spencer class for $\tilde{X}_\eta \to \mathrm{Spec} k(\eta)$. Then the Kodaira-Spencer class $\theta_\eta \in H^1(X_\eta, \Theta_{X_\eta/k(\eta)})$ for X_η coincides with $\tilde{\theta}_\eta$ by the natural injection $H^1(X_\eta, \Theta_{X_\eta/k(\eta)}) \to H^1(\tilde{X}_\eta, \Theta_{\tilde{X}_\eta/k(\eta)})$. Let $i_\eta : U_\eta \to X_\eta$ be the embedding of the regular part. Since $(i_\eta)_*\Omega^2_{U_\eta/k(\eta)} \cong (\pi_\eta)_*\Omega^2_{\tilde{X}_\eta/k(\eta)}$, by [Fl], ω^0_η extends to

$$\omega_\eta \in \Gamma(X_\eta, (\pi_\eta)_*\Omega^2_{\tilde{X}_\eta/k(\eta)}).$$

(iv) We have a pairing map:

$$H^0(X_\eta, (\pi_\eta)_*\Omega^2_{\tilde{X}_\eta/k(\eta)}) \times H^1(X_\eta, (\pi_\eta)_*\Theta_{\tilde{X}_\eta/k(\eta)}) \to H^1(X_\eta, (\pi_\eta)_*\Omega^1_{\tilde{X}_\eta/k(\eta)}).$$

Denote by $i(\theta_\eta)(\omega_\eta)$ the image of $(\omega_\eta, \theta_\eta)$ by this pairing map. By pulling back L_η by π_η, one can define a class $[L_\eta] \in H^1(X_\eta, (\pi_\eta)_*\Omega^1_{\tilde{X}_\eta k(\eta)})$. Let us consider the exact sequence

$$H^1_\Sigma(X_\eta, (\pi_\eta)_*\Omega^1_{\tilde{X}_\eta/k(\eta)}) \to H^1(X_\eta, (\pi_\eta)_*\Omega^1_{\tilde{X}_\eta/k(\eta)}) \to H^1(U_\eta, \Omega^1_{U_\eta/k(\eta)}),$$

where $\Sigma := X_\eta \setminus U_\eta$. Since $(\pi_\eta)_*\Omega^1_{\tilde{X}_\eta/k(\eta)} \cong (i_\eta)_*\Omega^1_{U_\eta/k(\eta)}$ by [Fl], it is a reflexive sheaf. A reflexive sheaf on X_η is locally written as the kernel of a homomorphism from a free sheaf to a torsion free sheaf. Since X_η is Cohen-Macaulay and $\mathrm{Codim}(\Sigma \subset X_\eta) \geq 2$, we have $H^1_\Sigma(X_\eta, (\pi_\eta)_*\Omega^1_{\tilde{X}_\eta/k(\eta)}) = 0$. We already know in (i) that $[L_\eta]^0 = i(\theta^0_\eta)(\omega^0_\eta)$ in $H^1(U_\eta, \Omega^1_{U_\eta/k(\eta)})$. Therefore, by the exact sequence, we see that

$$[L_\eta] = i(\theta_\eta)(\omega_\eta).$$

(v) Consider the pairing map

$$H^0(\tilde{X}_\eta, \Omega^2_{\tilde{X}_\eta/k(\eta)}) \times H^1(\tilde{X}_\eta, \Theta_{\tilde{X}_\eta/k(\eta)}) \to H^1(\tilde{X}_\eta, \Omega^1_{\tilde{X}_\eta/k(\eta)}).$$

By the construction of $\tilde{X}_\eta$, the Kodaira-Spencer class $\tilde{\theta}_\eta$ of $\tilde{X}_\eta \to \mathrm{Spec} k(\eta)$ coincides with the Kodaira-Spencer class θ_η. Hence $(\omega_\eta, \tilde{\theta}_\eta)$ is sent to $(\pi_\eta)^*[L_\eta]$ by the pairing map.

Now we shall prove (a). For a proper regular curve C defined over $k(\eta)$, assume that there is a $k(\eta)$-morphism $\iota : C \to \tilde{X}_\eta$. We shall prove that $(\pi_\eta) \circ \iota(C)$ is a point. By the compatibility of the Kodaira-Spencer classes, we have

$$i(\theta_C)(\iota^*\omega_\eta) = \iota^*(i(\omega_\eta)(\tilde{\theta}_\eta)).$$

The left hand side is zero because $\iota^*\omega_\eta = 0$. The right hand side is $\iota^*(\pi_\eta)^*[L_\eta]$ as we just remarked above. If $\pi_\eta \circ \iota(C)$ is not a point, then this is not zero because L_η is ample; but this is a contradiction.

Next we shall prove (b). We shall derive a contradiction assuming that $g_T : \mathcal{X}^L \to \mathcal{Y}^L$ is a divisorial birational contraction. We put $\mathcal{E} := \mathrm{Exc}(g_T)$. By the assumption, there is another convex symplectic variety X^+ over Y, and X and X^+ are isomorphic in codimension one over Y. Let $F \subset X$ (resp. $F^+ \subset X^+$) be the locus where the birational map $X - - \to X^+$ is not an isomorphism. Then $\mathrm{codim}(F \subset X) \geq 2$. We shall prove that $(L, \bar{C}) > 0$ for any proper irreducible curve $\bar{C}$ which is not contained in F. Let $\bar{C}$ be such a curve. Take a common resolution $\mu : Z \to X$ and $\mu^+ : Z \to X^+$. We may assume that $\mathrm{Exc}(\mu)$ is a union of irreducible divisors, say $\{E_i\}$. Since X and X^+ are isomorphic in codimension one, $\mathrm{Exc}(\mu) = \mathrm{Exc}(\mu^+)$. On can write $(\mu^+)^*L^+ = \mu^*L - \Sigma a_i E_i$ with *non-negative* integers a_i. In fact, if $a_j < 0$ for some j, then E_j should be a fixed component of the linear system $|(\mu^+)^*L^+|$; but this is a contradiction since L^+ is (very) ample. One can find a proper curve D on Z such that $\mu(D) = \bar{C}$ and such that $C^+ := \mu^+(D)$ is an irreducible curve on X^+. (i.e. $\mu^+(D)$ is not reduced to a point.) Note that D is not contained in any E_i. Then

$$(L, \bar{C}) = (\mu^*L, D) = ((\mu^+)^*L^+ + \Sigma a_i E_i, D) > 0.$$

Let us consider all effective 1-cycles on X which are contracted to points by g and are obtained as the limit of effective 1-cycles on X^L_η. Since $\mathcal{E}$ has codimension 1 in $\mathcal{X}^L$, one can find such an effective 1-cycle whose support intersects F at most in finite points. In other words, there is a flat family $\mathcal{C} \to T$ of proper curves in $\mathcal{X}^L/T$ in such a way that any irreducible component of $\mathcal{C}_0 := \mathcal{C} \cap X$ is not contained in F. Let $\bar{C}_\eta$ be the generic fiber of $\mathcal{C} \to T$. Take a regular proper curve C over $k(\eta)$ and a $k(\eta)$-morphism $\iota : C \to \tilde{X}_\eta$ so that $\pi_\eta \circ \iota(C) = \bar{C}_\eta$. By the definition of $\mathcal{C}$, $(L_\eta, \bar{C}_\eta) > 0$. Now one can get a contradiction by using this curve C in the similar way to (a). $\square$

Corollary 25. *Let Y be an affine symplectic variety with a good $\mathbf{C}^*$-action and assume that the Poisson structure of Y is positively weighted. Let*

$$X \xrightarrow{f} Y \xleftarrow{f'} X'$$

be a diagram such that,

1. f (resp. f') is a crepant, birational, projective morphism.

2. X (resp. X') has only terminal singularities.

3. X (resp. X') is $\mathbf{Q}$-factorial.

Then both X and X' have locally trivial deformations to an affine variety Y_t obtained as a Poisson deformation of Y. In particular, X and X' have the same kind of singularities.

Proof. (i) By Step 1 of the proof of Proposition A.7, the $\mathbf{C}^*$-action of Y lifts to X and X'. So we are in the situation of section 4. Since Y is a symplectic variety, outside certain locus at least of codimension 4 (say $\bar{\Sigma}$), its singularity is locally isomorphic to the product $(\mathbf{C}^{n-2}, 0) \times (S, 0)$ (as an analytic space). Here $(S, 0)$ is the germ of a rational double point singularity of a surface (cf. [Ka 2]). We put $\bar{V} := Y - \bar{\Sigma}$. Since f and f' are both (unique) minimal resolutions of rational double points over $\bar{V}$, $f^{-1}(\bar{V}) \cong (f')^{-1}(\bar{V})$.

(ii) Fix an ample line bundle L of X and let $\{X_n\}$ be the twistor deformation associated with L. This induces a formal deformation $\{Y_n\}$ of Y. Let L' be the proper transform of L by $X - - \to X'$. Since X' is $\mathbf{Q}$-factorial, we may assume that L' is a line bundle of X'.[5] Let $\{X'_n\}$ be the twistor deformation of X' associated with L'. This induces a formal deformation $\{Y'_n\}$ of Y.

Lemma 26. *The formal deformation $\{Y'_n\}$ coincides with $\{Y_n\}$.*

Proof. The formal deformation $\{X_n\}$ of X induces a formal deformation of $W := f^{-1}(\bar{V})$, say $\{W_n\}$. The deformation induces a formal deformation $\{\bar{V}_n\}$ of $\bar{V}$ by $\bar{V}_n := \mathrm{Spec}\Gamma(W_n, \mathcal{O}_{W_n})$ because $R^1(f|_W)_*\mathcal{O}_W = 0$ and

[5] The twistor deformation associated with L and the one associated with $L^{\otimes m}$ are essentially the same. The latter one is obtained from the first one just by changing the parameters t by mt.

$(f|_W)_*\mathcal{O}_W = \mathcal{O}_{\bar{V}}$ (cf. [Wa]). Since $\bar{V} = Y - \bar{\Sigma}$ with $\mathrm{codim}(\bar{\Sigma} \subset Y) \geq 4$, the formal deformation $\{\bar{V}_n\}$ of $\bar{V}$ extends uniquely to that of Y (cf. Proposition 13, (1)). This extended deformation is nothing but $\{Y_n\}$. On the other hand, the formal deformation $\{X'_n\}$ of X' induces a formal deformation of $W' := (f')^{-1}(\bar{V})$, say $\{W'_n\}$. As remarked in (i), $W \cong W'$. Moreover, by Corollary 10, the Poisson deformations of W (resp. W') are controlled by the cohomology $H^2(W^{an}, \mathbf{C})$ (resp. $H^2((W')^{an}, \mathbf{C})$) because $H^i(\mathcal{O}_W) = 0$ for $i = 1, 2$ (resp. $H^i(\mathcal{O}_{W'}) = 0$ for $i = 1, 2$). Since L' is the proper transform of L, $[L|_W]$ is sent to $[L'|_{W'}]$ by the natural identification $H^2(W^{an}, \mathbf{C}) \cong H^2(W'^{an}, \mathbf{C})$. This implies that $\{W_1\}$ and $\{W'_1\}$ coincide. By the construction of $\{X_n\}$ (resp. $\{X'_n\}$), L (resp. L') extends uniquely to L_n (resp. L'_n). Then $[L_1|_W] \in H^2(W^{an}, S_1)$ is sent to $[L'_1|_{W'}] \in H^2(((W')^{an}, S_1)$, which implies that W_2 and $W'{}_2$ coincide. By the similar inductive process, one concludes that $\{W_n\}$ and $\{W'_n\}$ coincide. The formal deformation $\{W'_n\}$ of W' induces a formal deformation $\{\bar{V}'_n\}$ of $\bar{V}$, which coincides with $\{\bar{V}_n\}$. So the extended deformation $\{Y'_n\}$ also coincides with $\{Y_n\}$.

(iii) Let

$$\mathcal{X}^L \to \mathcal{Y} \leftarrow (\mathcal{X}')^L$$

be the algebraizations of

$$\{X_n\} \to \{Y_n\} \leftarrow \{X'_n\}$$

over T. Let $\eta \in T$ be the generic point. Then, by Proposition 24, (a), $X^L_\eta \cong Y^L_\eta$. Since X is $\mathbf{Q}$-factorial, we have:

Lemma 27. *X^L_η is also $\mathbf{Q}$-factorial.*

Proof. Let D be a Weil divisor of X^L_η. One can extend D to a Weil divisor $\bar{D}$ of $\mathcal{X}^L$ by taking its closure. The restriction of $\bar{D}$ to X defines a Weil divisor $\bar{D}|_X$. Note that the support of $\bar{D}|_X$ is $\bar{D} \cap X$ and the multiplicity on each irreducible component is well determined because $\bar{D}$ is a Cartier divisor at a regular point of X. Let $m > 0$ be an integer such that $m(\bar{D}|_X)$ is a Cartier divisor. Let $\mathcal{O}(m\bar{D})$ be the reflexive sheaf associated with $m\bar{D}$ and let $\mathcal{O}(m\bar{D}|_X)$ be the line bundle associated with $m\bar{D}|_X$. By [K-M, Lemma (12.1.8)],

$$\mathcal{O}(m\bar{D}) \otimes_{\mathcal{O}_{\mathcal{X}^L}} \mathcal{O}_X = \mathcal{O}(m\bar{D}|_X).$$

In particular, $\mathcal{O}(m\bar{D})$ is a line bundle around X. Therefore, $m\bar{D}$ is a Cartier divisor on some Zariski open neighborhood of $X \subset \mathcal{X}^L$. Let Z be the non-Cartier locus of $m\bar{D}$. Since $\mathcal{O}(m\bar{D})$ is fixed by the $\mathbf{C}^*$-action on $\mathcal{X}^L$, Z is stable under the $\mathbf{C}^*$-action. Since $f_T : \mathcal{X}^L \to \mathcal{Y}$ is a projective morphism, $f_T(Z)$ is a closed subset of $\mathcal{Y}$. Since $Y \cap f_T(Z) = \emptyset$ and Y has a good $\mathbf{C}^*$-action, $f_T(Z)$ should be empty; hence Z should be also empty. □

Since X and X' are both crepant partial resolutions (with terminal singularities) of Y, they are isomorphic in codimension one. Now one can apply Proposition 24, (b) to the twistor deformation $\{X'_n\}$ of X'. Then we conclude that $(X')^{L'}_\eta \to Y_\eta$ is a *small* birational projective morphism. On the other hand, $Y_\eta (\cong X^L_\eta)$ is $\mathbf{Q}$-factorial by Lemma 27. These imply that $(X')^L_\eta \cong Y_\eta$. By Theorem 19, $\mathcal{X}^L \to T$ and $(\mathcal{X}')^{L'} \to T$ are locally trivial deformations of X and X' respectively. □

Corollary 28. *Let Y be an affine symplectic variety with a good $\mathbf{C}^*$-action. Assume that the Poisson structure of Y is positively weighted, and Y has only terminal singularities. Let $f : X \to Y$ be a crepant, birational, projective morphism such that X has only terminal singularities and such that X is $\mathbf{Q}$-factorial. Then the following are equivalent.*

(a) *X is non-singular.*
(b) *Y is smoothable by a Poisson deformation.*

Proof. First of all, the $\mathbf{C}^*$-action of Y lifts to X by Step 1 of the proof of Proposition A.7. Secondly, by Corollary A.10, X^{an} is $\mathbf{Q}$-factorial. We regard X and Y as Poisson schemes. The Poisson deformation functors PD_X and PD_Y have pro-representable hulls R_X and R_Y respectively (Theorem 14). We put $U := (X)_{reg}$ and $V := Y_{reg}$. Then, by Lemma 12, $\mathrm{HP}^2(U) = H^2(U^{an}, \mathbf{C})$ and $\mathrm{HP}^2(V) = H^2(V^{an}, \mathbf{C})$. Note that, by Proposition 13, they coincide with $\mathrm{PD}_X(\mathbf{C}[\epsilon])$ and $\mathrm{PD}_Y(\mathbf{C}[\epsilon])$, respectively. By the proof of [Na, Proposition 2], we see that

$$(*) : H^2(U^{an}, \mathbf{C}) \cong H^2(V^{an}, \mathbf{C}).$$

Let $l > 0$ be the weight of Poisson structure on Y. Then one can get universal $\mathbf{C}^*$-equivariant Poisson deformations $\mathcal{X}$ and $\mathcal{Y}$ over the same affine base $B :=$ Spec $\mathbf{C}[t_1, ..., t_m]$, where $m = h^2(U^{an}, \mathbf{C})$ and each t_i has weight l. By Theorem 17 $\mathcal{X} \to B$ is a locally trivial deformations of X. The birational projective morphism f induces a birational projective B-morphism

$$f_B : \mathcal{X} \to \mathcal{Y}.$$

Let η be the generic point of B. Take the generic fibers over η. Then we have

$$\mathcal{X}_\eta \xrightarrow{f_\eta} \mathcal{Y}_\eta.$$

Every twistor deformation of X associated with an ample line bundle L determines a (non-closed) point $\zeta_L \in B$. By Proposition 24, f_{ζ_L} is an isomorphism. This implies that f_η is an isomorphism. Therefore, $\mathcal{Y}_\eta$ is regular if and only if X is non-singular. $\square$

§5. General Cases

Let X be a convex symplectic variety with terminal singularities. Let $\{X_n\}$ be a twistor deformation for $L \in \mathrm{Pic}(X)$. We put $Y_n := \mathrm{Spec}\Gamma(X_n, \mathcal{O}_{X_n})$ and $Y^L_\infty := \mathrm{Spec} \lim \Gamma(X_n, \mathcal{O}_{X_n})$. As in §3, $\{X_n\}$ is algebraized to $g_\infty : X^L_\infty \to Y^L_\infty$ over T_∞, where $T_\infty :=$ Spec $\mathbf{C}[[t]]$. We do not know, however, as in §3, that $\{X_n\}$ can be algebraized to $g_T : \mathcal{X}^L \to \mathcal{Y}^L$ over $T :=$ Spec $\mathbf{C}[t]$. Let $\eta_\infty \in$ Spec $\mathbf{C}[[t]]$ be the generic point and let $g_{\eta_\infty} : X^L_{\eta_\infty} \to Y^L_{\eta_\infty}$ be the morphism between the generic fibers induced by g_∞.

Proposition 29. (a) *If L is ample, then $g_{\eta_\infty} : X^L_{\eta_\infty} \to Y^L_{\eta_\infty}$ is an isomorphism.*

(b) *Let X^+ be another convex symplectic variety over Y with terminal singularities and assume that L becomes the proper transform of an ample line bundle L^+ on X^+. Then g_{η_∞} is a small birational morphism; in other words,* $\mathrm{codimExc}(g_{\eta_\infty}) \geq 2$.

Proof. The idea of the proof is the same as Proposition 24. But we need more delicate argument because neither X^L_∞ or Y^L_∞ is of finite type over T_∞. First of all, we should replace the usual differential sheaves $\Omega^i_{X^L_\infty/T_\infty}$ ($i \geq 1$), $\Omega^1_{X^L_\infty/\mathbf{C}}$, and $\Omega^1_{T_\infty/\mathbf{C}}$ respectively by $\hat{\Omega}^i_{X^L_\infty/T_\infty}$, $\hat{\Omega}^1_{X^L_\infty/\mathbf{C}}$, and $\hat{\Omega}^1_{T_\infty/\mathbf{C}}$. Here $\hat{\Omega}^i_{X^L_\infty/T_\infty}$ is a coherent sheaf on $X^L_{\eta_\infty}$ determined as the limit of the formal sheaves $\{\Omega^i_{X_n/T_n}\}$, $\hat{\Omega}^1_{X^L_\infty/\mathbf{C}}$ is a coherent sheaf on X^L_∞ determined as the limit of $\{\Omega^1_{X_{n+1}/\mathbf{C}}|_{X_n}\}$ and $\hat{\Omega}^1_{T_\infty/\mathbf{C}}$ is a coherent sheaf on T_∞ determined as the limit of $\{\Omega^1_{T_{n+1}/\mathbf{C}}|_{T_n}\}$. Now the Kodaira-Spencer class $\theta_{T_\infty} \in \mathrm{Ext}^1(\hat{\Omega}^1_{X^L_\infty/T_\infty}, \mathcal{O}_{X^L_\infty})$ for $X^L_\infty \to T_\infty$ is the extension class of the exact sequence

$$0 \to (f_\infty)^*\hat{\Omega}^1_{T_\infty/\mathbf{C}} \to \hat{\Omega}^1_{X^L_\infty/\mathbf{C}} \to \hat{\Omega}^1_{X^L_\infty/T_\infty} \to 0.$$

Then, as in Proposition 24, we can construct a good resolution $\pi_\infty : \tilde{X}^L_\infty \to X^L_\infty$ of X^L_∞. Let E_∞ be the exceptional locus of g_∞. Assume that $f_\infty(E_\infty)$ contains a generic point $\eta_\infty \in T_\infty$. By cutting E_∞ by g_∞-very ample divisors and by the pull-back of suitable divisors on Y_∞, we can find an integral subscheme $\bar{C}_\infty \subset X^L_\infty$ of dimension 2 such that $g_\infty(\bar{C}_\infty) \to T_\infty$ is a finite surjective morphism. Note that $\bar{C}_\infty \to T_\infty$ is a flat projective morphism with fiber dimension 1. Take a desingularization $C_\infty \to \bar{C}_\infty$ which factors through $\tilde{X}^L_\infty$. We put $C_n := C_\infty \times_{T_\infty} T_n$, and $C_{\eta_\infty} := C_\infty \times_{T_\infty} \mathrm{Spec}\ k(\eta_\infty)$. Then C_{η_∞} is a proper regular curve over $k(\eta_\infty)$. Moreover, one can define $\hat{\Omega}^1_{C_\infty/\mathbf{C}}$ as the limit of the formal sheaf $\{\Omega^1_{C_n/\mathbf{C}}\}$. Then, the Kodaira-Spencer class $\theta_{C_{\eta_\infty}}$ for $C_{\eta_\infty} \to \mathrm{Spec}\ k(\eta_\infty)$ is well-defined as an element of $H^1(C_{\eta_\infty}, \Theta_{C_{\eta_\infty}/k(\eta_\infty)})$. Then, the final argument in the proof of Proposition 24 is valid in our case. □

The same argument of Proposition 25 now yields:

Corollary 30. *Let Y be an affine symplectic variety. Let*

$$X \xrightarrow{f} Y \xleftarrow{f'} X'$$

be a diagram such that,

1. f (resp. f') is a crepant, birational, projective morphism.
2. X (resp. X') has only terminal singularities.
3. X (resp. X') is $\mathbf{Q}$-factorial.

Then, there is a flat deformation

$$X_\infty \to Y_\infty \leftarrow X'_\infty$$

over $T_\infty :=$ Spec$\mathbf{C}[[t]]$ of the original diagram $X \to Y \leftarrow X'$ such that
(i)*$X_\infty \to T_\infty$ and $X'_\infty \to T_\infty$ are both locally trivial deformations, and*
(ii) *the generic fibers are all isomorphic*:

$$X_\eta \cong Y_\eta \cong X'_\eta$$

for the generic point $\eta \in T_\infty$.

In Corollary 30, X_η (res. X'_η) is not of finite type over $k(\eta)$. So, at this moment, it is not clear how the singularities of X are related to those of X'. However, one can say more when X is smooth:

Corollary 31. *With the same assumption as Corollary* 30*, if X is non-singular, then X' is also non-singular.*

Proof. Since X is non-singular, X_∞ is formally smooth over $\mathbf{C}$. Since $X_\eta \cong Y_\eta$, Y_∞ is formally smooth over $\mathbf{C}$ outside Y. By [Ar 2, Theorem 3.9] (see also [Hi], [Ri], [Ka 3]), for each closed point $p \in Y$, there is an etale map

$Z_\infty \to Y_\infty$ whose image contains $p \in Y_\infty$, and $Z_\infty \to T_\infty$ is algebraized to $\mathcal{Z} \to T$. Here $T = \mathrm{Spec}\ \mathbf{C}[t]$. The completion $\hat{Z}$ of $\mathcal{Z}$ along the closed fiber coincides with Z_∞. The diagram

$$X_\infty \to Y_\infty \leftarrow X'_\infty$$

is pulled back by the map $Z_\infty \to Y_\infty$ to

$$X_\infty \times_{Y_\infty} Z_\infty \to Z_\infty \leftarrow X'_\infty \times_{Y_\infty} Z_\infty.$$

Take generic fibers of this diagram over T_∞. Then three generic fibers are all isomorphic. Hence, the formal completion of the diagram along the closed fibers (over $0 \in T_\infty$) gives two "formal modifications" in the sense of [Ar 3]. By [Ar 3], there exists a diagram of algebraic spaces of finite type over $\mathbf{C}$:

$$\mathcal{X} \to \mathcal{Z} \leftarrow \mathcal{X}'$$

which extends such formal modifications. Take the closed fibers of this diagram over $0 \in T$. Then $\mathcal{X}_0$ is non-singular since $\mathcal{X}_0$ is etale over X. On the other hand, $\mathcal{X}_0$ and $\mathcal{X}'_0$ both have locally trivial deformations to a common affine variety $\mathcal{Z}_t$ $(t \neq 0)$ by the diagram. Therefore, $\mathcal{X}'_0$ is non-singular. Since $\mathcal{X}'_0$ is etale over X' and the image of this etale map contains $(f')^{-1}(p)$ by the construction, X' is non-singular at every point $q \in X'$ with $f'(q) = p$. Since $p \in Y$ is an arbitrary closed point, X' is non-singular. □

§6. Examples

Example 32. Assume that $\mathcal{O}_x \subset \mathfrak{sl}(n)$ is the orbit containing an nilpotent element x of Jordan type $\mathbf{d} := [d_1, ..., d_k]$. Let $[s_1, ..., s_m]$ be the dual partition of $\mathbf{d}$, that is, $s_i := \sharp\{j; d_j \geq i\}$. Let $P \subset SL(n)$ be the parabolic subgroup of flag type $(s_1, ..., s_m)$. Define $F := SL(n)/P$. Note that $h^1(F, \Omega^1_F) = m - 1$. Let

$$\tau_1 \subset \cdots \subset \tau_{m-1} \subset \mathbb{C}^n \otimes_{\mathbb{C}} \mathcal{O}_F$$

be the universal subbundles on F. A point of the cotangent bundle T^*F of F is expressed as a pair (p, ϕ) of $p \in F$ and $\phi \in \mathrm{End}(\mathbb{C}^n)$ such that

$$\phi(\mathbb{C}^n) \subset \tau_{m-1}(p), \cdots, \phi(\tau_2(p)) \subset \tau_1(p), \phi(\tau_1(p)) = 0.$$

The Springer resolution

$$s : T^*F \to \bar{\mathcal{O}}_x$$

is defined as $s((p,\phi)) := \phi$. Therefore, T^*F is a smooth convex symplectic variety. Let $\mathcal{E}$ be the universal extension of $\mathcal{O}_F$ by Ω^1_F. In other words, $\mathcal{E}$ fits in the exact sequence

$$0 \to \Omega^1_F \to \mathcal{E} \xrightarrow{\eta} \mathcal{O}_F^{m-1} \to 0,$$

and the induced map $H^0(F, \mathcal{O}_F^{m-1}) \to H^1(F, \Omega^1_F)$ is an isomorphism. The locally free sheaf $\mathcal{E}$ can be constructed as follows. For $p \in F_\sigma$, we can choose a basis of $\mathbf{C}^n$ such that $\Omega^1_F(p)$ consists of the matrices of the following form

$$\begin{pmatrix} 0 & * \cdots & * \\ 0 & 0 \cdots & * \\ \cdots & & \cdots \\ 0 & 0 \cdots & 0 \end{pmatrix}.$$

Then $\mathcal{E}(p)$ is the vector subspace of $\mathfrak{sl}(n)$ consisting of the matrices A of the following form

$$\begin{pmatrix} a_1 & * \cdots & * \\ 0 & a_2 \cdots & * \\ \cdots & & \cdots \\ 0 & 0 \cdots & a_m \end{pmatrix},$$

where $a_i := a_i I_{s_i}$ and I_{s_i} is the identity matrix of the size $s_i \times s_i$. Since $A \in \mathfrak{sl}(n)$, $\Sigma_i s_i a_i = 0$. Here we define the map $\eta(p) : \mathcal{E}(p) \to \mathbb{C}^{\oplus m-1}$ as $\eta(p)(A) := (a_1, a_2, \cdots, a_{m-1})$. Let $\mathbf{A}(\mathcal{E}^*) := \mathrm{Spec}_F \mathrm{Sym}^{\cdot}(\mathcal{E}^*)$ be the vector bundle over F associated with $\mathcal{E}$. Then we have an exact sequence of vector bundles

$$0 \to T^*F \to \mathbf{A}(\mathcal{E}^*) \to F \times \mathbf{C}^{n-1} \to 0.$$

The last homomorphism in the exact sequence gives a map

$$f : \mathbf{A}(\mathcal{E}^*) \to \mathbf{C}^{m-1},$$

where $f^{-1}(0) = T^*F$. This is a universal Poisson deformation of the Poisson scheme T^*F (with respect to the canonical symplectic 2-form). In fact, by Proposition 1.4.14 of [C-G], there is a relative symplectic 2-form of f extending the canonical symplectic 2-form on T^*F; hence f is a Poisson deformation. Let $p : T^*F \to F$ be the canonical projection. Then we have a commutative diagram of exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & p^*\Omega^1_F & \longrightarrow & p^*\mathcal{E} & \longrightarrow & p^*\mathcal{O}_F^{m-1} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Theta_{T^*F} & \longrightarrow & \Theta_{\mathbf{A}(\mathcal{E}^*)}|_{T^*F} & \longrightarrow & N_{T^*F/\mathbf{A}(\mathcal{E}^*)} & \longrightarrow & 0 \end{array} \tag{20}$$

Let T be the tangent space of the base space $\mathbf{C}^{m-1}$ of f at $0 \in \mathbf{C}^{m-1}$. The Kodaira-Spencer map θ_f of f is given as the composite

$$T \to H^0(T^*F, N_{T^*F/\mathbf{A}(\mathcal{E}^*)}) \to H^1(T^*F, \Theta_{T^*F}).$$

On the other hand, if one identifies T with $H^0(F, \mathcal{O}_F^{m-1})$, then one has a map

$$T \cong H^0(F, \mathcal{O}_F^{m-1}) \to H^1(F, \Omega_F^1).$$

By the construction, the Kodaira-Spencer map is factored by this map:

$$T \to H^1(F, \Omega_F^1) \to H^1(T^*F, \Theta_{T^*F}).$$

The first map is an isomorphism by the definition of $\mathcal{E}$. The second map is an injection. In fact, let $S \subset T^*F$ be the zero section. Then $N_{S/T^*F} \cong \Omega_S^1$ and the composite $H^1(F, \Omega_F^1) \to H^1(T^*F, \Theta_{T^*F}) \to H^1(S, \Omega_S^1)$ is an isomorphism. Therefore, the Kodaira-Spencer map θ_f is an injection. Since f is a Poisson deformation of T^*F, the Kodaira-Spencer map θ_f is factored by the "Poisson Kodaira-Spencer map " θ_f^P:

$$T \xrightarrow{\theta_f^P} H^2(T^*F, \mathbf{C}) \to H^1(T^*F, \Omega_{T^*F}^1).$$

Hence θ_f^P is also injective. Since $\dim T = h^2(T^*F, \mathbf{C}) = m-1$, θ_f^P is actually an isomorphism.

More generally, let G be a complex simple Lie group and $\mathcal{O}$ be a nilpotent orbit in $\mathfrak{g} := \mathrm{Lie}(G)$. Assume that the closure $\bar{\mathcal{O}}$ of $\mathcal{O}$ admits a Springer resolution $\mu : T^*(G/P) \to \bar{\mathcal{O}}$ for some parabolic subgroup $P \subset G$. One can identify $T^*(G/P)$ with the adjoint bundle $G \times^P n(P)$, where $n(P)$ is the nilradical of $\mathfrak{p} := \mathrm{Lie}(P)$. Let $r(P)$ be the solvable radical of $\mathfrak{p}$ and let $m(P)$ be the Levi-factor of $\mathfrak{p}$. We put $\mathfrak{k}(P) := \mathfrak{g}^{m(P)}$, where

$$\mathfrak{g}^{m(P)} := \{x \in \mathfrak{g}; [x, y] = 0, y \in m(P)\}.$$

In [Na 4, §7], we have defined a flat deformation of $T^*(G/P)$ as

$$G \times^P r(P) \to \mathfrak{k}(P).$$

Then this becomes a universal Poisson deformation of $T^*(G/P)$.

Example 33. Let $\mathcal{O}$ be the nilpotent orbit in $\mathfrak{sl}(3)$ of Jordan type $[1, 2]$. Then the closure $\bar{\mathcal{O}}$ has two different Springer resolutions

$$T^*(SL(3)/P_{1,2}) \to \bar{\mathcal{O}} \leftarrow T^*(SL(3)/P_{2,1}),$$

where $P_{1,2}$ and $P_{2,1}$ are parabolic subgroups of $SL(3)$ of flag type $(1,2)$ and $(2,1)$ respectively. We put $X^+ := T^*(SL(3)/P_{1,2})$ and $X^- := T^*(SL(3)/P_{2,1})$. Then X^+ and X^- are both isomorphic to the cotangent bundle of $\mathbf{P}^2$. We call the diagram a Mukai flop. Let $G \subset SL(3)$ be the finite group of order 3 generated by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta & 0 \\ 0 & 0 & \zeta^2 \end{pmatrix},$$

where ζ is a primitive 3-rd root of unity. Then G acts on $\bar{\mathcal{O}}$ by the adjoint action. Since the Kostant-Kirillov 2-form on $\mathcal{O}$ is $SL(3)$-invariant, the G-action lifts to symplectic actions on X^+ and X^-. Divide $\bar{\mathcal{O}}$, X^+ and X^- by these G-action, we get the diagram of a singular flop:

$$X^+/G \to \bar{\mathcal{O}}/G \leftarrow X^-/G.$$

Here X^+/G (resp. X^-/G) has 3 isolated quotient (terminal) singularities. This is a typical example of Corollary 25.

§7. Appendix

(A.1) Let $Y := \mathrm{Spec}\, R$ be an affine variety over $\mathbf{C}$. A $\mathbf{C}^*$-action on Y is a homomorphism $\mathbf{C}^* \to \mathrm{Aut}_{\mathbf{C}}(R)$ induced from a $\mathbf{C}$-algebra homomorphism

$$R \to R \otimes_{\mathbf{C}} \mathbf{C}[t, 1/t].$$

More exactly, a $\mathbf{C}$-valued point of $\mathbf{C}^*$ is regarded as a surjection of $\mathbf{C}$-algebras:

$$\sigma : \mathbf{C}[t, 1/t] \to \mathbf{C}.$$

Then

$$R \to R \otimes_{\mathbf{C}} \mathbf{C}[t, 1/t] \stackrel{id \otimes \sigma}{\to} R$$

is an element of $\mathrm{Aut}_{\mathbf{C}}(R)$. If this correspondence gives a homomorphism $\mathbf{C}^* \to \mathrm{Aut}_{\mathbf{C}}(R)$, we say that R (or Y) has a $\mathbf{C}^*$-action. A $\mathbf{C}^*$-action on Y is called good if there is a maximal ideal m_R of R fixed by the action and if $\mathbf{C}^*$ has only positive weight on m_R. Next let us consider the case where Y is the spectrum of a local complete $\mathbf{C}$-algebra R with $R/m_R = \mathbf{C}$. A $\mathbf{C}^*$-action on Y is then a homomorphism $\mathbf{C}^* \to \mathrm{Aut}_{\mathbf{C}}(R)$ induced from a $\mathbf{C}$-algebra homomorphism

$$R \to R \hat{\otimes}_{\mathbf{C}} \mathbf{C}[t, 1/t],$$

where $R\hat{\otimes}_{\mathbf{C}}\mathbf{C}[t, 1/t]$ is the completion of $R\otimes_{\mathbf{C}}\mathbf{C}[t, 1/t]$ with respect to the ideal $m_R(R\otimes_{\mathbf{C}}\mathbf{C}[t, 1/t])$. Then the $\mathbf{C}^*$-action is called good if $\mathbf{C}^*$ has only positive weight on the maximal ideal of R.

Lemma (A.2). *Let (A, m) be a complete local $\mathbf{C}$-algebra with a good $\mathbf{C}^*$-action. Assume that $A/m = \mathbf{C}$. Let R be the $\mathbf{C}$-vector subspace of A spanned by all eigen-vectors in A. Then R is a finitely generated $\mathbf{C}$-algebra with a good $\mathbf{C}^*$-action. Moreover, $\hat{R} = A$ where $\hat{R}$ is the completion of R with the maximal ideal m_R.*

Proof. Since A/m^k $(k \geq 1)$ are finite dimensional $\mathbf{C}$-vector spaces, they are direct sum of eigen-spaces with non-negative weights:

$$A/m^k = \oplus_w (A/m^k)^w.$$

The natural maps $(A/m^k)^w \to (A/m^{k-1})^w$ are surjections for all k. Since m/m^2 is also decomposed into the direct sum of eigen-spaces, one can take eigen-vectors $\bar{\phi}_i$, $(i = 1, 2, ..., l)$ as a generator of m/m^2. We put $w_i := wt(\bar{\phi}_i) > 0$. One can lift $\bar{\phi}_i$ to $\phi_i \in \lim(A/m^k)^{w_i}$ by the surjections above. Since A is complete, $\phi_i \in A$ and $wt(\phi_i) = w_i$. Put $w_{min} := min\{w_1, ..., w_l\} > 0$. We shall prove that $R = \mathbf{C}[\phi_1, ..., \phi_l]$. Let $\psi \in A$ be an eigen-vector with weight w. Take an integer k_0 such that $\psi \in m^{k_0}$ and $\psi \notin m^{k_0+1}$. Since every element of m^{k_0}/m^{k_0+1} can be written as a homogeneous polynomial of $\phi = (\bar{\phi}_1, ..., \bar{\phi}_l)$ of degree k_0, we see that

$$\psi \equiv f_{k_0}(\phi_1, ..., \phi_l) \pmod{m^{k_0+1}}$$

for some homogeneous polynomial f_{k_0} of degree k_0. We continue the similar approximation by replacing ψ with $\psi - f_{k_0}(\phi)$. Finally, for any given k, we have an approximation

$$\psi \equiv f_{k_0}(\phi) + ... + f_{k-1}(\phi) \pmod{m^k}.$$

Assume here that $k > w/w_{min}$. We set $\psi' := \Sigma_{k_0 \leq i \leq k-1} f_i(\phi)$. Assume that $\psi - \psi' \in m^r$ and $\psi - \psi' \notin m^{r+1}$ with some $r \geq k$. Since $\psi - \psi'$ has weight w, $[\psi - \psi'] \in m^r/m^{r+1}$ also has weight w. On the other hand, every non-zero eigen-vector in m^r/m^{r+1} has weight at least rw_{min}. Hence $w \geq rw_{min}$, but this contradicts that $r \geq k > w/w_{min}$. Therefore, $\psi = \psi'$ mod m^r for any r. Thus,

$$\psi = f_{k_0}(\phi) + ... + f_{k-1}(\phi).$$

This implies that $R = \mathbf{C}[\phi_1, ..., \phi_l]$. Let $m_R \subset R$ be the maximal ideal generated by ϕ_i's. Let R_k be the $\mathbf{C}$-vector subspace of m^k $(\subset A)$ spanned by the eigen-vectors. The argument above shows that $R_k = (m_R)^k$. Since $R_k = \oplus_w \lim(m^k/m^{k+i})^w$, we conclude that $(m_R)^k = \oplus_w \lim(m^k/m^{k+i})^w$. We now have

$$R/(m_R)^k = \oplus_w \lim(A/m^i)^w / \oplus_w \lim(m^k/m^{k+i})^w =$$

$$\oplus_w \{\lim(A/m^i)^w / \lim(m^k/m^{k+i})^w\} = \oplus_w (A/m^k)^w = A/m^k.$$

Here the 2-nd last equality holds because $\{(m^k/m^{k+i})^w\}_i$ satisfies the Mittag-Leffler condition. This implies that $\hat{R} = A$. □

(A.3) Let R be a integral domain finitely generated over $\mathbf{C}$ or a complete local $\mathbf{C}$-algebra with residue field $\mathbf{C}$. Assume that R has a good $\mathbf{C}^*$-action. Let M be a finite R-module. We say that M has an equivariant $\mathbf{C}^*$-action if, for each $\sigma \in \mathbf{C}^*$, we are given a map

$$\phi_\sigma : M \to M$$

with the following properties:

(1) ϕ_σ is a $\mathbf{C}$-linear map.
(2) $\phi_\sigma(rx) = \sigma(r)\phi_\sigma(x)$ for $r \in R$ and $x \in M$.
(3) $\phi_{\sigma\tau} = \phi_\sigma \circ \phi_\tau$ for $\sigma, \tau \in \mathbf{C}^*$.
(4) $\phi_1 = id$.

We say that a non-zero element $x \in M$ is an eigen-vector if there exists an integer w such that $\phi_\sigma(x) = \sigma^w x$ for all $\sigma \in G$.

Let M and N be R-modules with equivariant $\mathbf{C}^*$-actions. Then an R-homomorphism $f : M \to N$ is an equivariant map if f is compatible with both $\mathbf{C}^*$-actions.

Lemma (A.4). *Let A and R be the same as Lemma* (A.2)*. Let M be a finite A-module with an equivariant $\mathbf{C}^*$-action. Define M_R to be the $\mathbf{C}$-vector subspace of M spanned by the eigen-vectors of M. Then M_R is a finite R-module with an equivariant $\mathbf{C}^*$-action. Moreover, $M_R \otimes_R A = M$.*

Proof. The idea is the same as Lemma (A.2). The finite dimensional $\mathbf{C}$-vector space M/m^kM is the direct sum of eigen-spaces. Thus, for each weight w,

$$(M/m^kM)^w \to (M/m^{k-1}M)^w$$

is surjective. Let $\bar{x}_i$ $(i = 1, ..., r)$ be the eigen-vectors which generate M/mM. We lift $\bar{x}_i$ to $x_i \in \hat{M}$ by the surjections above. Since $\hat{M} = M$, x_i are eigen-vectors of M. We set $u_i := wt(x_i)$ and $u_{min} := min\{u_1, ..., u_r\}$. We shall prove that M_R is generated by $\{x_i\}$ as an R-module. Let $y \in M$ be an eigen-vector with weight u. Take an integer k_0 such that $y \in m^{k_0}M$ and $y \notin m^{k_0+1}M$. Let us consider the surjection

$$m^{k_0}/m^{k_0+1} \otimes M/mM \to m^{k_0}M/m^{k_0+1}M.$$

As in the proof of Lemma (A.2), every element of m^{k_0}/m^{k_0+1} is written as a homogeneous polynomial of ϕ_1, ..., ϕ_l of degree k_0, where ϕ_i are certain eigen-vectors contained in R. We put $w_{min} := min\{wt(\phi_1), ..., wt(\phi_l)\} > 0$. On the other hand, M/mM is spanned by x_i's. Thus,

$$y \equiv \Sigma r_i(\phi)x_i \bmod m^{k_0+1}M,$$

where r_i are homogeneous polynomials of degree k_0 such that $wt(r_i(\phi))+u_i = u$. We write g_{k_0} for the right-hand side for short. Now, we have $y-g_{k_0} \in m^{k_0+1}M$. By replacing y with $y - g_{k_0}$, we continue the similar approximation. Finally, for any k, we have an approximation:

$$y \equiv g_{k_0} + g_{k_0+1} + ... + g_{k-1} \bmod m^kM.$$

By the construction, $y - \Sigma_{k_0 \leq i \leq k-1} g_i$ is an eigen-vector with weight u. In particular, $[y-\Sigma_{k_0\leq i\leq k-1}g_i] \in m^kM/m^{k+1}M$ has weight u. On the other hand, every non-zero eigen-vector of $m^kM/m^{k+1}M$ has weight at least $kw_{min}+u_{min}$. If we take k sufficiently large, then $kw_{min} + u_{min} > u$. This implies that $[y - \Sigma_{k_0\leq i\leq k-1}g_i] = 0$. Repeating the same, we conclude that, for any $r > k$,

$$y \equiv \Sigma_{k_0\leq i\leq k-1}g_i \bmod m^rM.$$

This implies that, in M,

$$y = \Sigma_{k_0\leq i\leq k-1}g_i.$$

Thus, M_R is generated by $\{x_i\}$ as an R-module. Let M_k be the subspace of M spanned by the eigen-vectors in m^kM. Then the argument above shows that $M_k = (m_R)^kM_R$. Then, by the same argument as Lemma (A.2), $M_R/(m_R)^kM_R = M/m^kM$; hence $M_R \otimes_R A = M$. In order to prove that M_R has an equivariant $\mathbf{C}^*$-action, we have to check that $\phi_\sigma(M_R) \subset M_R$ for all $\sigma \in \mathbf{C}^*$ (cf. (A.3)); but it is straightforward. □

Proposition (A.5). *Let A be a local complete $\mathbf{C}$-algebra with residue field $\mathbf{C}$ and with a good $\mathbf{C}^*$-action. Let $f : X \to \mathrm{Spec}(A)$ be a $\mathbf{C}^*$-equivariant projective morphism and let L be an f-ample, $\mathbf{C}^*$-linearized line bundle. Let R be the same as Lemma* (A.2). *Then there is a $\mathbf{C}^*$-equivariant projective* *morphism* $f_R : X_R \to \mathrm{Spec}(R)$ *and a* $\mathbf{C}^*$*-linearized,* f_R*-ample line bundle* L_R*, such that* $X_R \times_{\mathrm{Spec}(R)} \mathrm{Spec}(A) \cong X$ *and* $L_R \otimes_R A \cong L$.

Proof. We put $A_i := \Gamma(X, L^{\otimes i})$ for $i \geq 0$. Then, $X = \mathrm{Proj}_A \oplus_{i\geq 0} A_i$. If necessary, by taking a suitable multiple $L^{\otimes m}$, we may assume that $A_* := \oplus_{i\geq 0} A_i$ is generated by A_1 as an $A_0 (= A)$-algebra. By Lemma (A.4), we take a finite R-module $A_{i,R}$ such that $A_{i,R} \otimes_R A = A_i$. The multiplication map $A_i \otimes_{A_0} A_j \to A_{i+j}$ induces a map $A_{i,R} \otimes_R A_{j,R} \to A_{i+j,R}$; hence $(A_*)_R := \oplus_{i\geq 0} A_{i,R}$ becomes a graded R-algebra. We shall check that $(A_*)_R$ is a finitely generated R-algebra. In order to do this, we only have to prove that $(A_R)_*$ is generated by $A_{1,R}$ as an R-algebra since $A_{1,R}$ is a finite R-module. Let us consider the n-multiplication map

$$m_n : \overbrace{A_{1,R} \otimes_R ... \otimes_R A_{1,R}}^{n} \to A_{n,R}.$$

Let M be the cokernel of this map. Since m_n is a $\mathbf{C}^*$-equivariant map, M is a finite R-module with an equivariant $\mathbf{C}^*$-action. Taking the tensor product $\otimes_R A$ with m_n, we get the n-multiplication map for A_*; but this is surjective by the assumption. Therefore, $\hat{M} := M \otimes_R A = 0$. The support of M is a closed subset of $\mathrm{Spec}(R)$, stable under the $\mathbf{C}^*$-action. Since $\hat{M} = 0$, this closed subset does not contain the origin $0 \in R$; hence it must be empty because R has a good $\mathbf{C}^*$-action. Finally it is clear that $(A_R)_* \otimes_R A = A_*$ by the construction. □

Proposition (A.6)**.** *Let* $f : X \to \mathrm{Spec}(A)$ *and* $f_R : X_R \to \mathrm{Spec}(R)$ *be the same as Lemma* (A.5)*. Let* F *be a coherent sheaf of* X *with a* $\mathbf{C}^*$*-linearization. Then there is a* $\mathbf{C}^*$*-linearized coherent sheaf* F_R *of* X_R *such that* $F_R \otimes_R A = F$.

Proof. We put $\mathcal{O}_X(1) := (\widetilde{\oplus_{i\geq 0} A_i})[1]$. Then the coherent sheaf F can be written as

$$F = \oplus_{i\geq 0} \widetilde{\Gamma(X, F(i))}.$$

Let us write M_i for $\Gamma(X, F(i))$. By Lemma (A.4), there is a finite R-module $M_{i,R}$ such that $M_{i,R} \otimes_R A = M_i$. We define

$$F_R := \oplus_{i\geq 0} \widetilde{M_{i,R}}.$$

We shall prove that $(M_*)_R := \oplus_{i\geq 0} M_{i,R}$ is a finite $(A_*)_R$-module. There is an integer n_0 such that, for any $i \geq n_0$, and for any $j \geq 0$, the multiplication map

$$\overbrace{A_1 \otimes_{A_0} ... \otimes_{A_0} A_1}^{j} \otimes_{A_0} M_i \to M_{i+j}$$

is surjective. For the same i, j, let us consider the R-linear map

$$\overbrace{A_{1,R} \otimes_R \dots \otimes_R A_{1,R}}^{j} \otimes_R M_{i,R} \to M_{i+j,R}.$$

Let N be the cokernel of this map. Since this R-linear map is compatible with the $\mathbf{C}^*$-action on R, N is a finite R-module with an equivariant $\mathbf{C}^*$-action. By the choice of i and j, $\hat{N} := N \otimes_R A$ is zero. This implies that $N = 0$. $\square$

Proposition (A.7). *Let Y be an affine symplectic variety. Assume that Y has a good $\mathbf{C}^*$-action with a fixed point $0 \in Y$. Assume that, in the analytic category, Y^{an} admits a crepant, projective, partial resolution $\bar{f} : \mathcal{X} \to Y^{an}$ such that $\mathcal{X}$ has only terminal singularities. Then, in the algebraic category, Y admits a crepant, projective, partial resolution $f : X \to Y$ such that $X^{an} = \mathcal{X}$ and $f^{an} = \bar{f}$.*

Proof. (STEP 1): We shall prove that the $\mathbf{C}^*$-action of Y^{an} lifts to $\mathcal{X}$. Since Y^{an} is symplectic, one can take a closed subset Σ of Y^{an}, stable under the $\mathbf{C}^*$-action and $\mathrm{codim}(\Sigma \subset Y^{an}) \geq 4$, such that the singularities of $Y^{an} - \Sigma$ are local trivial deformations of two dimensional rational double points. We put $Y_0 := Y^{an} - \Sigma$. Since $\bar{f}$ is the minimal resolution over Y_0, the $\mathbf{C}^*$-action on Y_0 extends to $\mathcal{X}_0 := \bar{f}^{-1}(Y_0)$. Note that, in $\mathcal{X}$, $\mathcal{X} - \mathcal{X}_0$ has codimension at least two by the semi-smallness of $\bar{f}$ ([Na 3]). The $\mathbf{C}^*$-action defines a holomorphic map

$$\sigma^0 : \mathbf{C}^* \times \mathcal{X}_0 \to \mathcal{X}_0,$$

and this extends to a meromorphic map

$$\sigma : \mathbf{C}^* \times \mathcal{X} - - \to \mathcal{X}.$$

Let us prove that $\sigma_t : \mathcal{X} - - \to \mathcal{X}$, which is an isomorphism in codimension one, is actually an isomorphism everywhere for each $t \in \mathbf{C}^*$. Let $\mathcal{L}$ be an $\bar{f}$-ample line bundle on $\mathcal{X}$. We put $\mathcal{L}^0_t := (\sigma^0)^*\mathcal{L}|_{\{t\}\times\mathcal{X}_0}$. Since $\mathrm{Pic}(\mathcal{X}_0)$ is discrete, $\mathcal{L}^0_t$ are all isomorphic to $\mathcal{L}|_{X_0}$. Since $\mathcal{L}|_{X_0}$ extends to the line bundle $\mathcal{L}$ on $\mathcal{X}$, $(\sigma^0)^*\mathcal{L}$ extends to a line bundle on $\mathbf{C}^* \times \mathcal{X}$, say $\sigma^*\mathcal{L}$ by abuse of notation. The line bundle $\mathcal{L}_t := \sigma^*\mathcal{L}|_{\{t\}\times\mathcal{X}}$ coincides with the proper transform of $\mathcal{L}$ by σ_t. Since $\mathcal{L}_0 (= \mathcal{L})$ is $\bar{f}$-ample and $\mathrm{Pic}(\mathcal{X}/Y^{an}) := \mathrm{Pic}(\mathcal{X})/\bar{f}^*\mathrm{Pic}(Y^{an})$ is discrete, $\mathcal{L}_t$ are all $\bar{f}$-ample. This implies that σ_t are all isomorphisms and σ is a holomorphic map. One can check that σ gives a $\mathbf{C}^*$-action because it already becomes a $\mathbf{C}^*$-action on $\mathcal{X}_0$.

(STEP 2): Let Y_n be the n-th infinitesimal neighborhood of Y^{an} at 0, which becomes an affine scheme with a unique point 0. We put $\mathcal{X}_n := \mathcal{X} \times_{Y^{an}}$

Y_n^{an}. By GAGA, there are projective schemes X_n over Y_n such that $(X_n)^{an} = \mathcal{X}_n$. Fix an $\bar{f}$-ample line bundle $\mathcal{L}$ on $\mathcal{X}$. Again by GAGA, it induces line bundles L_n on X_n. The $\mathbf{C}^*$-action on $\mathcal{X}$ induces a $\mathbf{C}^*$-action on $\mathcal{X}_n$ for each n. This action induces an algebraic $\mathbf{C}^*$-action of X_n. In fact, the $\mathbf{C}^*$-action of $\mathcal{X}$ originally comes from an algebraic $\mathbf{C}^*$-action on Y, the holomorphic action map

$$\mathbf{C}^* \times \mathcal{X} \to \mathcal{X}$$

extends to a meromorphic map

$$\mathbf{P}^1 \times \mathcal{X} - - \to \mathcal{X}.$$

Thus, the holomorphic action map

$$\mathbf{C}^* \times \mathcal{X}_n \to \mathcal{X}_n$$

extends to a meromorphic map

$$\mathbf{P}^1 \times \mathcal{X}_n - - \to \mathcal{X}_n.$$

Thus, by GAGA, we have a rational map

$$\mathbf{P}^1 \times X_n - - \to X_n$$

which restricts to an algebraic $\mathbf{C}^*$-action on X_n. Let us regard $\{X_n\}$ and $\{Y_n\}$ as formal schemes and $\{f_n : X_n \to Y_n\}$ as a projective equivariant morphism of formal schemes with $\mathbf{C}^*$-actions. Put $\hat{A} := \lim \mathcal{O}_{Y_n,0}$ and $\hat{Y} := \mathrm{Spec}(\hat{A})$. Then, by [EGA III], Theoreme 5.4.5, the projective morphism of formal schemes can be algebraized to a projective equivariant morphism of schemes

$$\hat{f} : \hat{X} \to \hat{Y}.$$

The affine scheme $\hat{Y}$ admits a $\mathbf{C}^*$-action coming from the original $\mathbf{C}^*$-action on Y, which is compatible with the $\mathbf{C}^*$-action on $\{Y_n\}$. The $\mathbf{C}^*$-action on $\{X_n\}$ naturally lifts to $\hat{X}$ in such a way that $\hat{f}$ becomes a $\mathbf{C}^*$-equivariant morphism. In fact, let

$$\sigma_n : \mathbf{C}^* \times X_n \to X_n$$

be the $\mathbf{C}^*$-action on X_n. Let us consider the morphism (of formal schemes):

$$id \times \{\sigma_n\} : \mathbf{C}^* \times \{X_n\} \to \mathbf{C}^* \times \{X_n\}.$$

Here we regard the first factor (resp. the second factor) as a $\mathbf{C}^* \times \{Y_n\}$- formal scheme by $id \times (\{f_n\} \circ \{\sigma_n\})$ (resp. $id \times \{f_n\}$). Then the morphism above is a $\mathbf{C}^* \times \{Y_n\}$-morphism. By [ibid, Theoreme 5.4.1], this morphism of formal schemes extends to a $\mathbf{C}^* \hat{\times} \hat{Y}$-morphism [6]

$$\mathbf{C}^* \hat{\times} \hat{X} \to \mathbf{C}^* \hat{\times} \hat{X},$$

where the first factor (resp. the second factor) is regarded as a $\mathbf{C}^* \hat{\times} \hat{Y}$-scheme by $id \hat{\times} \hat{f} \circ \hat{\sigma}$ (resp. $id \hat{\times} \hat{f}$). The extended morphism gives a $\mathbf{C}^*$-action

$$\mathbf{C}^* \hat{\times} \hat{X} \to \mathbf{C}^* \hat{\times} \hat{X} \xrightarrow{p_2} \hat{X}.$$

Moreover, $\{L_n\}$ is algebraized to an $\hat{f}$-ample line bundle $\hat{L}$ on $\hat{X}$ ([ibid, Theoreme 5.4.5]). Since $\mathcal{L}$ is fixed by the $\mathbf{C}^*$-action on $\mathcal{X}$, $\hat{L}$ is also fixed by the $\mathbf{C}^*$-action on $\hat{X}$.

Lemma (A.8). *Let $\hat{f} : \hat{X} \to \hat{Y}$ be a $\mathbf{C}^*$-equivariant projective morphism where $\hat{Y} = \mathrm{Spec}(\hat{A})$ with a complete local $\mathbf{C}$-algebra $\hat{A}$ with $\hat{A}/m = \mathbf{C}$. Assume that $\hat{f}_*\mathcal{O}_{\hat{X}} = \mathcal{O}_{\hat{Y}}$. Let $\hat{L}$ be an $\hat{f}$-ample line bundle on $\hat{X}$ fixed by the $\mathbf{C}^*$-action. Then $\hat{L}^{\otimes m}$ can be $\mathbf{C}^*$-linearized for some $m > 0$. Moreover, in this case, any $\mathbf{C}^*$-fixed line bundle M on $\hat{X}$ is $\mathbf{C}^*$-linearized after taking a suitable multiple of M.*

Proof. We only have to deal with an $\hat{f}$-ample line bundle $\hat{L}$. In fact, let M be an arbitrary line bundle on $\hat{X}$ fixed by the $\mathbf{C}^*$-action. Then $M \otimes \hat{L}^{\otimes r}$ becomes $\hat{f}$-ample for a sufficiently large r. If we could prove the lemma for $\hat{f}$-ample line bundles, then $M^{\otimes m} \otimes \hat{L}^{\otimes rm}$ is $\mathbf{C}^*$-linearized. Since $\hat{L}^{\otimes rm}$ is also $\mathbf{C}^*$-linearized, $M^{\otimes m}$ is $\mathbf{C}^*$-linearized. We assume that $\hat{L}$ is $\hat{f}$-very ample and $\hat{X}$ is embedded into $\mathbf{P}_{\hat{A}}(H^0(\hat{X}, \hat{L}))$ as a $\hat{Y}$-scheme, where $H^0(\hat{X}, \hat{L})$ is a free $\hat{A}$-module of finite rank, say n. Since $\mathbf{C}^*$ acts on $\hat{A}$, we regard $\mathbf{C}^*$ as a subgroup of the automorphism group of the $\mathbf{C}$-algebra $\hat{A}$. Let $\sigma \in \mathbf{C}^*$ and let M be an $\hat{A}$-module. Then a $\mathbf{C}$-linear map $\phi : M \to M$ is called a twisted $\hat{A}$-linear map if there exists $\sigma \in \mathbf{C}^*$ and $\phi(ax) = \sigma(a)\phi(x)$ for $a \in \hat{A}$ and for $x \in M$. Now let us consider the case $M = H^0(\hat{X}, \hat{L})$, which is a free $\hat{A}$-module of rank n. We define $G(n, \hat{A})$ to be the group of all twisted $\hat{A}$-linear bijective maps from $H^0(\hat{X}, \hat{L})$ onto itself. One can define a surjective homomorphisms $G(n, \hat{A}) \to \mathbf{C}^*$ by sending $\phi \in G(n, \hat{A})$ to the associated twisting element $\sigma \in \mathbf{C}^*$. Note that this homomorphism admits a canonical splitting $\iota : \mathbf{C}^* \to G(n, \hat{A})$ defined by $\iota(\sigma)(x_1, ..., x_n) := (\sigma(x_1), ..., \sigma(x_n))$. There is an exact sequence

$$1 \to GL(n, \hat{A}) \to G(n, \hat{A}) \to \mathbf{C}^* \to 1.$$

[6] $\hat{\times}$ means the formal product. Let B be the completion of $\hat{A}[t, 1/t]$ by the ideal $m\hat{A}[t, 1/t]$ where $m \subset \hat{A}$ is the maximal ideal. Then $\mathbf{C}^* \hat{\times} \hat{Y} = \mathrm{Spec}(B)$. The scheme $\mathbf{C}^* \hat{\times} \hat{X}$ is defined as the fiber product of $\mathbf{C}^* \times \hat{X} \to \mathbf{C}^* \times \hat{Y}$ and $\mathbf{C}^* \hat{\times} \hat{Y} \to \mathbf{C}^* \times \hat{Y}$.

Let us denote by $\hat{A}^*$ the multiplicative group of units of $\hat{A}$. One can embed $\hat{A}^*$ diagonally in $GL(n, \hat{A})$; hence in $G(n, \hat{A})$. Then $\hat{A}^*$ is a normal subgroup of $G(n, \hat{A})$. The group $PG(n, \hat{A}) := G(n, \hat{A})/\hat{A}^*$ acts faithfully on $H^0(\hat{X}, \hat{L}) - \{0\}/\hat{A}^*$. On the other hand, define $S(n, \hat{A})$ to be the subgroup of $G(n, \hat{A})$ generated by $SL(n, \hat{A})$ and $\iota(\mathbf{C}^*)$ There are two exact sequences

$$1 \to SL(n, \hat{A}) \to S(n, \hat{A}) \to \mathbf{C}^* \to 1,$$

and

$$1 \to PGL(n, \hat{A}) \to PG(n, \hat{A}) \to \mathbf{C}^* \to 1.$$

Since $\hat{A}$ is a complete local ring and its residue is an algebraically closed field with characteristic 0, the canonical map $SL(n, \hat{A}) \to PGL(n, \hat{A})$ is a surjection; hence the composed map $S(n, \hat{A}) \to G(n, \hat{A}) \to PG(n, \hat{A})$ is surjective.

Let us start the proof. Note that $H^0(\hat{X}, \hat{L}) - \{0\}/\hat{A}^*$ is identified with the space of Cartier divisors whose associated line bundle is $\hat{L}$. Since $\hat{L}$ is fixed by the $\mathbf{C}^*$-action, the $\mathbf{C}^*$ action on $\hat{X}$ induces a $\mathbf{C}^*$ action on $H^0(\hat{X}, \hat{L}) - \{0\}/\hat{A}^*$. This action gives a splitting

$$\alpha : \mathbf{C}^* \to PG(n, \hat{A})$$

of the exact sequence above. We want to lift the map α to $S(n, \hat{A})$. We put $H := \varphi^{-1}(\alpha(\mathbf{C}^*))$, where $\varphi : S(n, \hat{A}) \to PG(n, \hat{A})$ is the quotient map. Since $\mathrm{Ker}(\varphi) = \mu_n$, H is an etale cover of $\mathbf{C}^*$. Now H acts on $H^0(\hat{X}, \hat{L})$. Then H naturally acts on the n-th symmetric product $\mathrm{S}^n(H^0(\hat{X}, \hat{L}))$, where μ_n acts trivially. Therefore, we get a $\mathbf{C}^*$-action on $S^n(H^0(\hat{X}, \hat{L}))$. This $\mathbf{C}^*$-action induces a $\mathbf{C}^*$-linearization of $\mathcal{O}_{\mathbf{P}(H^0(\hat{X}, \hat{L}))}(n)$. Since $\hat{L}^{\otimes n}$ is the pull-back of this line bundle by the $\mathbf{C}^*$-equivariant embedding $\hat{X} \to \mathbf{P}(H^0(\hat{X}, \hat{L}))$, $\hat{L}^{\otimes n}$ has a $\mathbf{C}^*$-linearization. $\square$

By the lemma above, $\hat{L}^{\otimes m}$ is $\mathbf{C}^*$-linearized for some $m > 0$. Now one can write

$$\hat{X} = \mathrm{Proj}_{\hat{A}} \oplus_{n \geq 0} \hat{f}_* \hat{L}^{\otimes nm},$$

where each $\hat{f}_* \hat{L}^{\otimes nm}$ is an $\hat{A}$-module with $\mathbf{C}^*$-action. Since Y has a good $\mathbf{C}^*$-action, there exists a projective $\mathbf{C}^*$-equivariant morphism $f : X \to Y$ such that $X \times_Y \hat{Y} = \hat{X}$ by Proposition (A.5).

(STEP 3): We shall finally show that $X^{an} = \mathcal{X}$ and $f^{an} = \bar{f}$. The formal neighborhoods of X^{an} and $\mathcal{X}$ along $f^{-1}(0)$ are the same. By [Ar], the bimeromorphic map $X^{an} -- \to \mathcal{X}$ is an isomorphism over a small open neighborhood

U of $0 \in Y^{an}$. But, since Y^{an} has a good $\mathbf{C}^*$-action and this action lifts to both X^{an} and $\mathcal{X}$, the bimeromorphic map must be an isomorphism over Y^{an}. $\square$

Proposition (A.9)**.** *Let $Y = \mathrm{Spec}(A)$ be an affine variety with a good $\mathbf{C}^*$-action and let $f : X \to Y$ be a birational projective morphism with X normal. Assume that Y has only rational singularities, and X is $\mathbf{Q}$-factorial. Then X^{an} is $\mathbf{Q}$-factorial.*

Proof. Let $g : Z \to Y$ be a $\mathbf{C}^*$-equivariant projective resolution. Let $0 \in Y$ be the fixed origin of the $\mathbf{C}^*$-action and let $\hat{Y} := \mathrm{Spec}(\hat{A})$ where $\hat{A}$ is the completion of A at 0. We put $\hat{Z} := Z \times_Y \hat{Y}$ and denote by $\hat{g} : \hat{Z} \to \hat{Y}$ the induced morphism. Since Y has only rational singularities, $\mathrm{Pic}(Z^{an}) \cong H^2(Z^{an}, \mathbf{Z})$, which is discrete. Hence every element $\mathcal{L} \in \mathrm{Pic}(Z^{an})$ is fixed by the $\mathbf{C}^*$-action. Take an arbitrary line bundle $\mathcal{L}$. We shall prove that, for some $m > 0$, $\mathcal{L}^{\otimes m}$ comes from an algebraic line bundle. As in the proof of Proposition 26, $\mathcal{L}$ defines a line bundle $\hat{L}$ on $\hat{X}$. By Lemma (A.8), $\hat{L}^{\otimes m}$ is $\mathbf{C}^*$-linearized for some m. By Proposition (A.6), $\hat{L}^{\otimes m}$ extends to a $\mathbf{C}^*$-linearized line bundle M on Z. By the construction, there is an open neighborhood U of $0 \in Y^{an}$ such that $M^{an}|_{(g^{an})^{-1}(U)} \cong \mathcal{L}^{\otimes m}|_{(g^{an})^{-1}(U)}$. Since Y^{an} has a good $\mathbf{C}^*$-action, one can assume that $H^2(Z^{an}, \mathbf{Z}) \cong H^2((g^{an})^{-1}(U), \mathbf{Z})$; this implies that $\mathrm{Pic}(Z^{an}) \cong \mathrm{Pic}((g^{an})^{-1}(U))$. Thus, $M^{an} \cong \mathcal{L}^{\otimes m}$. Let us take a common resolution of Z and X: $h_1 : W \to Z$ and $h_2 : W \to X$. Let D be an irreducible (analytic) Weil divisor of X^{an}. Take an irreducible component D' of $(h_2^{an})^{-1}(D)$ such that $(h_2^{an})(D') = D$. We put $\bar{D} := (h_1^{an})(D')$. We first assume that $\bar{D}$ is a divisor of Z^{an}. Then the line bundle $\mathcal{O}_{Z^{an}}(r\bar{D})$ becomes algebraic for some $r > 0$. Hence $\mathcal{O}_{W^{an}}(rD')$ is algebraic. Finally, the direct image $(h_2^{an})_*\mathcal{O}_{W^{an}}(rD')$ is algebraic, and its double dual is also algebraic. Thus we conclude that $\mathcal{O}_{X^{an}}(rD)$ is an algebraic reflexive sheaf of rank 1. We next assume that $\bar{D}$ is not a divisor. Then D' is an exceptional divisor of h_1. In this case, $\mathcal{O}_{W^{an}}(D')$ is algebraic, and the same argument as the first case shows that $\mathcal{O}_{X^{an}}(D)$ is algebraic. $\square$

Corollary (A.10)**.** *Let Y be an affine symplectic variety with a good $\mathbf{C}^*$-action. Then the following hold.*

(i) *If $f : X \to Y$ is a $\mathbf{Q}$-factorial terminalization, then X^{an} is $\mathbf{Q}$-factorial as an analytic space.*

(ii)*If $\bar{f} : \mathcal{X} \to Y^{an}$ is a $\mathbf{Q}$-factorial terminalization as an analytic space, then there is a projective birational morphism $f : X \to Y$ such that $X^{an} = \mathcal{X}$ and $f^{an} = \bar{f}$.*

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Added in the proof: In the proof of Theorem 14, it is claimed that the T^0-lifting property implies the pro-representability of PD. But, the argument here is not correct. One can find a correct argument in [Na 6].