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FLOW BY MEAN CURVATURE OF CONVEX SURFACES INTO SPHERES

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1. Introduction

The motion of surfaces by their mean curvature has been studied by Brakke [1] from the viewpoint of geometric measure theory. Other authors investigated the corresponding nonparametric problem [2], [5], [9]. A reason for this interest is that evolutionary surfaces of prescribed mean curvature model the behavior of grain boundaries in annealing pure metal.

In this paper we take a more classical point of view: Consider a compact, uniformly convex n-dimensional surface $M = M_0$ without boundary, which is smoothly imbedded in $\mathbf{R}^{n+1}$. Let M_0 be represented locally by a diffeomorphism

$$F_0\colon \mathbf{R}^n \supset U \to F_0(U) \subset M_0 \subset \mathbf{R}^{n+1}.$$

Then we want to find a family of maps $F(\cdot, t)$ satisfying the evolution equation

$$(1)\qquad \begin{aligned} &\frac{\partial}{\partial t}F(\vec{x}, t) = \Delta_t F(\vec{x}, t), \qquad \vec{x} \in U,\\ &F(\cdot, 0) = F_0, \end{aligned}$$

where Δ_t is the Laplace-Beltrami operator on the manifold M_t, given by $F(\cdot, t)$. We have

$$\Delta_t F(\vec{x}, t) = -H(\vec{x}, t) \cdot \nu(\vec{x}, t),$$

where $H(\cdot, t)$ is the mean curvature and $\nu(\cdot, t)$ is the outer unit normal on M_t. With this choice of sign the mean curvature of our convex surfaces is always positive and the surfaces are moving in the direction of their inner unit normal. Equation (1) is parabolic and the theory of quasilinear parabolic differential equations guarantees the existence of $F(\cdot, t)$ for some short time interval.

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We want to show here that the shape of M_t approaches the shape of a sphere very rapidly. In particular, no singularities will occur before the surfaces M_t shrink down to a single point after a finite time. To describe this more precisely, we carry out a normalization: For any time t, where the solution $F(\cdot, t)$ of (1) exists, let $\psi(t)$ be a positive factor such that the manifold $\tilde{M}_t$ given by

$$\tilde{F}(\vec{x}, t) = \psi(t) \cdot F(\vec{x}, t)$$

has total area equal to $|M_0|$, the area of M_0:

$$\int_{\tilde{M}_t} d\tilde{\mu} = |M_0| \quad \text{for all } t.$$

After choosing the new time variable $\tilde{t}(t) = \int_0^t \psi^2(\tau)\, d\tau$ it is easy to see that $\tilde{F}$ satisfies

$$\begin{aligned} \frac{\partial}{\partial \tilde{t}} \tilde{F}(\vec{x}, \tilde{t}) &= \tilde{\Delta}_{\tilde{t}} \tilde{F}(\vec{x}, \tilde{t}) + \frac{1}{n} \tilde{h}_{\tilde{t}} \tilde{F}(\vec{x}, \tilde{t}), \\ \tilde{F}(\cdot, 0) &= F_0, \end{aligned} \tag{2}$$

where

$$\tilde{h} = \int_{\tilde{M}} \tilde{H}^2\, d\tilde{\mu} \Big/ \int_{\tilde{M}} d\tilde{\mu}$$

is the mean value of the squared mean curvature on $\tilde{M}_t$ (see §9 below).

1.1 Theorem. *Let $n \geqslant 2$ and assume that M_0 is uniformly convex, i.e., the eigenvalues of its second fundamental form are strictly positive everywhere. Then the evolution equation* (1) *has a smooth solution on a finite time interval $0 \leqslant t < T$, and the M_t's converge to a single point $\mathfrak{O}$ as $t \to T$. The normalized equation* (2) *has a solution $\tilde{M}_{\tilde{t}}$ for all time $0 \leqslant \tilde{t} < \infty$. The surfaces $\tilde{M}_{\tilde{t}}$ are homothetic expansions of the M_t's, and if we choose $\mathfrak{O}$ as the origin of $\mathbf{R}^{n+1}$, then the surfaces $\tilde{M}_{\tilde{t}}$ converge to a sphere of area $|M_0|$ in the C^∞-topology as $\tilde{t} \to \infty$.*

Remarks. (i) The convergence of $\tilde{M}_{\tilde{t}}$ in any C^k-norm is exponential.

(ii) The corresponding one-dimensional problem has been solved recently by Gage and Hamilton (see [4]).

The approach to Theorem 1.1 is inspired by Hamiltons paper [6]. He evolved the metric of a compact three-dimensional manifold with positive Ricci curvature in direction of the Ricci curvature and obtained a metric of constant curvature in the limit. The evolution equations for the curvature quantities in our problem turn out to be similar to the equations in [6] and we can use many of the methods developed there.

In §3 we establish evolution equations for the induced metric, the second fundamental form and other important quantities. In the next step a lower bound independent of time for the eigenvalues of the second fundamental form is proved. Using this, the Sobolev inequality and an iteration method we can show in §5 that the eigenvalues of the second fundamental form approach each other. Once this is established we obtain a bound for the gradient of the mean curvature and then long time existence for a solution of (2). The exponential convergence of the metric then follows from evolution equations for higher derivatives of the curvature and interpolation inequalities.

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2 Notation and preliminary results

In the following vectors on M will be denoted by $X = \{X^i\}$, covectors by $Y = \{Y_i\}$ and mixed tensors by $T = \{T^{ij}_{kl}\}$. The induced metric and the second fundamental form on M will be denoted by $g = \{g_{ij}\}$ and $A = \{h_{ij}\}$. We always sum over repeated indices from 1 to n and we use brackets for the inner product on M:

$$\left\langle T^i_{jk}, S^i_{jk}\right\rangle = g_{is}g^{jr}g^{ku}T^i_{jk}S^s_{ru}, \qquad |T|^2 = \left\langle T^i_{jk}, T^i_{jk}\right\rangle.$$

In particular we use the following notation for traces of the second fundamental form on M:

$$H = g^{ij}h_{ij}, \qquad |A|^2 = g^{ij}g^{kl}h_{ik}h_{jl},$$
$$C = g^{ij}g^{kl}g^{mn}h_{ik}h_{lm}h_{nj}, \qquad Z = HC - |A|^4.$$

By $(\cdot, \cdot)$ we denote the ordinary inner product in $\mathbf{R}^{n+1}$. If M is given locally by some F as in the introduction, the metric and the second fundamental form on M can be computed as follows:

$$g_{ij}(\vec{x}) = \left(\frac{\partial F(\vec{x})}{\partial x_i}, \frac{\partial F(\vec{x})}{\partial x_j}\right), \quad h_{ij}(\vec{x}) = -\left(\nu(\vec{x}), \frac{\partial^2 F(\vec{x})}{\partial x_i \partial x_j}\right), \qquad \vec{x} \in \mathbf{R}^n,$$

where $\nu(\vec{x})$ is the outer unit normal to M at $F(\vec{x})$. The induced connection on M is given by

$$\Gamma^k_{ij} = \frac{1}{2}g^{kl}\left(\frac{\partial}{\partial x_i}g_{jl} + \frac{\partial}{\partial x_j}g_{il} - \frac{\partial}{\partial x_l}g_{ij}\right)$$

so that the covariant derivative on M of a vector X is

$$\nabla_j X^i = \frac{\partial}{\partial x_j}X^i + \Gamma^i_{jk}X^k.$$

The Riemann curvature tensor, the Ricci tensor and scalar curvature are given by Gauss' equation

$$R_{ijkl} = h_{ik}h_{jl} - h_{il}h_{jk},$$
$$R_{ik} = Hh_{ik} - h_{il}g^{lj}h_{jk},$$
$$R = H^2 - |A|^2.$$

With this notation we obtain, for the interchange of two covariant derivatives,

$$\nabla_i\nabla_j X^h - \nabla_j\nabla_i X^h = R^h_{ijk}X^k = \left(h_{lj}h_{ik} - h_{lk}h_{ij}\right)g^{hl}X^k,$$
$$\nabla_i\nabla_j Y_k - \nabla_j\nabla_i Y_k = R_{ijkl}g^{lm}Y_m = \left(h_{ik}h_{jl} - h_{il}h_{jk}\right)g^{lm}Y_m.$$

The Laplacian ΔT of a tensor T on M is given by

$$\Delta T^i_{jk} = g^{mn}\nabla_m\nabla_n T^i_{jk},$$

whereas the covariant derivative of T will be denoted by $\nabla T = \{\nabla_l T^i_{jk}\}$. Now we want to state some consequences of these relations, which are crucial in the forthcoming sections. We start with two well-known identities.

2.1 Lemma. (i) $\Delta h_{ij} = \nabla_i\nabla_j H + Hh_{il}g^{lm}h_{mj} - |A|^2 h_{ij}$.
(ii) $\frac{1}{2}\Delta|A|^2 = \langle h_{ij}, \nabla_i\nabla_j H\rangle + |\nabla A|^2 + Z$.

Proof. The first identity follows from the Codazzi equations $\nabla_i h_{kl} = \nabla_k h_{il} = \nabla_l h_{ik}$ and the formula for the interchange of derivatives quoted above, whereas (ii) is an immediate consequence of (i).

The obvious inequality $|\nabla H|^2 \leqslant n|\nabla A|^2$ can be improved by the Codazzi equations.

2.2 Lemma. (i) $|\nabla A|^2 \geqslant 3/(n+2)\cdot|\nabla H|^2$.
(ii) $|\nabla A|^2 - |\nabla H|^2/n \geqslant 2(n-1)|\nabla A|^2/3n$.

Proof. Similar as in [6, Lemma 11.6] we decompose the tensor ∇A:

$$\nabla_i h_{jk} = E_{ijk} + F_{ijk},$$

where

$$E_{ijk} = \frac{1}{n+2}\left(\nabla_i H \cdot g_{jk} + \nabla_j H g_{ik} + \nabla_k H \cdot g_{ij}\right).$$

Then we can easily compute that $|E|^2 = 3|\nabla H|^2/(n+2)$ and

$$\left\langle E_{ijk}, F_{ijk}\right\rangle = \left\langle E_{ijk}, \nabla_i h_{jk} - E_{ijk}\right\rangle = 0,$$

i.e., E and F are orthogonal components of ∇A. Then

$$|\nabla A|^2 \geqslant |E|^2 = \frac{3}{n+2}|\nabla H|^2,$$

which proves the lemma.

If M_{ij} is a symmetric tensor, we say that M_{ij} is nonnegative, $M_{ij} \geqslant 0$, if all eigenvalues of M_{ij} are nonnegative. In view of our main assumption that all eigenvalues of the second fundamental form of M_0 are strictly positive, there is some $\varepsilon > 0$ such that the inequality

$$h_{ij} \geqslant \varepsilon H g_{ij} \tag{3}$$

holds everywhere on M_0. It will be shown in §4 that this lower bound is preserved with the same ε for all M_t as long as the solution of (1) exists. The relation (3) leads to the following inequalities, which will be needed in §5.

2.3 Lemma. *If $H > 0$, and (3) is valid with some $\varepsilon > 0$, then*

(i) $Z \geqslant n\varepsilon^2 H^2(|A|^2 - H^2/n)$.

(ii) $|\nabla_i h_{kl} \cdot H - \nabla_i H \cdot h_{kl}|^2 \geqslant \frac{1}{2}\varepsilon^2 H^2 |\nabla H|^2$.

Proof. (i) This is a pointwise estimate, and we may assume that $g_{ij} = \delta_{ij}$ and

$$h_{ij} = \begin{pmatrix} \kappa_1 & & & 0 \\ & \kappa_2 & & \\ & & \ddots & \\ 0 & & & \kappa_n \end{pmatrix}.$$

In this setting we have

$$\begin{aligned} Z = HC - |A|^4 &= \left(\sum_{i=1}^n \kappa_i\right)\left(\sum_{j=1}^n \kappa_j^3\right) - \left(\sum_{i=1}^n \kappa_i^2\right)^2 \\ &= \sum_{i<j}^n \left(\kappa_i \kappa_j^3 + \kappa_j \kappa_i^3\right) - \sum_{i<j}^n 2\kappa_i^2 \kappa_j^2 \\ &= \sum_{i<j}^n \kappa_i \kappa_j (\kappa_i - \kappa_j)^2 \geqslant \varepsilon^2 H^2 \sum_{i<j}^n (\kappa_i - \kappa_j)^2, \end{aligned}$$

and the conclusion follows since

$$|A|^2 - \frac{1}{n} H^2 = \frac{1}{n} \sum_{i<j}^n (\kappa_i - \kappa_j)^2.$$

(ii) We have

$$\begin{aligned} &|\nabla_i h_{kl} \cdot H - \nabla_i H \cdot h_{kl}|^2 \\ &\quad = \left|\nabla_i h_{kl} \cdot H - \tfrac{1}{2}(\nabla_i H \cdot h_{kl} + \nabla_k H \cdot h_{il}) - \tfrac{1}{2}(\nabla_i H \cdot h_{kl} - \nabla_k H \cdot h_{il})\right|^2 \\ &\quad = \left|\nabla_i h_{kl} \cdot H - \tfrac{1}{2}(\nabla_i H \cdot h_{kl} + \nabla_k H \cdot h_{il})\right|^2 + \tfrac{1}{4}\left|\nabla_i H \cdot h_{kl} - \nabla_k H \cdot h_{il}\right|^2 \\ &\quad \geqslant \tfrac{1}{4}\left|\nabla_i H \cdot h_{kl} - \nabla_k H \cdot h_{il}\right|^2, \end{aligned}$$

since $\nabla_i h_{kl}$ is symmetric in (i, k) by the Codazzi equations. Now we have only to consider points where the gradient of the mean curvature does not vanish. Around such a point we introduce an orthonormal frame $e_1, \cdots, e_n$ such that $e_1 = \nabla H / |\nabla H|$. Then

$$\nabla_i H = \begin{cases} |\nabla H|, & i = 1, \\ 0, & i \geqslant 2, \end{cases}$$

in these coordinates. Therefore

$$\begin{aligned} \frac{1}{4} \sum_{i,k,l=1}^{n} & (\nabla_i H \cdot h_{kl} - \nabla_k H \cdot h_{il})^2 \\ & \geqslant \tfrac{1}{4}(\nabla_1 H \cdot h_{22} - \nabla_2 H \cdot h_{12})^2 + \tfrac{1}{4}(\nabla_2 H \cdot h_{12} - \nabla_1 H \cdot h_{22})^2 \\ & = \tfrac{1}{2} h_{22}^2 |\nabla H|^2 \geqslant \tfrac{1}{2}\varepsilon^2 H^2 |\nabla H|^2, \end{aligned}$$

since any eigenvalue, and thus any trace element of h_{ij} is greater than εH.

3. Evolution of metric and curvature

In this and the following sections we investigate equation (1) which is easier to handle than the normalized equation (2). The results will be converted to the normalized equation in §9.

3.1 Theorem. *The evolution equation* (1) *has a solution M_t for a short time with any smooth compact initial surface $M = M_0$ at $t = 0$.*

This follows from the fact that (1) is strictly parabolic (see for example [3, III.4]). From now on we will assume that (1) has a solution on the interval $0 \leqslant t < T$.

Equation (1) implies evolution equations for g and A, which will be derived now.

3.2 Lemma. *The metric of M_t satisfies the evolution equation*

$$\frac{\partial}{\partial t} g_{ij} = -2 H h_{ij}. \tag{4}$$

Proof. The vectors $\partial F / \partial x_i$ are tangential to M, and thus

$$\left(\nu, \frac{\partial F}{\partial x_i} \right) = 0, \qquad h_{ij} = \left(\frac{\partial}{\partial x_i} \nu, \frac{\partial F}{\partial x_j} \right) = \left(\frac{\partial}{\partial x_j} \nu, \frac{\partial F}{\partial x_i} \right).$$

From this we obtain

$$\begin{aligned}\frac{\partial}{\partial t}g_{ij} &= \frac{\partial}{\partial t}\left(\frac{\partial F}{\partial x_i}, \frac{\partial F}{\partial x_j}\right)\\ &= \left(\frac{\partial}{\partial x_i}(-H\nu), \frac{\partial F}{\partial x_j}\right) + \left(\frac{\partial F}{\partial x_i}, \frac{\partial}{\partial x_j}(-H\nu)\right)\\ &= -H\left(\frac{\partial}{\partial x_i}\nu, \frac{\partial F}{\partial x_j}\right) - H\left(\frac{\partial F}{\partial x_i}, \frac{\partial}{\partial x_j}\nu\right)\\ &= -2Hh_{ij}.\end{aligned}$$

3.3 Lemma. *The unit normal to M_t satisfies $\partial\nu/\partial t = \nabla H$.*

Proof. This is a straightforward computation:

$$\begin{aligned}\frac{\partial}{\partial t}\nu &= \left(\frac{\partial}{\partial t}\nu, \frac{\partial F}{\partial x_i}\right)\frac{\partial F}{\partial x_j}g^{ij} = -\left(\nu, \frac{\partial}{\partial t}\frac{\partial F}{\partial x_i}\right)\frac{\partial F}{\partial x_j}g^{ij}\\ &= \left(\nu, \frac{\partial}{\partial x_i}(H\nu)\right)\frac{\partial F}{\partial x_j}g^{ij} = \frac{\partial}{\partial x_i}H \cdot \frac{\partial F}{\partial x_j}g^{ij} = \nabla H.\end{aligned}$$

Now we can prove

3.4 Theorem. *The second fundamental form satisfies the evolution equation*

$$\frac{\partial}{\partial t}h_{ij} = \Delta h_{ij} - 2Hh_{il}g^{lm}h_{mj} + |A|^2 h_{ij}.$$

Proof. We use the Gauss-Weingarten relations

$$\frac{\partial^2 F}{\partial x_i \partial x_j} = \Gamma_{ij}^k \frac{\partial F}{\partial x_k} - h_{ij}\nu, \qquad \frac{\partial}{\partial x_j}\nu = h_{jl}g^{lm}\frac{\partial F}{\partial x_m}$$

to conclude

$$\begin{aligned}\frac{\partial}{\partial t}h_{ij} &= -\frac{\partial}{\partial t}\left(\frac{\partial^2 F}{\partial x_i \partial x_j}, \nu\right)\\ &= \left(\frac{\partial^2}{\partial x_i \partial x_j}(H\nu), \nu\right) - \left(\frac{\partial^2 F}{\partial x_i \partial x_j}, \frac{\partial}{\partial x_l}H\frac{\partial F}{\partial x_m}g^{lm}\right)\\ &= \frac{\partial^2}{\partial x_i \partial x_j}H + H\left(\frac{\partial}{\partial x_i}\left(h_{jm}g^{ml}\frac{\partial F}{\partial x_l}\right), \nu\right)\\ &\quad - \left(\Gamma_{ij}^k\frac{\partial F}{\partial x_k} - h_{ij}\nu, \frac{\partial}{\partial x_l}H \cdot \frac{\partial F}{\partial x_m}g^{lm}\right)\\ &= \frac{\partial^2}{\partial x_i \partial x_j}H - \Gamma_{ij}^k\frac{\partial}{\partial x_k}H + Hh_{jm}g^{ml}\left(\Gamma_{il}^\sigma\frac{\partial F}{\partial x_\sigma} - h_{il}\nu, \nu\right)\\ &= \nabla_i\nabla_j H - Hh_{il}g^{lm}h_{mj}.\end{aligned}$$

Then the theorem is a consequence of Lemma 2.1.

3.5 Corollary. *We have the evolution equations:*

(i) $\frac{\partial}{\partial t} H = \Delta H + |A|^2 H,$

(ii) $\frac{\partial}{\partial t}|A|^2 = \Delta|A|^2 - 2|\nabla A|^2 + 2|A|^4,$

(iii) $$\frac{\partial}{\partial t}\left(|A|^2 - \frac{1}{n}H^2\right) = \Delta\left(|A|^2 - \frac{1}{n}H^2\right) - 2\left(|\nabla A|^2 - \frac{1}{n}|\nabla H|^2\right) + 2|A|^2\left(|A|^2 - \frac{1}{n}H^2\right).$$

Proof. We get, from Lemma 3.2,

$$\frac{\partial}{\partial t} H = \frac{\partial}{\partial t}\left(g^{ij}h_{ij}\right) = g^{ij}\frac{\partial}{\partial t}h_{ij} + 2Hg^{ik}g^{jl}h_{kl}h_{ij},$$

and the first identity follows from Theorem 3.4. To prove the second equation, we calculate

$$\begin{aligned}\frac{\partial}{\partial t}A^2 &= \frac{\partial}{\partial t}\left(g^{ik}g^{jl}h_{ij}h_{kl}\right)\\ &= -4Hg^{im}g^{kn}h_{mn}g^{jl}h_{ij}h_{kl}\\ &\quad + 2g^{ik}g^{jl}h_{kl}\left(\Delta h_{ij} - 2Hh_{im}g^{mn}h_{nj} + |A|^2 h_{ij}\right)\\ &= 2g^{ik}g^{jl}h_{kl}\Delta h_{ij} + 2|A|^4,\end{aligned}$$

$$\Delta|A|^2 = g^{kl}\nabla_k\nabla_l\left(g^{pq}g^{mn}h_{pm}h_{qn}\right) = 2g^{pq}g^{mn}h_{pm}\Delta h_{qn} + 2|\nabla A|^2.$$

The last identity follows from (ii) and

$$\frac{\partial}{\partial t}H^2 = 2H\left(\Delta H + |A|^2 H\right) = \Delta H^2 - 2|\nabla H|^2 + 2|A|^2 H^2.$$

3.6 Corollary. (i) *If $d\mu_t = \mu_t(\vec{x})\,dx$ is the measure on M_t, then $\mu = \sqrt{\det g_{ij}}$ and $\partial\mu_t/\partial t = -H^2 \cdot \mu_t$. In particular the total area $|M_t|$ of M_t is decreasing.*

(ii) *If the mean curvature of M_0 is strictly positive everywhere, then it will be strictly positive on M_t as long as the solution exists.*

Proof. The first part of the corollary follows from Lemma 3.2, whereas the second part is a consequence of the evolution equation for H and the maximum principle.

4. Preserving convexity

We want to show now that our main assumption, that is inequality (3), remains true as long as the solution of equation (1) exists. For this purpose we need the following maximum principle for tensors on manifold, which was proved in [6, Theorem 9.1]:

Let u^k be a vector field and let g_{ij}, M_{ij} and N_{ij} be symmetric tensors on a compact manifold M which may all depend on time t. Assume that $N_{ij} = p(M_{ij}, g_{ij})$ is a polynomial in M_{ij} formed by contracting products of M_{ij} with itself using the metric. Furthermore, let this polynomial satisfy a null-eigenvector condition, i.e. for any null-eigenvector X of M_{ij} we have $N_{ij}X^iX^j \geqslant 0$. Then we have

4.1 Theorem (*Hamilton*). *Suppose that on* $0 \leqslant t < T$ *the evolution equation*

$$\frac{\partial}{\partial t}M_{ij} = \Delta M_{ij} + u^k\nabla_k M_{ij} + N_{ij}$$

holds, where $N_{ij} = p(M_{ij}, g_{ij})$ *satisfies the null-eigenvector condition above. If* $M_{ij} \geqslant 0$ *at* $t = 0$, *then it remains so on* $0 \leqslant t < T$.

An immediate consequence of Theorems 3.4 and 4.1 is

4.2 Corollary. *If* $h_{ij} \geqslant 0$ *at* $t = 0$, *then it remains so for* $0 \leqslant t < T$.

Proof. Set $M_{ij} = h_{ij}$, $u^k \equiv 0$ and $N_{ij} = -2Hh_{il}g^{lm}h_{mj} + |A|^2h_{ij}$.

We also have the following stronger result.

4.3 Theorem. *If* $\varepsilon Hg_{ij} \leqslant h_{ij} \leqslant \beta Hg_{ij}$, *and* $H > 0$ *at the beginning for some constants* $0 < \varepsilon \leqslant 1/n < \beta < 1$, *then this remains so on* $0 \leqslant t < T$.

Proof. To prove the first inequality, we want to apply Theorem 4.1 with

$$M_{ij} = \frac{h_{ij}}{H} - \varepsilon g_{ij}, \qquad u^k = \frac{2}{H}g^{kl}\nabla_l H,$$

$$N_{ij} = 2\varepsilon Hh_{ij} - 2h_{im}g^{ml}h_{lj}.$$

With this choice the evolution equation in Theorem 4.1 is satisfied since

$$\frac{\partial}{\partial t}\left(\frac{h_{ij}}{H}\right) = \frac{H\Delta h_{ij} - h_{ij}\Delta H}{H^2} - 2h_{im}g^{ml}h_{lj},$$

$$\Delta\left(\frac{h_{ij}}{H}\right) = \frac{H\Delta h_{ij} - h_{ij}\Delta H}{H^2} - \frac{2}{H}g^{kl}\nabla_k H\,\nabla_l\left(\frac{h_{ij}}{H}\right).$$

It remains to check that N_{ij} is nonnegative on the null-eigenvectors of M_{ij}. Assume that, for some vector $X = \{X^i\}$,

$$h_{ij}X^j = \varepsilon HX_i.$$

Then we derive

$$\begin{aligned} N_{ij}X^iX^j &= 2\varepsilon Hh_{ij}X^iX^j - 2h_{im}g^{ml}h_{lj}X^iX^j \\ &= 2\varepsilon^2H^2|X|^2 - 2\varepsilon^2H^2|X|^2 = 0. \end{aligned}$$

That the second inequality remains true follows in the same way after reversing signs.

5. The eigenvalues of A

In this section we want to show that the eigenvalues of the second fundamental form approach each other, at least at those points where the mean curvature tends to infinity (for the unnormalized equation (1)). Following the idea of Hamilton in [6], we look at the quantity

$$|A|^2 - \frac{1}{n}H^2 = \frac{1}{n}\sum_{i<j}^{n} (\kappa_i - \kappa_j)^2,$$

which measures how far the eigenvalues κ_i of A diverge from each other. We show that $|A|^2 - H^2/n$ becomes small compared to H^2.

5.1 Theorem. *There are constants $\delta > 0$ and $C_0 < \infty$ depending only on M_0, such that*

$$|A|^2 - \frac{1}{n}H^2 \leqslant C_0 H^{2-\delta},$$

for all times $0 \leqslant t < T$.

Our goal is to bound the function $f_\sigma = (|A|^2 - H^2/n)/H^{2-\sigma}$ for sufficiently small σ. We first need an evolution equation for f_σ.

5.2 Lemma. *Let $\alpha = 2 - \sigma$. Then, for any σ,*

$$\begin{aligned}\frac{\partial}{\partial t} f_\sigma = \Delta f_\sigma &+ \frac{2(\alpha-1)}{H} g^{pq} \nabla_p H \nabla_q f_\sigma \\ &- \frac{2}{H^{\alpha+2}} |H \nabla_i h_{kl} - \nabla_i H \cdot h_{kl}|^2 \\ &- \frac{(2-\alpha)(\alpha-1)}{H^{\alpha+2}}\left(|A|^2 - \frac{1}{n}H^2\right)|\nabla H|^2 + (2-\alpha)|A|^2 f_\sigma.\end{aligned}$$

Proof. We have, in view of the evolution equations for $|A|^2$ and H,

$$\begin{aligned}\frac{\partial}{\partial t} f_\sigma &= \frac{\partial}{\partial t}\left(\frac{|A|^2}{H^\alpha} - \frac{1}{n}H^{2-\alpha}\right) \\ &= \frac{H\Delta|A|^2 - \alpha|A|^2\Delta H}{H^{\alpha+1}} - \frac{(2-\alpha)}{n}H^{1-\alpha}\Delta H \\ &\quad - \frac{2}{H^\alpha}|\nabla A|^2 + (2-\alpha)|A|^2 f_\sigma.\end{aligned}$$

Furthermore

$$\nabla_i f_\sigma = \frac{H \nabla_i |A|^2 - \alpha |A|^2 \nabla_i H}{H^{\alpha+1}} - \frac{(2-\alpha)}{n} H^{1-\alpha} \nabla_i H,$$

$$\Delta f_\sigma = \frac{H \Delta |A|^2 - \alpha |A|^2 \Delta H}{H^{\alpha+1}} - \frac{(2-\alpha)}{n} H^{1-\alpha} \Delta H$$

$$(5) \qquad - \frac{2\alpha}{H^{\alpha+1}} \left\langle \nabla_i |A|^2, \nabla_i H \right\rangle + \alpha(\alpha+1) \frac{|A|^2}{H^{\alpha+2}} |\nabla H|^2$$

$$- \frac{1}{n} \frac{(2-\alpha)(1-\alpha)}{H^\alpha} |\nabla H|^2,$$

and now the conclusion of the lemma follows from reorganizing terms and the identity

$$|\nabla_i h_{kl} \cdot H - \nabla_i H \cdot h_{kl}|^2 = H^2 |\nabla A|^2 + |A|^2 |\nabla H|^2 - \left\langle \nabla_i |A|^2, \nabla_i H \right\rangle H.$$

Unfortunately the absolute term $(2-\alpha)|A|^2 f_\sigma$ in this evolution equation is positive and we cannot achieve our goal by the ordinary maximum principle. But from Theorem 4.3 and Lemma 2.3(ii) we get

5.3 Corollary. *For any σ the inequality*

$$(6) \qquad \frac{\partial}{\partial t} f_\sigma \leqslant \Delta f_\sigma + \frac{2(\alpha-1)}{H} \langle \nabla_i H, \nabla_i f_\sigma \rangle - \varepsilon^2 \frac{1}{H^\alpha} |\nabla H|^2 + \sigma |A|^2 f_\sigma$$

holds on $0 \leqslant t < T$.

The additional negative term in (6) will be exploited by the divergence theorem:

5.4 Lemma. *Let $p \geqslant 2$. Then for any $\eta > 0$ and any $0 \leqslant \sigma \leqslant \frac{1}{2}$ we have the estimate*

$$n\varepsilon^2 \int f_\sigma^p H^2 \, d\mu \leqslant (2\eta p + 5) \int \frac{1}{H^\alpha} f_\sigma^{p-1} |\nabla H|^2 \, d\mu$$

$$+ \eta^{-1}(p-1) \int f_\sigma^{p-2} |\nabla f_\sigma|^2 \, d\mu.$$

Proof. Let us denote by h_{ij}^0 the trace-free second fundamental form

$$h_{ij}^0 = h_{ij} - \frac{1}{n} g_{ij}.$$

In view of Lemma 2.1(ii), the identity (5) may then be rewritten as

$$\begin{aligned}\Delta f_\sigma = &\frac{2}{H^\alpha}\left\langle h_{ij}^0, \nabla_i\nabla_j H\right\rangle + \frac{2}{H^\alpha}Z\\ &+\frac{2}{H^{\alpha+2}}|\nabla_i h_{kl}\cdot H - \nabla_i H\cdot h_{kl}|^2 - \frac{\alpha}{H}f_\sigma\Delta H\\ &+\frac{(2-\alpha)(\alpha-1)}{H^2}f_\sigma|\nabla H|^2 - \frac{2(\alpha-1)}{H}\langle\nabla_i H, \nabla_i f_\sigma\rangle.\end{aligned}$$

Now we multiply the inequality

$$\begin{aligned}\Delta f_\sigma \geqslant &\frac{2}{H^\alpha}\left\langle h_{ij}^0, \nabla_i\nabla_j H\right\rangle + \frac{2}{H^\alpha}Z\\ &-\frac{2(\alpha-1)}{H}\langle\nabla_i H, \nabla_i f_\sigma\rangle - \frac{\alpha}{H}f_\sigma\cdot\Delta H\end{aligned}$$

by f_σ^{p-1} and integrate. Integration by parts yields

$$\begin{aligned}0 \geqslant\ &(p-1)\int f_\sigma^{p-2}|\nabla f_\sigma|^2\,d\mu + \int\frac{2}{H^\alpha}Zf_\sigma^{p-1}\,d\mu\\ &-2(\alpha-1)\int\frac{1}{H}f_\sigma^{p-1}\langle\nabla_i f_\sigma, \nabla_i H\rangle\,d\mu\\ &+2\alpha\int\frac{1}{H^{\alpha+1}}f_\sigma^{p-1}\left\langle h_{ij}^0, \nabla_i H\nabla_j H\right\rangle d\mu\\ &-\frac{(n-1)}{n}\int\frac{2}{H^\alpha}f_\sigma^{p-1}|\nabla H|^2\,d\mu\\ &-(p-1)\int\frac{2}{H^\alpha}f_\sigma^{p-2}\left\langle h_{ij}^0, \nabla_i H\cdot\nabla_j f_\sigma\right\rangle d\mu\\ &-\alpha\int\frac{1}{H^2}f_\sigma^p|\nabla H|^2\,d\mu + \alpha p\int\frac{1}{H}f_\sigma^{p-1}\langle\nabla_i H, \nabla_i f_\sigma\rangle\,d\mu,\end{aligned}$$

where we used the Codazzi equation. Now, taking the relations

$$ab \leqslant \frac{\eta}{2}a^2 + \frac{1}{2\eta}b^2, \qquad \alpha \leqslant 2, \tag{7}$$

$$f_\sigma \leqslant H^{2-\alpha}, \qquad \left|h_{ij}^0\right|^2 = \left(|A|^2 - \frac{1}{n}H^2\right) = f_\sigma H^\alpha$$

into account, we derive, for any $\eta > 0$,

$$\begin{aligned}\int\frac{1}{H^\alpha}f_\sigma^{p-1}Z\,d\mu \leqslant\ &(2\eta p + 5)\int\frac{1}{H^\alpha}f_\sigma^{p-1}|\nabla H|^2\,d\mu\\ &+\eta^{-1}(p-1)\int f_\sigma^{p-2}|\nabla f_\sigma|^2\,d\mu.\end{aligned}$$

The conclusion then follows from Lemma 2.3(i) and Theorem 4.3.

Now we can show that high L^p-norms of f_σ are bounded, provided σ is sufficiently small.

5.5 Lemma. *There is a constant $C_1 < \infty$ depending only on M_0, such that, for all*

$$p \geqslant 100\varepsilon^{-2}, \qquad \sigma \leqslant \frac{n}{8}\varepsilon^3 p^{-1/2}, \tag{8}$$

the inequality

$$\left(\int_{M_t} f_\sigma^p \, d\mu\right)^{1/p} \leqslant C_1$$

holds on $0 \leqslant t < T$.

Proof. We choose

$$C_1 := \left(|M_0| + 1\right) \sup_{\sigma \in [0, 1/2]} \left(\sup_{M_0} f_\sigma\right)$$

and it is then sufficient to show

$$\frac{\partial}{\partial t} \int f_\sigma^p \, d\mu \leqslant 0 \quad \text{on } 0 \leqslant t < T. \tag{9}$$

To accomplish this, we multiply inequality (6) by pf_σ^{p-1} and obtain

$$\begin{aligned} \frac{\partial}{\partial t} \int f_\sigma^p \, d\mu &+ p(p-1) \int f_\sigma^{p-2} |\nabla f_\sigma|^2 \, d\mu \\ &+ \varepsilon^2 p \int \frac{1}{H^\alpha} f_\sigma^{p-1} |\nabla H|^2 \, d\mu + \int H^2 f_\sigma^p \, d\mu \\ &\leqslant 2(\alpha - 1) p \int \frac{1}{H} f_\sigma^{p-1} |\nabla H| |\nabla f_\sigma| d\mu + \sigma p \int |A|^2 f_\sigma^p \, d\mu, \end{aligned}$$

where the last term on the left-hand side occurs due to the time dependence of $d\mu$ as stated in Corollary 3.6(i). In view of (7) we can estimate

$$\begin{aligned} 2(\alpha - 1) p &\int \frac{1}{H} f_\sigma^{p-1} |\nabla H| |\nabla f_\sigma| d\mu \\ &\leqslant \tfrac{1}{2} p(p-1) \int f_\sigma^{p-2} |\nabla f_\sigma|^2 \, d\mu + 2 \frac{p}{p-1} \int f_\sigma^{p-1} \frac{1}{H^\alpha} |\nabla H|^2 \, d\mu, \end{aligned}$$

and since $p - 1 \geqslant 100\varepsilon^{-2} - 1 \geqslant 4\varepsilon^{-2}$, $|A|^2 \leqslant H^2$, we conclude

$$\begin{aligned} \frac{\partial}{\partial t} \int f_\sigma^p \, d\mu + \tfrac{1}{2} p(p-1) \int f_\sigma^{p-2} |\nabla f_\sigma|^2 \, d\mu + \tfrac{1}{2}\varepsilon^2 p \int \frac{1}{H^\alpha} f_\sigma^{p-1} |\nabla H|^2 \, d\mu \\ \leqslant \sigma p \int H^2 f_\sigma^p \, d\mu. \end{aligned}$$

The assumption (8) on σ and Lemma 5.4 yield

$$\frac{\partial}{\partial t}\int f_\sigma^p\,d\mu + \tfrac{1}{2}p(p-1)\int f_\sigma^{p-2}|\nabla f_\sigma|^2\,d\mu + \tfrac{1}{2}\varepsilon^2 p\int \frac{1}{H^\alpha}f_\sigma^{p-1}|\nabla H|^2\,d\mu$$
$$\leqslant \frac{\varepsilon}{8}p^{1/2}(2\eta p + 5)\int \frac{1}{H^\alpha}f_\sigma^{p-1}|\nabla H|^2\,d\mu$$
$$+\frac{\varepsilon}{8}\eta^{-1}p^{1/2}(p-1)\int f_\sigma^{p-2}|\nabla f_\sigma|^2\,d\mu$$

for any $\eta > 0$. Then (9) follows if we choose $\eta = \varepsilon p^{-1/2}/4$.

5.6 Corollary. *If we assume*

$$p \geqslant \left(\frac{m}{n}\right)^2 2^8\varepsilon^{-6}, \qquad \sigma \leqslant \frac{n}{16}\varepsilon^3 p^{-1/2},$$

then we have

$$\left(\int H^m f_\sigma^p\,d\mu\right)^{1/p} \leqslant C_1$$

on $0 \leqslant t < T$.

Proof. This follows from Lemma 5.5 since

$$\left(\int H^m f_\sigma^p\,d\mu\right)^{1/p} = \left(\int f_{\sigma'}^p\,d\mu\right)^{1/p},$$

with

$$\sigma' = \sigma + \frac{m}{p} \leqslant \frac{n}{16}\varepsilon^3 p^{-1/2} + mp^{-1/2}\frac{n}{m}\frac{\varepsilon^3}{16} \leqslant \frac{n}{8}\varepsilon^3 p^{-1/2}.$$

We are now ready to bound f_σ by an iteration similar to the methods used in [2], [5]. We will need the following Sobolev inequality from [7].

5.7 Lemma. *For all Lipschitz functions v on M we have*

$$\left(\int_M |v|^{n/n-1}\,d\mu\right)^{n-1/n} \leqslant c(n)\left(\int_M |\nabla v|\,d\mu + \int_M H|v|\,d\mu\right).$$

Proof of Theorem 5.1. Multiply inequality (6) by $pf_{\sigma,k}^{p-1}$, where $f_{\sigma,k} = \max(f_\sigma - k, 0)$ for all $k \geqslant k_0 = \sup_{M_0} f_\sigma$, and denote by $A(k)$ the set where $f_\sigma > k$. Then we derive as in the proof of Lemma 5.5 for $p \geqslant 100\varepsilon^{-2}$

$$\frac{\partial}{\partial t}\int_{A(k)} f_{\sigma,k}^p\,d\mu + \frac{1}{2}p(p-1)\int_{A(k)}|\nabla f_\sigma|^2 f_{\sigma,k}^{p-2}\,d\mu$$
$$\leqslant \sigma p\int_{A(k)} H^2 f_{\sigma,k}^{p-1} f_\sigma\,d\mu.$$

On $A(k)$ we have

$$\tfrac{1}{2}p(p-1)f_{\sigma,k}^{p-2}|\nabla f_\sigma|^2 \geqslant \left|\nabla f_{\sigma,k}^{p/2}\right|^2,$$

and thus we obtain with $v = f_{\sigma,k}^{p/2}$

$$\frac{\partial}{\partial t}\int_{A(k)} v^2\,d\mu + \int_{A(k)} |\nabla v|^2\,d\mu \leqslant \sigma p\int_{A(k)} H^2 f_\sigma^p\,d\mu.$$

Let us agree to denote by c_n any constant which only depends on n. Then Lemma 5.7 and the Hölder inequality lead to

$$\left(\int_M v^{2q}\,d\mu\right)^{1/q} \leqslant c_n\int_M |\nabla v|^2\,d\mu + c_n\left(\int_{\operatorname{supp} v} H^n\,d\mu\right)^{2/n}\left(\int_M v^{2q}\,d\mu\right)^{1/q},$$

where

$$q = \begin{cases} n/(n-2), & n > 2,\\ < \infty, & n = 2.\end{cases}$$

Since $\operatorname{supp} v \subset A(k)$, we have in view of Corollary 5.6

$$\left(\int_{\operatorname{supp} v} H^n\,d\mu\right)^{2/n} \leqslant k^{-2p/n}\left(\int_{A(k)} H^n f_\sigma^p\,d\mu\right)^{2/n} \leqslant k^{-2p/n}C_1^{2p/n},$$

provided

$$p \geqslant 2^8\varepsilon^{-6}, \qquad \sigma \leqslant \frac{n}{16}\varepsilon^3 p^{-1/2}.$$

Thus, under this assumption we conclude for $k \geqslant k_1 = k_1(k_0, C_1, n, \varepsilon)$ that

$$\sup_{[0,T]}\int_{A(k)} v^2\,d\mu + c_n\int_0^T\left(\int_{A(k)} v^{2q}\,d\mu\right)^{1/q} dt$$
$$\leqslant \sigma p\int_0^T\int_{A(k)} H^2 f_\sigma^p\,d\mu\,dt.$$

Now we use interpolation inequalities for L^p-spaces

$$\left(\int_{A(k)} v^{2q_0}\,d\mu\right)^{1/q_0} \leqslant \left(\int_{A(k)} v^{2q}\,d\mu\right)^{a/q}\left(\int_{A(k)} v^2\,d\mu\right)^{(1-a)},$$

$$\frac{1}{q_0} = \frac{a}{q} + (1-a),$$

with $a = 1/q_0$ such that $1 < q_0 < q$. Then we have

$$\left(\int_0^T\int_{A(k)} v^{2q_0}\,d\mu\,dt\right)^{1/q_0} \leqslant c_n\sigma p\int_0^T\int_{A(k)} H^2 f_\sigma^p\,d\mu\,dt$$
$$\leqslant c_n\sigma p\|A(k)\|^{1-1/r}\left(\int_0^T\int_{A(k)} H^{2r} f_\sigma^{pr}\,d\mu\,dt\right)^{1/r},$$

where $r > 1$ is to be chosen and

$$\|A(k)\| = \int_0^T \int_{A(k)} d\mu\, dt.$$

Again using the Hölder inequality we obtain

$$\int_0^T \int_{A(k)} f_{\sigma,k}^p\, d\mu\, dt \leqslant c_n \sigma p \|A(k)\|^{2-1/q_0-1/r} \left(\int_0^T \int_{A(k)} H^{2r} f_\sigma^{pr}\, d\mu\, dt \right)^{1/r}.$$

If we now choose r so large that $2 - 1/q_0 - 1/r = \gamma > 1$, then r only depends on n and we may take

$$(10) \qquad p \geqslant r\varepsilon^{-6} 2^{10}, \qquad \sigma \leqslant \varepsilon^6 2^{-9} r^{-1/2}$$

such that by Corollary 5.6

$$|h - k|^p \|A(h)\| \leqslant C_2(n, C_1, \varepsilon) \|A(k)\|^\gamma$$

for all $h > k \geqslant k_1$. By a well-known result (see e.g. [8, Lemma 4.1]) we conclude

$$f_\sigma \leqslant k_1 + d, \qquad d^p = C_2 2^{p\gamma/(\gamma+1)} \|A(k_1)\|^{\gamma-1}$$

for some p and σ satisfying (10). Since

$$\int_{A(k_1)} d\mu \leqslant |M_t| \leqslant |M_0|$$

by Corollary 3.6(i), it remains only to show that T is finite.

5.8 Lemma. $T < \infty$.

Proof. The mean curvature H satisfies the evolution equation

$$\frac{\partial}{\partial t} H = \Delta H + H|A|^2 \geqslant \Delta H + \frac{1}{n} H^3.$$

Then let φ be the solution of the ordinary differential equation

$$\frac{\partial \varphi}{\partial t} = \frac{1}{n} \varphi^3, \qquad \varphi(0) = H_{\min}(0) > 0.$$

If we consider φ as a function on $M \times [0, T)$, we get

$$\frac{\partial}{\partial t}(H - \varphi) \geqslant \Delta(H - \varphi) + \frac{1}{n}(H^3 - \varphi^3)$$

such that by the maximum principle

$$H \geqslant \varphi \quad \text{on } 0 \leqslant t < T.$$

On the other hand φ is explicitly given by

$$\varphi(t) = \frac{H_{\min}(0)}{\sqrt{1 - (2/n) H_{\min}^2(0) \cdot t}}.$$

And since $\varphi \to \infty$ as $t \to (n/2)H_{\min}^{-2}(0)$, the result follows. Moreover, in the case that M_0 is a sphere, φ describes exactly the evolution of the mean curvature and so the bound $T \leqslant (n/2)H_{\min}^{-2}(0)$ is sharp. This completes the proof of Theorem 5.1.

6. A bound on $|\nabla H|$

In order to compare the mean curvature at different points of the surface M_t, we bound the gradient of the mean curvature as follows.

6.1 Theorem. *For any $\eta > 0$ there is a constant $C(\eta, M_0, n)$ such that*

$$|\nabla H|^2 \leqslant \eta H^4 + C(\eta, M_0, n).$$

Proof. First of all we need an evolution equation for the gradient of the mean curvature.

6.2 Lemma. *We have the evolution equation*

$$\frac{\partial}{\partial t}|\nabla H|^2 = \Delta|\nabla H|^2 - 2|\nabla^2 H|^2 + 2|A|^2|\nabla H|^2$$
$$+2\langle \nabla_i H \cdot h_{mj}, \nabla_j H \nabla h_{im}\rangle + 2H\langle \nabla_i H, \nabla_i |A|^2\rangle.$$

6.3 Corollary.

$$\frac{\partial}{\partial t}|\nabla H|^2 \leqslant \Delta|\nabla H|^2 - 2|\nabla^2 H|^2 + 4|A|^2|\nabla H|^2 + 2H\langle \nabla_i H, \nabla_i |A|^2\rangle.$$

Proof of Lemma 6.2. Using the evolution equations for H and g we obtain

$$\frac{\partial}{\partial t}|\nabla H|^2 = \frac{\partial}{\partial t}\left(g^{ij}\nabla_i H \nabla_j H\right)$$
$$= 2H\langle h_{ij}, \nabla_i H \cdot \nabla_j H\rangle + 2g^{ij}\nabla_i(\Delta H)\cdot \nabla_j H$$
$$+2g^{ij}\nabla_i\left(H|A|^2\right)\nabla_j H.$$

The result then follows from the relations

$$\Delta|\nabla H|^2 = 2g^{kl}\Delta(\nabla_k H)\cdot \nabla_l H + 2|\nabla^2 H|^2,$$
$$\Delta(\nabla_k H) = \nabla_k(\Delta H) + g^{ij}\nabla_i H\left(Hh_{kj} - h_{km}g^{mn}h_{nj}\right).$$

6.4 Lemma. *We have the inequality*

$$\frac{\partial}{\partial t}\left(\frac{|\nabla H|^2}{H}\right) \leqslant \Delta\left(\frac{|\nabla H|^2}{H}\right) + 3|A|^2\left(\frac{|\nabla H|^2}{H}\right) + 2\langle \nabla_i H, \nabla_i |A|^2\rangle.$$

Proof. We compute

$$\frac{\partial}{\partial t}\left(\frac{|\nabla H|^2}{H}\right) \leqslant \frac{H\Delta|\nabla H|^2 - |\nabla H|^2\Delta H}{H^2} - \frac{2}{H}|\nabla^2 H|^2$$
$$+3|A|^2\left(\frac{|\nabla H|^2}{H}\right) + 2\left\langle \nabla_i H, \nabla_i |A|^2\right\rangle,$$
$$\Delta\left(\frac{|\nabla H|^2}{H}\right) = \frac{H\Delta|\nabla H|^2 - |\nabla H|^2\Delta H}{H^2} + \frac{2}{H^3}|\nabla H|^4$$
$$-\frac{4}{H^3}\langle H\nabla_i\nabla_j H, \nabla_i H\nabla_j H\rangle,$$

and the result follows from Schwarz' inequality. We need two more evolution equations.

6.5 Lemma. *We have*

(i) $$\frac{\partial}{\partial t}H^3 = \Delta H^3 - 6H|\nabla H|^2 + 3|A|^2\cdot H^3,$$

(ii) $$\frac{\partial}{\partial t}\left(\left(|A|^2 - \frac{1}{n}H^2\right)H\right) \leqslant \Delta\left(\left(|A|^2 - \frac{1}{n}H^2\right)H\right) - \frac{2(n-1)}{3n}H|\nabla A|^2$$
$$+C_3|\nabla A|^2 + 3|A|^2 H\left(|A|^2 - \frac{1}{n}H^2\right),$$

with a constant C_3 depending on n, C_0 and δ, i.e., only on M_0.

Proof. The first identity is an easy consequence of the evolution equation for H. To prove the inequality (ii), we derive from Corollary 3.5(iii)

$$\frac{\partial}{\partial t}\left(\left(|A|^2 - \frac{1}{n}H^2\right)H\right) = \Delta\left(\left(|A|^2 - \frac{1}{n}H^2\right)H\right) - 2H\left(|\nabla A|^2 - \frac{1}{n}|\nabla H|^2\right)$$
$$-2\left\langle \nabla_i H, \nabla_i\left(|A|^2 - \frac{1}{n}H^2\right)\right\rangle$$
$$+3|A|^2 H\left(|A|^2 - \frac{1}{n}H^2\right).$$

Now, using Theorem 5.1 and (7) we estimate

$$2\left|\left\langle \nabla_i H, \nabla_i\left(|A|^2 - \frac{1}{n}H^2\right)\right\rangle\right| = 4|\langle \nabla_i H\cdot h^0_{kl}, \nabla_i h^0_{kl}\rangle|$$
$$\leqslant 4|\nabla H||h^0_{kl}|\,|\nabla A|$$
$$\leqslant 4nC_0^{1/2}H^{1-\delta/2}|\nabla A|^2$$
$$\leqslant \frac{2(n-1)}{3n}H|\nabla A|^2 + C(n, C_0, \delta)|\nabla A|^2,$$

and the conclusion follows from Lemma 2.2(ii).

We are now going to bound the function

$$f = \frac{|\nabla H|^2}{H} + N\left(|A|^2 - \frac{1}{n}H^2\right)H + NC_3|A|^2 - \eta H^3$$

for some large N depending only on n and $0 < \eta \leqslant 1$. From Lemmas 6.4 and 6.5 we obtain

$$\begin{aligned}\frac{\partial f}{\partial t} \leqslant \Delta f &+ 3|A|^2\left(\frac{|\nabla H|^2}{H}\right) + 2\left\langle \nabla_i H, \nabla_i |A|^2\right\rangle \\ &+ 6\eta H|\nabla H|^2 - N\frac{2(n-1)}{3n}H|\nabla A|^2 \\ &+ 2NC_3|A|^4 + 3N|A|^2 H\left(|A|^2 - \frac{1}{n}H^2\right) - 3\eta|A|^2 H^3.\end{aligned}$$

Since $(1/n)H^2 \leqslant |A|^2 \leqslant H^2$, $|\nabla H|^2 \leqslant n|\nabla A|^2$ and $\eta \leqslant 1$ we may choose N depending only on n so large that

$$\frac{\partial f}{\partial t} \leqslant \Delta f + 2NC_3H^4 + 3NH^3\left(|A|^2 - \frac{1}{n}H^2\right) - \frac{3}{n}\eta H^5.$$

By Theorem 5.1 we have

$$\begin{aligned}2NC_3H^4 + 3NH^3\left(|A|^2 - \frac{1}{n}H^2\right) &\leqslant 2NC_3H^4 + 3NC_0H^{5-\delta} \\ &\leqslant \frac{3}{n}\eta H^5 + C(\eta, \delta, n, C_0, C_3)\end{aligned}$$

and hence $\partial f/\partial t \leqslant \Delta f + C(\eta, M_0)$.

This implies that $\max f(t) \leqslant \max f(0) + C(\eta, M_0)t$, and since we already have a bound for T, f is bounded by some (possibly different) constant $C(\eta, M_0)$. Therefore

$$|\nabla H|^2 \leqslant \eta H^4 + C(\eta, M_0)H \leqslant 2\eta H^4 + \tilde{C}(\eta, M_0)$$

which proves Theorem 6.1 since η is arbitrary.

7. Higher derivatives of A

As in [6] we write $S * T$ for any linear combination of tensors formed by contraction on S and T by g. The mth iterated covariant derivative of a tensor T will be denoted by $\nabla^m T$. With this notation we observe that the time derivative of the Christoffel symbols Γ^i_{jk} is equal to

$$\begin{aligned}\frac{\partial}{\partial t}\Gamma^i_{jk} &= \frac{1}{2}g^{il}\left\{\nabla_j\left(\frac{\partial}{\partial t}g_{kl}\right) + \nabla_k\left(\frac{\partial}{\partial t}g_{jl}\right) - \nabla_l\left(\frac{\partial}{\partial t}g_{jk}\right)\right\} \\ &= -g^{il}\left\{\nabla_j(Hh_{kl}) + \nabla_k(Hh_{jl}) - \nabla_l(Hh_{jk})\right\} = A * \nabla A,\end{aligned}$$

in view of the evolution equation for $g = \{g_{ij}\}$. Then we may proceed exactly as in [6, §13] to conclude

7.1 Theorem. *For any m we have an equation*

$$\frac{\partial}{\partial t}|\nabla^m A|^2 = \Delta|\nabla^m A|^2 - 2|\nabla^{m+1}A|^2 + \sum_{i+j+k=m} \nabla_i A * \nabla_j A * \nabla_k A * \nabla_m A.$$

Now we need the following interpolation inequality which is proven in [6, §12].

7.2 Lemma. *If T is any tensor and if $1 \leqslant i \leqslant m-1$, then with a constant $C(n, m)$ which is independent of the metric g and the connection Γ we have the estimate*

$$\int |\nabla^i T|^{2m/i}\, d\mu \leqslant C \cdot \max_M |T|^{2(m/i-1)} \int |\nabla^m T|^2\, d\mu.$$

This leads to

7.3 Theorem. *We have the estimate*

$$\frac{d}{dt}\int_{M_t} |\nabla^m A|^2\, d\mu + 2\int |\nabla^{m+1}A|^2\, d\mu \leqslant C \cdot \max_{M_t} |A|^2 \int_M |\nabla^m A|^2\, d\mu,$$

where C only depends on n and the number of derivatives m.

Proof. By integrating the identity in Theorem 7.1 and using the generalised Hölder inequality we derive

$$\frac{d}{dt}\int_{M_t} |\nabla^m A|^2\, d\mu + 2\int_{M_t} |\nabla^{m+1}A|^2\, d\mu$$

$$\leqslant C\left\{\int_{M_t} |\nabla^i A|^{2m/i}\, d\mu\right\}^{i/2m} \left\{\int_{M_t} |\nabla^j A|^{2m/j}\, d\mu\right\}^{j/2m}$$

$$\cdot \left\{\int_{M_t} |\nabla^k A|^{2m/k}\, d\mu\right\}^{k/2m} \left\{\int_{M_t} |\nabla^m A|^2\, d\mu\right\}^{1/2},$$

with $i + j + k = m$. The interpolation inequality above gives

$$\left\{\int_{M_t} |\nabla^i A|^{2m/i}\, d\mu\right\}^{i/2m} \leqslant C \cdot \max|A|^{1-i/m} \left\{\int_{M_t} |\nabla^m A|^2\, d\mu\right\}^{i/2m},$$

and if we do the same with j and k, the theorem follows.

8. The maximal time interval

We already stated that equation (1) has a (unique) smooth solution on a short time interval if the uniformly convex, closed and compact initial surface M_0 is smooth enough. Moreover, we have

8.1 Theorem. *The solution of equation* (1) *exists on a maximal time interval* $0 \leqslant t < T < \infty$ *and* $\max_{M_t} |A|^2$ *becomes unbounded as t approaches T.*

Proof. Let $0 \leqslant t < T$ be the maximal time interval where the solution exists. We showed in Lemma 5.8 that $T < \infty$. Here we want to show that if $\max_{M_t}|A|^2 \leqslant C$ for $t \to T$, the surfaces M_t converge to a smooth limit surface M_T. We could then use the local existence result to continue the solution to later times in contradiction to the maximality of T.

In the following we suppose

$$\max_{M_t} |A|^2 \leqslant C \quad \text{on } 0 \leqslant t < T, \tag{11}$$

and assume that as in the introduction M_t is given locally by $F(\vec{x}, t)$ defined for $\vec{x} \in U \subset \mathbf{R}^n$ and $0 \leqslant t < T$. Then from the evolution equation (1) we obtain

$$|F(\vec{x}, \rho) - F(\vec{x}, \sigma)| \leqslant \int_\sigma^\rho H(\vec{x}, \tau)\, d\tau$$

for $0 \leqslant \sigma \leqslant \rho < T$. Since H is bounded, $F(\cdot, t)$ tends to a unique continuous limit $F(\cdot, T)$ as $t \to T$.

In order to conclude that $F(\cdot, t)$ represents a surface M_T, we use [6, Lemma 14.2].

8.2 Lemma. *Let g_{ij} be a time dependent metric on a compact manifold M for* $0 \leqslant t < T \leqslant \infty$. *Suppose*

$$\int_0^T \max_M \left|\frac{\partial}{\partial t} g_{ij}\right| dt \leqslant C < \infty.$$

Then the metrics $g_{ij}(t)$ for all different times are equivalent, and they converge as $t \to T$ uniformly to a positive definite metric tensor $g_{ij}(T)$ which is continuous and also equivalent.

Here we used the notation

$$\left|\frac{\partial}{\partial t} g_{ij}\right|^2 = g^{ik} g^{jl} \left(\frac{\partial}{\partial t} g_{ij}\right)\left(\frac{\partial}{\partial t} g_{kl}\right).$$

In our case all the surfaces M_t are diffeomorphic and we can apply Lemma 8.2 in view of Lemma 3.2, assumption (11) and the fact that $T < \infty$. It remains only to show that M_T is smooth. To accomplish this it is enough to prove that all derivatives of the second fundamental form are bounded, since the evolution equations (1) and (4) then imply bounds on all derivatives of F.

8.3 Lemma. *If* (11) *holds on* $0 \leqslant t < T$ *and* $T < \infty$, *then* $|\nabla^m A| \leqslant C_m$ *for all* m. *The constant* C_m *depends on* n, M_0 *and* C.

Proof. Theorem 7.3 immediately implies

$$\int_{M_t} |\nabla^m A|^2 \, d\mu \leqslant C_m,$$

since the inequality $\partial g/\partial t \leqslant cg$ on a finite time interval gives a bound on g in terms of its initial data. Then Lemma 7.2 yields

$$\int_{M_t} |\nabla^m A|^p \, d\mu \leqslant C_{m,p}$$

for all m and $p < \infty$. The conclusion of the lemma now follows if we apply a version of the Sobolev inequality in Lemma 5.7 to the functions $g_m = |\nabla^m A|^2$.

Thus the surfaces M_t converge to M_T in the C^∞-topology as $t \to T$. By Theorem 3.1 this contradicts the maximality of T and proves Theorem 8.1.

We now want to compare the maximum value of the mean curvature $H_{\max}$ to the minimum value $H_{\min}$ as t tends to T. Since $|A|^2 \leqslant H^2$, we obtain from Theorem 8.1 that $H_{\max}$ is unbounded as t approaches T.

8.4 Theorem. *We have* $H_{\max}/H_{\min} \to 1$ *as* $t \to T$.

Proof. We will follow Hamiltons idea to use Myer's theorem.

8.5 Theorem (*Myers*). *If* $R_{ij} \geqslant (n-1)Kg_{ij}$ *along a geodesic of length at least* $\pi K^{-1/2}$ *on* M, *then the geodesic has conjugate points.*

To apply the theorem we need

8.6 Lemma. *If* $h_{ij} \geqslant \varepsilon H g_{ij}$ *holds on* M *with some* $0 < \varepsilon \leqslant 1/n$, *then*

$$R_{ij} \geqslant (n-1)\varepsilon^2 H^2 g_{ij}.$$

Proof of Lemma 8.6. This is immediate from the identity

$$R_{ij} = H h_{ij} - h_{im} g^{mn} h_{nj}.$$

Now we obtain from Theorem 6.1 that for every $\eta > 0$ we can find a constant $c(\eta)$ with $|\nabla H| \leqslant \frac{1}{2}\eta^2 H^2 + C(\eta)$ on $0 \leqslant t < T$. Since $H_{\max}$ becomes unbounded as $t \to T$, there is some $\theta < T$ with $C(\eta) \leqslant \frac{1}{8}\eta^2 H_{\max}^2$ at $t = \theta$. Then

$$(12) \qquad |\nabla H| \leqslant \eta^2 H_{\max}^2$$

at time $t = \theta$. Now let x be a point on M_θ, where H assumes its maximum. Along any geodesic starting at x of length at most $\eta^{-1} H_{\max}^{-1}$ we have $H \geqslant (1-\eta) H_{\max}$. In view of Lemma 8.6 and Theorem 8.5 those geodesics then reach any point of M_θ if η is small and thus

$$(13) \qquad H_{\min} \geqslant (1-\eta) H_{\max} \quad \text{on } M_\theta.$$

Since $H_{\min}$ is nondecreasing we have

$$H_{\max}(t) \geqslant \tfrac{1}{2} H_{\max}(\theta) \quad \text{on } \theta \leqslant t < T,$$

and hence the inequalities (12) and (13) are true on all of $\theta \leqslant t < T$ which proves Theorem 8.4.

We need the following consequences of Theorem 8.4.

8.7 Theorem. *We have* $\int H_{\max}^2(\tau)\, d\tau = \infty$.

Proof. Look at the ordinary differential equation

$$\frac{\partial g}{\partial t} = H_{\max}^2 g, \qquad g(0) = H_{\max}.$$

We get a solution since $H_{\max}^2$ is continuous in t. Furthermore we have

$$\frac{\partial}{\partial t} H = \Delta H + |A|^2 H \leqslant \Delta H + H_{\max}^2 H,$$

and therefore

$$\frac{\partial}{\partial t}(H - g) \leqslant \Delta(H - g) + H_{\max}^2 (H - g).$$

So we obtain $H \leqslant g$ for $0 \leqslant t < T$ by the maximum principle, and $g \to \infty$ as $t \to T$. But now we have

$$\int_0^t H_{\max}^2(\tau)\, d\tau = \log\{ g(t)/g(0) \} \to \infty \quad \text{as } t \to T,$$

which proves Theorem 8.7.

8.8 Corollary. *If, as in the introduction, h is the average of the squared mean curvature*

$$h = \int_M H^2\, d\mu \Big/ \int_M d\mu,$$

then

$$\int_0^T h(\tau)\, d\tau = \infty.$$

Proof. This follows from Theorems 8.4 and 8.7 since $H_{\min}^2 \leqslant h \leqslant H_{\max}^2$.

8.9 Corollary. *We have* $|A|^2/H^2 - 1/n \to 0$ *as* $t \to T$.

Proof. This is a consequence of Theorem 5.1 since $H_{\min} \to \infty$ by Theorem 8.4.

Obviously M_{t_1} stays in the region of $\mathbf{R}^{n+1}$ which is enclosed by M_{t_2} for $t_1 > t_2$ since the surfaces are shrinking. By Theorem 8.4 the diameter of M_t tends to zero as $t \to T$. This implies the first part of Theorem 1.1.

9. The normalized equation

As we have seen in the last sections, the solution of the unnormalized equation

$$(1)\qquad \frac{\partial}{\partial t}F = \Delta F = -H\nu$$

shrinks down to a single point $\mathfrak{O}$ after a finite time. Let us assume from now on that $\mathfrak{O}$ is the origin of $\mathbf{R}^{n+1}$. Note that $\mathfrak{O}$ lays in the region enclosed by M_t for all times $0 \leqslant t < T$. We are going to normalize equation (1) by keeping some geometrical quantity fixed, for example the total area of the surfaces M_t. We could as well have taken the enclosed volume which leads to a slightly different normalized equation. As in the introduction multiply the solution F of (1) at each time $0 \leqslant t < T$ with a positive constant $\psi(t)$ such that the total area of the surface $\tilde{M}_t$ given by

$$\tilde{F}(\cdot, t) = \psi(t) \cdot F(\cdot, t)$$

is equal to the total area of M_0:

$$(14)\qquad \int_{\tilde{M}_t} d\tilde{\mu} = |M_0| \quad \text{on } 0 \leqslant t < T.$$

Then we introduce a new time variable by

$$\tilde{t}(t) = \int_0^t \psi^2(\tau)\, d\tau$$

such that $\partial\tilde{t}/\partial t = \psi^2$. We have

$$\tilde{g}_{ij} = \psi^2 g_{ij}, \qquad \tilde{h}^2_{ij} = \psi h_{ij},$$
$$\tilde{H} = \psi^{-1}H, \qquad |\tilde{A}|^2 = \psi^{-2}|A|^2,$$

and so on. If we differentiate (14) for time t, we obtain

$$(15)\qquad \psi^{-1}\frac{\partial\psi}{\partial t} = \frac{1}{n}\frac{\int H^2\, d\mu}{\int d\mu} = \frac{1}{n}h.$$

Now we can derive the normalized evolution equation for F on a different maximal time interval $0 \leqslant \tilde{t} < \tilde{T}$:

$$\frac{\partial\tilde{F}}{\partial\tilde{t}} = \frac{\partial\tilde{F}}{\partial t}\psi^{-2} = \psi^{-2}\left\{\frac{\partial\psi}{\partial t}F + \psi\frac{\partial F}{\partial t}\right\}$$
$$= -\tilde{H}\tilde{\nu} + \frac{1}{n}\tilde{h}\tilde{F}$$

as stated in (2). We can also compute the new evolution equations for other geometric quantities.

9.1 Lemma. *Suppose the expressions P and Q, formed from g and A, satisfy* $\partial P/\partial t = \Delta P + Q$, *and P has 'degree'* α, *that is,* $\tilde{P} = \psi^{\alpha} P$. *Then Q has degree* $(\alpha - 2)$ *and*

$$\frac{\partial \tilde{P}}{\partial \tilde{t}} = \tilde{\Delta}\tilde{P} + \tilde{Q} + \frac{\alpha}{n}\tilde{h}\tilde{P}.$$

Proof. We calculate with the help of (15)

$$\begin{aligned} \frac{\partial \tilde{P}}{\partial \tilde{t}} &= \psi^{-2}\left\{\alpha\psi^{\alpha-1}\frac{\partial \psi}{\partial t}P + \psi^{\alpha}\frac{\partial P}{\partial t}\right\} \\ &= \psi^{-2}\left\{\frac{\alpha}{n}h\tilde{P} + \psi^{\alpha}\Delta P + \psi^{\alpha}Q\right\} \\ &= \frac{\alpha}{n}\tilde{h}\tilde{P} + \tilde{\Delta}\tilde{P} + \tilde{Q}. \end{aligned}$$

The results in Theorem 4.3, Theorem 8.4 and Corollary 8.9 convert unchanged to the normalized equation, since at each time the whole configuration is only dilated by a constant factor.

9.2 Lemma. *We have*

(i) $\tilde{h}_{ij} \geqslant \varepsilon \tilde{H}\tilde{g}_{ij}$,

(ii) $\tilde{H}_{\max}/\tilde{H}_{\min} \to 1$ *as* $\tilde{t} \to \tilde{T}$,

(iii) $\dfrac{|\tilde{A}|^2}{\tilde{H}^2} \to \dfrac{1}{n}$ *as* $\tilde{t} \to \tilde{T}$.

Now we prove

9.3 Lemma. *There are constants* C_4 *and* C_5 *such that for* $0 \leqslant \tilde{t} < \tilde{T}$

$$0 < C_4 \leqslant \tilde{H}_{\min} \leqslant \tilde{H}_{\max} \leqslant C_5 < \infty.$$

Proof. The surface $\tilde{M}$ encloses a volume $\tilde{V}$ which is given by the divergence theorem

$$\tilde{V} = \frac{1}{n+1}\int_{\tilde{M}} \tilde{F}\tilde{\nu}\, d\tilde{\mu}.$$

Since the origin $\mathfrak{O}$ is in the region enclosed by $\tilde{M}_{\tilde{t}}$ for all times as well, we have that $\tilde{F}\tilde{\nu}$ is everywhere positive on $\tilde{M}_{\tilde{t}}$. By the isoperimetric inequality we have

$$\tilde{V}_{\tilde{t}} \leqslant c_n |\tilde{M}_{\tilde{t}}|^{n/(n-1)} = c_n |M_0|^{n/(n-1)}.$$

On the other hand we get from the first variation formula

$$|M_0| = |\tilde{M}_{\tilde{t}}| = \frac{1}{n}\int \tilde{H}(\tilde{F}\tilde{\nu})\, d\mu \leqslant \tilde{H}_{\max} \cdot \tilde{V}_{\tilde{t}},$$

which proves the first inequality in view of Lemma 9.2(ii). To obtain the upper bound we observe that in view of $\tilde{h}_{ij} \geqslant \varepsilon \tilde{H}_{\min} \tilde{g}_{ij}$ the enclosed volume $\tilde{V}$ can be estimated by the volume of a ball of radius $(\varepsilon \tilde{H}_{\min})^{-1}$:

$$\tilde{V}_{\tilde{t}} \leqslant c_n (\varepsilon \tilde{H}_{\min})^{-(n+1)}.$$

The first variation formula yields

$$\tilde{V}_t \geqslant \frac{1}{n+1}\tilde{H}_{\max}^{-1}\int (\tilde{F}\tilde{\nu})\, \tilde{H} d\tilde{\mu} \geqslant \frac{n}{n+1}\tilde{H}_{\max}^{-1}|M_0|,$$

which proves the upper bound again in view of Lemma 9.2(ii).

9.4 Corollary. $\tilde{T} = \infty$.

Proof. We have $d\tilde{t}/dt = \psi^2$ and $\tilde{H}^2 = \psi^{-2}H^2$ such that

$$\int_0^{\tilde{T}} \tilde{h}(\tilde{\tau})\, d\tilde{\tau} = \int_0^T h(\tau)\, d\tau = \infty$$

by Corollary 8.8. But by Lemma 9.3 we have $\tilde{h} \leqslant \tilde{H}_{\max}^2 \leqslant C_5^2$ and therefore $\tilde{T} = \infty$.

10. Convergence to the sphere

We want to show that the surfaces $\tilde{M}_{\tilde{t}}$ converge to a sphere in the C^∞-topology as $\tilde{t} \to \infty$. Let us agree in this section to denote by $\delta > 0$ and $C < \infty$ various constants depending on known quantities. We start with

10.1 Lemma. *There are constants $\delta > 0$ and $C < \infty$ such that*

$$\int_{\tilde{M}_{\tilde{t}}} |\tilde{A}|^2 - \frac{1}{n}\tilde{H}^2\, d\tilde{\mu}^2 \leqslant Ce^{-\delta\tilde{t}}.$$

Proof. Let $\tilde{f}$ be the function $\tilde{f} = |\tilde{A}|^2/\tilde{H}^2 - 1/n$ which has degree 0. Then we conclude as in the proof of Lemma 5.5 that, for some large p and a small δ depending on ε,

$$\frac{\partial}{\partial\tilde{t}}\int \tilde{f}^p\, d\tilde{\mu} \leqslant -\delta\int \tilde{f}^p|A|^2\, d\tilde{\mu} + \int (\tilde{h} - \tilde{H}^2)\tilde{f}^p\, d\tilde{\mu},$$

since $\partial/\partial\tilde{t}\, d\tilde{\mu} = (\tilde{h} - \tilde{H}^2)\, d\tilde{\mu}$. In view of Lemma 9.2(ii) and Lemma 9.3 we have for all times $\tilde{t}$ larger than some $\tilde{t}_0$

$$\frac{d}{d\tilde{t}}\int \tilde{f}^p\, d\tilde{\mu} \leqslant -\delta\int \tilde{f}^p\, d\tilde{\mu}$$

with a different δ. Thus

$$\int \tilde{f}^p \, d\tilde{\mu} \leqslant Ce^{-\delta\tilde{t}},$$

where C now depends on $\tilde{t}_0$ as well. The conclusion of the lemma then follows from the Hölder inequality $|\tilde{M}_{\tilde{t}}| = |M_0|$ and Lemma 9.3.

Now let us denote by $\tilde{\underline{h}}$ the mean value of the mean curvature on $\tilde{M}$:

$$\tilde{\underline{h}} = \int_{\tilde{M}} \tilde{H} \, d\tilde{\mu} \Big/ \int_{\tilde{M}} d\tilde{\mu}$$

10.2 Lemma. *We have*

$$\int (\tilde{H} - \tilde{\underline{h}})^2 \, d\tilde{\mu} = \int \tilde{H}^2 - \tilde{\underline{h}}^2 \, d\mu \leqslant Ce^{-\delta\tilde{t}}.$$

Proof. In view of the Poincaré inequality it is enough to show that $\int |\nabla\tilde{H}|^2 \, d\tilde{\mu}$ decreases exponentially. Note that the constant in the Poincaré inequality can be chosen independently of $\tilde{t}$ since we got control on the curvature in Lemma 9.2 and Lemma 9.3. Look at the function

$$\tilde{g} = \frac{|\nabla\tilde{H}|^2}{\tilde{H}} + N\left(|\tilde{A}|^2 - \frac{1}{n}\tilde{H}^2\right)\tilde{H},$$

where N is a large constant depending only on n. The degree of g is -3, and from the results in §6 we obtain

$$\frac{\partial\tilde{g}}{\partial\tilde{t}} \leqslant \tilde{\Delta}\tilde{g} + 3N|\tilde{A}|^2\tilde{H}\left(|\tilde{A}|^2 - \frac{1}{n}\tilde{H}^2\right) - \frac{3}{n}\tilde{h}\tilde{g}$$

for all times larger than some $\tilde{t}_1$. Here we used that the term

$$\left\langle \nabla_i\tilde{H}, \nabla_i\left(|\tilde{A}|^2 - \frac{1}{n}\tilde{H}^2\right)\right\rangle = 2\left\langle \nabla_i\tilde{H} \cdot \tilde{h}^0_{kl}, \nabla_i\tilde{h}^0_{kl}\right\rangle$$

becomes small compared to $H|\nabla A|^2$ as $\tilde{t} \to \infty$ since $|\tilde{h}^0_{kl}| = (|\tilde{A}|^2 - \tilde{H}^2/n)^{1/2}$ tends to zero. Now using Lemma 10.1 and $C_4 \leqslant \tilde{H} \leqslant C_5$ we conclude for $\tilde{t} \geqslant \tilde{t}_1$,

$$\frac{d}{d\tilde{t}}\int \tilde{g} \, d\tilde{\mu} \leqslant -\delta\int \tilde{g} \, d\tilde{\mu} + Ce^{-\delta\tilde{t}} + \int (\tilde{h} - \tilde{H}^2)\tilde{g} \, d\tilde{\mu}.$$

Since $(\tilde{h} - \tilde{H}^2) \to 0$ as $\tilde{t} \to \infty$ by Lemma 9.2(ii), we have for all t larger than some $\tilde{t}_2$

$$\frac{d}{d\tilde{t}}\left\{e^{\delta\tilde{t}}\int \tilde{g} \, d\tilde{\mu} - C\tilde{t}\right\} \leqslant 0,$$

and therefore

$$\int \frac{|\nabla\tilde{H}|^2}{\tilde{H}} d\tilde{\mu} \leqslant Ce^{-\delta\tilde{t}}$$

with some constants C and δ depending on $\tilde{t}_2$, and the conclusion follows from $C_4 \leqslant \tilde{H} \leqslant C_5$.

To bound higher derivatives of the curvature, we need another interpolation inequality [6, 12.7].

10.3 Lemma. *If T is any tensor on M, then with a constant $C = C(n, m)$ independent of the metric g and the connection Γ we have the estimate*

$$\int_M |\nabla^i T|^2 \, d\mu \leqslant C\left\{\int_M |\nabla^m T|^2 \, d\mu\right\}^{i/m}\left\{\int_M |T|^2 \, d\mu\right\}^{1-i/m}$$

for $0 \leqslant i \leqslant m$.

We start with Theorem 7.3. The estimate

$$\begin{aligned}(16)\qquad &\frac{d}{d\tilde{t}}\int_{\tilde{M}} |\nabla^m \tilde{A}|^2 \, d\tilde{\mu} + 2\int_{\tilde{M}} |\nabla^{m+1}\tilde{A}|^2 \, d\tilde{\mu} \\ &\leqslant C \cdot \max_{\tilde{M}} |\tilde{A}|^2 \int_{\tilde{M}} |\nabla^m \tilde{A}|^2 \, d\tilde{\mu}\end{aligned}$$

carries over to the normalized equation since both sides stretch by the same factor, and we have $\max|\tilde{A}| \leqslant C_5^2$. Let us now introduce the tensor $\tilde{E} = \{\tilde{E}_{ij}\}$ given by

$$\tilde{E}_{ij} = \tilde{h}_{ij} - \frac{1}{n}\tilde{\underline{h}}\tilde{g}_{ij}.$$

Then $\nabla^m \tilde{A} = \nabla^m \tilde{E}$ for all $m > 0$ and the right-hand side of (16) can be estimated by Lemma 10.3:

$$\int_{\tilde{M}} |\nabla^m \tilde{A}|^2 \, d\tilde{\mu} \leqslant C\left\{\int_{\tilde{M}} |\nabla^{m+1}\tilde{A}|^2 \, d\tilde{\mu}\right\}^{m/(m+1)}\left\{\int_{\tilde{M}} |\tilde{E}|^2 \, d\tilde{\mu}\right\}^{1/(m+1)}$$

By Young's inequality this is less than

$$C\eta\int_{\tilde{M}} |\nabla^{m+1}\tilde{A}|^2 \, d\tilde{\mu} + C\eta^{-m}\int_{\tilde{M}} |\tilde{E}|^2 \, d\tilde{\mu}$$

for any $\eta > 0$. Choosing η such that $C\eta \leqslant 2$ we derive from (16)

$$\frac{d}{d\tilde{t}}\int_{\tilde{M}} |\nabla^m \tilde{A}|^2 \, d\tilde{\mu} \leqslant C\int_{\tilde{M}} |\tilde{E}|^2 \, d\tilde{\mu}.$$

But

$$\begin{aligned}\int |\tilde{E}|^2 \, d\tilde{\mu} &= \int |\tilde{A}|^2 - \frac{2}{n}\tilde{H}\tilde{\underline{h}} + \frac{1}{n}\tilde{\underline{h}}^2 \, d\mu \\ &= \int_{\tilde{M}} |\tilde{A}|^2 - \frac{1}{n}\tilde{H}^2 \, d\tilde{\mu} + \frac{1}{n}\int_{\tilde{M}} (\tilde{H} - \tilde{\underline{h}})^2 \, d\tilde{\mu},\end{aligned}$$

and both integrals decrease exponentially by Lemmas 10.1 and 10.2. Thus we have proven

10.4 Lemma. *For every m we have $\int_{\tilde{M}} |\nabla^m \tilde{A}|^2 \, d\tilde{\mu} \leqslant C$ on $0 \leqslant \tilde{t} < \infty$ with a constant depending on m.*

From Lemma 7.2 we deduce immediately that higher L^p-norms of $|\nabla^m \tilde{A}|$ are bounded as well:

$$\int_{\tilde{M}} |\nabla^m \tilde{A}|^p \, d\tilde{\mu} \leqslant C_{m,p},$$

and a version of the Sobolev inequality in Lemma 5.7 applied to the function $\tilde{E}_m = |\nabla^m \tilde{A}|^2$ yields $\max_{\tilde{M}} |\nabla^m \tilde{A}| \leqslant C$ for a constant $C < \infty$ depending on m. Now we can prove

10.5 Theorem. *There are constants $\delta > 0$ and $C < \infty$ such that*

$$|\tilde{A}|^2 - \frac{1}{n}\tilde{H}^2 \leqslant Ce^{-\delta \tilde{t}}.$$

Proof. We denote by $\mathring{\tilde{A}}$ the traceless second fundamental form

$$\mathring{\tilde{A}} = \left\{\tilde{h}^0_{ij}\right\} = \left\{\tilde{h}_{ij} - \frac{1}{n}\tilde{H}\tilde{g}_{ij}\right\}$$

such that $|\mathring{\tilde{A}}|^2 = |\tilde{A}|^2 - \tilde{H}^2/n$. Since $|\nabla^m \tilde{A}|$ is bounded we conclude from Lemma 10.3

$$\int_{\tilde{M}} |\nabla^m \mathring{\tilde{A}}|^2 \, d\tilde{\mu} \leqslant C_m \left\{ \int |\tilde{A}|^2 - \frac{1}{n}\tilde{H}^2 \, d\tilde{\mu} \right\}^{1/(m+1)} \leqslant C_m e^{-\delta \tilde{t}}$$

in view of Lemma 10.1. Then we have from Lemma 7.2

$$\int_{\tilde{M}} |\nabla^m \mathring{\tilde{A}}|^p \, d\tilde{\mu} \leqslant C_{m,p} e^{-\delta \tilde{t}},$$

and the conclusion follows once again from the Sobolev inequality.

Theorem 10.5 is the crucial estimate from where we can proceed exactly as in Hamiltons paper [6, §17] to conclude

10.6 Lemma. *There are constants $\delta > 0$ and $C < \infty$ such that*

$$\text{(i)} \quad \tilde{H}_{\max} - \tilde{H}_{\min} \leqslant Ce^{-\delta \tilde{t}},$$

$$\text{(ii)} \quad \left| \tilde{h}_{ij}\tilde{H} - \frac{1}{n}\tilde{h}\tilde{g}_{ij} \right| \leqslant Ce^{-\delta \tilde{t}},$$

$$\text{(iii)} \quad \max_{\tilde{M}} |\nabla^m \tilde{A}| \leqslant C_m e^{-\delta_m \tilde{t}}, \qquad m > 0.$$

All surfaces $\tilde{M}_{\tilde{t}}$ stay in a bounded region around $\mathfrak{O}$ since Lemma 9.3 implies a bound on the diameter of $\tilde{M}_{\tilde{t}}$. Moreover, by Lemma 9.2(ii) and (iii) we can pinch $\tilde{M}_{\tilde{t}}$ arbitrarily close between an interior and an exterior sphere if $\tilde{t}$ is large. This already shows that $\tilde{M}_{\tilde{t}}$ converges to a sphere in some weak sense. We have the evolution equation

$$\frac{\partial}{\partial \tilde{t}}\tilde{g}_{ij} = \frac{2}{n}\tilde{h}\tilde{g}_{ij} - 2\tilde{H}\tilde{h}_{ij},$$

and we conclude from Lemma 10.6(ii) and Lemma 8.2 that the metrics $\tilde{g}_{ij}(\tilde{t})$ converge uniformly to a positive definite metric $\tilde{g}_{ij}(\infty)$ as $\tilde{t} \to \infty$. By Lemma 10.6(iii) the metrics also converge in the C^∞-topology and thus $\tilde{g}_{ij}(\infty)$ is smooth. Finally, $\tilde{g}_{ij}(\infty)$ is the metric of a sphere by Theorem 10.5. This completes the proof of Theorem 1.1.

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