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Flow equivalence of graph algebras

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Abstract. This paper explores the effect of various graphical constructions upon the associated graph C^*-algebras. The graphical constructions in question arise naturally in the study of flow equivalence for topological Markov chains. We prove that out-splittings give rise to isomorphic graph algebras, and in-splittings give rise to strongly Morita equivalent C^*-algebras. We generalize the notion of a delay as defined in (D. Drinen, *Preprint*, Dartmouth College, 2001) to form in-delays and out-delays. We prove that these constructions give rise to Morita equivalent graph C^*-algebras. We provide examples which suggest that our results are the most general possible in the setting of the C^*-algebras of arbitrary directed graphs.

1. *Introduction*

The purpose of this paper is to describe various constructions on a directed graph which give rise to equivalences between the associated graph C^*-algebras. The graphical constructions in question all have their roots in the theory of flow equivalence for topological Markov chains. Our results will unify the work of several authors over the last few years who have studied similar constructions for Cuntz–Krieger algebras and, more recently, graph C^*-algebras (see [**CK, MRS, Ash, D, DS, Br, B1**] amongst others).

The motivation for the graphical constructions we use lies in the theory of subshifts of finite type. A shift space (X, σ) over a finite alphabet $\mathcal{A}$ is a compact subset X of $\mathcal{A}^{\mathbb{Z}}$ invariant under the shift map σ. To a directed graph E with finitely many edges and no sources or sinks, one may associate a shift space X_E, called the edge shift of E, whose alphabet is the edge set of E (see [**LM**, Definition 2.2.5]). Edge shifts are examples of subshifts of finite type. Alternatively, to every square 0–1 matrix A with no zero rows or columns, one may associate a subshift of finite type X_A (see [**LM**, Definition 2.3.7]).

Two important types of equivalence between shift spaces are conjugacy and flow equivalence. Shift spaces $(\mathsf{X}, \sigma_\mathsf{X})$ and $(\mathsf{Y}, \sigma_\mathsf{Y})$ are conjugate ($\mathsf{X} \cong \mathsf{Y}$) if there is an isomorphism $\phi : \mathsf{X} \to \mathsf{Y}$ such that $\sigma_\mathsf{Y} \circ \phi = \phi \circ \sigma_\mathsf{X}$. The suspension of $(\mathsf{X}, \sigma_\mathsf{X})$ is

$$\mathsf{SX} := (\mathsf{X} \times \mathbb{R})/[(x, t+1) \sim (\sigma_\mathsf{X}(x), t)].$$

Shift spaces $(\mathsf{X}, \sigma_{\mathsf{X}})$ and $(\mathsf{Y}, \sigma_{\mathsf{Y}})$ are flow equivalent ($\mathsf{X} \sim \mathsf{Y}$) if there is a homeomorphism between SX and SY preserving the orientation of flow lines.

By [**LM**, Proposition 2.3.9], every subshift of finite type is conjugate to an edge shift X_E for some directed graph E. Since the edge connectivity matrix B_E of E is a 0–1 matrix such that $\mathsf{X}_E \cong \mathsf{X}_{B_E}$, every subshift of finite type is conjugate to a shift described by a 0–1 matrix. Conversely, every shift described by a 0–1 matrix A is conjugate to an edge shift: let E_A be the directed graph with vertex connectivity matrix A, then X_{E_A} is conjugate to X_A (see [**LM**, Exercise 1.5.6, Proposition 2.3.9]). Hence, subshifts of finite type are edge shifts or shifts associated to 0–1 matrices.

Conjugacy and flow equivalence for subshifts of finite type may be expressed in terms of 0–1 matrices: an elementary strong shift equivalence between square 0–1 matrices A, B is a pair (R, S) of 0–1 matrices such that $A = RS$ and $B = SR$. We say A and B are strong shift equivalent if there is a chain of elementary strong shift equivalences from A to B. From [**W**, Theorem A] (see also [**LM**, Theorem 7.2.7]), $\mathsf{X}_A \cong \mathsf{X}_B$ if and only if A and B are strong shift equivalent. By [**PS**], $\mathsf{X}_A \sim \mathsf{X}_B$ if and only if A and B are related via a chain of elementary strong shift equivalences and certain matrix expansions. Both these matrix operations have graphical interpretations: following [**LM**, Theorem 2.4.14 and Exercise 2.4.9], an elementary strong shift equivalence corresponds to either an in- or out-splitting of the corresponding graphs. Following [**D**, §3], the matrix expansions in [**PS**] correspond to an out-delay of the corresponding graph.

To a 0–1 matrix A with n non-zero rows and columns is associated a C^*-algebra generated by partial isometries $\{S_i\}_{i=1}^n$ with mutually orthogonal ranges satisfying

$$S_j^* S_j = \sum_{i=1}^{n} A(i, j) S_i S_i^*.$$

If the matrix A satisfies condition (I) the Cuntz–Krieger algebra $\mathcal{O}_A$ is unique up to isomorphism. Results about Cuntz–Krieger algebras may be expressed in terms of the underlying directed graph E_A associated to A (see [**EW**] and [**FW**], for instance). More recent results are expressed entirely in terms of E_A (see [**KPRR, KPR**] amongst others).

To a row-finite directed graph E with finitely many edges and no sources or sinks is associated the universal C^*-algebra, $C^*(E)$ generated by partial isometries $\{s_e : e \in E^1\}$ with mutually orthogonal ranges satisfying

$$s_e^* s_e = \sum_{s(f)=r(e)} s_f s_f^*.$$

If A is a square 0–1 matrix which satisfies condition (I) and E_A is the associated directed graph, then $\mathcal{O}_A \cong C^*(E_A)$ (see [**MRS**, Proposition 4.1]). On the other hand, if E satisfies condition (L) (every 1000 has an exit), then the associated edge connectivity matrix B_E satisfies condition (I) and $C^*(E) \cong \mathcal{O}_{B_E}$ (see [**KPRR**, Proposition 4.1]). There are similar equivalences between Cuntz–Krieger algebras associated to infinite 0–1 matrices which are row-finite and certain row-finite directed graphs (see [**PR**], [**BPRSz**, Theorem 3.1]).

By [**CK**, Proposition 2.17, Theorems 3.8 and 4.1], if A and B satisfy condition (I) and $\mathsf{X}_A \cong \mathsf{X}_B$, then $\mathcal{O}_A \cong \mathcal{O}_B$; moreover, if $\mathsf{X}_A \sim \mathsf{X}_B$, then $\mathcal{O}_A$ is stably isomorphic to $\mathcal{O}_B$. The aim of this paper is to show that the graphical procedures involved in flow equivalence and conjugacy for edge shifts may be applied to arbitrary graphs and give rise to isomorphisms or Morita equivalences of the corresponding graph C^*-algebras. Initial results in this direction were proved in [**D**] for graphs with no sinks and finitely many vertices: out-splittings lead to isomorphisms of the underlying graph C^*-algebras whilst the C^*-algebra of an in-split graph is isomorphic to the C^*-algebra of a certain out-delayed graph. Further partial results may be found in [**DS**] and [**Br**].

The paper is organized as follows. Section 2 describes the C^*-algebra of any directed graph and the gauge-invariant uniqueness result used to establish our results. Section 3 deals with out-split graphs, §4 with in- and out-delays and §5 with in-split graphs. Finally, §6 relates our results to those in [**B1**]. Our main results are:

(1) if E is a directed graph and F is a proper out-split graph formed from E, then $C^*(E) \cong C^*(F)$ (Theorem 3.2);

(2) if E is a directed graph and F is an out-delayed graph formed from E, then $C^*(E)$ is strongly Morita equivalent to $C^*(F)$ if and only if F arises from a proper out-delay (Theorem 4.2);

(3) if E is a directed graph and F is an in-delayed graph formed from E, then $C^*(E)$ is strongly Morita equivalent to $C^*(F)$ (Theorem 4.5);

(4) if E is a directed graph and F is an in-split graph formed from E, then $C^*(E)$ is strongly Morita equivalent to $C^*(F)$ if and only if F arises from a proper in-splitting (Corollary 5.4).

2. *The C^*-algebra of a directed graph*

Here we briefly set out some of the basic definitions and terminology which we use throughout this paper. A directed graph E consists of countable sets of vertices and edges E^0 and E^1 respectively, together with maps $r, s : E^1 \to E^0$ giving the direction of each edge. The maps r, s extend naturally to E^*, the collection of all finite paths in E. The graph E is called *row-finite* if every vertex emits a finite number of edges.

A Cuntz–Krieger E-family consists of a collection $\{s_e : e \in E^1\}$ of partial isometries with orthogonal ranges, and mutually orthogonal projections $\{p_v : v \in E^0\}$ satisfying

(i) $s_e^* s_e = p_{r(e)}$,

(ii) $s_e s_e^* \leq p_{s(e)}$,

(iii) if v emits finitely many edges then $p_v = \sum_{s(e)=v} s_e s_e^*$.

The graph C^*-algebra of E, $C^*(E)$ is the universal C^*-algebra generated by a Cuntz–Krieger E-family. An important property of a graph C^*-algebra is the existence of an action γ of $\mathbb{T}$, called the gauge action, which is characterized by

$$\gamma_z s_e = z s_e \quad \text{and} \quad \gamma_z p_v = p_v$$

where $\{s_e, p_v\} \subseteq C^*(E)$ is the canonical Cuntz–Krieger E-family and $z \in \mathbb{T}$. This gauge action is a key ingredient in the uniqueness theorem which we shall frequently use.

THEOREM 2.1. [**BHRSz**, Theorem 2.1] *Let E be a directed graph, $\{S_e, P_v\}$ be a Cuntz–Krieger E-family and $\pi : C^*(E) \to C^*(S_e, P_v)$ the homomorphism satisfying $\pi(s_e) = S_e$ and $\pi(p_v) = P_v$. Suppose that each P_v is non-zero, and that there is a strongly continuous action β of $\mathbb{T}$ on $C^*(S_e, P_v)$ such that $\beta_z \circ \pi = \pi \circ \gamma_z$ for all $z \in \mathbb{T}$. Then π is faithful.*

To apply Theorem 2.1, we exhibit a non-trivial Cuntz–Krieger E-family within a C^*-algebra B, which carries a suitable $\mathbb{T}$-action β.

Some results in this paper require the following result on the Morita equivalence of graph algebras. As in [**BHRSz**, Remark 3.1], we define the saturation $\Sigma H(S)$ of $S \subseteq E^0$ to be the union of the sequence $\Sigma_n(S)$ of subsets of E^0 defined inductively as follows:

$$\Sigma_0(S) := \{v \in E^0 \ : \ v = r(\mu) \text{ for some } \mu \in E^* \text{ with } s(\mu) \in S\}$$
$$\Sigma_{n+1}(S) := \Sigma_n(S) \cup \{w \in E^0 : 0 < |s^{-1}(w)| < \infty \text{ and } s(e) = w \text{ imply } r(e) \in \Sigma_n(S)\}.$$

We note that if E is row-finite, then $\Sigma H(S)$ is the saturation of the hereditary set $\Sigma_0(S)$ as defined in [**BPRSz**].

LEMMA 2.2. *Suppose that E is a directed graph, S a subset of E^0 and $\{s_e, p_v\}$ the canonical Cuntz–Krieger E-family. Let $P = \sum_{v \in S} p_v$. Then $P \in \mathcal{M}(C^*(E))$ and the corner $PC^*(E)P$ is full if and only if $\Sigma H(S) = E^0$.*

Proof. By [**PR**, Lemma 3.3.1], the sum $\sum_{v \in S} p_v$ converges to a projection $P \in \mathcal{M}(C^*(E))$. We claim that $PC^*(E)P \subseteq I_{\Sigma H(S)}$. Let $s_\mu s_\nu^*$ be a non-zero element of $PC^*(E)P$, then $s(\mu) \in S$ and so $p_{s(\mu)} \in I_{\Sigma H(S)}$. Thus, $s_\mu s_\nu^* = p_{s(\mu)} s_\mu s_\nu^* \in I_{\Sigma H(S)}$.

If $PC^*(E)P$ is full, then $I_{\Sigma H(S)} = C^*(E)$ and so $\Sigma H(S) = E^0$ by [**BHRSz**, §3]. Conversely, suppose that $\Sigma H(S) = E^0$ and $PC^*(E)P \subseteq I$ for some ideal I in $C^*(E)$. By [**BHRSz**, Lemma 3.2] $H_I = \{v : p_v \in I\}$ is a saturated hereditary subset of E^0 containing S and, hence, $\Sigma H(S)$. Thus, $C^*(E) = I_{\Sigma H(S)} \subseteq I$ and the result follows. □

3. *Out-splittings*

The following definitions are adapted from [**LM**, Definition 2.4.3]. Let $E = (E^0, E^1, r, s)$ be a directed graph. For each $v \in E^0$ which emits an edge, partition $s^{-1}(v)$ into disjoint non-empty subsets $\mathcal{E}_v^1, \dots, \mathcal{E}_v^{m(v)}$ where $m(v) \geq 1$ (if v is a sink, then we put $m(v) = 0$). Let $\mathcal{P}$ denote the resulting partition of E^1. We form the *out-split graph $E_s(\mathcal{P})$ from E using $\mathcal{P}$* as follows. Let

$$E_s(\mathcal{P})^0 = \{v^i : v \in E^0, 1 \leq i \leq m(v)\} \cup \{v : m(v) = 0\},$$
$$E_s(\mathcal{P})^1 = \{e^j : e \in E^1, 1 \leq j \leq m(r(e))\} \cup \{e : m(r(e)) = 0\},$$

and define $r_{E_s(\mathcal{P})}, s_{E_s(\mathcal{P})} : E_s(\mathcal{P})^1 \to E_s(\mathcal{P})^0$ for $e \in \mathcal{E}_{s(e)}^i$ by

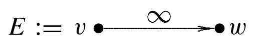
$$s_{E_s(\mathcal{P})}(e^j) = s(e)^i \text{ and } s_{E_s(\mathcal{P})}(e) = s(e)^i,$$
$$r_{E_s(\mathcal{P})}(e^j) = r(e)^j \text{ and } r_{E_s(\mathcal{P})}(e) = r(e).$$

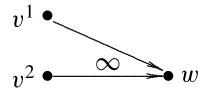
The partition $\mathcal{P}$ is *proper* if for every vertex v with infinite valency we have $m(v) < \infty$ and only one of the partition sets $\mathcal{E}_v^i$ is infinite.

Examples 3.1.

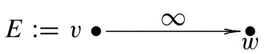
(i) The partitions which give rise to the out-splittings described in [**LM**, Figures 2.4.3–2.4.5] and [**D**, §4.1] are all examples of proper partitions.

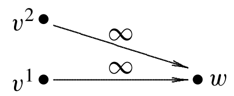
(ii) If we out-split at an infinite valence vertex, taking a partition $\mathcal{P}$ which has finitely many subsets, such as in



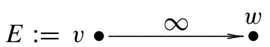
which splits at v to


then $\mathcal{P}$ is proper. If $\mathcal{P}$ has more than one infinite subset, such as in



which splits at v to $E_s(\mathcal{P}') :=$


then $\mathcal{P}$ is not proper. In this case, $C^*(E)$ is not Morita equivalent to $C^*(E_s(\mathcal{P}))$ since the latter has an additional ideal. If $\mathcal{P}$ has infinitely many subsets, such as in



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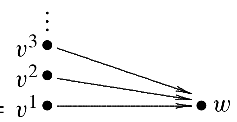


then $\mathcal{P}$ is not proper. Again, $C^*(E)$ and $C^*(E_s(\mathcal{P}))$ are not Morita equivalent: The former is non-simple and the latter simple.

THEOREM 3.2. *Let E be a directed graph, $\mathcal{P}$ a partition of E^1 and $E_s(\mathcal{P})$ the out-split graph formed from E using $\mathcal{P}$. If $\mathcal{P}$ is proper, then $C^*(E) \cong C^*(E_s(\mathcal{P}))$.*

Proof. Let $\{s_f, p_w : f \in E_s(\mathcal{P})^1, w \in E_s(\mathcal{P})^0\}$ be a Cuntz–Krieger $E_s(\mathcal{P})$-family. For $v \in E^0$ and $e \in E^1$, set $Q_v = p_v$ if $m(v) = 0$, $T_e = s_e$ if $m(r(e)) = 0$,

$$Q_v = \sum_{1 \le i \le m(v)} p_{v^i} \text{ if } m(v) \neq 0 \quad \text{and} \quad T_e = \sum_{1 \le j \le m(r(e))} s_{e^j} \text{ if } m(r(e)) \neq 0.$$

Because $\mathcal{P}$ is proper, $m(v) < \infty$ for all $v \in E^0$ and all of these sums are finite. We claim that $\{T_e, Q_v : e \in E^1, v \in E^0\}$ is a Cuntz–Krieger E-family in $C^*(E_s(\mathcal{P}))$.

The Q_vs are non-zero mutually orthogonal projections since they are sums of projections satisfying the same properties. The partial isometries T_e for $e \in E^1$ have mutually orthogonal ranges since they consist of sums of partial isometries with mutually orthogonal ranges. For $e \in E^1$, it is easy to see that $T_e^* T_e = Q_{r(e)}$ and $T_e T_e^* \le Q_{s(e)}$.

For $e \in E^1$ with $m(r(e)) \neq 0$, then since $r_{E_s(\mathcal{P})}(e^j) \neq r_{E_s(\mathcal{P})}(e^k)$ for $j \neq k$, we have

$$T_e T_e^* = \Bigg(\sum_{1 \le j \le m(r(e))} s_{e^j} \Bigg) \Bigg(\sum_{1 \le k \le m(r(e))} s_{e^k} \Bigg)^* = \sum_{1 \le j \le m(r(e))} s_{e^j} s_{e^j}^*. \tag{3.1}$$

If $m(r(e)) = 0$, then $T_e T_e^* = s_e s_e^*$. For $v \in E^0$ and $1 \le i \le m(v)$, put $\mathcal{E}_{1,v}^i = \{e \in \mathcal{E}_v^i : m(r(e)) \ge 1\}$ and $\mathcal{E}_{0,v}^i = \{e \in \mathcal{E}_v^i : m(r(e)) = 0\}$. If $v \in E^0$ has finite valency and is not a sink, then $s^{-1}(v) = \bigcup_{i=1}^{m(v)} \mathcal{E}_v^i$ and for $1 \le i \le m(v)$, we have

$$s_{E_s(\mathcal{P})}^{-1}(v^i) = \{e^j : e \in \mathcal{E}_{1,v}^i, 1 \le j \le m(r(e))\} \cup \{e \in E^1 : m(r(e)) = 0\}.$$

Hence, using (3.1), we may compute

$$Q_v = \sum_{1\le i\le m(v)} p_{v^i} = \sum_{1\le i\le m(v)} \sum_{e\in\mathcal{E}^i_{1,v}} \sum_{1\le j\le m(r(e))} s_{e^j}s^*_{e^j} + \sum_{1\le i\le m(v)} \sum_{e\in\mathcal{E}^i_{0,v}} s_e s_e^*$$
$$= \sum_{1\le i\le m(v)} \sum_{e\in\mathcal{E}^i_v} T_eT_e^* = \sum_{e:s(e)=v} T_eT_e^*,$$

completing the proof of our claim, since vertices $v \in E^0$ with $m(v) = 0$ are sinks.

Let $\{t_e, q_v\}$ be the canonical generators of $C^*(E)$, then by the universal property of $C^*(E)$ there is a homomorphism $\pi : C^*(E) \to C^*(E_s(\mathcal{P}))$ taking t_e to T_e and q_v to Q_v. To prove that π is onto, we show that the generators of $C^*(E_s(\mathcal{P}))$ lie in $C^*(T_e, Q_v)$. If $w = v^j \in E_s(\mathcal{P})^0$ has finite valency or w is a sink, then $p_w \in C^*(T_e, Q_v)$. If v^j has infinite valency, then without loss of generality we suppose $j = 1$. Since $\mathcal{P}$ is proper, it follows that $v^2, \dots, v^{m(v)}$ have finite valency, so $p_{v^2}, \dots p_{v^{m(v)}} \in C^*(T_e, Q_v)$ and, hence,

$$p_{v^1} = Q_v - p_{v^2} - \cdots - p_{v^{m(v)}} \in C^*(T_e, Q_v).$$

If $e^j \in E_s(\mathcal{P})^1$, then $m(r(e)) \neq 0$. Since $p_{r(e)^j} \in C^*(T_e, Q_v)$, we have $s_{e^j} = T_e p_{r(e)^j} \in C^*(T_e, Q_v)$. If $e \in E_s(\mathcal{P})^1$, then $m(r(e)) = 0$ and so $s_e = T_e \in C^*(T_e, Q_v)$.

Since π commutes with the canonical gauge action on each C^*-algebra and as $Q_v \neq 0$ for all $v \in E^0$, it follows from Theorem 2.1 that π is injective, and the result follows. □

Remarks 3.3.

(i) The *maximal out-splitting* $\tilde{E}$ of E is formed from a partition $\mathcal{P}$ of E^1 which admits no refinements. For a graph $E = (E^0, E^1, r, s)$ without sinks, $\tilde{E}$ is isomorphic to the dual graph $\widehat{E} = (E^1, E^2, r', s')$ (where $r'(ef) = f$ and $s'(ef) = e$). Since the out-splitting is maximal and there are no sinks, we have

$$\tilde{E}^0 = \{v^e : s(e) = v\} \quad \text{and} \quad \tilde{E}^1 = \{e^f : s(f) = r(e)\}.$$

The maps $v^e \mapsto e$ and $e^f \mapsto ef$ induce an isomorphism from $\tilde{E}$ to $\widehat{E}$. We define the dual graph $\widehat{E}$ of any directed graph E to be its maximal out-split graph $\tilde{E}$. Since a maximal out-splitting is proper if and only E is row-finite, we may now use Theorem 3.2 to generalize [**BPRSz**, Corollary 2.5] to any row-finite graph.

(ii) Brenken defines a graph algebra $G^*(E)$ which, under certain conditions, is isomorphic to $C^*(E)$. By [**Br**, Theorem 3.4], $G^*(E)$ and the C^*-algebra of its out-splitting are isomorphic if the graph satisfies a certain properness condition (see [**Br**, Definition 3.2]). This result only applies to $C^*(E)$ when E is row-finite.

(iii) For row-finite graphs with finitely many vertices and no sinks, a proof of Theorem 3.2 may be deduced from [**D**, §4.1]. In [**DS**, §6] similar results are proved for an 'explosion' which is a particular example of an out-splitting operation.

(iv) The C^*-algebra of an out-split graph is isomorphic to the C^*-algebra of an ultragraph (see [**T**]). Given a directed graph E and a partition $\mathcal{P}$, define the ultragraph $\mathcal{G}(\mathcal{P}) = (G^0, \mathcal{G}^1, r', s')$ as follows. Put $G^0 = E_s(\mathcal{P})^0$, $\mathcal{G}^1 = E^1$, $s'(e) = s(e)^i$ if $e \in \mathcal{E}^i_{s(e)}$, $r'(e) = r(e)$ if $m(r(e)) = 0$ and

$$r'(e) = \{r(e)^i : 1 \le i \le m(r(e))\} \quad \text{if } m(r(e)) \neq 0.$$

We claim that if $\mathcal{P}$ is proper, then $C^*(\mathcal{G}(\mathcal{P})) \cong C^*(E_s(\mathcal{P}))$. When $\mathcal{P}$ is proper, $\mathcal{G}^0$ is the set of finite subsets of $E_s(\mathcal{P})^0$. Let $\{p_A, s_e : A \in \mathcal{G}^0, e \in \mathcal{G}^1\}$ be a Cuntz–Krieger $\mathcal{G}(\mathcal{P})$-family. For $w \in E_s(\mathcal{P})^0$, set $Q_w = p_w$ and for $f^j \in E_s(\mathcal{P})^1$, put $T_{f^j} = s_f p_{r(f)^j}$. Then $\{Q_w, T_{f^j}\}$ is a Cuntz–Krieger $E_s(\mathcal{P})$-family in which each $Q_w \neq 0$. Let $\{t_{f^j}, q_w\}$ be the canonical generators of $C^*(E_s(\mathcal{P}))$, then by the universal property of $C^*(E_s(\mathcal{P}))$ there is a map $\pi : C^*(E_s(\mathcal{P})) \to C^*(\mathcal{G}(\mathcal{P}))$ sending t_{f^j} to T_{f^j} and q_w to Q_w. As each $A \in \mathcal{G}^0$ is finite, $C^*(T_{f^j}, Q_w)$ contains each generator of $C^*(\mathcal{G}(\mathcal{P}))$, so π is onto. By [**T**, §2] there is an appropriate action of $\mathbb{T}$ on $C^*(\mathcal{G}(\mathcal{P}))$, so π is injective by Theorem 2.1, proving our claim.

(v) Let Γ act freely on the edges of a row-finite graph E, then the induced Γ-action on the dual graph $\widehat{E}$ is free on its vertices. The isomorphism $C^*(E) \cong C^*(\widehat{E})$ is Γ-equivariant, so $C^*(E) \times \Gamma \cong C^*(\widehat{E}) \times \Gamma$ and by [**KQR**, Corollary 3.3] we have

$$C^*(E) \times \Gamma \cong C^*(\widehat{E}/\Gamma) \otimes \mathcal{K}(\ell^2(\mathbb{Z}_2)). \tag{3.2}$$

For instance, there is a free action of $\mathbb{Z}_2$ on the edges of graph B_2 which consists of a single vertex and two edges (the 'flip-flop automorphism' of $C^*(B_2) \cong \mathcal{O}_2$ described by [**Ar**]). Equation (3.2) shows that $\mathcal{O}_2 \times \mathbb{Z}_2$ is Morita equivalent to $\mathcal{O}_2$.

4. *Delays*

Let $E = (E^0, E^1, r, s)$ be a directed graph. A map $d_s : E^0 \cup E^1 \to \mathbb{N} \cup \{\infty\}$ such that:

(i) if $w \in E^0$ is not a sink then $d_s(w) = \sup\{d_s(e) : s(e) = w\}$;

(ii) if $d_s(x) = \infty$ for some x then either x is a sink or x emits infinitely many edges;

is called a *Drinen source-vector*. Note that only vertices are allowed to have an infinite d_s-value; moreover, if $d_s(v) = \infty$ and v is not a sink, then there are edges with source v and arbitrarily large d_s-value. From these data, we construct a new graph as follows. Let

$$d_s(E)^0 = \{v^i : v \in E^0, 0 \leq i \leq d_s(v)\}$$

and

$$d_s(E)^1 = E^1 \cup \{f(v)^i : 1 \leq i \leq d_s(v)\},$$

and for $e \in E^1$ define $r_{od}(e) = r(e)^0$ and $s_{od}(e) = s(e)^{d_s(e)}$. For $f(v)^i$, define $s_{od}(f(v)^i) = v^{i-1}$ and $r_{od}(f(v)^i) = v^i$. The resulting directed graph $d_s(E)$ is called the *out-delayed graph of E* for the Drinen source-vector d_s.

In the out-delayed graph the original vertices correspond to those vertices with superscript 0; the edge $e \in E^1$ is delayed from leaving $s(e)^0$ and arriving at $r(e)^0$ by a path of length $d_s(e)$. The Drinen source vector d_s is *strictly proper* if, whenever v has infinite valency, there is no v^i with infinite valency unless $i = d_s(v) < \infty$. A Drinen source-vector d_s which gives rise to an out-delayed graph $d_s(E)$ which may be constructed using a finite sequence of strictly proper Drinen source-vectors is said to be *proper*.

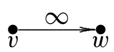
Examples 4.1.

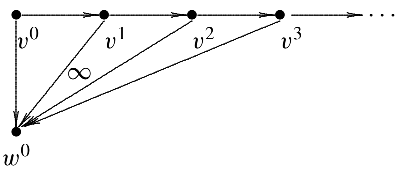
(i) The notion of an out-delay in the context of graph C^*-algebras was first introduced in [**CK**, §4] and subsequently generalized in [**D**]. The graphs shown in [**D**, §3.1] are all examples of out-delays for some proper Drinen source-vector where all the edges out of a given vertex are delayed by the same amount.

(ii) The Drinen–Tomforde desingularization of a graph described in [**DT**, Definition 2.2] is an example of a out-delay with a proper Drinen source-vector: if v has infinite valency, then the edges with source v may be written as $\{e_i : i \in \mathbb{N}\}$; we set $d_s(v) = \infty$ and $d_s(e_i) = i$ for $i \in \mathbb{N}$. If v has finite valency, then we set $d_s(v) = 0$ (and so $d_s(e) = 0$ for all $e \in s^{-1}(v)$). If v is a sink, then we put $d_s(v) = \infty$. The resulting graph $d_s(E)$ is row-finite with no sinks.

(iii) Putting $d_s(v) = \infty$ for a sink adds an infinite tail to the sink. If $d_s(v) = \infty$ for all sinks and $d_s(v) = 0$ for all vertices which emit edges, then $d_s(E)$ is the graph E with tails added to all sinks (cf. [**RS**, Lemma 1.4]).

(iv) Consider the following graph E:

$E :=$ 

with out-delay $d_s(E) :=$ 

The Drinen source-vector for this out-delay is not proper since vertex v_1 has infinite valency. Moreover, the C^*-algebra $C^*(d_s(E))$ is not Morita equivalent to $C^*(E)$ since the former C^*-algebra has two proper ideals and the latter only one.

THEOREM 4.2. *Let E be a directed graph and $d_s : E^0 \cup E^1 \to \mathbb{N} \cup \{\infty\}$ be a Drinen source-vector. Then $C^*(d_s(E))$ is strongly Morita equivalent to $C^*(E)$ if and only if d_s is proper.*

Proof. Without loss of generality, we may assume that $d_s : E^0 \cup E^1 \to \mathbb{N} \cup \{\infty\}$ is an essentially proper Drinen source-vector. Let $\{s_f, p_w : f \in d_s(E)^1, w \in d_s(E)^0\}$ be a Cuntz–Krieger $d_s(E)$-family. For $e \in E^1$ and $v \in E^0$ define $Q_v = p_{v^0}$ and

$$T_e = s_{f(s(e))^1} \dots s_{f(s(e))^{d_s(e)}} s_e \text{ if } d_s(e) \neq 0 \quad \text{and} \quad T_e = s_e \text{ otherwise.}$$

We claim that $\{T_e, Q_v : e \in E^1, v \in E^0\}$ is a Cuntz–Krieger E-family. The Q_v's are non-zero mutually orthogonal projections since the p_v's are. The T_e's are partial isometries with mutually orthogonal ranges since they are products of partial isometries with this property. For $e \in E^1$, it is routine to check that $T_e^* T_e = Q_{r(e)}$ and $T_e T_e^* \leq Q_{s(e)}$.

If $v \in E^0$ is neither a sink nor has infinite valence, then $d_s(v) < \infty$. If $d_s(v) = 0$, then we certainly have $Q_v = \sum_{s(e)=v} T_e T_e^*$. Otherwise, for $0 \leq j \leq d_s(v) - 1$, we have,

$$p_{v^j} = \sum_{s(e)=v, d_s(e)=j} s_e s_e^* + s_{f(v)^{j+1}} p_{v^{j+1}} s_{f(v)^{j+1}}^*, \tag{4.1}$$

and since we must have some edges with $s(e) = v$ and $d_s(e) = d_s(v)$, we have

$$p_{v^{d_s(v)}} = \sum_{s(e)=v, d_s(e)=d_s(v)} s_e s_e^*. \tag{4.2}$$

Using (4.1) recursively and (4.2) when $j = d_s(v) - 1$, we see that

$$Q_v = p_{v^0} = \sum_{s(e)=v, d_s(e)=0} T_e T_e^* + \dots + \sum_{s(e)=v, d_s(e)=d_s(v)} T_e T_e^* = \sum_{s(e)=v} T_e T_e^*,$$

and this establishes our claim.

Let $\{t_e, q_v\}$ be the canonical generators of $C^*(E)$, then by the universal property of $C^*(E)$ there is a homomorphism $\pi : C^*(E) \to C^*(d_s(E))$ which takes t_e to T_e and q_v to Q_v. It remains to show that $C^*(T_e, Q_v)$ is a full corner in $C^*(d_s(E))$.

Let α denote the strongly continuous $\mathbb{T}$-action satisfying, for $z \in \mathbb{T}$,

$$\alpha_z s_e = z s_e, \quad \alpha_z s_{f(v)^i} = s_{f(v)^i} \text{ for } 1 \le i \le d_s(v) \quad \text{and} \quad \alpha_z p_{v^i} = p_{v^i} \text{ for } 0 \le i \le d_s(v).$$

It is straightforward to check that $\pi \circ \gamma = \alpha \circ \pi$ where γ is the usual gauge action of $\mathbb{T}$ on $C^*(E)$ and it follows from Theorem 2.1 that π is injective.

By Lemma 2.2, the sum $\sum_{v \in E^0} p_{v^0}$ converges to a projection $P \in \mathcal{M}(C^*(d_s(E)))$. We claim that $C^*(T_e, Q_v)$ is equal to $PC^*(d_s(E))P$. Note that if $Ps_\mu s_\nu^* P = s_\mu s_\nu^* \neq 0$, then $s_{od}(\mu) = v^0$ and $s_{od}(\nu) = w^0$ for some $v, w \in E^0$, and $r_{od}(\mu) = r_{od}(\nu)$.

If $\mu = \nu = v^0$, then $s_\mu s_\nu^* = p_{v^0} = Q_v \in C^*(\{T_e, Q_v\})$. If $r_{od}(\mu) = u^0$ for some $u \in E^0$, then there are paths $\alpha, \beta \in E^*$ with $r(\alpha) = r(\beta) = u$ such that $s_\mu s_\nu^* = T_\alpha T_\beta^*$ and so $s_\mu s_\nu^* \in C^*(T_e, Q_v)$. Suppose now that $r_{od}(\mu) \notin E^0$. Then $r_{od}(\mu) = u^q$ for some $u \in E^0$ and $0 < q \le d_s(u)$ and we can write

$$s_\mu s_\nu^* = T_\alpha s_{f(u)^1} \cdots s_{f(u)^q} s_{f(u)^q}^* \cdots s_{f(u)^1}^* T_\beta^*$$

for some $\alpha, \beta \in E^*$. Suppose $q > 1$, then since d_s is proper, $f(u)^{q-1}$ has finite valency. If there are no edges in E with $s(e) = u$ and $d_s(e) = q - 1$, then

$$s_\mu s_\nu^* = T_\alpha s_{f(u)^1} \cdots s_{f(u)^{q-1}} s_{f(u)^{q-1}}^* \cdots s_{f(u)^1}^* T_\beta^*. \tag{4.3}$$

If there are a finite number of edges $e_1, \ldots, e_l \in E^1$ with $s(e_i) = u$ and $d_s(e_i) = q - 1$ for $i = 1, \ldots, l$, then

$$\begin{aligned} s_\mu s_\nu^* &= T_\alpha s_{f(u)^1} \ldots s_{f(u)^{q-1}} (p_{u^{q-1}} - s_{e_1} s_{e_1}^* - \cdots - s_{e_l} s_{e_l}^*) s_{f(u)^{q-1}}^* \ldots s_{f(u)^1}^* T_\beta^* \\ &= T_\alpha s_{f(u)^1} \ldots s_{f(u)^{q-1}} s_{f(u)^{q-1}}^* \ldots s_{f(u)^1}^* T_\beta^* - \sum_{i=1}^{l} T_\alpha T_{e_i} T_{e_i}^* T_\beta^*. \end{aligned} \tag{4.4}$$

Our new expression for $s_\mu s_\nu^*$ may now be analysed as in (4.3) or (4.4), reducing the value of q until all $s_{f(u)^i}$ terms are removed. Then

$$s_\mu s_\nu^* = T_\alpha T_\beta^* - \sum_{s(e)=u, d_s(e) \le q-1} T_{\alpha e} T_{\beta e}^* \in C^*(T_e, Q_v),$$

completing the proof of our claim.

Since $\Sigma H(\{v^0 : v \in E^0\}) = d_s(E)^0$, it follows from Lemma 2.2 that $PC^*(d_s(E))P$ is a full corner in $C^*(d_s(E))$ and, hence, $C^*(d_s(E))$ and $C^*(E)$ are strongly Morita equivalent.

If d_s is not proper, then there are at least two vertices $f(v)^i, f(v)^j$ with $0 \le i \le j \le d_s(v)$ emitting infinitely many edges. In this case there is an ideal generated by $p_{f(v)^i}$ in $C^*(d_s(E))$ which was not present in $C^*(E)$. □

We are grateful to Daniel Gow and Tyrone Crisp for pointing out an error in an earlier version of Theorem 4.2 (see also [**CG**]).

Remarks 4.3.

(i) Theorem 4.2 significantly generalizes the results in [**D**, §3.1]. Drinen shows a limited number of graph groupoid isomorphisms for row-finite graphs with finitely many vertices and no sinks in which each edge is equally delayed.

(ii) The desingularization of a non-row-finite graph is an example of an out-delay (see Examples 4.1(ii)). Moreover, any out-delay of a non-row-finite graph using a proper Drinen source vector with $d_s(v) = \infty$ for all vertices of infinite valency provides an example of a row-finite graph $d_s(E)$ whose C^*-algebra is Morita equivalent to $C^*(E)$. It follows by [**B2**, Corollary 4.6] that if E satisfies condition (K) (every vertex lies either on no loops or on at least two loops), then $Prim(C^*(E))$ is the primitive ideal space of some AF-algebra.

(iii) If $d_s : E^0 \cup E^1 \to \mathbb{N} \cup \{\infty\}$ is a Drinen source-vector, then E is a deformation retract of $d_s(E)$ (see [**St**, §3.3]). The construction of an out-delayed graph replaces each vertex v with $d_s(v) \geq 1$ by the tree $\{v^i : 0 \leq i \leq d_s(v), f(v)^i : 1 \leq i \leq d_s(v)\}$ which may be contracted to the root v^0 and identified with v (see also [**GT**, §1.5.5]). In particular $\pi_1(d_s(E)) \cong \pi_1(E)$ and the universal covering tree T of E is a deformation retract of the universal covering tree T' of $d_s(E)$. It follows that the boundary ∂T of T is homeomorphic to the boundary $\partial T'$ of T' (see [**KP**, §4]). Hence, the Morita equivalence between $C^*(E)$ and $C^*(d_s(E))$ could be obtained for row-finite graphs with no sinks using the Kumjian–Pask description of $C^*(E)$ as a crossed product of $C_0(\partial T)$ by $\pi_1(E)$ (see [**KP**, Corollary 4.14]).

We now turn our attention to in-delays where edges are delayed from arriving at their range. Let $E = (E^0, E^1, r, s)$ be a graph. A map $d_r : E^0 \cup E^1 \to \mathbb{N} \cup \{\infty\}$ satisfying

(i) if w is not a source, then $d_r(w) = \sup\{d_r(e) : r(e) = w\}$ and

(ii) if $d_r(x) = \infty$, then x is either a source or receives infinitely many edges

is called a *Drinen range-vector.* We construct a new graph $d_r(E)$ called the *in-delayed graph of E* for the Drinen range-vector d_r as follows:

$$d_r(E)^0 = \{v_i : v \in E^0, 0 \leq i \leq d_r(v)\}$$

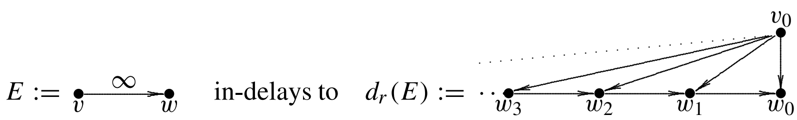
and

$$d_r(E)^1 = E^1 \cup \{f(v)_i : 1 \leq i \leq d_r(v)\},$$

and for $e \in E^1$, we define $r_{id}(e) = r(e)_{d_r(e)}$ and $s_{id}(e) = s(e)_0$. For $f(v)_i$, we define $s_{id}(f(v)_i) = v_i$ and $r_{id}(f(v)_i) = v_{i-1}$.

Examples 4.4.

(i) Consider the following graph E, with edges $\{e_i : i \geq 0\}$ from v to w. If we set $d_r(e_i) = i$, $d_r(v) = 0$ and $d_r(w) = \infty$, then



(ii) Observe that putting $d_r(v) = \infty$ for a source adds an infinite 'head' to the source. If $d_r(v) = \infty$ for all sources and $d_r(v) = 0$ for all vertices which receive edges, then $d_r(E)$ is the graph E with heads added to all sources (cf. [**RS**, Lemma 1.4]).

THEOREM 4.5. *If $d_r : E^0 \to \mathbb{N} \cup \{\infty\}$ is a Drinen range-vector, then $C^*(d_r(E))$ is strongly Morita equivalent to $C^*(E)$.*

Proof. Let $d_r : E^0 \cup E^1 \to \mathbb{N} \cup \{\infty\}$ be a Drinen range-vector and $\{s_e, p_v : e \in d_r(E)^0, v \in d_r(E)^1\}$ be a Cuntz–Krieger $d_r(E)$-family. For $v \in E^0$ let $Q_v = p_{v_0}$ and for $e \in E^1$, put

$$T_e = s_e s_{f(r(e))_{d_r(e)}} \dots s_{f(r(e))_1} \quad \text{if } d_r(e) \neq 0 \quad \text{and} \quad T_e = s_e \text{ otherwise.}$$

It is straightforward to check that $\{T_e, Q_v\}$ is a Cuntz–Krieger E-family in $C^*(d_r(E))$ in which all the projections Q_v are non-zero. Let $\{t_e, q_v\}$ be the canonical generators of $C^*(E)$, then by the universal property of $C^*(E)$ there is a homomorphism $\pi : C^*(E) \to C^*(d_r(E))$ satisfying $\pi(t_e) = T_e$ and $\pi(q_v) = Q_v$. It remains to show that $C^*(T_e, Q_v)$, the image of π, is a full corner in $C^*(d_r(E))$.

Let α be the strongly continuous $\mathbb{T}$-action α on $C^*(d_r(E))$ satisfying, for $z \in \mathbb{T}$,

$$\alpha_z(s_e) = z s_e, \alpha_z(s_{f(v)^i}) = s_{f(v)^i} \text{ for } 1 \leq i \leq d_r(v)$$

and

$$\alpha_z(p_{v^i}) = p_{v^i} \text{ for } 0 \leq i \leq d_r(v).$$

It is straightforward to check that $\pi \circ \gamma = \alpha \circ \pi$ where γ is the usual gauge action on $C^*(E)$ and it follows from Theorem 2.1 that π is injective.

By Lemma 2.2, the sum $\sum_{v \in E^0} p_{v_0}$ converges to a projection $P \in \mathcal{M}(C^*(d_r(E)))$. We claim that $C^*(\{T_e, Q_v\})$ is equal to $PC^*(d_r(E))P$. Note that if $Ps_\mu s_\nu^* P = s_\mu s_\nu^* \neq 0$, then $s_{id}(\mu) = v_0$ and $s_{id}(\nu) = w_0$ for some $v, w \in E^0$ and $r_{id}(\mu) = r_{id}(\nu)$.

If $r_{id}(\mu) \in E^0$, then $s_\mu s_\nu^* = T_\alpha T_\beta^*$ for some paths $\alpha, \beta \in E^*$ and, hence, $s_\mu s_\nu^* \in C^*(\{T_e, Q_v\})$. Suppose $r_{id}(\mu) \notin E^0$. Then $r_{id}(\mu) = r(e)_q$ for some $e \in E^1$ with $1 \leq q \leq d_r(e)$ and there are $\alpha, \beta \in E^*$ such that $s_\mu s_\nu^* = T_\alpha s_e p_{f(r(e))_{d_r(e)}} s_e^* T_\beta^*$ if $q = d_r(e)$ and

$$s_\mu s_\nu^* = T_\alpha s_e s_{f(r(e))_{d_r(e)}} \dots s_{f(r(e))_{q+1}} p_{f(r(e))_q} s^*_{f(r(e))_{q+1}} \dots s_e^* T_\beta^*,$$

otherwise. Since the vertices $f(r(e))_i$ for $2 \leq i \leq d_r(r(e))$ emit exactly one edge each, we have $p_{f(r(e))_i} = s_{f(r(e))_{i-1}} s^*_{f(r(e))_{i-1}}$ and, hence, we decrease q in the expression for $s_\mu s_\nu^*$ until we have $s_\mu s_\nu^* = T_{\alpha e} T_{\beta e}^* \in C^*(T_e, Q_v)$ as required.

It remains to check that the corner is full. To see this, we note that $\Sigma H(E^0) = d_r(E)^0$ and apply Lemma 2.2. Our result follows. □

Remarks 4.6.

(i) Using in-delays we can convert row-finite graphs into locally finite graphs (i.e. graphs where every vertex receives and emits finitely many edges). If E is row-finite and $v \in E^0$ receives edges $\{e_i : i \in \mathbb{N}\}$, set $d_r(v) = \infty$ and $d_r(e_i) = i$. If v is a source we put $d_r(v) = \infty$ and if v receives finitely many edges we set $d_r(v) = 0$. Evidently, $d_r : E^0 \cup E^1 \to \mathbb{N}$ is proper. The resulting graph $d_r(E)$ is then locally finite with no sources. Thus, combining Theorems 4.2 and 4.5, we can show that, for any graph E, there is a locally finite graph with no sinks and sources F such that $C^*(E)$ is strongly Morita equivalent to $C^*(F)$.

(ii) An in-delay at a vertex v with $d_r(v) \geq 1$ replaces $v \in E^0$ by the tree $\{v_i : 0 \leq i \leq d_r(v), f(v)_i : 1 \leq i \leq d_r(v)\}$ where v is identified with the leaf v_0. In combination with Remarks 4.3(iii) it seems that we may get similar Morita equivalence results if we replace vertices with more general trees (i.e. contractible graphs) where the original vertex lies within the tree itself.

(iii) Not every in-delay can be expressed as an out-delay. To see this, observe that for the graph E used in Examples 3.1 there can be no out-delay which corresponds to the in-delay described in Example 4.4. It should not be difficult to find examples where the graph contains no sources and sinks.

5. *In-splittings*

The following is adapted from [**LM**, Definition 2.4.7]: let $E = (E^0, E^1, r, s)$ be a directed graph. For each $v \in E^0$ with $r^{-1}(v) \neq \emptyset$ partition the set $r^{-1}(v)$ into disjoint non-empty subsets $\mathcal{E}_1^v, \ldots, \mathcal{E}_{m(v)}^v$ where $m(v) \geq 1$ (if v is a source then we put $m(v) = 0$). Let $\mathcal{P}$ denote the resulting partition of E^1. We form the *in-split graph* $E_r(\mathcal{P})$ from E using the partition $\mathcal{P}$ as follows. Let

$$E_r(\mathcal{P})^0 = \{v_i : v \in E^0, 1 \leq i \leq m(v)\} \cup \{v : m(v) = 0\},$$
$$E_r(\mathcal{P})^1 = \{e_j : e \in E^1, 1 \leq j \leq m(s(e))\} \cup \{e : m(s(e)) = 0\},$$

and define $r_{E_r(\mathcal{P})}, s_{E_r(\mathcal{P})} : E_r(\mathcal{P})^1 \to E_r(\mathcal{P})^0$ by

$$s_{E_r(\mathcal{P})}(e_j) = s(e)_j \text{ and } s_{E_r(\mathcal{P})}(e) = s(e)$$
$$r_{E_r(\mathcal{P})}(e_j) = r(e)_i \text{ and } r_{E_r(\mathcal{P})}(e) = r(e)_i \quad \text{where } e \in \mathcal{E}_i^{r(e)}.$$

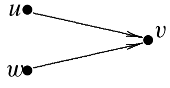
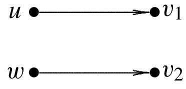
Partition $\mathcal{P}$ is *proper* if for every vertex v which is a sink or emits infinitely many edges we have $m(v) = 0, 1$. That is, we cannot in-split at a sink or vertex with infinite valency.

To relate the graph algebras of a graph and its in-splittings, we use a variation of the method introduced in [**D**, §4.2]: if $E_r(\mathcal{P})$ is the in-split graph formed from E using the partition $\mathcal{P}$, then we may define a Drinen range-vector $d_{r,\mathcal{P}} : E^0 \cup E^1 \to \mathbb{N} \cup \{\infty\}$ by $d_{r,\mathcal{P}}(v) = m(v) - 1$ if $m(v) \geq 1$ and $d_{r,\mathcal{P}}(v) = 0$ otherwise. For $e \in \mathcal{E}_i^{r(e)}$, we put $d_{r,\mathcal{P}}(e) = i - 1$. Hence, if v receives $n \geq 2$ edges, then we create an in-delayed graph in which v is given delay of size $m(v) - 1$ and all edges with range v are given a delay one less than their label in the partition of $r^{-1}(v)$. If v is a source or receives only one edge, then there is no delay attached to v.

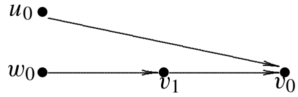
Examples 5.1.

(i) Examples of proper in-splittings are found in [**LM**, Figure 2.4.6] and [**D**, §4.2].

(ii) An in-splitting is not proper if we in-split at a sink, such as for

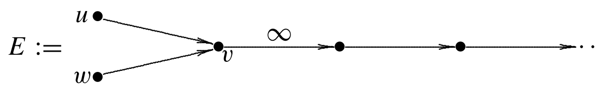
$E :=$  which in-splits at v to give $E_r(\mathcal{P}) :=$ 

The associated in-delayed graph is

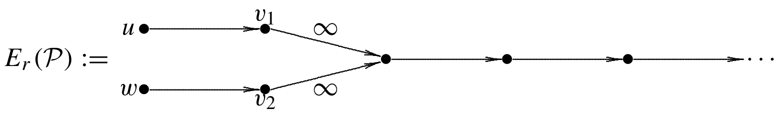
$d_{r,\mathcal{P}}(E) :=$ 

As $C^*(E_r(\mathcal{P}))$ has two ideals and $C^*(d_{r,\mathcal{P}}(E))$ one, they are not Morita equivalent.

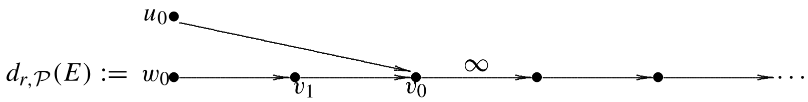
(iii) In-splittings at infinite valence vertices are not proper, such as in



which in-splits at v to give



The associated in-delayed graph is



In this case $C^*(E_r(\mathcal{P}))$ has two ideals, whereas $C^*(d_r(E))$ only has one. Thus, these algebras are not Morita equivalent.

Remark 5.2. If $\mathcal{P}$ is proper, then every vertex v which is either a sink or a vertex of infinite valency occurs only as v or v_1 in $E_r(\mathcal{P})$ and only as v_0 in $d_{r,\mathcal{P}}(E)$. In particular, if $\mathcal{P}$ is a proper partition and v is a sink or infinite valence vertex, then there are no edges of the form e_j for $j \geq 2$ with $s(e) = v$ in $E_r(\mathcal{P})$ and no edges of the form $f(v)_i$ in $d_{r,\mathcal{P}}(E)$.

THEOREM 5.3. *Let E be a directed graph, $\mathcal{P}$ a partition of E^1, $E_r(\mathcal{P})$ the in-split graph formed from E using $\mathcal{P}$ and $d_{r,\mathcal{P}} : E^0 \cup E^1 \to \mathbb{N} \cup \{\infty\}$ the Drinen range-vector defined as above. Then $C^*(E_r(\mathcal{P})) \cong C^*(d_{r,\mathcal{P}}(E))$ if and only if $\mathcal{P}$ is proper.*

Proof. Let $\{s_f, p_w : f \in d_{r,\mathcal{P}}(E)^1, w \in d_{r,\mathcal{P}}(E)^0\}$ be a Cuntz–Krieger $d_{r,\mathcal{P}}(E)$-family. To simplify our definitions, for $v \in E^0$ we put $s_{f(v)_0} = p_{v_0}$. For $e \in E_r(\mathcal{P})^1$, we define $T_e = s_e$. For $e_j \in E_r(\mathcal{P})^1$ with $1 \leq j \leq m(s(e))$, we define

$$T_{e_j} = s_{f(s(e))_{j-1}} \cdots s_{f(s(e))_1} s_e.$$

For $1 \leq i \leq m(v)$, define $Q_{v_i} = p_{v_{i-1}}$; if $m(v) = 0$, define $Q_v = p_{v_0}$. Then $\{T_g, Q_u : g \in E_r(\mathcal{P})^1, u \in E_r(\mathcal{P})^0\}$ is a Cuntz–Krieger $E_r(\mathcal{P})$-family with $Q_u \neq 0$ for all u.

Let $\{t_g, q_u\}$ be the canonical generators of $C^*(E_r(\mathcal{P}))$. By the universal property of $C^*(E_r(\mathcal{P}))$, there is a homomorphism $\pi : C^*(E_r(\mathcal{P})) \to C^*(d_{r,\mathcal{P}}(E))$ such that $\pi(t_g) = T_g$ and $\pi(q_u) = Q_u$. We claim that π is surjective, that is $\{T_g, Q_u\}$ generates $C^*(d_{r,\mathcal{P}}(E))$.

For $w \in d_r\mathcal{P}(E)^0$, we have $p_w \in C^*(T_g, Q_u)$ by definition. For $e \in E_r(\mathcal{P})^1$, we have $s_e = T_e \in C^*(T_g, Q_u)$. Since $\mathcal{P}$ is proper, by Remark 5.2 there are no edges in $d_{r,\mathcal{P}}(E)$ of the form $f(r(e))_j$ with $r(e)$ a sink. In particular, every edge $f(v)_j$ in $d_{r,\mathcal{P}}(E)$ is of the form $f(s(e))_j$ where $s(e)$ has finite valency. For $1 \leq j \leq m(s(e)) - 1 = d_{r,\mathcal{P}}(s(e))$,

$$T_{e_{j+1}} T_{e_j}^* = s_{f(s(e))_j} \cdots s_{f(s(e))_1} s_e s_e^* s_{f(s(e))_1} \cdots s_{f(s(e))_{j-1}}$$

and since $v = s(e)$ has finite valency, we have

$$\sum_{s(e)=v} T_{e_{j+1}} T_{e_j}^* = s_{f(s(e))_j} \dots s_{f(s(e))_1} \Big(\sum_{s(e)=v_0} s_e s_e^* \Big) s_{f(s(e))_1}^* \dots s_{f(s(e))_{j-1}}^*$$
$$= s_{f(s(e))_j} \dots s_{f(s(e))_1} p_{v_0} s_{f(s(e))_1}^* \dots s_{f(s(e))_{j-1}}^* = s_{f(s(e))_j}.$$

Then $s_{f(s(e))_j} \in C^*(T_g, Q_u)$ and our claim follows.

For $z \in \mathbb{T}$ define an action α on $C^*(d_{r,\mathcal{P}}(E))$ by $\alpha_z(p_v) = p_v$ for $v \in d_{r,\mathcal{P}}(E)^0$, $\alpha_z(s_e) = zs_e$ for $e \in E_r(\mathcal{P})^1$, and $\alpha_z(s_{f(v)_i}) = s_{f(v)_i}$ for $1 \leq i \leq d_{r,\mathcal{P}}(v)$. Since $\gamma \circ \pi = \pi \circ \alpha$ where γ is the usual gauge action on $C^*(E_r(\mathcal{P}))$, by Theorem 2.1 $C^*(d_{r,\mathcal{P}}(E)) \cong C^*(E_r(\mathcal{P}))$.

If $\mathcal{P}$ is not proper, then there is a non-trivial in-splitting at a sink or a vertex of infinite valency. The graph $E_r(\mathcal{P})$ will have at least one more sink or vertex of infinite valency than $d_{r,\mathcal{P}}(E)$ and, hence, $C^*(E_r(\mathcal{P}))$ will have more ideals than $C^*(d_{r,\mathcal{P}}(E))$. □

Applying Theorems 5.3 and 4.5, we have the following.

COROLLARY 5.4. *Let E be a directed graph, $\mathcal{P}$ a partition of E^1 and $E_r(\mathcal{P})$ the in-split graph formed from E using $\mathcal{P}$, then $C^*(E_r(\mathcal{P}))$ is strongly Morita equivalent to $C^*(E)$ if and only if $\mathcal{P}$ is proper.*

6. *Connections with strong shift equivalence*

In [**B1**] the following definition (which generalizes one given in [**Ash**]) was given for *elementary strong shift equivalence* of directed graphs which contain no sinks.

Definition 6.1. Let $E_i = (E_i^0, E_i^1, r^i, s^i)$ for $i = 1, 2$ be directed graphs. Suppose there is a directed graph $E_3 = (E_3^0, E_3^1, r_3, s_3)$ such that
(a) $E_3^0 = E_1^0 \cup E_2^0$ and $E_1^0 \cap E_2^0 = \emptyset$,
(b) $E_3^1 = E_{12}^1 \cup E_{21}^1$ where $E_{ij}^1 := \{e \in E_3^1 : s_3(e) \in E_i^0, r_3(e) \in E_j^0\}$ and
(c) for $i = 1, 2$ there are range and source-preserving bijections $\theta_i : E_i^1 \to E_3^2(E_i^0, E_i^0)$ where for $i \in \{1, 2\}$, $E_3^2(E_i^0, E_i^0) := \{\alpha \in E_3^2 : s_3(\alpha) \in E_i^0, r_3(\alpha) \in E_i^0\}$.
Then we say that E_1 and E_2 are *elementary strong shift equivalent* ($E_1 \sim_{ES} E_2$) via E_3.

The equivalence relation $\sim_S$ on directed graphs generated by elementary strong shift equivalence is called *strong shift equivalence*. Row-finite graphs which are strong shift equivalent have Morita equivalent C^*-algebras (see [**B1**, Theorem 5.2]).

PROPOSITION 6.2. *Let E be a directed graph with no sinks and $E_s(\mathcal{P})$ be an out-split graph formed from E using $\mathcal{P}$. Then $E \sim_{ES} E_s(\mathcal{P})$.*

Proof. One constructs a bipartite graph E_3 in the following manner. Let $E_3^0 = E^0 \cup E_s(\mathcal{P})^0$. For each $v \in E^0$, draw an edge e_v^i to the corresponding split vertices $v^i \in E_s(\mathcal{P})^0$ with $s(e_v^i) = v$ and $r(e_v^i) = v^i$. For each set of edges $\{e^i\}_{i=1}^{m(r(e))} \subseteq E_s(P)^1$ with $s(e^i) = v^i$ and $r(e^i) \in \{w^j\}_{j=1}^{m(w)}$, draw an edge $e_{v,w}^i$ with $s(e_{v,w}^i) = v^i$ and $r(e_{v,w}^i) = w$. The graph E_3 satisfies the conditions of Definition 6.1 and, hence, $E \sim_{ES} E_s(\mathcal{P})$ via E_3. □

In a similar manner, we may show the following.

PROPOSITION 6.3. *Let E be a directed graph with no sinks and $E_r(\mathcal{P})$ be an in-split graph formed from E using $\mathcal{P}$. Then $E \sim_{ES} E_r(\mathcal{P})$.*

Remark 6.4. Proposition 6.3 and [**B1**, Theorem 5.2] enable us to give another proof that the C^*-algebras of a row-finite directed graph and its in-splittings are Morita equivalent. We have analogous results for in-amalgamations and out-amalgamations as they are the reverse operations of in-splittings and out-splittings.

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