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FLOW FIELD AT COLLAPSE OF A CAVITY

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Abstract

Employing Rayleigh's method, the collapse of a vaporous bubble in an incompressible liquid with surface tension is analysed. The expressions of time versus radius, bubble-wall velocity and pressure developed at collapse are thus introduced.

Finally, the numerical solution of velocity and pressure field in the liquid surrounding the cavity is also given.

I. Process of Bubble Collapse

The collapse of a spherical bubble in an inviscid incompressible liquid at rest at infinity is discussed here. Suppose that the initial radius of bubble is R_0, the radius at time t is R, the bubble-wall velocity at time t is U and the radial distance from bubble center to any point is r. The velocity u and velocity potential at any point in liquid can be expressed as[1]

$$\phi=\frac{UR^2}{r};\qquad u=\frac{UR^2}{r^2} \tag{1.1}$$

The expression for the kinetic energy of the entire body of liquid at time t is given by

$$(KE)=\frac{\rho}{2}\int_R^\infty u^2\cdot 4\pi r^2dr=2\pi\rho U^2R^3 \tag{1.2}$$

where ρ is the density of liquid.

The external work done on the system as the bubble is collapsing from initial radius R_0 to radius R consists of:

The work of pressure p_∞ done on the system at infinity

$$W_{p\infty}=\frac{4\pi}{3}(R_0^3-R^3)p_\infty \tag{1.3}$$

The work of internal pressure p done on the system

$$W_p=\int_R^{R_0} 4\pi R^2pdR \tag{1.4}$$

Since bubble collapses very rapidly, the change of its volume can be regarded as an adiabatic process. Hence

$$p=p_1\left(\frac{R_0}{R}\right)^{3\gamma}+p_v-\frac{2\sigma}{R} \tag{1.5}$$

in which p_1 is air pressure inside the bubble at intial time ($R=R_0$, $\dot{R}=0$), $p_v=p_v(T)$ is vapor pressure inside the bubble, $\sigma=\sigma(T)$ is surface tension of liquid, γ is the gas constant (adiabatic), T is temperature of liquid.

An expression for the work of pressure p done on the system can be obtained from Eq. (1.4) by substituting (1.5) into (1.4) and performing the integration. This gives

$$W_p=\frac{4\pi p_1}{3(1-\gamma)}\left[R_0^3-R^3\left(\frac{R_0}{R}\right)^{3\gamma}\right]+\frac{4\pi p}{3}(R_0^3-R^3)-4\pi\sigma(R_0^2-R^2) \tag{1.6}$$

If the liquid is inviscid as well as incompressible, the work done appears as kinetic energy. Therefore, combining Eq. (1.2), Eq. (1.3) with Eq. (1.6), the following equation will be given

$$2\pi\rho U^2R^3=\frac{4\pi p_\infty}{3}(R_0^3-R^3)-\frac{4\pi p_1}{3(1-\gamma)}\left[R_0^3-R^3\left(\frac{R_0}{R}\right)^{3\gamma}\right]-\frac{4\pi p_v}{3}(R_0^3-R^3)+4\pi\sigma(R_0^2-R^2) \tag{1.7}$$

Which gives

$$U=\left\{\frac{2}{3\rho}(p_\infty-p_v)\left[\left(\frac{R_0}{R}\right)^3-1\right]-\frac{2p_1}{3(1-\gamma)\rho}\cdot\left[\left(\frac{R_0}{R}\right)^3-\left(\frac{R_0}{R}\right)^{3\gamma}\right]+\frac{2\sigma}{\rho R}\left[\left(\frac{R_0}{R}\right)^2-1\right]\right\}^{1/2} \tag{1.8}$$

Giving initial radius R_0, temperature T and pressure p_∞, the bubble-wall velocity U at any radius R can be obtained by means of Eq. (1.8) (in Fig. 1 and Table 1, the results of computaton are shown).

Table 1 The computation results of bubble-wall velocity ($R_0=3.556$mm., $T=15$℃ .)

	$p_\infty=0.03$ (kg/cm²)	$p_\infty=0.05$ (kg/cm²)	$p_\infty=0.1$ (kg/cm²)	$p_\infty=0.3$ (kg/cm²)	$p_\infty=0.5$ (kg/cm²)	$p_\infty=0.7$ (kg/cm²)	$p_\infty=1.0$ (kg/cm²)
R(mm)	U(m/s)	U(m/s)	U(m/s)	U(m/s)	U(m/s)	U(m/s)	U(m/s)
3.50	0.103	0.092	0.057	0.132	0.196	0.243	0.301
3.00	0.238	0.647	1.151	2.226	2.929	3.494	4.201
2.50	1.041	1.895	3.140	5.911	7.748	9.226	11.079
2.00	2.564	4.348	7 053	13.157	17.218	20.490	24.594
1.50	6.110	10.054	16.139	29.968	39.188	46.618	55.944
1.00	17.164	27.776	44.313	82.051	107.243	127.554	153.046
0.50	83.833	134.273	213.384	394.395	516.327	612.843	735.254
0.10	2640.57	4208.02	6674.16	12324.47	16100.9	19146.5	22969.7

It can be found from Fig. 1 that the bubble-wall velocity rises rapidly with the R reducing. If $R=0.1$mm. the bubble-wall velocity could reach the quantity of 10^5m/s. Alsc, in the whole process of bubble collapse, the increases of pressure p_∞ will lead to the considerable increases of collapse velocity.

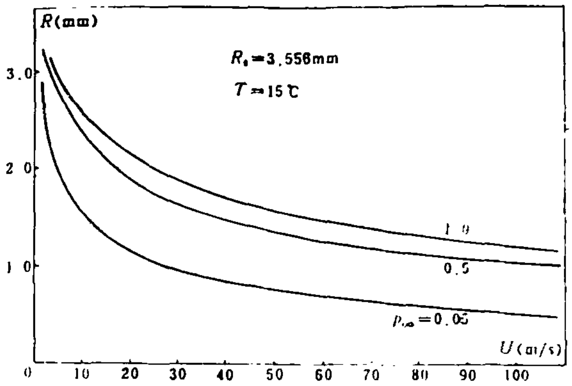

Fig. 1 $U-R$ curve

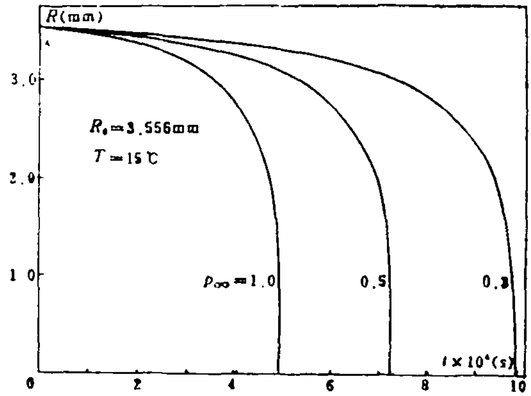

Fig. 2 $R-t$ curve

Table 2 The computation results of collapsing time ($R_0=3.556$mm., $T=15$℃ .)

	$p_\infty=0.03$ (kg/cm²)	$p_\infty=0.05$ (kg/cm²)	$p_\infty=0.1$ (kg/cm²)	$p_\infty=0.3$ (kg/cm²)	$p_\infty=0.5$ (kg/cm²)	$p_\infty=0.7$ (kg/cm²)	$p_\infty=1.0$ (kg/cm²)
R(mm)	t(s)	t(s)	t(s)	t(s)	t(s)	t(s)	t(s)
3.40	7.5673×10^{-4}	1.0365×10^{-3}	1.6043×10^{-3}	3.7761×10^{-4}	2.6982×10^{-4}	2.2121×10^{-4}	1.8114×10^{-4}
3.00	3.9455×10^{-3}	3.5365×10^{-3}	2.3821×10^{-3}	7.5042×10^{-4}	5.5021×10^{-4}	4.5534×10^{-4}	3.7530×10^{-4}
2.50	4.9154×10^{-3}	3.9904×10^{-3}	2.6474×10^{-3}	8.8959×10^{-4}	6.5619×10^{-4}	5.4426×10^{-4}	4.4931×10^{-4}
2.00	5.2279×10^{-3}	4.1684×10^{-3}	2.7560×10^{-3}	9.4755×10^{-4}	7.0045×10^{-4}	5.8145×10^{-4}	4.8028×10^{-4}
1.50	5.3589×10^{-3}	4.2468×10^{-3}	2.8045×10^{-3}	9.7365×10^{-4}	7.2039×10^{-4}	5.9821×10^{-4}	4.9424×10^{-4}
1.00	5.4111×10^{-3}	4.2788×10^{-3}	2.8245×10^{-3}	9.8441×10^{-4}	7.2863×10^{-4}	6.0513×10^{-4}	5.0001×10^{-4}
0.50	5.4268×10^{-3}	4.2886×10^{-3}	2.8306×10^{-3}	9.8773×10^{-4}	7.3116×10^{-4}	6.0727×10^{-4}	5.0179×10^{-4}
1.10	5.4287×10^{-3}	4.2897×10^{-3}	2.8314×10^{-3}	9.8813×10^{-4}	7.3147×10^{-4}	6.0752×10^{-4}	5.0200×10^{-4}
0.001	5.4287×10^{-3}	4.2897×10^{-3}	2.8314×10^{-3}	9.8813×10^{-4}	7.3147×10^{-4}	6.0752×10^{-4}	5.0200×10^{-4}

Suppose $U=dR/dt$, from Eq. (1.8), the time t required for a bubble to collapse from R_0 to R can be expressed as

$$t=\int_R^{R_0}\left\{\frac{2}{3\rho}\left(p_\infty-p_v\right)\left[\left(\frac{R_0}{R}\right)^3-1\right]-\frac{2p_1}{3(1-\gamma)\rho}\left[\left(\frac{R_0}{R}\right)^3-\left(\frac{R_0}{R}\right)^{3\gamma}\right]+\frac{2\sigma}{\rho R}\left[\left(\frac{R_0}{R}\right)^2-1\right]\right\}^{-1/2}dR \qquad (1.9)$$

By doing numerical integrations to formula (1.9), we found the relationship of time versus bubble radius under different p_∞ (See Fig. 2 or Table 2).

Because the collapse velocity increases with the increase of p_∞, the time of collapse on Fig. 2 will be cut short simultaneously. We see from Table 2 that under general pressure, the period of the bubble collapsing completely (i.e. from collapse to the state of free gas nucleus) is about 10^{-3} seconds. Therefore, the collapse of bubble is transient.

Fig. 3 illustrates under $p_\infty=0.5$ atm. the measured data and calculated results by Rayleigh's method and by the method given in this paper.

Fig. 3 shows that the results derived from the method mentioned by this paper are more coincident with the measured results than that from Rayleigh's method. It is well-known that the following assumptions are taken in Rayleigh's method, i.e. considering the bubble as an empty one and omitting the impacts of the pressure inside the bubble and the impacts of the surface tension of it. Because of these assumptions, Rayleigh's method has more divergence from the measureed results.

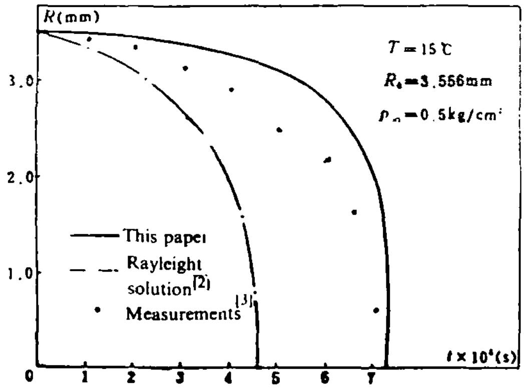

Fig. 3 Comparison of $R-t$ curve

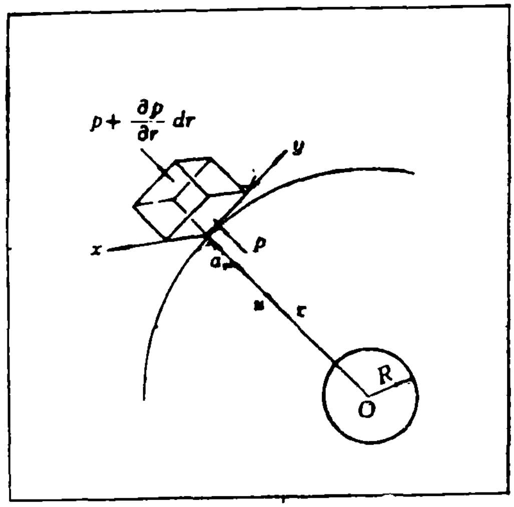

Fig. 4

II. Determination of Velocity Field and Pressure Field

1. Determination of velocity field

The bubble-wall velocity U and bubble radius R at any moment of collapse can be obtained by means of Fig. 1 and Fig. 2. Substituting this U and R into Eq. (1.1), we get the velocity of any point in liquid (its radial distance is r) at this moment.

$$u=\frac{R^2U}{r^2} \tag{2.1}$$

For cavitation zone, we can use the Biot-Savart theorem to determine the velocity of any point, which is similar to the determination of the induced velocity of line vortices.

2. Determination of pressure field

Consider an element in the liquid in Fig. 4, where the radial distance is r, the pressure is p, the velocity is u, the acceleration is a_r .

$$\because \quad a_r=-\frac{du}{dt}=-\frac{\partial u}{\partial t}-u\frac{\partial u}{\partial r}$$

On the element, we have

$$\rho dxdydr\cdot a_r=\left(p+\frac{\partial p}{\partial r}dr\right)dxdy-pdxdy$$

$$\therefore \quad a_r=-\frac{\partial u}{\partial t}-u\frac{\partial u}{\partial r}=\frac{1}{\rho}\frac{\partial p}{\partial r} \tag{2.2}$$

Then, by (1.1), we have $u=UR^2/r^2$. Substituting this equation in (2.2), we get

$$\frac{p_\infty R_0^3}{\rho r^2R^2}+\frac{2U^2R^4}{r^5}-\frac{2RU^2}{r^2}=\frac{1}{\rho}\frac{\partial p}{\partial r} \tag{2.3}$$

Integrating (2.3) with respect to radial distance from $r=\infty$ to r, the final result is

$$p=\left(\frac{2R}{r}-\frac{R^4}{2r^4}\right)U^2\rho-\frac{p_\infty R_0^3}{rR^2}+p_\infty \tag{2.4}$$

Let $R=R_0$ (i.e. the moment when collapse starts) then

$$p=p_\infty\left(1-\frac{R_0}{r}\right) \tag{2.5}$$

If the values of p_∞, R_0 and T are given, substituting (1.8) into (2.4), the value of pressure at any point in liquid at any time can be obtained.

For the outside of bubble-wall (where $r=R$), formula (2.4) can be simplified as

$$p=\left(2-\frac{1}{2}\right)U^2\rho-\frac{p_\infty R_0^3}{R^3}+p_\infty \tag{2.6}$$

If $p_\infty=0.05\text{kg/cm}^2$, $T=15°\text{C}$, $R_0=3.556\text{mm}$, $r=R=0.1\text{mm}$, from Table 1, we get $U^2=1.7707\times10^{13}\ \text{mm}^2/\text{s}^2$. Substituting these values into (2.5), we can see that the pressure p in the outside of the bubble-wall is 268668.8 kg/cm^2, when bubble collapses to $R=0.1\text{mm}$.

In the discussion above, the effects of viscosity is omitted. The results of computations indicate the appreciable effects of the viscosity in retarding both growth and collapse for viscosities much greater than that of water[4].

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