BACHELOR

Flow in porous media

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Bachelor Project

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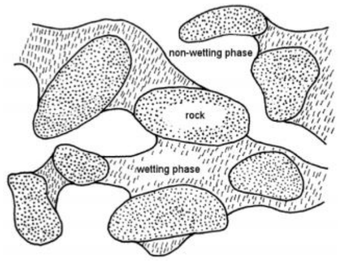

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1 Motivation

The flow in porous media has a lot of real life applications. For instance geostructures; oil recovery, CO_2 storage, or the bio system; drug release in human tissue and technology; fuel cells.
Various partial differential equations (PDEs) describe the flow in porous media. These PDEs describe a velocity field for a certain liquid or gas flowing through the porous medium. Some suitable applications for these PDEs are:

- Oil recovery; deep beneath the surface, there is a lot of oil. Oil is often used to fuel our cars, trucks and planes. To recover the oil, we need to inject a certain fluid into the oil source. The fluid flows through the source and stimulates the oil to flow as well. When the oil is in motion, it will be easier to recover it.
  Many scientists belief there would not be enough oil for the human population over time. But if we analyze the flow, we might come to a method to recover more oil from the source.
- CO_2-sequestration; the greenhouse effect is a hot topic nowadays. This effect is empowered by the emission of CO_2 from our cars, trucks, planes and industries into the atmosphere. If we reduce the amount of CO_2 in the atmosphere, the greenhouse effect reduces and thereby the climate change reduces in speed. As we focus on storing CO_2 in the subsurface. Analyzing the gas flow in the specific porous medium might answer the question where to store the gas best.
- Healing; people get sick. To recover we use medicinal drugs. But how progresses drug release in human tissue? The blood, including drugs, flows through the porous human tissue. Analyzing this flow might give us the reason to use other drugs and maybe take them at an other, more suitable, time.

These are just a few of the many applications for these PDEs. We will explain some simplifications of the models constructed over the years in the next section. Our goal in this project is to find an explicit solution for a certain simplified case and construct a test for some numerical codes.

Abstract

The flow in porous media is hard to describe. Based on a representative elementary volume, we use conservation laws of porous media to construct one-phase models (which models the flow through a porous medium where only one liquid or gas is present) and two-phase models. Both lead to the same partial differential equation for the saturation of a phase, which has an equilibrium and a non-equilibrium form. We analytically solve the equilibrium form using similarity solutions, this gives us useful results. For the non-equilibrium form we use a numerical approach to find a similarity solution. With the results we can say how the water distributes in some porous media.

2 Introduction

In this project we work on the Darcy scale. It is also called the representative elementary volume (REV) scale. On the right, we can see an example of such a Darcy scale volume with some voids filled with two phases (wetting and non-wetting) in a solid matrix.
In Figure 1 the porosity is defined as the ratio of the volume of the voids to the total volume

$$\phi = \frac{V_{\text{voids}}}{V_{\text{REV}}} \qquad (\phi \in [0,1])\,. \tag{2.1}$$

In the figure we also see a wetting phase and a non-wetting phase. A phase is a fluid or gas, different phases can be of the same substance (for example liquid water and water steam). It is enough to remember that 'wetting' and 'non-wetting' denote different phases.

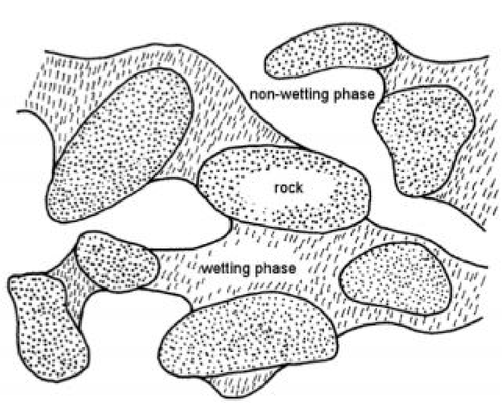

Figure 1: The Darcy scale

The saturation of a phase is the ratio of the volume of that phase to the volume of the voids, i.e.,

$$S_w = \frac{V_w}{V_{\text{voids}}} \qquad (S_w \in [0,1]) \tag{2.2}$$

for the wetting phase. Here it is obvious that for the two-phase model $S_w + S_n = 1$, for the wetting and non-wetting phase. In this project the saturation is an unknown function of space and time. Finding this function is the main purpose of the project. We have found some laws and equations to help us on our quest. The flux is the amount of a phase going in or out of the medium at some point per unit time. According to Darcy's law, the flux depends on the capillary pressure, the viscosity and the relative permeability of the phase. The flux also depends on the hydraulic conductivity. The relative permeability is defined as the ratio of the effective permeability K_α ($[m^2s^{-1}]$) to the intrinsic permeability K_i ($[m^2s^{-1}]$),

$$k_{r\alpha} = \frac{K_\alpha}{K_i}, \qquad \text{for the phase } \alpha\,. \tag{2.3}$$

When the capillary pressure and the relative permeability are large, the flux would be large. Concluding these variables enhance each other. On the other hand if the viscosity is large, the flux is small, it is like pushing honey or water through a tube. These relations return in a more general notation of Darcy's law

$$q = \frac{-k}{\mu} \nabla P\,. \tag{2.4}$$

The variables can be found below, more explanation follows during the project. In this project, the hydraulic conductivity, viscosity and the porosity are assumed to be known constants. The capillary pressure and the relative permeability are experimental stated functions.
By conservation laws for porous media we can construct a partial differential equation for the saturation of a phase. The main purpose is to find a solution for this equation.

2.1 Table of variables

symbol	**definition**	**dimension**
q_α	the flux of a certain phase with the abbreviation α	$[\frac{m^2}{s}]$
K	the hydraulic conductivity	$[\frac{m}{s}]$
μ_α	the viscosity of phase α	$[\frac{kg}{ms}]$
p_α	the pressure of phase α	$[Pa \equiv \frac{kg}{ms^2}]$
p_c	the capillary pressure	$[Pa \equiv \frac{kg}{ms^2}]$
S_α	the saturation of the phase α	[-]
$k_{r\alpha}$	the relative permeability	[-]
ϕ	the porosity	[-]

3 Conservation laws for porous media

To simplify the model we only focus on one direction, the horizontal direction. There are two important laws for modeling the flow through porous media [1].

$$\begin{array}{lll} 1. \text{ Darcy's law} & : & q_\alpha = \frac{-K}{\mu_\alpha} k_{r\alpha}(S_\alpha)\partial_x p_\alpha \,, \\ 2. \text{ Law of conservation of mass} & : & \phi\partial_t(S_\alpha) + \partial_x(q_\alpha) = 0 \,. \end{array} \tag{3.1}$$

Combining these result in the PDE

$$\phi\partial_t S_\alpha = \partial_x \left(\frac{K}{\mu_\alpha} k_{r\alpha}(S_\alpha)\partial_x p_\alpha \right) \,. \tag{3.2}$$

There is a difference in the two-phase and the one-phase model for the capillary pressure (p_c).

$$\begin{array}{lll} \text{Two-phase} & : & p_c = p_{\alpha_2} - p_{\alpha_1} \,, \\ \text{One-phase} & : & p_c = -p_\alpha \,. \end{array} \tag{3.3}$$

Where for the two-phase model, α_1 is seen as the main fluid (the fluid we are the most interested in). α_2 is the other phase.
In the one-phase model $\alpha_1 = \alpha = w$, the wetting phase. The phase α_2 is air, it has endless mobility and the air-pressure is constant. This phase will be there when needed. That is why only the wetting phase is obtained in the capillary pressure.

The standard model, in equilibrium, is where the capillary pressure is a function of the saturation. In the non-equilibrium model, the capillary pressure is also a function of time derivative saturation (according to Hassanizadeh and Gray [2]). The phase α is called the wetting phase explicitly.

$$\begin{array}{lll} \text{Standard} & : & p_c = p_c(S_w) \,, \\ \text{Non-equilibrium} & : & p_c = p_c(S_w) + \tau(S_w)\partial_t S_w \,. \end{array} \tag{3.4}$$

4 One-phase models

In this project, we are most interested in the non-equilibrium model. The equilibrium model only tells us what happens when the situation reaches equilibrium. However, in real life we probably would not have the time to wait for this equilibrium state. In the work of DiCarlo [3], there is experimental evidence that non-equilibrium effects need to be considered. That is why it is more important for us to focus on the non-equilibrium model.

We assume K and μ_w to be positive real constants. We can rewrite the model equation (3.2) as follows

$$\phi \partial_t S_w = \frac{K}{\mu_w} \partial_x \left(k_{rw}(S_w) \partial_x p_c \right) . \tag{4.1}$$

Now we put the model in dimensionless form by introducing the reference quantities for length L, time T, pressure P, and τ $\tilde{\tau}$ and (re)define

$$u := S_w, \qquad \tilde{x} := \frac{x}{L}, \qquad \tilde{t} := \frac{t}{T}, \qquad \tilde{p_c} := \frac{1}{P} p_c, \qquad \tilde{\tau} := \frac{\tau}{\tilde{\tau}} . \tag{4.2}$$

Then (4.1) will become

$$\frac{\phi}{T} \partial_t u = \frac{K}{\mu_w} \frac{P}{L^2} \partial_x \left(k_{rw}(u) \partial_x p_c \right) . \tag{4.3}$$

The two models for the capillary pressure also change

$$\begin{aligned} &\text{Standard} && : \quad p_c = p_c(u), \\ &\text{Non-equilibrium} && : \quad p_c = p_c(u) + \tfrac{\tilde{\tau}}{P} \tfrac{1}{T} \tau(u) \partial_t u . \end{aligned} \tag{4.4}$$

We will focus on the non-equilibrium model, which result in the PDE

$$\partial_t u = N_c \partial_x \left[k_{rw}(u) \partial_x \left(p_c(u) + \tau_c \tau(u) \partial_t u \right) \right] , \tag{4.5}$$

with the real constants

$$N_c := \frac{TKP}{\mu_w L^2 \phi} \quad \text{and} \quad \tau_c := \frac{\tilde{\tau}}{PT} . \tag{4.6}$$

4.1 Elimination of N_c from equation (4.5)

We will eliminate the N_c from equation (4.5) by redefining

$$\tilde{t} := \frac{t}{N_c} , \qquad \tilde{x} := \frac{x}{N_c} , \qquad \tilde{\tau_c} := \frac{\tau_c}{N_c} . \tag{4.7}$$

This gives us

$$\frac{\partial}{\partial t}(-) = \frac{1}{N_c} \frac{\partial}{\partial \tilde{t}}(-) , \qquad \frac{\partial}{\partial x}(-) = \frac{1}{N_c} \frac{\partial}{\partial \tilde{x}}(-) , \qquad \tau_c = N_c \tilde{\tau_c} . \tag{4.8}$$

When we substitute this in the equation, we get

$$\begin{aligned} \frac{1}{N_c} \frac{\partial}{\partial \tilde{t}} u &= N_c \frac{1}{N_c} \frac{\partial}{\partial \tilde{x}} \left[k_{rw}(u) \frac{1}{N_c} \frac{\partial}{\partial \tilde{x}} \left(p_c(u) + N_c \tilde{\tau_c} \tau(u) \frac{1}{N_c} \frac{\partial}{\partial \tilde{t}} u \right) \right] \\ &= \frac{1}{N_c} \frac{\partial}{\partial \tilde{x}} \left[k_{rw}(u) \frac{\partial}{\partial \tilde{x}} \left(p_c(u) + \tilde{\tau_c} \tau(u) \frac{\partial}{\partial \tilde{t}} u \right) \right] \end{aligned} \tag{4.9}$$

and by this we have

$$\frac{\partial}{\partial \tilde{t}} u = \frac{\partial}{\partial \tilde{x}} \left[k_{rw}(u) \frac{\partial}{\partial \tilde{x}} \left(p_c(u) + \tilde{\tau_c} \tau(u) \frac{\partial}{\partial \tilde{t}} u \right) \right] . \tag{4.10}$$

Now, we redefine

$$t = \tilde{t} , \qquad x = \tilde{x} , \qquad \tau_c = \tilde{\tau_c} \tag{4.11}$$

and find

$$\partial_t u = \partial_x \left[k_{rw}(u) \partial_x \left(p_c(u) + \tau_c \tau(u) \partial_t u \right) \right] . \tag{4.12}$$

This equation has one real parameter τ_c. We only assume it is finite.

5 Two-phase models

For the two-phase models we will explicitly use the water and oil (wetting and non-wetting was also possible). We want to describe the saturation of two phases now. We assume there are no other phases in the homogeneous porous medium. This result in four equations

$$\phi\partial_t S_w + \partial_x q_w = 0\,, \tag{5.1}$$
$$\phi\partial_t S_o + \partial_x q_o = 0\,, \tag{5.2}$$
$$S_w + S_o = 1\,, \tag{5.3}$$
$$p_c = p_o - p_w\,. \tag{5.4}$$

Taking the sum of (5.1) and (5.2), combined with (5.3), we get

$$\partial_x\left(q_w + q_o\right) = 0\,, \tag{5.5}$$

which means

$$q_w + q_o = \text{ constant in } x\,. \tag{5.6}$$

To simplify the model we assume the sum of the fluxes to be constant in time as well. We rewrite

$$q := q_w + q_o = \left[-K\frac{k_{rw}(S_w)}{\mu_w}\partial_x p_w - K\frac{k_{ro}(S_o)}{\mu_o}\partial_x(p_c + p_w)\right] \tag{5.7}$$

to

$$-K\left(\frac{k_{rw}(S_w)}{\mu_w} + \frac{k_{ro}(S_o)}{\mu_o}\right)\partial_x p_w = K\frac{k_{ro}(S_o)}{\mu_o}\partial_x p_c + q\,. \tag{5.8}$$

We substitute the flux from (3.1) in (5.1)

$$\phi\partial_t S_w - \frac{K}{\mu_w}\partial_x\left(k_{rw}(S_w)\partial_x p_w\right) = 0\,. \tag{5.9}$$

The combination of this expression and expression (5.8) leads to

$$\begin{aligned}
-K\tfrac{k_{rw}(S_w)}{\mu_w}\partial_x p_w &= \tfrac{k_{rw}(S_w)}{\mu_w}\frac{1}{\frac{k_{rw}(S_w)}{\mu_w}+\frac{k_{ro}(S_o)}{\mu_o}}\left[-K\left(\tfrac{k_{rw}(S_w)}{\mu_w}+\tfrac{k_{ro}(S_o)}{\mu_o}\right)\right]\partial_x p_w \\
\text{or}\qquad -K\tfrac{k_{rw}(S_w)}{\mu_w}\partial_x p_w &= f(S_w)\left[q + K\tfrac{k_{ro}(S_o)}{\mu_o}\partial_x p_c\right], \qquad (5.10)\\
\text{here}\qquad f(S_w) &= \tfrac{k_{rw}(S_w)}{\mu_w}\frac{1}{\frac{k_{rw}(S_w)}{\mu_w}+\frac{k_{ro}(S_o)}{\mu_o}}\,.
\end{aligned}$$

Equation (5.9) will change form to

$$\phi\partial_t S_w + \partial_x\left[qf(S_w) + K\frac{k_{ro}(S_o)}{\mu_o}f(S_w)\partial_x p_c\right] = 0\,. \tag{5.11}$$

Below we consider the case $q = 0$. Then (5.11) becomes

$$\phi\partial_t S_w = -\partial_x\left[K\frac{k_{ro}(S_o)}{\mu_o}f(S_w)\partial_x p_c\right]\,. \tag{5.12}$$

Here we also focus on the non-equilibrium model where

$$p_c := p_c(S_w) + \tau_c\tau(S_w)\partial_t S_w\,. \tag{5.13}$$

We can eliminate the constants from the equation analog to the one-phase model (as in **4.1**) and using (5.3) will get us an equation only depending on S_w

$$\partial_t S_w = \partial_x\left[k_{ro}(1 - S_w)f(S_w)\partial_x\left(p_c(S_w) + \tau_c\tau(S_w)\partial_t S_w\right)\right]. \tag{5.14}$$

6 Similarity solutions

The two equations (4.12), of the one-phase model, and (5.14), of the two-phase model, can be written as

$$\partial_t u = \partial_x \left[H(u) \partial_x \left(p_c(u) + \tau_c \tau(u) \partial_t u \right) \right] , \tag{6.1}$$

here u denotes the water saturation S_w and the function $H(u)$ has a different meaning for the one− and two-phase models

$$\begin{array}{ll} \text{One-phase} & H(u) := k_{rw}(u), \\ \text{Two-phase} & H(u) := k_{ro}(1-u)f(u). \end{array} \tag{6.2}$$

We will search for similarity solutions obtained for particular choices for $\tau(u)$, $p_c(u)$ and $H(u)$ (more suitable for one-phase models)

$$H(u) = u^m , \quad p_c(u) = u^n , \quad \tau(u) = u^l , \tag{6.3}$$

where m, n, and l are real constants and m, $n > 0$, typically in porous media literature. There are several methods for finding a solution to this equation. We assume the existence of similarity solutions. Existence and uniqueness of the so-called weak solutions to (6.1) are investigated in [4], [5], [6], [7], [8] and [9].
For the saturation we assume

$$u(x,t) = t^\alpha f(\eta) \quad \text{and } \eta = xt^\beta, \quad \beta < 0. \tag{6.4}$$

For now we only assume the function f to be finite and positive for every η.
We will considerate two cases. One where conservation of mass holds, which leads to $\alpha = \beta$, and one where $\alpha = 0$.

6.1 Conservation of mass

The problem we have, for a certain positive real constant C, is given by

$$\mathcal{P} = \begin{cases} \partial_t u = \partial_x \left[H(u) \partial_x \left(p_c(u) + \tau_c \tau(u) \partial_t u \right) \right] , & t > 0 , \quad -\infty < x < \infty , \\ u(x,0) = 0 , & x \neq 0 , \\ \displaystyle\int_{-\infty}^{\infty} u(x,t) \mathrm{d}x = 2C , & t > 0 . \end{cases} \tag{6.5}$$

The problem $\mathcal{P}$ can be interpreted as a water-injection at the point $x = 0$. The water distributes in the same manner in both the positive and the negative direction (so u is symmetric). Therefore we only focus on $x > 0$. Another explanation follows. Since

$$y := -x , \quad \frac{\partial}{\partial y}(-) = -\frac{\partial}{\partial x}(-) \tag{6.6}$$

provides the same equation as (6.1), $\mathcal{P}$ is symmetric in x. It is enough to consider the problem on the interval $(0, \infty)$.
We have u defined as in (6.4). By conservation of mass we get

$$\begin{aligned} C &= \int_{\mathbb{R}_+} u(x,t) \mathrm{d}x \\ &= \int_{\mathbb{R}_+} t^\alpha f(\eta) t^{-\beta} \mathrm{d}\eta \\ &= \int_{\mathbb{R}_+} t^{\alpha-\beta} f(\eta) \mathrm{d}\eta \end{aligned} \tag{6.7}$$

$$\tag{6.8}$$

This is true for every value of t, from which follows

$$\begin{aligned} C &= t^{\alpha-\beta}\int_{\mathbb{R}_+} f(\eta)\mathrm{d}\eta \\ &= t^{\alpha-\beta}C. \end{aligned} \tag{6.9}$$

By the above we conclude that $\alpha = \beta$ and continue assuming

$$u(x,t) = t^{\alpha} f(\eta) \quad \text{and } \eta = xt^{\alpha}, \quad \alpha < 0\,. \tag{6.10}$$

To simplify our work we split (6.1) up to three terms a, b and c.

$$\begin{aligned} \underbrace{\partial_t u}_{a} &= \underbrace{\partial_x \left[H(u)\partial_x p_c(u)\right]}_{b} + \underbrace{\partial_x \left[H(u)\partial_x \left(\tau_c \tau(u)\partial_t u\right)\right]}_{c\,.} \end{aligned} \tag{6.11}$$

First we construct a

$$\begin{aligned} \partial_t u &= \alpha t^{\alpha-1} f + \alpha t^{\alpha} x t^{\alpha-1} f' \\ &= \alpha t^{\alpha-1}(f + \eta f') \\ &= \alpha t^{\alpha-1}(\eta f)'\,. \end{aligned} \tag{6.12}$$

Secondly we construct term b, which is more difficult. Therefore we use

$$\frac{\partial}{\partial x}(-) \equiv t^{\alpha}\frac{d}{d\eta}(-) \tag{6.13}$$

in combination with (6.3) and find

$$\begin{aligned} \partial_x p_c(u) &= \partial_x u^n \\ &\equiv t^{\alpha}\left(t^{n\alpha} f^n\right)' \\ &= t^{(n+1)\alpha}\left(f^n\right)'\,. \end{aligned} \tag{6.14}$$

as we continue using (6.3) we get

$$\begin{aligned} H(u)\partial_x p_c(u) &= u^m \partial_x u^n \\ &\equiv t^{(m+n+1)\alpha} f^m \left(f^n\right)' \end{aligned} \tag{6.15}$$

and eventually we find

$$\begin{aligned} b &= \partial_x \left[H(u)\partial_x p_c(u)\right] \\ &\equiv t^{(m+n+2)\alpha}\left[f^m \left(f^n\right)'\right]'\,. \end{aligned} \tag{6.16}$$

Finally, we will also construct c step by step

$$\begin{aligned} \tau_c \tau(u)\partial_t u &= \tau_c u^l \partial_t u \\ &\equiv \tau_c t^{l\alpha} f^l \alpha t^{\alpha-1}(\eta f)' \\ &= \tau_c \alpha t^{(l+1)\alpha-1} f^l (\eta f)'\,, \end{aligned} \tag{6.17}$$

then the second step

$$\begin{aligned} \partial_x(\tau_c \tau(u)\partial_t u) &\equiv t^{\alpha}\left(\tau_c \alpha t^{(l+1)\alpha-1} f^l (\eta f)'\right)' \\ &= \tau_c \alpha t^{(l+2)\alpha-1}\left(f^l (\eta f)'\right)'\,, \end{aligned} \tag{6.18}$$

the third step leads us to

$$\begin{aligned} H(u)\partial_x(\tau_c\tau(u)\partial_t u) &= u^m\partial_x(\tau_c\tau(u)\partial_t u) \\ &\equiv \tau_c\alpha t^{(m+l+2)\alpha-1} f^m \left(f^l(\eta f)'\right)' \end{aligned} \tag{6.19}$$

and in the last step we find c

$$\begin{aligned} c &= \partial_x\left[H(u)\partial_x(\tau_c\tau(u)\partial_t u)\right] \\ &\equiv t^\alpha\left[\tau_c\alpha t^{(m+l+2)\alpha-1} f^m\left(f^l(\eta f)'\right)'\right]' \\ &= \tau_c\alpha t^{(m+l+3)\alpha-1}\left[f^m\left(f^l(\eta f)'\right)'\right]' . \end{aligned} \tag{6.20}$$

For (6.1), we now find

$$\begin{array}{ccccc} \underbrace{\partial_t u} & = & \underbrace{\partial_x\left[H(u)\partial_x p_c(u)\right]} & + & \underbrace{\partial_x\left[H(u)\partial_x\left(\tau_c\tau(u)\partial_t u\right)\right]} \\ a & = & b & + & c \\ \overbrace{\alpha t^{\alpha-1}(\eta f)'} & = & \overbrace{t^{(m+n+2)\alpha}\left[f^m\left(f^n\right)'\right]'} & + & \overbrace{\tau_c\alpha t^{(m+l+3)\alpha-1}\left[f^m\left(f^l(\eta f)'\right)'\right]'} . \end{array} \tag{6.21}$$

6.1.1 Equilibrium case ($\tau_c = 0$)

We consider the equilibrium model where $\tau_c = 0$. Here equation (6.1) becomes

$$\partial_t u = \partial_x\left(H(u)\partial_x p_c(u)\right) . \tag{6.22}$$

Using (6.10) we have only $a = b$ as in (6.21), which gives us

$$\alpha t^{\alpha-1}(\eta f)' = t^{(m+n+2)\alpha}\left[f^m\left(f^n\right)'\right]' . \tag{6.23}$$

If there exists a similarity solution, t has to be eliminated from the equation. We have

$$\alpha - 1 = (m+n+2)\alpha\,, \tag{6.24}$$

which leads to

$$\alpha = \frac{-1}{m+n+1} < 0\,. \tag{6.25}$$

Now we have

$$\alpha(\eta f)' = \left[f^m\left(f^n\right)'\right]' . \tag{6.26}$$

After integration we get

$$\alpha\eta f = f^m\left(f^n\right)' + B\,, \quad \text{for every } \eta\,, \tag{6.27}$$

here B is a real integration constant. We know $f(0)$ is bounded, thus

$$\lim_{\eta\to 0}\eta f(\eta) = 0\,. \tag{6.28}$$

Because f is symmetric we have the next proposition.

Proposition 1.

$$\lim_{\eta\to 0} f'(\eta) = 0\,. \tag{6.29}$$

Proof. By symmetry of problem $\mathcal{P}$ we have symmetry for f

$$f(\eta) = f(-\eta)\,. \tag{6.30}$$

This provides

$$\lim_{\eta\to 0} f'(\eta) = \begin{cases} \lim\limits_{h\searrow 0} \dfrac{f(0)-f(-h)}{h} & (f'(0-)) \, , \\ \lim\limits_{h\searrow 0} \dfrac{f(h)-f(0)}{h} & (f'(0+)) \, . \end{cases} \tag{6.31}$$

Then we have

$$\begin{aligned} f'(0+) &= f'(0-) \\ &= \lim_{h\searrow 0} \frac{f(0)-f(-h)}{h} \\ &= \lim_{h\searrow 0} \frac{f(0)-f(h)}{h} \\ &= \lim_{h\searrow 0} -\frac{f(h)-f(0)}{h} \\ &= -f'(0+) \, . \end{aligned} \tag{6.32}$$

$$\tag{6.33}$$

This means $f'(0+) = \lim_{\eta\to 0} f'(\eta) = 0$. □

By (6.27), (6.28) and proposition 1, we conclude $B = 0$ and

$$\alpha\eta f = n f^{m+n-1} f' \, . \tag{6.34}$$

This leaves us

$$\begin{aligned} f &= 0 & &, \text{ or} \\ \alpha\eta &= n f^{m+n-2} f' & &, \text{ if } f > 0 \, . \end{aligned} \tag{6.35}$$

Integrating if $1 < m+n \neq 2$ provides

$$\begin{aligned} & f = 0 \, , \quad \text{or} \\ & -\tfrac{1}{2(m+n+1)}\eta^2 = \tfrac{n}{m+n-1} f^{m+n-1} - D \, , \quad \text{if } f > 0 \text{ and } D = \tfrac{n}{m+n-1} f^{m+n-1}(0) \, . \end{aligned} \tag{6.36}$$

Finally we have

$$f(\eta) = \begin{cases} \sqrt[m+n-1]{\frac{m+n-1}{n}\left(D - \frac{1}{2(m+n+1)}\eta^2\right)} & , \, |\eta| < \sqrt{2(m+n+1)D} \, , \\ 0 & , \text{ otherwise.} \end{cases} \tag{6.37}$$

Now, we consider the case that $m + n = 2$. Which means that (6.34) becomes

$$-\frac{1}{3}\eta f = n f f' \, , \tag{6.38}$$

this clearly means

$$f(\eta) = \begin{cases} f(0) - \frac{1}{6n}\eta^2 & , \, |\eta| < \sqrt{6nf(0)} \, , \\ 0 & , \text{ otherwise.} \end{cases} \tag{6.39}$$

When $m + n = 1$, (6.34) becomes

$$-\frac{1}{2}\eta f = n f' \, , \tag{6.40}$$

here we have

$$f(\eta) = f(0) \exp\left(-\frac{1}{4n}\eta^2\right) , \text{ for every } \eta \, . \tag{6.41}$$

If $m+n<1$, we know by (6.34) that $f \neq 0$. This provides

$$\alpha\eta = n f^{m+n-2} f' \,. \tag{6.42}$$

Integrating gives us

$$-\frac{1}{2(m+n+1)}\eta^2 = \frac{n}{m+n-1} f^{m+n-1} - D\,, \text{ and } D = \frac{n}{m+n-1} f^{m+n-1}(0)\,. \tag{6.43}$$

This result in

$$f(\eta) = \left(\frac{1-m-n}{n}\frac{1}{2(m+n+1)}\eta^2 + f^{m+n-1}(0)\right)^{-\frac{1}{1-m-n}}\,, \text{ for every } \eta\,. \tag{6.44}$$

For every situation we have

$$u(x,t) = t^{\frac{-1}{m+n+1}} f(\eta) \text{ with } \eta = x t^{\frac{-1}{m+n+1}}\,, \text{ for } t>0 \text{ and } x \in \mathbb{R}\,. \tag{6.45}$$

In the next figures, the results are shown for different values of m and n. For each figure holds $0 < t \leq 250$, $-25 \leq x \leq 25$ and $f(0) = 1$. The figures are constructed in Mathematica (see **A**).

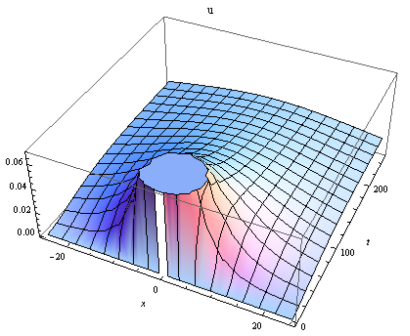

Figure 2: $u(x, t)$, $m = n = 0.25$.

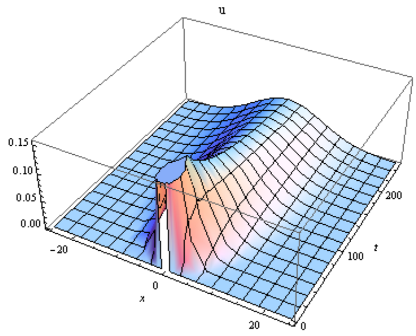

Figure 3: $u(x, t)$, $m = 0.75$, $n = 0.25$.

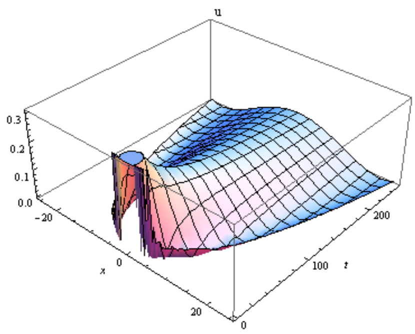

Figure 4: $u(x, t)$, $m = n = 0.75$.

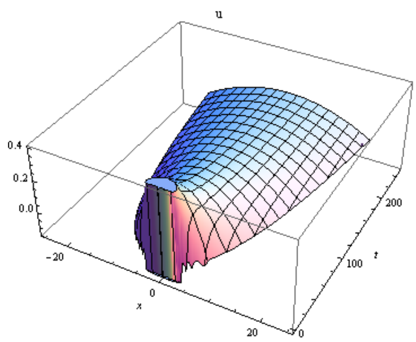

Figure 5: $u(x, t)$, $m = 0.5$, $n = 1.5$.

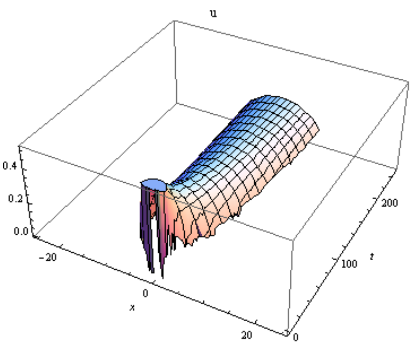

Figure 6: $u(x, t)$, $m = 1.75$, $n = 1.25$.

In Figure 2 holds $m + n < 1$, therefore we have similarity solution (6.44) for f. In this figure we see that u is defined for every x, which is true because of the solution we have found. Comparing to the other figures this figure has relatively high values for u when t and x are close to zero. But notice that there is no value of u greater than 0.05 shown.

We conclude in this situation that the water distributes extremely fast in every direction compared to the other four situations.

In Figure 3, u is also defined for every x, which can be confirmed by the similarity solution (6.41) found for this case ($m+n=1$). Actually we notice that this case is more or less the same case as a standard diffusion equation

$$\partial_t u = n\partial_{xx} u\,, \tag{6.46}$$

which has exactly the same solution.

We found similarity solution (6.37) for the case where $1 < m+n < 2$, which provides Figure 4. In this figure we see only illustrated a bounded area on which the function is defined. But notice that the similarity solution is zero for every η outside the bounded area, this is not illustrated. This also holds for the next two figures. Strange about this figure are the high values of u when t is close to zero and x is not. This is probably a result of rounding errors when we take a certain power of the difference of two numbers which are almost the same.

Figure 5 illustrates the saturation u when $m+n=2$. In this case we found the similarity solution (6.39) which has also a bounded region for η where $f > 0$. The similarity solution in this case is a simple expression. This results in a clean continuous function as we see in the figure. Notice that the saturation is zero for every x and t where no value is shown for u.

In Figure 6 holds $m+n > 2$. Therefore we have again similarity solution (6.37). We notice that of the five situations shown, this is the case where the water distributes very slowly in both directions. Because it distributes this slowly, the saturation is higher than in the other other situations for x and t around zero. Notice that by the similarity solution, it holds that the saturation should be zero for x and t outside the shown region for which u is illustrated. It looks like the solution is not continuous, this is a result of floating point errors.

6.1.2 Non-equilibrium ($\tau_c \neq 0$)

If there exists a similarity solution t must be eliminated from equation (6.21), which provides

$$\alpha - 1 = (m+n+2)\alpha = (m+l+3)\alpha - 1\,.$$

We conclude

$$\alpha = \frac{-1}{m+n+1} = \frac{-1}{n-l-1} < 0\,, \quad m = -2-l \quad \text{and} \quad n > l+1\,. \tag{6.47}$$

This result in

$$-\frac{1}{n-l-1}(\eta f)' = \left[f^m\left(f^n\right)'\right]' - \frac{\tau_c}{n-l-1}\left[f^m\left(f^l(\eta f)'\right)'\right]'\,.$$

The equation depends only on η, therefore we can integrate both sides and find

$$A - \frac{1}{n-l-1}\eta f = f^m\left(f^n\right)' - \frac{\tau_c}{n-l-1}f^m\left(f^l(\eta f)'\right)'\,,$$

here A is an integration constant, real. As we use $m = -l-2$, we find

$$Af^{l+2} - \frac{1}{n-l-1}\eta f^{l+3} = \left(f^n\right)' - \frac{\tau_c}{n-l-1}\left(f^l(\eta f)'\right)'\,. \tag{6.48}$$

6.1.3 Assumptions on the limits

We assume that at the moment before the injection of the water, the saturation of water equals zero at every point. We get the condition

$$\lim_{t\to 0} u(t,x) = 0\,,\ \text{for every } x > 0\,. \tag{6.49}$$

Here $x = 0$ is stated as the injection point. The water distributes in the same manner in both the positive and the negative direction (so u is symmetric). Therefore we only focus on $x > 0$.
We want to know what condition (6.49) means for the limits of the saturation and $f(\eta)$ (for every $x > 0$). Here we use that

$$u(x,t) = t^\alpha f(\eta) \quad \text{and } \eta = xt^\alpha, \quad \alpha < 0 \tag{6.50}$$

and

$$t^\alpha = \frac{\eta}{x}\,. \tag{6.51}$$

We get for the limits

$$\lim_{t\to 0} u(t,x) = \lim_{\eta\to\infty} \eta f(\eta)\frac{1}{x} = \frac{1}{x}\lim_{\eta\to\infty} \eta f(\eta) = 0\,. \tag{6.52}$$

We can conclude

$$\lim_{\eta\to\infty} f(\eta) = 0 \tag{6.53}$$

from the last limit. Therefore f has an asymptotic behavior as η goes to infinity, or there exists an $\eta_0 > 0$ from which f becomes the zero-function. In other words

$$\begin{array}{llll} \text{Asymptotic} & \eta \to \infty & : & f(\eta) \sim \eta^{-z}\,, \quad z > 1\,, \\ \text{Non-asymptotic} & \eta \to \eta_0 & : & f(\eta) \sim (\eta_0 - \eta)^z\,, \quad z > 0\,, \quad \text{and} \quad f(\eta \geq \eta_0) \equiv 0. \end{array} \tag{6.54}$$

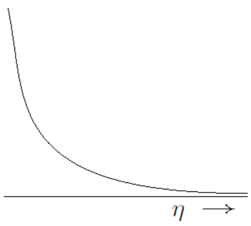

Figure 7: Asymptotic behavior for $\eta \to \infty$

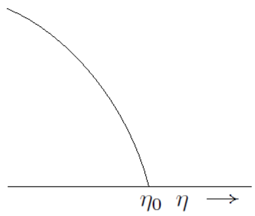

Figure 8: Non-asymptotic behavior for η around η_0

For both situations also follows that the function f is bounded and not zero for $\eta = 0$ and by proposition 1

$$\lim_{\eta\to 0} f'(\eta) = 0\,. \tag{6.55}$$

Proposition 2. *The integration constant A in equation (6.48) is zero.*

Proof. We will compute the values of the terms of equation (6.48). At first we compute

$$-\frac{1}{n-l-1}\eta f^{l+3}\,, \tag{6.56}$$

where f^{l+3} is bounded and not zero at $\eta = 0$, because f is bounded and not zero. This means the product equals zero.
Next we compute

$$(f^n)' = nf^{n-1}f'\,, \tag{6.57}$$

which also becomes zero at $\eta = 0$, because f to a certain power is bounded and $f'(0) = 0$.
Finally we compute the last term

$$\begin{aligned}
-\frac{\tau_c}{n-l-1}\left(f^l(\eta f)'\right)' &= -\frac{\tau_c}{n-l-1}\left(f^{l+1} + \eta f^l f'\right)' \\
&= -\frac{\tau_c}{n-l-1}\left(f^{l+1} + \frac{\eta}{l+1}\left(f^{l+1}\right)'\right)' \\
&= -\frac{\tau_c}{n-l-1}\left(\left(f^{l+1}\right)' + \frac{1}{l+1}\left(f^{l+1}\right)' + \frac{\eta}{l+1}\left(f^{l+1}\right)''\right) \\
&= -\frac{\tau_c}{n-l-1}\left(\frac{l+2}{l+1}\left(f^{l+1}\right)' + \frac{\eta}{l+1}\left(f^{l+1}\right)''\right). \qquad (6.58)
\end{aligned}$$

By (6.57) we can conclude that the first of above two terms equals zero at $\eta = 0$. As $f > 0$ and bounded and $f'(0) = 0$, we conclude f'' is bounded close to $\eta = 0$. By this we conclude the second product to equal zero as well at $\eta = 0$.
Now only left of equation (6.48) is

$$Af^{l+2} = 0\,, \qquad (6.59)$$

where f^{l+2} is bounded. Which means the integration constant must be zero ($A = 0$). □

From now on we follow

$$-\frac{1}{n-l-1}\eta f^{l+3} = (f^n)' - \frac{\tau_c}{n-l-1}\left(f^l(\eta f)'\right)'. \qquad (6.60)$$

6.1.4 Asymptotic behavior

We will investigate the situation where the function f has an asymptotic behavior for $\eta \to \infty$,

$$f(\eta) \sim \eta^{-z}\,, \quad z > 1 \text{ for } \eta \to \infty\,. \qquad (6.61)$$

The first term of equation (6.60) becomes

$$\begin{aligned}
-\frac{1}{n-l-1}\eta f^{l+3} &\sim -\frac{1}{n-l-1}\eta\left(\eta^{-z}\right)^{l+3} \\
&= -\frac{1}{n-l-1}\eta^{1-z(l+3)}\,, \qquad (6.62)
\end{aligned}$$

the first term on the right side becomes

$$\begin{aligned}
(f^n)' &\sim \left(\eta^{-zn}\right)' \\
&= -zn\eta^{-zn-1} \qquad (6.63)
\end{aligned}$$

and the last term will be

$$\begin{aligned}
-\frac{\tau_c}{n-l-1}\left(f^l(\eta f)'\right)' &\sim -\frac{\tau_c}{n-l-1}\left(\eta^{-zl}(\eta^{1-z})'\right)' \\
&= -\frac{\tau_c}{n-l-1}\left((1-z)\eta^{-(l+1)z}\right)' \\
&= \frac{\tau_c}{n-l-1}(1-z)(l+1)z\eta^{-(l+1)z-1}\,. \qquad (6.64)
\end{aligned}$$

The largest power on the right side dominates the right side. We know $n > l+1$ (because $\alpha < 0$. The second term dominates, which means

$$1 - z(l+3) \approx -(l+1)z - 1\,, \quad \text{for} \quad \eta \to \infty\,, \qquad (6.65)$$

concluding $z \approx 1$. But if $z \approx 1$, then the second term on the right side equals zero and is certainly not the dominating term. Therefore we can not say that f has this kind of asymptotic behavior.

6.1.5 Non-asymptotic

We will investigate the situation where the function f does not have an asymptotic behavior for $\eta \to \infty$. In other words we assume

$$\eta \to \eta_0 \quad : \quad f(\eta) \sim (\eta_0 - \eta)^z\,, \quad z > 0\,, \quad \text{and} \quad f(\eta \geq \eta_0) \equiv 0\,. \tag{6.66}$$

We will compute equation (6.60) term by term again. On the left side we have

$$\begin{aligned} -\frac{1}{n-l-1}\eta f^{l+3} &\sim -\frac{1}{n-l-1}\eta(\eta_0-\eta)^{z(l+3)} \\ &= \frac{1}{n-l-1}\Big((\eta_0-\eta)^{z(l+3)+1} - \eta_0(\eta_0-\eta)^{z(l+3)}\Big)\,. \end{aligned} \tag{6.67}$$

The first term on te right becomes

$$\begin{aligned} (f^n)' &\sim [(\eta_0-\eta)^{zn}]' \\ &= -zn(\eta_0-\eta)^{zn-1}\,. \end{aligned} \tag{6.68}$$

The last term is a bit more difficult (notice the sign changing at every time we derive) and is

$$\begin{aligned} -\frac{\tau_c}{n-l-1}\left(f^l(\eta f)'\right)' &\sim -\frac{\tau_c}{n-l-1}\left((\eta_0-\eta)^{zl}\left(\eta(\eta_0-\eta)^z\right)'\right)' \\ &= \frac{\tau_c}{n-l-1}\Big((\eta_0-\eta)^{zl}\left((\eta_0-\eta)^{z+1} - \eta_0(\eta_0-\eta)^z\right)'\Big)' \\ &= -\frac{\tau_c}{n-l-1}\left((\eta_0-\eta)^{zl}\left((z+1)(\eta_0-\eta)^z - z\eta_0(\eta_0-\eta)^{z-1}\right)\right)' \\ &= -\frac{\tau_c}{n-l-1}\Big((z+1)(\eta_0-\eta)^{(l+1)z} - z\eta_0(\eta_0-\eta)^{(l+1)z-1}\Big)' \\ &= \frac{\tau_c}{n-l-1}\Big(z(z+1)(l+1)(\eta_0-\eta)^{(l+1)z-1} - z((l+1)z-1)\eta_0(\eta_0-\eta)^{(l+1)z-2}\Big)\,. \end{aligned} \tag{6.69}$$

For this behavior we also check for the dominating term on the right side. The power of the dominating term on the right side has to be close to the power on the left side. Both the powers $(l+1)z-1$ and $(l+1)z-2$ lead to a negative z. We can say

$$z(l+3) \approx zn - 1\,, \text{ concluding } z \approx \frac{1}{n-l-3}\,. \tag{6.70}$$

Clearly $n > l+3$, because z has to be positive. Now consider the situation $l+3 < n < l+4$. Then we have $z > 1$ which means

$$\begin{aligned} \lim_{\eta\nearrow\eta_0} f'(\eta) &= \lim_{\eta\nearrow\eta_0} [(\eta_0-\eta)^z]' \\ &= \lim_{\eta\nearrow\eta_0} -z(\eta_0-\eta)^{z-1} = 0\,. \end{aligned} \tag{6.71}$$

Concluding we should have a smooth change to $f \equiv 0$ around η_0. Now we consider $n = l+4$, in other words $z \approx 1$. Which means

$$\lim_{\eta\nearrow\eta_0} f'(\eta) \sim \lim_{\eta\nearrow\eta_0} [\eta_0-\eta]' = -1\,. \tag{6.72}$$

Concluding we would not have a smooth change to $f \equiv 0$, but more a kind of a kink. At last we consider $n > l+4$. We have $0 < z < 1$, which means $z-1 < 0$ and find

$$\begin{aligned} \lim_{\eta\nearrow\eta_0} f'(\eta) &= \lim_{\eta\nearrow\eta_0} [(\eta_0-\eta)^z]' \\ &= \lim_{\eta\nearrow\eta_0} -z(\eta_0-\eta)^{z-1} = -\infty\,. \end{aligned} \tag{6.73}$$

Here we conclude absolutely no smooth change to $f \equiv 0$, but a change from vertical to horizontal. We will check below if we can find any numerical evidence for this behavior.

6.1.6 Numerical approach

By symmetry we only focus on $\eta \geq 0$. We try a numerical approach for (6.60), therefore we compute every term

$$-\frac{1}{n-l-1}\eta f^{l+3} = nf^{n-1}f' - \frac{\tau_c}{n-l-1}\left(l\eta f^{l-1}f' + (2+l)f^l f' + \eta f^l f''\right) . \tag{6.74}$$

For a moment we prefer α and rewrite the equation to

$$\alpha\eta f^{l+3} = nf^{n-1}f' + \alpha\tau_c\left(l\eta f^{l-1}f' + (2+l)f^l f' + \eta f^l f''\right) . \tag{6.75}$$

When f, η, α and τ_c are not zero we find

$$f''(\eta) = \frac{\alpha\eta f^{l+3}(\eta) - \left[nf^{n-1}f'(\eta) + \alpha\tau_c\left(l\eta f^{l-1}(\eta) + (2+l)f^l(\eta)\right)f'(\eta)\right]}{\alpha\tau_c\eta f^l(\eta)} . \tag{6.76}$$

We will use this as an approximation for f'' in our program. We choose a real constant $h > 0$, very small. Then we use the simple approximation for f' as follows

$$f'(\eta) \approx \frac{f(\eta+h) - f(\eta)}{h} . \tag{6.77}$$

At last we choose for f a second order Taylor approximation

$$f(\eta+h) \approx f(\eta) + \frac{f'(\eta)}{1!}h + \frac{f''(\eta)}{2!}h^2 , \tag{6.78}$$

which leads to

$$f(\eta+h) \approx f(\eta) + f'(\eta)h + \frac{1}{2}f''(\eta)h^2 . \tag{6.79}$$

The program we will use to numerically approach the solution of (6.60) is written in Matlab (see **B**). By iteration it calculates up-following values of the functions f, f' and f''. As f reaches zero the iteration stops. We need initial values for the three functions. We know $f(0)$ is bounded, therefore we assume $f(0) = 1$. By Proposition 1 we have $f'(0) = 0$. Now we only need an initial value for f''. We have Proposition 3.

Proposition 3.

$$\lim_{\eta\to 0} f''(\eta) = \frac{1}{\tau_c} . \tag{6.80}$$

Proof. We check f'' when $\eta \to 0$ and by (6.76) we find

$$\begin{aligned}
\lim_{\eta\to 0} f''(\eta) &= \lim_{\eta\to 0} \frac{\alpha\eta f^{l+3}(\eta) - \left[nf^{n-1}f'(\eta) + \alpha\tau_c\left(l\eta f^{l-1}(\eta)f'(\eta) + (2+l)f^l(\eta)f'(\eta)\right)\right]}{\alpha\tau_c\eta f^l(\eta)} \\
&= \lim_{\eta\to 0} \frac{\alpha\eta f^{l+3}(\eta)}{\alpha\tau_c\eta f^l(\eta)} \qquad \text{(Proposition 1)} \\
&= \lim_{\eta\to 0} \frac{\alpha\eta}{\alpha\tau_c\eta} \qquad (f(0) = 1) \\
&= \frac{1}{\tau_c} , \quad \left(\text{or} \quad \frac{1}{\tau_c}f^3(0) \quad \text{if} \quad f(0) \neq 1\right) .
\end{aligned} \tag{6.81}$$

□

Finally we get the initial system

$$\begin{aligned}
f(0) &= 1 , \\
f'(0) &= 0 , \\
f''(0) &= \frac{1}{\tau_c} .
\end{aligned} \tag{6.82}$$

We start making small steps of length h. This means in the j-th step we have approximations for $f(\eta = jh) \approx f_j$ and calculate approximations for $f(\eta + h = (j+1)h) \approx f_{j+1}$ as follows. We first calculate f_{j+1} by a second order Taylor approximation as in (6.79)

$$f_{j+1} = f_j + f_j' h + \frac{1}{2} f_j'' h^2 \,.$$

Then we calculate f_{j+1}' as in (6.77)

$$f_{j+1}' = \frac{f_{j+1} - f_j}{h} \,.$$

Afterwards we calculate f_{j+1}'' by (6.76)

$$f_{j+1}'' = \frac{\alpha h(j+1) f_{j+1}^{l+3} - \left[n f_{j+1}^{n-1} f_{j+1}' + \alpha \tau_c \left(lh(j+1) f_{j+1}^{l-1} + (2+l) f_{j+1}^{l} \right) f_{j+1}' \right]}{\alpha \tau_c h(j+1) f_{j+1}^{l}} \,.$$

The iteration will stop when $f < 0.000001$.
Before we start creating results we need to keep in mind that m, n, > 0 and $m = -l - 2$, therefore $l < -2$. We will check a few situations.

1. $\tau_c = -0.6 < 0$ and $l + 1 < n < l + 3$,
2. $\tau_c = -0.6 < 0$ and $l + 3 = n$,
3. $\tau_c = -0.6 < 0$ and $l + 3 < n < l + 4$,
4. $\tau_c = -0.6 < 0$ and $l + 4 = n$,
5. $\tau_c = -0.6 < 0$ and $l + 4 < n$,
6. $\tau_c = -2 < 0$ and $l + 4 < n$,
7. $\tau_c > 0$ and $l + 4 < n$.

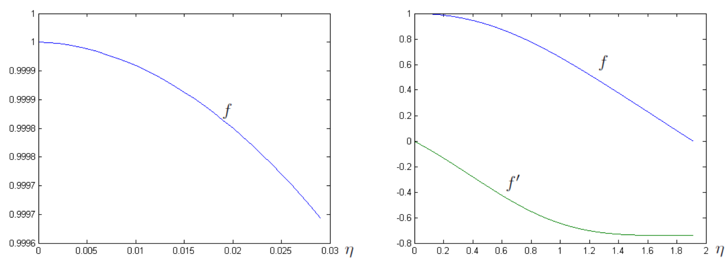

Figure 9: Starting- and global view of the numerical approximation of f (and f') for $\tau_c = -0.6$, $n = 0.3$ and $l = -2.3$ as in 1.

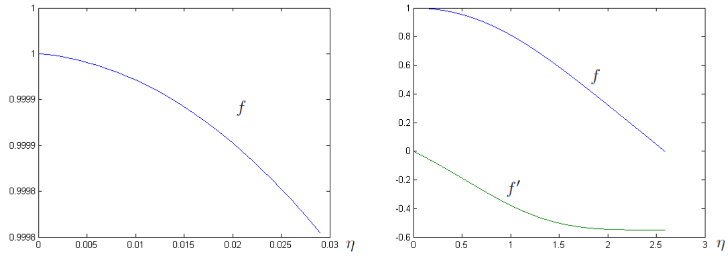

Figure 10: Starting- and global view of the numerical approximation of f (and f') for $\tau_c = -0.6$, $n = 0.7$ and $l = -2.3$ as in 2.

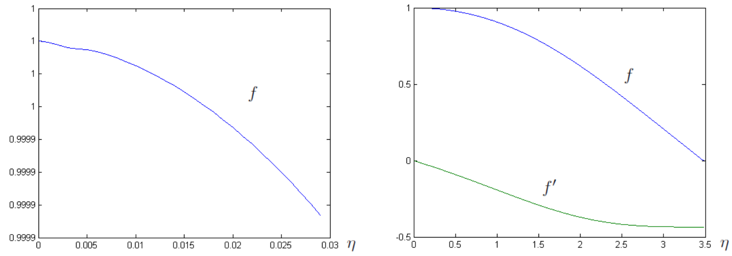

Figure 11: Starting- and global view of the numerical approximation of f (and f') for $\tau_c = -0.6$, $n = 1.3$ and $l = -2.3$ as in 3.

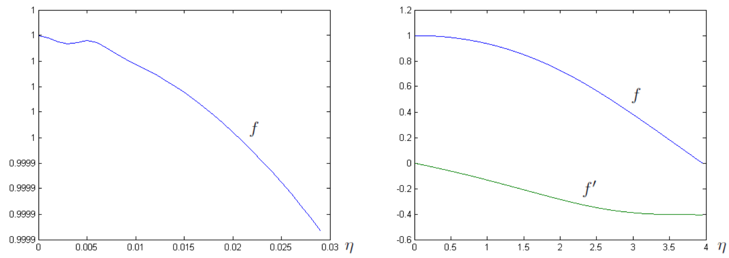

Figure 12: Starting- and global view of the numerical approximation of f (and f') for $\tau_c = -0.6$, $n = 1.7$ and $l = -2.3$ as in 4.

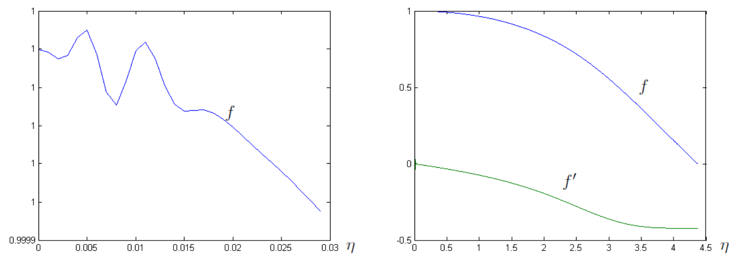

Figure 13: Starting- and global view of the numerical approximation of f (and f') for $\tau_c = -0.6$, $n = 2.3$ and $l = -3.3$ as in 5.

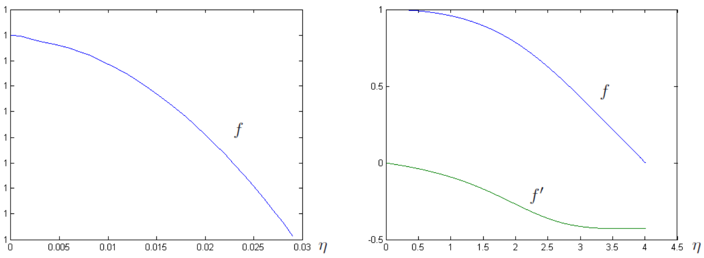

Figure 14: Starting- and global view of the numerical approximation of f (and f') for $\tau_c = -2$, $n = 2.3$ and $l = -3.3$ as in 6.

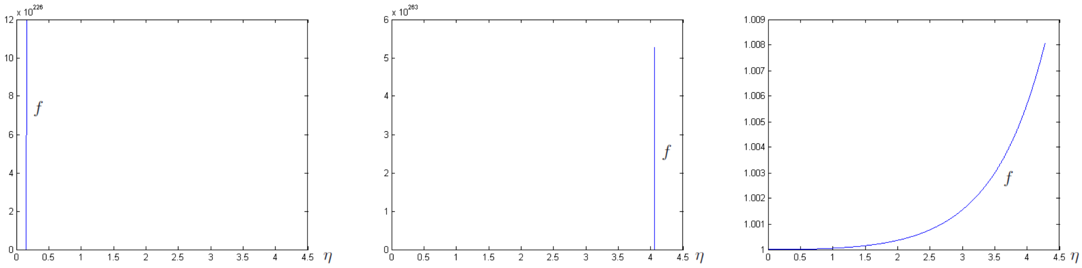

Figure 15: Global view of the numerical approximation of f for $\tau_c > 0$, $n = 2.3$ and $l = -2.3$ as in 7. Respectively $\tau_c = 1$, $\tau_c = 100$ and $\tau_c = 10000$.

As we look at the global view when $\tau_c < 0$, the values of the variables do not matter that much. We see that f' nearly reaches a constant value. More or less we see the behavior discussed in **6.1.5**. Regardless of the difference in n and l, we do not notice different behaviors. Actually we only find a constant descend to zero. We see when the difference in n and l grows, f reaches zero at a higher value of η. Meaning when n and l differ less from each other, the water will distribute slower. The change in τ_c is hardly noticeable in the global view.

As we focus on the starting view. Most noticeable is the strange behavior when the difference in n and l grows. This is probably the result of rounding errors which grow as the powers grow. As you can see in the vertical axis, the changes are actually very small. When we take an absolute greater value for τ_c (as in Figure 14), we see the problem is not that bad anymore because the non-equilibrium term in (6.74) gets more impact on the equation.

In Figure 15 we have τ_c positive. This means that the function will grow at the start. But because f would become zero for $\eta \to \infty$, the function should descend, at least once, after a certain value of η. Unfortunately the function explodes. When τ_c grows, it takes longer for the function to explode.

6.2 The case where $\alpha = 0$

We will now try to solve (6.1) where

$$u(x,t) = f(\eta)\,, \quad \eta = xt^{\beta}\,, \quad \beta < 0\,. \tag{6.83}$$

Therefore we get

$$\partial_t u = \beta t^{-1} \eta f' \tag{6.84}$$

and

$$\frac{\partial}{\partial x}(-) = t^{\beta} \frac{\mathrm{d}}{\mathrm{d}\eta}(-)\,. \tag{6.85}$$

The first term on the right in (6.1) becomes

$$\partial_x \left(H(u)\partial_x p_c(u)\right) \quad = \quad t^{2\beta} \left[f^m \left(f^n\right)'\right]' . \tag{6.86}$$

The last term will be

$$\partial_x \left[H(u)\partial_x \left(\tau_c \tau(u)\partial_t u\right)\right] = t^{2\beta-1} \left[f^m \left(\tau_c f^l \beta\eta f'\right)'\right]' . \tag{6.87}$$

If there exists a similarity solution t must be eliminated from the equation. This leads to

$$-1 = 2\beta = 2\beta - 1 , \tag{6.88}$$

which can not be true. Therefore there exists no similarity solution of this kind.

7 Conclusion and discussion

For the one − and two-phase models we found one equation. This equation had a equilibrium − and a non-equilibrium form. For the equilibrium form we analytically found similarity solutions. By the results of these solutions we could conclude how, for example, the water would distribute itself after being injected if there would be an equilibrium situation.

For the non-equilibrium form we used a numerical approach. by these result we could conclude how, for example, the water would distribute itself after being injected if there would be an equilibrium situation.

These results could be used by other engineers as a verification for their findings in the field. Unfortunately this is not true. This project needs more research to be applicable in the real world, because everything is based on a very small scaled homogeneous medium.

A Mathematica code for certain figures

A.1 Figure 2

$\alpha = -1/(m + n + 1);$
$\eta = x * t\hat{}\alpha;$
$b = 1;$
$m = 0.25;$
$n = 0.25;$
$f[\eta] = ((1 - m - n)/(2n(m + n + 1))\eta\hat{}2 + b\hat{}(m + n - 1))\hat{}(-1/(1 - m - n));$
$u = t\hat{}\alpha * f[\eta];$
$\text{Plot3D}[u, \{x, -25, 25\}, \{t, 0, 250\},$
$\quad \text{AxesLabel} \to \{x, t\}, \text{PlotLabel} \to \text{u}]$

A.2 Figure 3

$\alpha = -1/(m + n + 1);$
$\eta = x * t\hat{}\alpha;$
$b = 1;$
$m = 0.75;$
$n = 0.25;$
$f[\eta] = \text{Exp}[-1/(2n)\eta\hat{}2];$
$u = t\hat{}\alpha * f[\eta];$
$\text{Plot3D}[u, \{x, -25, 25\}, \{t, 0, 250\},$
$\quad \text{AxesLabel} \to \{x, t\}, \text{PlotLabel} \to \text{u}]$

A.3 Figure 4

$\alpha = -1/(m + n + 1);$
$\eta = x * t\hat{}\alpha;$
$b = 1;$
$m = 0.75;$
$n = 0.75;$
$f[\eta] = ((m + n - 1)/n(n/(m + n - 1)b\hat{}(m + n - 1) - 1/(2n(m + n + 1))\eta\hat{}2))\hat{}(1/(m + n - 1));$
$u = t\hat{}\alpha * f[\eta];$
$\text{Plot3D}[u, \{x, -25, 25\}, \{t, 0, 250\},$
$\quad \text{RegionFunction} \to \text{Function}[\{x, t\}, \text{Abs}[x * t\hat{}\alpha] \leq \text{Sqrt}[2b(m + n + 1)n/(m + n - 1)]],$
$\quad \text{AxesLabel} \to \{x, t\}, \text{PlotLabel} \to \text{u}]$

A.4 Figure 5

$\alpha = -1/(m + n + 1);$
$\eta = x * t\hat{}\alpha;$
$b = 1;$
$m = 0.5;$
$n = 1.5;$
$f[\eta] = b - 1/(6n)\eta\hat{}2; u = t\hat{}\alpha * f[\eta];$
$\text{Plot3D}[u, \{x, -25, 25\}, \{t, 0, 250\},$
$\quad \text{RegionFunction} \to \text{Function}[\{x, t\}, \eta\hat{}2 \leq 6n * b],$
$\quad \text{AxesLabel} \to \{x, t\}, \text{PlotLabel} \to \text{u}]$

A.5 Figure 6

$\alpha = -1/(m+n+1);$
$\eta = x * t^\wedge\alpha;$
$b = 1;$
$m = 1.75;$
$n = 1.25;$
$f[\eta] = ((m+n-1)/n(n/(m+n-1)b^\wedge(m+n-1) - 1/(2n(m+n+1))\eta^\wedge 2))^\wedge(1/(m+n-1));$
$u = t^\wedge\alpha * f[\eta];$
Plot3D$[u, \{x, -25, 25\}, \{t, 0, 250\},$
RegionFunction $\to$ Function$[\{x, t\}, \text{Abs}[x * t^\wedge\alpha] \leq \text{Sqrt}[2b(m+n+1)n/(m+n-1)]],$
AxesLabel $\to \{x, t\}$, PlotLabel $\to$ u]

B MATLAB code for numerical approximation

The first three lines set the variables. Maximal number of steps and the length of the plotting interval differ per image because of the variables.

```
tauc = -0.6;            % or -0.6
N = 0.3;             % or  1.3
L = -2.3;            % or -2.3
a = -1/(N-L-1);      % the constant befor \eta f^{l+3}
b = -tauc/(N-L-1);  % the constant befor (f^l(\eta f)')'
dx = 0.001;          % the small interval h

f = [];
f = [f 1];

df = [];
df = [df 0];

ddf = [];
ddf = [ddf 1/tauc];

t = [];
t = [t 0];

max = 52000;
MAX = 3000;           % number of steps

for k = 2:MAX      % all steps until f reached 0, or maximal number of steps is 30 (start image)
    t(k) = (k-1)*dx;
    f(k) = f(k-1) + df(k-1)*dx + ddf(k-1)*dx*dx/2;
    df(k) = (f(k)-f(k-1))/dx;
    ddf(k) = (a*t(k)*f(k)^(L+3)-(N*f(k)^(N-1)+b*L*t(k)+b*(2+L))*df(k))/(b*t(k)*f(k)^L);
    if f(k)<0.000001
       k
        plot(t,f)
        return

    end
end

plot(t,f)
```

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