Open access • Journal Article • DOI:10.1017/S0143385704000197

Flow invariant subsets for geodesic flows of manifolds with non-positive curvature — Source link

Bert Reinold

Institutions: University of Zurich

Published on: 01 Dec 2004 - Ergodic Theory and Dynamical Systems (Cambridge University Press)

Topics: Non-positive curvature, Unit tangent bundle, Geodesic, Invariant (mathematics) and Manifold

Flow invariant subsets for geodesic flows of manifolds with non-positive curvature

Reinold, B

Abstract: Consider a closed, smooth manifold M of non-positive curvature. Write p:UM→M for the unit tangent bundle over M and let > denote the subset consisting of all vectors of higher rank. This subset is closed and invariant under the geodesic flow on UM. We define the structured dimension s-dim > which, essentially, is the dimension of the set p(>) of base points of > . The main result of this paper holds for manifolds with s-dim > <dim M/2: for every >0, there is an -dense, flow invariant, closed subset Ξ UM > such that p(Ξ)=M.

DOI: https://doi.org/10.1017/S0143385704000197

Originally published at:
Reinold, B (2004). Flow invariant subsets for geodesic flows of manifolds with non-positive curvature. Ergodic Theory and Dynamical Systems, 24(6):1981-1990.
DOI: https://doi.org/10.1017/S0143385704000197

Ergod. Th. & Dynam. Sys. (2004), **24**, 1981–1990
DOI: 10.1017/S0143385704000197

Flow invariant subsets for geodesic flows of manifolds with non-positive curvature

BERT REINOLD

Institut für Mathematik, Universität Zürich, Winterthurer Str. 190, CH-8057 Zürich, Switzerland
(e-mail: reinold@math.unizh.ch)

(*Received* 16 *November* 2003 *and accepted in revised form* 4 *February 2004*)

Abstract. Consider a closed, smooth manifold M of non-positive curvature. Write $p : UM \to M$ for the unit tangent bundle over M and let $\mathcal{R}_>$ denote the subset consisting of all vectors of higher rank. This subset is closed and invariant under the geodesic flow ϕ on UM. We will define the structured dimension s-dim $\mathcal{R}_>$ which, essentially, is the dimension of the set $p(\mathcal{R}_>)$ of base points of $\mathcal{R}_>$.

The main result of this paper holds for manifolds with s-dim $\mathcal{R}_> < \dim M/2$: for every $\epsilon > 0$, there is an ϵ-dense, flow invariant, closed subset $\Xi_\epsilon \subset UM\backslash\mathcal{R}_>$ such that $p(\Xi_\epsilon) = M$.

1. *Introduction*

The aim of this paper is to generalize a result† of Burns and Pollicott for manifolds of constant negative curvature to the case of non-positively curved manifolds of rank one. They showed that, on a compact surface M of constant negative curvature, it is possible to construct a closed proper subset Ξ of the unit tangent bundle UM which is invariant under the geodesic flow and full in the sense that its image under the base-point projection $p(\Xi)$ is the whole surface M. This means that there is an open subset $\mathcal{W}$ of UM such that, for every point p in M, we can find a geodesic passing through p such that the velocity field of the geodesic avoids that subset $\mathcal{W}$. By the construction, $\mathcal{W}$ was always some neighbourhood of a non-recurrent vector.

For dimension bigger than three, Schroeder improved this result to manifolds with curvature $K < -1$ in [**7**]. He even proved that a neighbourhood like $\mathcal{W}$ can be found for every vector, no matter whether it is recurrent or not.

The next generalization took place in [**3**]. Buyalo and Schroeder considered a compact manifold of rank one and proved the existence of a neighbourhood like $\mathcal{W}$ for every vector of rank one.

The result of this paper is that even for vectors of higher rank a neighbourhood $\mathcal{W}$ can be avoided, provided that the set of vectors of higher rank satisfies some dimensional condition. The main result is the following.

† The proof is unpublished but the result is stated in [**4**].

THEOREM 1.1. *Let M be a compact manifold of non-positive curvature. Suppose the s-dimension of the set of vectors of higher rank $\mathcal{R}_>$ is bounded by*

$$\text{s-dim}\,\mathcal{R}_> < \frac{\dim M}{2}.$$

Then, for every $\epsilon > 0$, there is a closed, flow invariant, full, ϵ-dense subset Ξ_ϵ of the unit tangent bundle UM consisting only of vectors of rank one.

This result was obtained as part of the author's thesis [**6**]. The proof as presented in this paper has been considerably shortened to explain the main ideas. For further details, please refer to the original thesis.

2. *Notation*

Throughout this text, M will denote a complete, compact, Riemannian manifold of non-positive curvature. The unit tangent bundle with the base-point projection is denoted by $p : UM \to M$. The Riemannian metric on M induces a Riemannian metric (the Sasaki metric) on the unit tangent bundle UM. Both metrics induce distances, denoted by $d(\cdot,\cdot)$. With respect to the metric on UM, we will write $\mathcal{W}_\epsilon(v)$ for an ϵ-neighbourhood of a vector $v \in UM$.

For an element $v \in UM$, γ_v denotes the unique geodesic with initial condition $\dot\gamma_v(0) = v$. The geodesic flow ϕ_t is defined by $\phi_t(v) := \dot\gamma_v(t)$. We will say that a geodesic γ *avoids* an η-neighbourhood of the vector $v \in UM$, if $d(\dot\gamma, v) \geq \eta$, i.e. the distance of v to the velocity field $\dot\gamma$ is bounded below by η.

For a geodesic γ, the set of all parallel Jacobi fields along that geodesic is a vector space. By $\text{rank}(\gamma)$, we denote the dimension of this vector space. For a unit vector v, define $\text{rank}(v) := \text{rank}(\gamma_v)$. The rank of a manifold is defined to be the minimal rank of its tangent vectors. From a result by Ballmann [**2**, Appendix 1], closed manifolds of rank greater than one are quotients of either products or locally symmetric spaces. We will, therefore, always suppose that M is of rank one, i.e. there is a geodesic γ such that the only parallel Jacobi fields along γ are multiples of its velocity field $\dot\gamma$. The set of all vectors of rank one is denoted by $\mathcal{R}_1$, and the set of all vectors of higher rank by $\mathcal{R}_>$. Both these sets are flow invariant; $\mathcal{R}_1$ is open, $\mathcal{R}_>$ closed. If M is a real analytic manifold, then both sets are subanalytic (see [**5**] for a definition). This motivates the definition of the structured dimension.

For any subset $\mathcal{R} \subset UM$, a *support* of $\mathcal{R}$ is a finite union $Z = \bigcup Z_i$ of closed submanifolds $Z_i \subset M$ (called the strata of Z) such that $\mathcal{R} \subset UZ := \bigcup UZ_i$, i.e. all elements of $\mathcal{R}$ are tangent to some stratum in Z. The dimension of a support is defined to be the maximal dimension of one of its strata. By $\text{s-dim}\,\mathcal{R}$ we denote the minimal dimension of a support of $\mathcal{R}$ and call this the *structured dimension* of $\mathcal{R}$. A support Z of $\mathcal{R}$ is called an *s-support* of $\mathcal{R}$ if $\dim Z = \text{s-dim}\,\mathcal{R}$, i.e. if Z realizes the structured dimension of $\mathcal{R}$. Note that $0 \leq \text{s-dim}\,\mathcal{R} \leq \dim M$ for *any* subset of UM. For example, for a submanifold $Y \subset M$, the unit tangent bundle UY has structured dimension $\text{s-dim}\,UY = \dim Y$. From now on, Z will always denote an s-support of $\mathcal{R}_>$.

$\pi : X \to M$ is the universal covering of M by the Hadamard manifold X. The covering maps $\pi : X \to M$ and $d\pi : UX \to UM$ induce Riemannian metrics on X and UX.

With respect to these metrics, geodesics are mapped to geodesics and the rank does not change. Therefore, we will use the same notation in X as in M. Throughout this paper, we will mostly work with X but the fact that X has the compact quotient M will be essential.

We will be concerned with some structures in the unit tangent spaces which we will call spheres and denote by $\mathcal{S}_r(p)$. If $S_r(p)$ is the sphere of radius r around the point p in X, then the set of outward normal vectors to this sphere will be denoted by $\mathcal{S}_r(p)$. Another way to describe this set is via the identity $\mathcal{S}_r(p) = \phi_r(U_pX)$. For fixed $r \geq 0$, the unit tangent bundle is foliated into these spheres. For $r \to \infty$, the spheres $\mathcal{S}_r(\gamma_v(-r))$ converge to the strong unstable leaf of v, i.e. to the outward pointing normal field to the horosphere at $\gamma_v(-\infty)$ through $p(v)$. If we only consider ϵ-neighbourhoods of v inside these spheres, then the convergence is uniform.

3. *Construction*

We want to construct a closed, flow invariant, ϵ-dense, subset Ξ_ϵ of UM which consists only of vectors of rank one and is full in the sense that $p(\Xi_\epsilon) = M$. For fixed η, define a flow invariant, closed set of rank one vectors by

$$\hat{\Xi}_\eta := \overline{\{u \in UM \mid \gamma_u \text{ avoids an } \eta\text{-neighbourhood of } \mathcal{R}_>\}}.$$

We will see that, for $\eta = \eta(\epsilon)$ small enough, the set $\Xi_\epsilon := \hat{\Xi}_\eta$ is ϵ-dense and full in the previous sense. For the moment, we will not focus on the fact that Ξ_ϵ is ϵ-dense. To prove that Ξ_ϵ is full, we construct, for every point $o \in M$, a geodesic γ through o which avoids the η-neighbourhood of $\mathcal{R}_>$. Then $\dot{\gamma}(0) \in \Xi_\epsilon$ and, hence, $o \in p(\Xi_\epsilon)$.

For the construction, we work in UX. Given a starting point $o \in X$, begin by constructing a sequence v_i of vectors in U_oX such that the geodesic segments $\gamma_{v_i}|_{[0,t_i]}$ avoid a neighbourhood of $\mathcal{R}_>$ and $t_i \to \infty$. By the hyperbolicity† of the vectors of rank one, the limit $v_\infty := \lim v_i$ will exist and the geodesic ray $\gamma_{v_\infty}|_{\mathbb{R}_+}$ will avoid a neighbourhood of $\mathcal{R}_>$.

Now instead of just considering one vector v_0, consider a compact, low-dimensional manifold Y together with a smooth map $V_0 : Y \to U_oX$ and define $t_{-1} := 0$. Recursively, we define a sequence of smooth maps $V_i : Y \to U_oX$ and times $t_i \to \infty$, such that the V_i converge to a continuous map V_∞ sufficiently close to our original map V_0.

Suppose V_i and t_{i-1} are given. We define

$$t_i := \min\{t \geq t_{i-1} + B \mid d(\phi_t \circ V_i(Y), UZ) \leq 11\epsilon\}$$

and deform V_i into V_{i+1} as follows, where C and τ are constants provided by Lemma 4.1. Deform the smooth map $\phi_{t_i} \circ V_i : Y \to \mathcal{S}_{t_i}(o)$ slightly, as explained in Proposition 4.1, to a C-close map into $\mathcal{S}_{t_i}(o)$ whose image is τ-far away from UZ. Composing the resulting map with ϕ_{-t_i} gives a map $V_{i+1} : Y \to U_oX$. If $t_i = \infty$ for some i, then define $V_j := V_i$ and $t_j := t_{i-1} + (j - i + 1)B$ for all $j \geq i$. By this construction, we have the following immediate properties.

(1) The image of $\phi_{t_j} \circ V_{j+1}$ is τ-far from UZ (and, hence, from $\mathcal{R}_>$). We say that V_{j+1} is *τ-far from UZ at time t_j*.

† Compare this with §5.

(2) The maps $\phi_{t_j} \circ V_j$ and $\phi_{t_j} \circ V_{j+1}$ are C-close. We say that V_j and V_{j+1} are *C-close at time t_j*.

(3) Two times t_j and t_{j+1} of deformations are separated by at least time B, i.e. the deformations happen at discrete, distant times.

In §5, we will see that, by (3), the widening property of the vectors of rank one will guarantee that the deformation at time t_j will not displace the image of $\phi_t \circ V_j$ much for earlier times $0 < t < t_j$. Therefore, the geodesic rays γ_{V_j} will avoid $\mathcal{R}_>$ for longer and longer times and the smooth maps $V_j : Y \to U_oX$ will converge to a continuous map $V_\infty : Y \to U_oX$ for $j \to \infty$.

4. *Perturbations on the Spheres $\mathcal{S}_r$*

Given the s-support Z, we want to describe a deformation Ψ_r of UX that respects the r-sphere foliation (i.e. $\Psi_r(\mathcal{S}_r(o)) \subset \mathcal{S}_r(o)$ for all $r > r_0$ and $o \in X$) and such that the displacement by Ψ_r is small but moves every vector away from UZ effectively. For the moment, suppose that Z consists of only one stratum, hence Z is a closed submanifold of UM. By the remark at the end of this section, we will see that this restriction has no effect on Proposition 4.1.

For every time $r > r_0$ and small ϵ, we define a perturbation $\Psi_{r,\epsilon}$ of $UX \backslash p^{-1}Z$ that respects the foliation of UX into spheres of radius r and such that the distance of the image of $\Psi_{r,\epsilon}$ to UZ is at least ϵ. The idea is illustrated in Figure 1 and works as follows. For any vector $v \in UX \backslash p^{-1}Z$ with $d(v, UZ) < 4\epsilon$, let γ denote the shortest geodesic† from Z to $p := p(v)$. Use parallel transport along γ to move v away from Z to a point q which is at distance $2\epsilon - d(v, UZ)/2$ and write $\bar{v}$ for this vector. Find the vector w in U_qX such that $\gamma_w(\mathbb{R}_-)$ goes through $o := \gamma_v(-r)$ (hence, o is the centre of the sphere containing v). Pull back w by the geodesic flow to a vector on $\mathcal{S}_r(o)$ and call this image $\Psi_{r,\epsilon}(v)$.

For v far away from UZ, the vector is only slightly displaced and, hence, stays away from UZ. For v close to UZ, the parallel transport moves the vector very effectively away from UZ. The readjustment afterwards is comparatively small and, hence, the resulting vector is far away from UZ. This is summed up in the following lemma.

LEMMA 4.1. *Given $r_0 > 0$, we can find constants C, τ, ϵ such that for all $\lambda \in]0, 1]$ the maps $\Psi_{r,\lambda\epsilon}$ have the following properties for all $r \geq r_0$:*

(1) *$\Psi_{r,\lambda\epsilon} : UX \backslash p^{-1}Z \to UX$ is continuous and respects the r-sphere foliation of UX.*

(2) *The image of $\Psi_{r,\lambda\epsilon}$ has no intersection with the $\lambda\tau$-neighbourhood of UZ:*
$$d(\Psi_{r,\lambda\epsilon}(v), UZ) \geq \lambda\tau \quad \text{for all } v \in UX \backslash p^{-1}Z.$$

(3) *The displacement by $\Psi_{r,\lambda\epsilon}$ is bounded by the global constant λC:*
$$d(\Psi_{r,\lambda\epsilon}(v), v) \leq \lambda C \quad \text{for all } v \in UX \backslash p^{-1}Z.$$

Remark. If Z is a manifold with boundary, some problems might arise at the boundary, since the parallel transport could point in the same direction as the geodesic flow and, hence, will be undone in the last step of the construction. However, these problems can be overcome if we use a slightly bigger submanifold $Z' \supset Z$ for the construction of $\Psi_{r,\epsilon}$. Hence, Lemma 4.1 holds in this case, too.

† If ϵ is small enough, γ is unique.

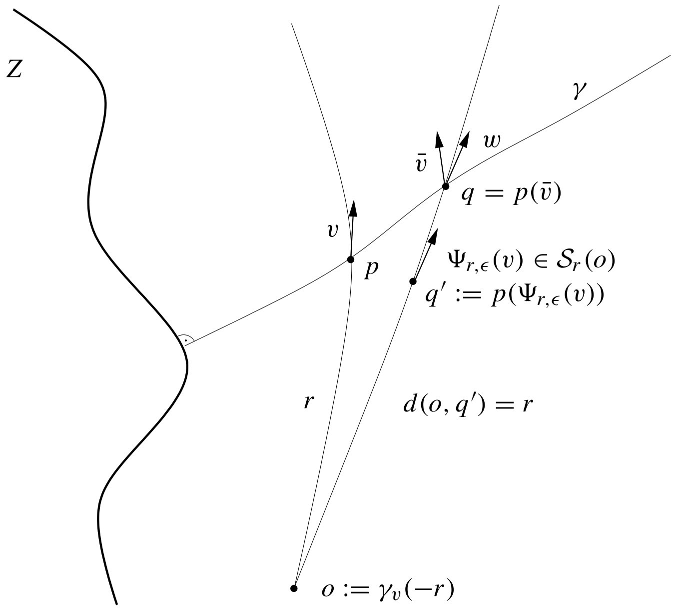

FIGURE 1. Construction of $\Psi_{r,\epsilon}$.

As a result we can deform submanifolds of spheres away from a stratum Z.

PROPOSITION 4.1. *For $r_0 > 0$ given, there are constants $C, \tau > 0$ such that for any $\lambda \in]0, 1]$, $r > r_0$, $o \in X$ and manifold Y with*

$$\dim Y < \dim X - \dim Z$$

we can deform any smooth map $c : Y \to \mathcal{S}_r(o)$ into a smooth map $c_\lambda : Y \to \mathcal{S}_r(o)$ such that

- *c_λ is λC-close to c, i.e.*

$$d(c(x), c_\lambda(x)) \leq \lambda C \quad \textit{for all } x \in Y.$$

- *c_λ avoids a $\lambda\tau$-neighbourhood of UZ, i.e.*

$$d(c_\lambda(x), UZ) \geq \lambda\tau \quad \textit{for all } x \in Y.$$

Proof. Choose $\tilde{C}$, $\tilde{\epsilon}$ and $\tilde{\tau}$ as provided by Lemma 4.1 for $r_0/2$. Without loss of generality, we can assume that $\tilde{\tau} < r_0/2$ since the constants are scalable. Fix these constants and consider any $r \geq r_0$ and $\lambda \in]0, 1]$. Choose $\tau' > 0$ such that on big spheres (radius $> r_0/2$) vectors are $\tilde{\tau}\lambda/6$-close if their base points are τ'-close. Now fix $o \in X$ and consider the differentiable map

$$\rho : Z \backslash \{o\} \longrightarrow \mathbb{R}$$
$$p \longmapsto d(o, p),$$

which measures the radius in polar coordinates around o. By Sard's theorem, the set of regular values of ρ is dense in $\mathbb{R}$. Hence, we can choose a regular value $r' \in]r - \tilde{\tau}\lambda/6, r + \tilde{\tau}\lambda/6[$. The inverse image $Z_{r'} := \rho^{-1}(r')$ is the intersection of Z with the sphere $S_{r'}(o)$ in X. Since r' is regular, this intersection $Z_{r'}$ is a smooth submanifold of dimension at most $\dim Z - \dim \mathbb{R} = \dim Z - 1$ in the sphere $S_{r'}(o)$ by the Preimage Theorem. Next note that the geodesic flow $\phi_{r'-r}$ identifies the two spheres $\mathcal{S}_r(o) \to \mathcal{S}_{r'}(o)$, moving every vector by exactly $|r' - r| < \tilde{\tau}/6\lambda$. Furthermore, we have natural identifications of spheres with centre o in X and in UX, given by the gradient of ρ: $\operatorname{grad} \rho|_{S_r(o)} : S_r(o) \to \mathcal{S}_r(o)$ and $\operatorname{grad} \rho|_{S_{r'}(o)} : S_{r'}(o) \to \mathcal{S}_{r'}(o)$.

Now $y := p \circ \phi_{r'-r} \circ c : Y \to S_{r'}(o)$ is a smooth map of Y into the sphere $S_{r'}(o)$. By Thom's Transversality Theorem, we can find a τ'-close smooth map $y' : Y \to S_{r'}(o)$ which avoids $Z_{r'}$. Write α for the homotopy of $S_{r'}(o)$ with $\alpha_0 = \mathrm{id}_{S_{r'}(o)}$ and $\alpha_1 \circ y = y'$. Consider the map

$$c_\lambda := \phi_{r-r'} \circ \Psi_{r,\lambda\epsilon} \circ \operatorname{grad} \rho \circ \alpha_1 \circ p \circ \phi_{r'-r} \circ c : Y \to \mathcal{S}_r(o).$$

This looks quite monstrous at first glance but we will explain it step by step.

- $\phi_{r'-r} \circ c$ is $\tilde{\tau}\lambda/6$-close to c, since $r' - r < \tilde{\tau}\lambda/6$.
- The base-point distance between $\phi_{r'-r} \circ c$ and $\operatorname{grad} \rho \circ \alpha_1 \circ p \circ \phi_{r'-r} \circ c$ is just the distance between y and y', which is smaller than τ' by definition of α. But τ'-close base points imply $\tilde{\tau}\lambda/6$-close radial vectors and, hence, $\operatorname{grad} \rho \circ \alpha_1 \circ p \circ \phi_{r'-r} \circ c = \operatorname{grad} \rho \circ y'$ is $\tilde{\tau}\lambda/6$-close to $\phi_{r'-r} \circ c$ and $\tilde{\tau}\lambda/3$-close to c.
- y' avoids Z and, hence, $\operatorname{grad} \rho \circ y'$ maps Y into $UX \backslash p^{-1}Z$ and, hence, the map $\Psi_{r,\lambda\epsilon} \circ \operatorname{grad} \rho \circ y'$ is well defined. By Lemma 4.1, it is $\tilde{C}\lambda$-close to $\operatorname{grad} \rho \circ y'$ and, hence, $(\tilde{C}\lambda + \tilde{\tau}\lambda/3)$-close to our original map c. Again by Proposition 4.1, it avoids a $\tilde{\tau}\lambda$-neighbourhood of UZ.
- Projecting this map back to $\mathcal{S}_r(o)$ by the geodesic flow $\phi_{r-r'}$ gives a further displacement of $|r - r'| \leq \tilde{\tau}\lambda/6$ which leaves us with the properties that c_λ is $(\tilde{C}\lambda + \tilde{\tau}\lambda/3 + \tilde{\tau}\lambda/6)$-close to c and avoids a $(\tilde{\tau}\lambda - \tilde{\tau}\lambda/6)$-neighbourhood of UZ.

This ends the proof if we set $C := \tilde{C} + \tilde{\tau}/2$ and $\tau := \tilde{\tau}/2$. □

Remark. Proposition 4.1 stays true if Z is not a submanifold of M but a finite union $Z = \bigcup Z_i$ of closed submanifolds of M and $\dim Y < \dim X - \max \dim Z_i$. To see this, pick C_i, τ_i for each of the submanifolds and then choose λ_{i+1} so small that $\lambda_{i+1}C_{i+1} < \tau_i/8$ and $\lambda_{i+1}\tau_{i+1} < \tau_i/4$. Now, after a finite number of displacements, we avoid all Z_i since the $(i+1)$st displacement will not undo the previous ones.

5. *Hyperbolicity*

It is a well-known fact that, in hyperbolic space, geodesics originating in the same point diverge qualitatively faster than in Euclidean space. Ballmann [**1**] introduced the term hyperbolic geodesic for geodesic segments where the distance between close geodesic segments is less than μ times the Hausdorff distance for some $\mu \in]0, 1[$. Buyalo and Schroeder [**3**] use a similar definition to define a hyperbolic vector and show that every vector of rank one is, indeed, hyperbolic. As a result, a compact set of rank-one vectors has a widening property explained in the following lemma and illustrated in Figure 2.

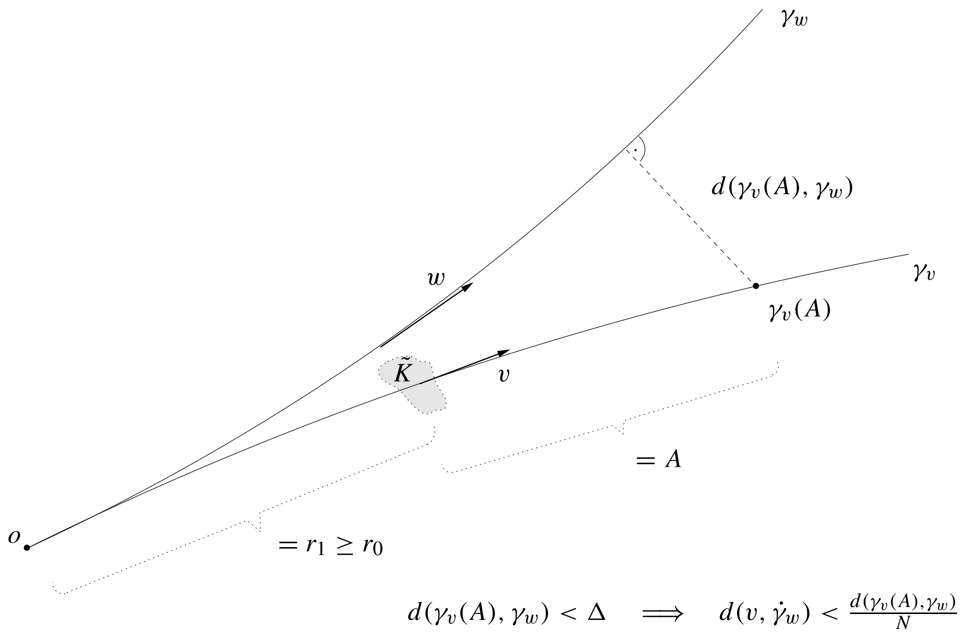

FIGURE 2. Widening property of $\tilde{K}$.

LEMMA 5.1. *Consider a compact set $K \subset UM$ consisting of vectors of rank one and $\tilde{K} := d\pi^{-1}K$. Given $N \in \mathbb{N}$ and $\Delta, r_0 > 0$, there is a distance $A > 0$ such that for any point $o \in X$ and vectors $v \in \tilde{K}$ and $w \in UX$ with $\gamma_v(-r_1) = o = \gamma_w(-r_2)$ ($r_1 > r_0$, $r_2 > 0$), the inequality $d(\gamma_v(A), \gamma_w) \leq \Delta$ implies $d(v, \dot{\gamma}_w) < d(\gamma_v(A), \gamma_w)/N$.*

The important point here is that $A = A(K, N, \Delta, r_0)$ does not depend on the time r_1. To illustrate the widening property, suppose K, N, Δ, r_0 are given and A is the constant provided by the lemma. Consider two close geodesics γ and σ starting in $o \in X$. Suppose that outside the ball of radius r_0 around o there is a point $\gamma(t)$ ($t > r_0$), where the tangent vector $v := \dot{\gamma}(t)$ to γ lies in $\tilde{K}$. Now if σ meets the ball of radius Δ around the point $\gamma(t + A)$, then the velocity field $\dot{\sigma}$ is Δ/N-close to v.

Roughly speaking, we can say that A is the time span after which the distance between close geodesics widens by a factor of N. Note that, in Euclidean space, it is impossible to find such A which works for all times $t > r_0$, since the distance between geodesics grows linearly.

6. *Choosing the right constants*

Suppose that s-dim $\mathcal{R}_> < \dim M - 1$ and Z is an s-support of $\mathcal{R}_>$. Fix $r_0 > 2$ and find constants τ, C by Proposition 4.1 for Z. Define new constants

$$\epsilon := \frac{\tau}{10} \quad \text{and} \quad \delta := 10\frac{C}{\tau},$$

fix $N \in \mathbb{N}$ such that $N > 1 + \frac{1}{2}\delta$, fix

$$\frac{\delta}{N-1} < \Lambda < 2N - \delta$$

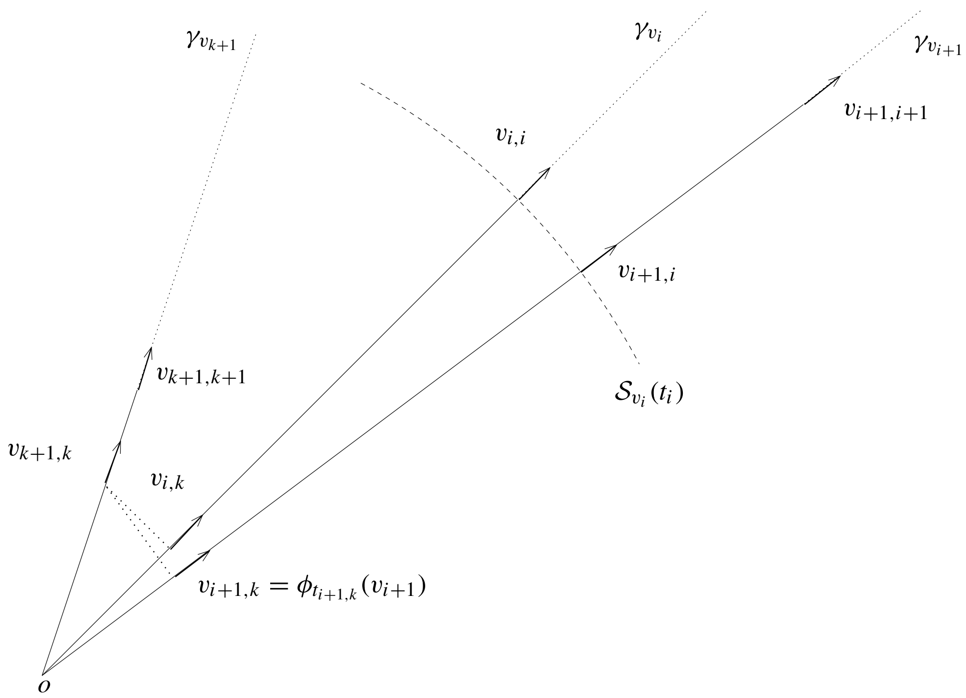

FIGURE 3. Construction of $v_{i+1,i}$ and $t_{i+1,k}$.

and choose $\Delta > (\delta + \Lambda)\epsilon$. Since Proposition 4.1 holds for $\lambda\tau$ and λC if $\lambda \in\ ]0, 1]$, we can use the same constants δ, N, Λ and Δ when working with $\epsilon' < \epsilon$. Fix A as provided by Lemma 5.1 for the compact set $K_\epsilon := UM \backslash \mathcal{W}_\epsilon(UZ)$ and $B > A + 8\epsilon$. Note that, for all $t \in \mathbb{R}$, the compact set K_ϵ consists only of vectors of rank one while the compact set $\mathcal{R}_>$ consists of vectors of higher rank. Hence, these sets are disjoint and we can define

$$\beta := \min_{t \in [-B,B]} d(\phi_t(K_\epsilon), \mathcal{R}_>)/\epsilon.$$

Given a smooth map $V_0 : Y \to U_o X$, we construct the sequences $t_i \in \mathbb{R}_+$ and $V_i : Y \to U_o X$ as explained in §3. We consider one point $x \in Y$ and its images $v_i = V_i(x) \in U_o X$.

As illustrated in Figure 3, define $t_{i,k}$ and $v_{i,k}$ for $0 \le k \le i - 1$ by the equation

$$d(\phi_{t_{i,k}}(v_i), \phi_{t_k}(v_{k+1})) = d(\dot{\gamma}_{v_i}, \phi_{t_k}(v_{k+1})),$$

i.e. $v_{i,k} := \phi_{t_{i,k}}(v_i)$ is the vector, tangent to the geodesic γ_{v_i} which is closest to the vector $v_{k,k} := \phi_{t_k}(v_{k+1})$. Now we know that

$$\begin{aligned} \beta\epsilon &< d(v_{i,i}, \mathcal{R}_>) \\ & \qquad d(v_{i+1,i}, v_{i,i}) \le \delta\epsilon \\ 10\epsilon &\le d(v_{i+1,i}, \mathcal{R}_>) \end{aligned}$$

and, step by step, we can prove that $v_{i+1,i} = v_i \in K_\epsilon$,

$$\beta\epsilon \le d(\phi_t(v_i), \mathcal{R}_>) \quad \text{for } t_{i-1} - B \le t \le t_{i-1} + B$$
$$11\epsilon < d(\phi_t(v_i), UZ) \quad \text{for } t_{i-1} + B \le t < t_i$$

and, for $0 \le k < i$,

$$d(v_{i,k}, v_{k,k}) < (\delta + \Lambda)\epsilon$$
$$d(v_{i,k}, v_{k+1,k}) < 2\epsilon$$

and $v_{i,k} \in K_\epsilon$. Furthermore,

$$t_{i,i} := t_i = t_{i,i-1} \quad \text{and} \quad |t_k - t_{i,k}| \le 4\epsilon$$

and, finally,

$$\beta\epsilon < d(\phi_t(v_i), \mathcal{R}_>) \quad \text{for } 0 \le t \le t_i (i \ge 1) \tag{1}$$
$$d(v_{i+1,i-k}, \dot{\gamma}_{v_i}) < \frac{\delta}{N^k}\epsilon. \tag{2}$$

By (2), the v_i converge to some v_∞ with

$$d(v_0, v_\infty) \le \frac{\delta}{N(N-1)}\epsilon$$

and by (1) for v_∞, we have

$$\beta\epsilon \le d(\dot{\gamma}_{v_\infty}(\mathbb{R}_+), \mathcal{R}_>).$$

Since none of the estimations depended on v_0, the convergence is uniform on Y and, hence, the maps $V_i : Y \to U_o X$ converge to a continuous map $V_\infty : Y \to U_o X$. Now we can prove the following proposition.

PROPOSITION 6.1. *There are constants ϵ and c such that, for any $\eta < \epsilon$, there is an $\eta' < \eta$ such that, for any compact manifold Y with*

$$\dim Y < \dim X - \text{s-dim}\,\mathcal{R}_>,$$

for any $o \in X$ and any continuous map $V_0 : Y \to U_o X$, we can find a map $V_\infty : Y \to U_o X$ which is $c\eta$-close to V_0 and defines a family of geodesic rays avoiding an η'-neighbourhood of $\mathcal{R}_>$, i.e.

$$d(\dot{\gamma}_{V_\infty(Y)}(\mathbb{R}_+), \mathcal{R}_>) \ge \eta'.$$

Proof. Take all the constants we had before. Note that, by definition, δ and N are independent of ϵ. So we can define the global constant $c := \delta/N(N-1)$. Choose β, B, A and Δ replacing ϵ by $\eta < \epsilon$. Write $\eta' := \beta\eta$ to get the desired result. □

For the special case where $\dim Y = 0$ we get the following corollary.

COROLLARY 6.1. *Let M denote a compact manifold of non-positive curvature with* s-dim $\mathcal{R}_> < \dim M$. *Then there is an $\eta' > 0$ such that, for every point $o \in M$, we can find a geodesic ray γ, starting in o whose tangent field avoids an η'-neighbourhood of the set $\mathcal{R}_>$ of all vectors of higher rank.*

7. *From rays to geodesics*

In Proposition 6.1, choose $\eta < \epsilon/c$. To find a complete geodesic, consider two complementary submanifolds N_1 and N_2 of U_oX which intersect in exactly two antipodal points. Deform the inclusion $N_1 \hookrightarrow U_oX$ of one of them so that it avoids $\mathcal{R}_>$ for all $r > 0$ and the inclusion $N_2 \hookrightarrow U_oX$ of the other one so that it avoids $\mathcal{R}_>$ for all $r < 0$. If ϵ is smaller than π, then the images of the deformed maps still intersect in at least two points. Pick one of these points of intersection. It defines a complete geodesic γ_o which avoids $\mathcal{R}_>$ at all times. The deformation of the N_i works if $\dim N_i < \dim X - \text{s-dim}\,\mathcal{R}_>$. We need two complementary submanifolds, i.e. $\dim N_1 + \dim N_2 = \dim U_oX = \dim M - 1$. Combining these two inequalities we see that the construction works if $\text{s-dim}\,\mathcal{R}_> < \dim M/2$. Since the choice of constants is independent of o, we can find such γ_o for all $o \in X$. The preimage under $d\pi$ of the union of the velocity fields of all such geodesics

$$d\pi^{-1}\bigg(\bigcup_{o\in X} \dot{\gamma}_o\bigg)$$

is full and contained in $\hat{\Xi}_{\eta'}$. Hence, $\hat{\Xi}_{\eta'}$ is full. To see that $\hat{\Xi}_{\eta'}$ is ϵ-dense, note that for any $v \in U_oX$ we can choose N_1 and N_2 such that they intersect in v and $-v$. Then ϵ-close to v and $-v$ the images of the deformed maps will intersect and we can suppose that $d(\dot{\gamma}_o(0), v) < \epsilon$. Theorem 1.1 follows.

Acknowledgement. Most ideas in this paper originated in stimulating discussions with my supervisor Viktor Schroeder. I am deeply indebted to him for raising my interest in this subject and for many helpful remarks. The definition of the structured dimension is due to him and without this notion the result of this paper would have been hardly communicable.

References

[1] W. Ballmann. *Lectures on Spaces of Non-positive Curvature*. Birkhäuser, Basel, 1995. With an appendix by Misha Brin.

[2] W. Ballmann, M. Gromov and V. Schroeder. *Manifolds of Non-positive Curvature (Progress in Mathematics, 61)*. Birkhäuser, Boston, MA, 1985.

[3] S. Buyalo and V. Schroeder. Invariant subsets of rank 1 manifolds. *Manuscripta Math.* **107**(1) (2002), 73–88.

[4] S. Buyalo, V. Schroeder and M. Walz. Geodesics avoiding open subsets in surfaces of negative curvature. *Ergod. Th. & Dynam. Sys.* **20**(4) (2000), 991–1006.

[5] H. Hironaka. Subanalytic sets. *Number Theory, Algebraic Geometry and Commutative Algebra, in Honor of Yasuo Akizuki*. Kinokuniya, Tokyo, 1973, pp. 453–493.

[6] B. Reinold. Vectors of higher rank on a Hadamard manifold with compact quotient. *Dissertation*, Universität Zürich, 2003. arXiv:math.DG/0311025.

[7] V. Schroeder. Bounded geodesics in manifolds of negative curvature. *Math. Z.* **235**(4) (2000), 817–828.