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## Research

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Flow of a new class of non-Newtonian fluids in tubes of non-circular cross-sections

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Fluids described by constitutive relations wherein the symmetric part of the velocity gradient is a function of the stress can be used to describe the flows of colloids and suspensions. In this paper, we consider the flow of a fluid obeying such a constitutive relation in a tube of elliptic and other non-circular cross-sections with the view towards determining the velocity field and the stresses that are generated at the boundary of the tube. As tubes are rarely perfectly circular, it is worthwhile to study the structure of the velocity field and the stresses in tubes of non-circular cross-section. After first proving that purely axial flows are possible, that is, there are no secondary flows as in the case of many viscoelastic fluids, we determine the velocity profile and the shear stresses at the boundaries. We find that the maximum shear stress is attained at the co-vertex of the ellipse. In general tubes of non-circular crosssection, the maximum shear stress occurs at the point on the boundary that is closest to the centroid of the cross-section.

This article is part of the theme issue 'Rivlin's legacy in continuum mechanics and applied mathematics'.

## 1. Introduction

Rivlin made pioneering contributions over a wide swathe of continuum mechanics and electrodynamics. Of relevance to this particular work are his myriad contributions to non-Newtonian fluid mechanics: the development of the hierarchy of Rivlin-Ericksen fluid models [1]; integral constitutive relations for materials with memory [2,3]; observing an analogy between the
characteristics of the flow of viscoelastic fluids and the turbulent flow of a Newtonian fluid in a tube and using the same to develop models for the turbulent flows of Newtonian fluids [4]; the solution of the flow of non-Newtonian fluids and the stability of the flows in a variety of flow domains [5-13]; experiments to determine whether there are 'pressure hole errors' [14]; and the determination of normal stresses that develop due to shear [15], among other issues. As with most pioneering contributions, these path-breaking works have been improved upon, as some of the ideas were incomplete or not completely correct, but these early forays of Rivlin have spurred and catalysed the development of the field. ${ }^{1}$

Rivlin, in different studies, in conjunction with Green, Pipkin and Langlois, considered the flows of non-Newtonian fluids in tubes of non-circular cross-sections. While in tubes of circular cross-section uniaxial flows are possible in most non-Newtonian fluids, such is not the case in general with regard to their flows in tubes of non-circular cross-section. In the case of simple fluids, whose viscometric functions are linearly independent, flowing in pipes of infinite length of non-circular cross-section, Ericksen [16] conjectured that 'it is impossible except perhaps when the cylinder is made up of portions of planes and right circular cylinders' to have purely rectilinear flow. His conjecture was only partially correct, for later Fosdick \& Serrin [17] proved the following result: 'Suppose an incompressible simple fluid moves rectilinearly and steadily in a fixed straight tube whose cross-section is a bounded and connected open set. Assume also that the adherence condition or the slip condition is satisfied at the tube wall. Under these circumstances, if the material functions $\phi$ and $\mu$ satisfy appropriate analyticity and monotonicity conditions, ${ }^{2}$ and if $\phi$ is not proportional to $\mu$ for small shear rates, then the cross-section of the tube must be either circular or the annulus between two concentric circles.' Langlois \& Rivlin [6] using a perturbation analysis studied the flows of a large class of viscoelastic fluids in tubes of elliptic cross-section and showed that secondary flows (flows in the plane of the cross-section) arise in addition to the flow along the axis of the tube. Such secondary flows arise by virtue of the normal stresses that develop due to shear. Rivlin used a perturbation analysis based on the smallness of the pressure gradient along the axis of the cylinder and the slowness of the flow, rather than a perturbation parameter that depended on the geometry which was a measure of the non-circularity of the cross-section, to study the flows of Rivlin-Ericksen fluids. ${ }^{3}$ Rivlin-Ericksen fluids fall under the category of what

[^0]is referred to as fluids of the differential type [20]. In such fluids, the stress is expressed in terms of the symmetric part of the velocity gradient and its various material time derivatives. Such fluids are incapable of stress relaxation among other non-Newtonian characteristics exhibited by most viscoelastic fluids (see Dunn \& Rajagopal [21] for a detailed critical analysis of the status of such models). Jones [22] used a material parameter as the perturbation parameter to study the problem of flow in a tube of infinite length of non-circular cross-section. Such a perturbation limits the study to fluids that are only slightly non-Newtonian as the perturbation parameter that determines the departure from Newtonian behaviour is assumed to be small.

A study of the counterpart to Ericksen's conjecture for fluids to that in elasticity under the assumption of small driving force has been carried out by Fosdick \& Kao [23], and in the case of finite driving force but for small deviations from the circularity of the boundaries has been carried out by Mollica \& Rajagopal [24]. It is also worth mentioning that both experimental and theoretical studies of secondary flows associated with the turbulent flows of fluids in tubes of non-circular cross-section, which have structural characteristics similar to the flow of non-Newtonian fluids, have been carried out by Nikuradse [25], Speziale [26] and Huang \& Rajagopal [27].

The classical Navier-Stokes fluid and classical power-law fluids are very special subclasses of fluids of differential type of grade, whose constitutive expression for the stress is a function of the symmetric part of the velocity gradient. Neither of the subclasses exhibit stress relaxation or the development of normal stress differences in shear flows (such normal stress differences are at the heart of phenomena like rod climbing, die swell, etc.). Since secondary flows in viscoelastic fluids are a manifestation of the normal stress differences that develop due to the shearing of the fluid, they do not occur in classical power-law fluids.

The classical incompressible power-law fluid is defined through the following constitutive relation:

$$
\begin{equation*}
\mathbb{T}=-p \mathbb{I}+\mu_{0}\left[\beta+\alpha|\mathbb{D}|^{2}\right]^{n} \mathbb{D}, \tag{1.1}
\end{equation*}
$$

where $|\cdot|$ is the Frobenius norm, $\mathbb{T}$ is the Cauchy stress, $\mathbb{D}$ is the symmetric part of the velocity gradient, $-p \mathbb{I}$ is the indeterminate part of the stress due to the constraint of incompressibility and $\mu_{0}, \alpha, \beta$ and $n$ are constants. When $n=0$, the model reduces to the Navier-Stokes model (note that in this case $\mu_{0}$ is twice the viscosity that is usually used in the representation of the model).

Rate-type models for describing the behaviour of non-Newtonian fluids have a hoary past, starting with the seminal work of Maxwell [28]. This work of Maxwell was followed by the rate-type models developed by Burgers [29]. A systematic three-dimensional approach for the development of frame-indifferent rate-type models was provided by Oldroyd [30]. The rate-type equations developed by Burgers and Oldroyd and others, in general, are implicit models wherein one has a relationship between the stress and its time derivative and the symmetric part of the velocity gradient and its time derivative. However, recently Rajagopal [31,32] developed implicit algebraic constitutive relations in that one has an implicit relationship between the stress and the symmetric part of the velocity gradient (see also Rajagopal \& Saccomandi [33] for a discussion of general algebraic implicit models). A subclass of such models are fluids whose symmetric part of the velocity gradient is given in terms of the stress, and a further subclass are fluids referred to as generalized stress power-law fluids [34] where the symmetric part of the velocity gradient is given as a power law of the stress. In fact, for an infinite range of values of the power-law exponent, the expression for the velocity gradient in terms of the stress cannot be inverted, that is, one cannot obtain an expression for the stress in terms of the velocity gradient. This is made evident when we consider a simple one-dimensional shear flow. The relationship between the norm of the deviatoric part of the stress in terms of the norm of the symmetric part of the velocity gradient is portrayed in figure 1 for such a flow. We note that for all values of the power-law exponent

[^1]

Figure 1. Behaviour of the stress power-law model in simple shear flow.
$n \in(-\infty,-1 / 2]$, the stress cannot be expressed in terms of the symmetric part of the velocity gradient (figure 1) [34]. Le Roux \& Rajagopal [35] generalized the constitutive relations developed by Málek et al. [34]. The model developed by Le Roux \& Rajagopal [35] and its generalization by Perlácová \& Průs̆a [36] can be used to describe the response of colloids and suspensions [37,38].

In this paper, we study the flow of a generalized stress power-law fluid in tubes of non-circular cross-sections, with much of the study being devoted to flow in a tube with elliptic cross-section. As mentioned earlier, it is well known that in a traditional incompressible power-law fluid wherein the deviatoric part of the stress has a power-law relationship to the symmetric part of the velocity gradient, a purely unidirectional flow along the axis of the tube is possible. However, since we are considering a generalized stress power-law fluid, which for values of $n \in(-\infty,-1 / 2$ ] cannot even be inverted (figure 2), we first have to determine if secondary flow is possible in such fluids for any specific values for $n \in(-\infty,-1 / 2]$. We show that secondary flows are not possible in a stress power-law fluid. Since in simple shear flows normal stress differences that are the cause for the development of secondary flow do not develop, we do not expect such secondary motion. Thus, our analysis might seem laborious, unedifying and best left out. We, however, include the same for completeness of the study. Having established this result, we then go ahead and determine the velocity field for different values of the power-law index $n$, and different values for the ratio of the lengths of the minor and major axes of the elliptic cross-section.

We use the finite-element method to numerically solve the governing partial differential equations by appealing to a splitting algorithm. ${ }^{4}$

In addition to plotting the velocity profile across the cross-section, we also determine the location of the maximum value for the shear stress. We find that this maximum value occurs, in the case of the elliptic cross-section, at the co-vertex, while in the case of arbitrary cross-sections, it occurs at the point on the circumference that is closest to the centroid of the cross-section.

The organization of the paper is as follows. In the next section, we record the basic kinematics and introduce the constitutive expression for a stress power-law fluid. This is followed in $\S 3$ by the statement of the problem and the development of the governing equations. In $\S 4$, we present the results and provide a discussion of the same.

[^2]

Figure 2. Deviatoric stress versus symmetric part of the velocity gradient.

## 2. Problem statement

Let $\mathbf{v}$ denote the velocity, $\mathbb{D}$ the symmetric part of the velocity gradient, $\mathbb{T}$ the Cauchy stress and $\mathbb{T}_{\delta}$ the deviatoric part of the Cauchy stress. Furthermore, let us suppose the constitutive relation for the fluid has the following structure [35]:

$$
\begin{equation*}
\mathbb{D}=\left\{\alpha\left[1+\beta\left|\mathbb{T}_{\delta}\right|^{2}\right]^{n}+\gamma\right\} \mathbb{T}_{\delta} \tag{2.1}
\end{equation*}
$$

where $\alpha, \beta, \gamma$ and $n$ are constants.
We are interested in studying flows of this new class of non-Newtonian fluids in tubes of noncircular cross-sections. We will initially assume that the flow comprises a primary axial flow with secondary flow on the cross-section of the tube. For values of $n$ for which the model is invertible, we know that purely axial flow is possible. We first show that secondary flow does not occur for any value of the exponent $n$.

## (a) Assumptions

To start with, we will assume that the flow is steady $(\partial \mathbf{v} / \partial t=\mathbf{0})$, that the velocity field is fully developed (which eliminates the dependence on the axial direction) and that velocity and stress can be expressed in the form ${ }^{5}$

$$
\begin{equation*}
\mathbf{v}=\mathbf{v}_{0}+\epsilon \mathbf{v}_{1}, \tag{2.2}
\end{equation*}
$$

where $\mathbf{v}_{0}$ is the primary axial flow (figure 3 ) and $\mathbf{v}_{1}$ is the secondary flow (in the plane perpendicular to the primary flow), and

$$
\begin{equation*}
\mathbb{T}=\mathbb{T}_{0}+\epsilon \mathbb{T}_{1} \tag{2.3}
\end{equation*}
$$

where $\mathbb{T}_{0}$ is the primary stress and $\mathbb{T}_{1}$ is the stress tensor that arises as a consequence of the secondary motion.

As stated in the introduction, when considering implicit constitutive relations, or one wherein the symmetric part of the velocity gradient is a function of the stress, we need to solve the balance

[^3]

Figure 3. Axial flow for elliptic, L-shaped and irregular cross-sections.
of linear momentum

$$
\begin{equation*}
\rho \frac{\mathrm{d} \mathbf{v}}{\mathrm{~d} t}=\operatorname{div} \mathbb{T}+\rho \mathbf{b}, \tag{2.4}
\end{equation*}
$$

and the balance of mass for an incompressible fluid given by

$$
\begin{equation*}
\operatorname{div} \mathbf{v}=0 \tag{2.5}
\end{equation*}
$$

and the constitutive relation (2.1) simultaneously. For simplicity of the analysis, we will ignore the body forces.

## (b) Non-dimensional equations

In order to render the equations dimensionless, we introduce the following dimensionless variables:

$$
\left.\begin{array}{l}
\mathbf{x}^{*}=\frac{1}{L} \mathbf{x}, \quad t^{*}=\frac{V}{L} t, \quad \mathbf{v}^{*}=\frac{1}{V} \mathbf{v},  \tag{2.6}\\
\mathbb{D}^{*}=\frac{L}{V} \mathbb{D}, \quad \mathbb{T}^{*}=\frac{\alpha L}{V} \mathbb{T}, \quad \mathbb{T}_{\delta}^{*}=\frac{\alpha L}{V}\left(\mathbb{T}^{*}\right)_{\delta}
\end{array}\right\}
$$

where $L$ is a characteristic length, say the major axis of the ellipse, $V$ is the characteristic maximum speed and $\alpha$ is the characteristic inverse of the dynamic viscosity.

Now, neglecting body forces, equations (2.1), (2.4) and (2.5) become

$$
\begin{align*}
\mathbb{D}^{*} & =\left[\left(1+R_{2}\left|\mathbb{T}_{\delta}^{*}\right|^{2}\right)^{n}+R_{3}\right] \mathbb{T}_{\delta}^{*}  \tag{2.7}\\
\frac{\mathrm{~d} \mathbf{v}^{*}}{\mathrm{~d} t^{*}} & =\frac{1}{R_{1}} \operatorname{div} \mathbb{T}^{*} \tag{2.8}
\end{align*}
$$

and

$$
\begin{equation*}
\operatorname{div} \mathbf{v}^{*}=0, \tag{2.9}
\end{equation*}
$$

where $R_{1}=\alpha \rho V L, R_{2}=\beta V^{2} / \alpha^{2} L^{2}$ and $R_{3}=\gamma / \alpha$. For the sake of simplicity of notation, from now on we will drop the asterisk.

The non-dimensional number $R_{1}$ is the Reynolds number, and $R_{2}$ and $R_{3}$ are both measures of the ratio of the non-Newtonian viscosities to the Newtonian viscosity.

## (c) Equations governing the primary and secondary flows

By substituting equations (2.2) and (2.3) into equations (2.8) and (2.9) and taking into account the assumptions previously discussed, one obtains that

$$
\begin{equation*}
\left[\operatorname{grad}\left(\mathbf{v}_{0}+\epsilon \mathbf{v}_{1}\right)\right]\left(\mathbf{v}_{0}+\epsilon \mathbf{v}_{1}\right)=\frac{1}{R_{1}} \operatorname{div}\left(\mathbb{T}_{0}+\epsilon \mathbb{T}_{1}\right) \tag{2.10}
\end{equation*}
$$

and

$$
\begin{equation*}
\operatorname{div}\left(\mathbf{v}_{0}+\epsilon \mathbf{v}_{1}\right)=0 . \tag{2.11}
\end{equation*}
$$

Now, if we substitute (2.2) and (2.3) into (2.7) and use the linearity of the trace operator we obtain:

$$
\begin{equation*}
\mathbb{D}_{0}+\epsilon \mathbb{D}_{1}=\left[\left(1+R_{2}\left|\mathbb{T}_{0 \delta}+\epsilon \mathbb{T}_{1 \delta}\right|^{2}\right)^{n}+R_{3}\right]\left(\mathbb{T}_{0 \delta}+\epsilon \mathbb{T}_{1 \delta}\right) . \tag{2.12}
\end{equation*}
$$

Next, since the primary flow $\mathbf{v}_{0}$ is assumed to be along the axis, it has only one component that is along the axial direction, and since we assume the tube to be infinite in length, that component depends only on the coordinates in the plane normal to the axis, and thus (grad $\left.\mathbf{v}_{0}\right) \mathbf{v}_{0}=\mathbf{0}$. Similarly, it is easy to see that $\left(\operatorname{grad} \mathbf{v}_{1}\right) \mathbf{v}_{0}=\mathbf{0}$. Moreover, because of the nature of the secondary flow, the tractions associated with both the primary and the secondary Cauchy stress tensors are perpendicular (figure 4), which then leads to the relation:

$$
\begin{equation*}
\left|\mathbb{T}_{0 \delta}+\epsilon \mathbb{T}_{1 \delta}\right|^{2}=\left|\mathbb{T}_{0 \delta}\right|^{2}+\epsilon^{2}\left|\mathbb{T}_{1 \delta}\right|^{2} \approx\left|\mathbb{T}_{0 \delta}\right|^{2} \tag{2.13}
\end{equation*}
$$

Rearranging terms that appear in equations (2.10) and (2.11) and making use of equation (2.12) and using the definition of trace of a tensor leads to the following equations:

$$
\begin{align*}
& \epsilon\left(\operatorname{grad} \mathbf{v}_{0}\right) \mathbf{v}_{1}+\epsilon^{2}\left(\operatorname{grad} \mathbf{v}_{1}\right) \mathbf{v}_{1} \\
& \quad=\frac{1}{R_{1}}\left[\operatorname{div} \mathbb{T}_{0 \delta}+\operatorname{grad}\left(\frac{1}{3} \operatorname{tr} \mathbb{T}_{0}\right) \mathbb{I}\right]+\epsilon \frac{1}{R_{1}}\left[\operatorname{div} \mathbb{T}_{1 \delta}+\operatorname{grad}\left(\frac{1}{3} \operatorname{tr} \mathbb{T}_{1}\right) \mathbb{I}\right],  \tag{2.14}\\
& \mathbb{D}_{0}+\epsilon \mathbb{D}_{1}=\left[\left(1+R_{2}\left|\mathbb{T}_{0 \delta}\right|^{2}\right)^{-n}+R_{3}\right] \mathbb{T}_{0 \delta}+\epsilon\left[\left(1+R_{2}\left|\mathbb{T}_{0 \delta}\right|^{2}\right)^{-n}+R_{3}\right] \mathbb{T}_{1 \delta} \tag{2.15}
\end{align*}
$$

and $\quad \operatorname{div} \mathbf{v}_{0}+\epsilon \operatorname{div} \mathbf{v}_{1}=0$.

Finally, equating the terms of the polynomial in the variable $\epsilon$ from the three equations recorded above, and neglecting the only second-order term ( $\left.\left(\operatorname{grad} \mathbf{v}_{1}\right) \mathbf{v}_{1}\right)$ that appears, the problem of finding the secondary flow leads to two coupled systems of partial differential equations given by

$$
\left.\begin{array}{l}
-\operatorname{grad}\left(\frac{1}{3} \operatorname{tr} \mathbb{T}_{0}\right)=\operatorname{div} \mathbb{T}_{0 \delta},  \tag{2.17}\\
\mathbb{D}_{0}=\left[\left(1+R_{2}\left|\mathbb{T}_{0 \delta}\right|^{2}\right)^{n}+R_{3}\right] \mathbb{T}_{0 \delta,} \\
\operatorname{div} \mathbf{v}_{0}=0
\end{array}\right\}
$$



Figure 4. Primary shear traction (big arrow) and secondary shear and normal traction (small arrow). The axial direction is along the $z$-coordinate.

$$
\left.\begin{array}{l}
-\operatorname{grad}\left(\frac{1}{3} \operatorname{tr} \mathbb{T}_{1}\right)=-\left(\operatorname{grad} \mathbf{v}_{0}\right) \mathbf{v}_{1}+\operatorname{div} \mathbb{T}_{1 \delta},  \tag{2.18}\\
\mathbb{D}_{1}=\left[\left(1+R_{2}\left|\mathbb{T}_{0 \delta}\right|^{2}\right)^{n}+R_{3}\right] \mathbb{T}_{1 \delta}, \\
\operatorname{div} \mathbf{v}_{1}=0
\end{array}\right\}
$$

Note that the system of equations (2.18) is linear in both $\mathbf{v}_{1}$ and $\mathbb{T}_{1 \delta}$, with one of its terms depending on $\mathbb{T}_{0 \delta}$ known from the solution of the problem at zeroth order. Also, it is clear that the system of partial differential equations (PDEs) for the secondary flow can only be solved once the solution for the primary flow is obtained.

Now that we have set up the appropriate system of PDEs, we will develop the theoretical scheme necessary for solving them.

## 3. Mathematical statement of the problem

Since we assumed fully developed flow, we can focus our analysis on what happens on a fixed cross-sectional area of the tube. Thus in the case of a tube with elliptic cross-section (given $a>0$, $b>0)$ let $\Omega=\left\{(x, y) \in \mathbb{R}: x^{2} / a^{2}+y^{2} / b^{2}<1\right\} \subset \mathbb{R}^{2}$ be the domain. Then the associated system of PDEs is given by

$$
\left.\begin{array}{ll}
\frac{\partial}{\partial x}\left(\frac{1}{3} \operatorname{tr} \mathbb{T}_{0}\right)=0 & \text { in } \Omega,  \tag{3.1}\\
\frac{\partial}{\partial y}\left(\frac{1}{3} \operatorname{tr} \mathbb{T}_{0}\right)=0 & \text { in } \Omega, \\
\frac{\partial}{\partial z}\left(\frac{1}{3} \operatorname{tr} \mathbb{T}_{0}\right)=\frac{\partial\left(\mathbb{T}_{0 \delta}\right)_{x z}}{\partial x}+\frac{\partial\left(\mathbb{T}_{0 \delta}\right)_{y z}}{\partial y} & \text { in } \Omega, \\
\frac{\partial\left(\mathbf{v}_{0}\right)_{z}}{\partial x}=2 G\left(\left(\mathbb{T}_{0 \delta}\right)_{x z}\left(\mathbb{T}_{0 \delta}\right)_{y z}\right)\left(\mathbb{T}_{0 \delta}\right)_{x z} & \text { in } \Omega, \\
\frac{\partial\left(\mathbf{v}_{0}\right)_{z}}{\partial y}=2 G\left(\left(\mathbb{T}_{0 \delta}\right)_{x z}\left(\mathbb{T}_{0 \delta}\right)_{y z}\right)\left(\mathbb{T}_{0 \delta}\right)_{y z} & \text { in } \Omega \\
\left(\mathbf{v}_{0}\right)_{z}=0 & \text { on } \partial \Omega,
\end{array}\right\}
$$

where $\mathbf{v}_{0}=\left(0,0,\left(\mathbf{v}_{0}\right)_{z}\right)$ and $\left(\mathbb{T}_{0 \delta}\right)_{x z}$ and $\left(\mathbb{T}_{0 \delta}\right)_{y z}$ are the only non-null elements of the stress tensor $\mathbb{T}_{0 \delta}$. It is easy to see that $\left(\frac{1}{3} \operatorname{tr} \mathbb{T}_{0}\right)$ is a function of $z$ only, and since the flow is fully developed the
system becomes

$$
\left.\begin{array}{ll}
\frac{\partial\left(\mathbb{T}_{0 \delta}\right)_{x z}}{\partial x}+\frac{\partial\left(\mathbb{T}_{0 \delta}\right)_{y z}}{\partial y}=\lambda & \text { in } \Omega,  \tag{3.2}\\
\frac{\partial\left(\mathbf{v}_{0}\right)_{z}}{\partial x}=2 G\left(\left(\mathbb{T}_{0 \delta}\right)_{x z},\left(\mathbb{T}_{0 \delta}\right)_{y z}\right)\left(\mathbb{T}_{0 \delta}\right)_{x z} & \text { in } \Omega, \\
\frac{\partial\left(\mathbf{v}_{0}\right)_{z}}{\partial y}=2 G\left(\left(\mathbb{T}_{0 \delta}\right)_{x z}\left(\mathbb{T}_{0 \delta}\right)_{y z}\right)\left(\mathbb{T}_{0 \delta}\right)_{y z} & \text { in } \Omega, \\
\left(\mathbf{v}_{0}\right)_{z}=0 & \text { on } \partial \Omega,
\end{array}\right\}
$$

with $G\left(\left(\mathbb{T}_{0 \delta}\right)_{x z},\left(\mathbb{T}_{0 \delta}\right)_{y z}\right)=\left[1+2 R_{2}\left(\left(\mathbb{T}_{0 \delta}\right)_{x z}^{2}+\left(\mathbb{T}_{0 \delta}\right)_{y z}^{2}\right)\right]^{n}+R_{3}$.

## (a) Variational formulation

We now proceed to seek a weak solution to the problem governed by equations (2.2) and (2.3).
Let us define a vector field $\mathbf{T}=\left(\left(\mathbb{T}_{0 \delta}\right)_{x z},\left(\mathbb{T}_{0 \delta}\right)_{y z}\right)$ and note that if we take a smooth function $v=\left(\mathbf{v}_{0}\right)_{z}$ with compact support in $\Omega$ and a smooth two-dimensional vector field $\mathbf{T}$, it follows that

$$
\begin{equation*}
\int_{\Omega} \operatorname{div}(v \mathbf{T}) \mathrm{d} x=\int_{\partial \Omega} v \mathbf{T} \cdot \hat{\mathbf{n}} \mathrm{~d} S=\int_{\Omega} \operatorname{grad} v \cdot \mathbf{T} \mathrm{~d} x+\int_{\Omega} v \operatorname{div} \mathbf{T} \mathrm{~d} x . \tag{3.3}
\end{equation*}
$$

This clearly leads to the relation

$$
\begin{equation*}
\int_{\Omega}-\operatorname{grad} v \cdot \mathbf{T} \mathrm{~d} x=\int_{\Omega} v \operatorname{div} \mathbf{T} \mathrm{~d} x \tag{3.4}
\end{equation*}
$$

The previous equality holds for any $v$ and $\mathbf{T}$ since they are arbitrary.
Thus, the problem (3.2) can be written as

$$
\left.\begin{array}{ll}
\operatorname{div} \mathbf{T}=\lambda & \text { in } \Omega,  \tag{3.5}\\
\operatorname{grad} v=2 G\left(|\mathbf{T}|^{2}\right) \mathbf{T} & \text { in } \Omega, \\
v=0 & \text { on } \partial \Omega .
\end{array}\right\}
$$

Now, let $w$ be a smooth function with compact support in $\Omega$ and $\mathbf{S}$ be a smooth twodimensional vector field. Forming the scalar product of the first and second equations of (3.5) with $w$ and $\mathbf{S}$, respectively, and integrating the result over $\Omega$ yields:

$$
\left.\begin{array}{l}
\int_{\Omega}(\operatorname{div} \mathbf{T}) w \mathrm{~d} x=\int_{\Omega} \lambda w \mathrm{~d} x,  \tag{3.6}\\
\int_{\Omega} \operatorname{grad} v \cdot \mathbf{S} \mathrm{~d} x=\int_{\Omega} 2 G\left(|\mathbf{T}|^{2}\right) \mathbf{T} \cdot \mathbf{S} \mathrm{d} x .
\end{array}\right\}
$$

Using the property (3.4) we end up with

$$
\left.\begin{array}{l}
-\int_{\Omega} \mathbf{T} \cdot \operatorname{grad} w \mathrm{~d} x=\int_{\Omega} \lambda w \mathrm{~d} x  \tag{3.7}\\
\int_{\Omega} \operatorname{grad} v \cdot \mathbf{S} \mathrm{~d} x=\int_{\Omega} 2 G\left(|\mathbf{T}|^{2}\right) \mathbf{T} \cdot \mathbf{S} \mathrm{d} x,
\end{array}\right\}
$$

which is true for any $w$ and $\mathbf{S}$ as previously defined. Finally, by invoking some classical results about density, we know that (3.7) holds for $v \in H_{0}^{1}(\Omega)$ and $\mathbf{T} \in\left(L^{2}(\Omega)\right)^{2}$.

## (b) Theoretical approach

The core analytical approach to solving the problem that corresponds to the flow having the specific form given by equations (2.2) and (2.3) is given below. It is important to bear in mind that, for the fully nonlinear problem, other solutions might exist. We are only proving the existence and uniqueness of flows given by (2.2) and (2.3).

Let us define the operator $G:\left(L^{2}(\Omega)\right)^{2} \mapsto\left(L^{2}(\Omega)\right)^{2}$ such that

$$
\begin{equation*}
G(|\mathbf{T}|) \mathbf{T}=2\left\{\left(1+2 R_{2}|\mathbf{T}|^{2}\right)^{n}+R_{3}\right\} \mathbf{T} \tag{3.8}
\end{equation*}
$$

which clearly is not linear, but it still has some useful properties that can help in our task of finding a solution. In fact, it is easy to see that

$$
\begin{equation*}
\left\|\left\{\left(1+2 R_{2}|\mathbf{T}|^{2}\right)^{n}+R_{3}\right\} \mathbf{T}\right\|_{L^{2}(\Omega)} \leq\left(1+R_{3}\right)\|\mathbf{T}\|_{L^{2}(\Omega)}, \tag{3.9}
\end{equation*}
$$

i.e. $G(|\mathbf{T}|) \mathbf{T} \in\left(L^{2}(\Omega)\right)^{2}$, and also

$$
\begin{equation*}
R_{3}\|\mathbf{T}\|_{L^{2}(\Omega)}^{2} \leq \int_{\Omega}\left\{\left(1+2 R_{2}|\mathbf{T}|^{2}\right)^{n}+R_{3}\right\} \mathbf{T} \cdot \mathbf{T} \mathrm{d} x \tag{3.10}
\end{equation*}
$$

Now, we take T, $\mathbf{S} \in\left(L^{2}(\Omega)\right)^{2} \bigcap\left(C^{1}(\Omega)\right)^{2}$ and let $t>0$ to calculate the following:

$$
\begin{equation*}
\int_{\Omega}\left[G\left(|\mathbf{T}+t \mathbf{S}|^{2}\right)(\mathbf{T}+t \mathbf{S})-G\left(|\mathbf{T}|^{2}\right) \mathbf{T}\right] \cdot[(\mathbf{T}+t \mathbf{S})-\mathbf{T}] \mathrm{d} x \tag{3.11}
\end{equation*}
$$

Rearranging terms and dividing by $1 / t^{2}$ yields

$$
\begin{equation*}
\int_{\Omega} \frac{1}{t}\left[G\left(|\mathbf{T}+t \mathbf{S}|^{2}\right)-G\left(|\mathbf{T}|^{2}\right)\right] \mathbf{T} \cdot \mathbf{S} \mathrm{d} x+\int_{\Omega} G\left(|\mathbf{T}+t \mathbf{S}|^{2}\right) \mathbf{S} \cdot \mathbf{S} \mathrm{d} x \tag{3.12}
\end{equation*}
$$

and by letting $t \rightarrow 0$ we obtain

$$
\begin{equation*}
\int_{\Omega}\left(\operatorname{grad} G\left(|\mathbf{T}|^{2}\right) \cdot \mathbf{T}\right)(\mathbf{T} \cdot \mathbf{S}) \mathrm{d} x+\int_{\Omega} G\left(|\mathbf{T}|^{2}\right) \mathbf{S} \cdot \mathbf{S} \mathrm{d} x \tag{3.13}
\end{equation*}
$$

Finally, by calculating the actual value of the differential of $G$, substituting it back into equation (3.13) and by the theorem of monotonicity of differentiable mappings, we find that the monotonicity of the operator holds if, for any $\mathbf{T}, \mathbf{S} \in\left(L^{2}(\Omega)\right)^{2}$,

$$
\begin{equation*}
\int_{\Omega}|\mathbf{S}|^{2}\left[4 R_{2} n\left(1+2 R_{2}|\mathbf{T}|^{2}\right)^{n-1}|\mathbf{T}|^{2}+G\left(|\mathbf{T}|^{2}\right)\right] \mathrm{d} x \geq 0 \tag{3.14}
\end{equation*}
$$

The critical point when $|\mathbf{T}|=0$ gives the final condition that the operator is monotonic if

$$
\begin{equation*}
R_{3} \geq 2\left(\frac{2 n-2}{2 n+1}\right)^{n-1} \tag{3.15}
\end{equation*}
$$

It turns out that this is the same condition for monotonicity as found by Le Roux \& Rajagopal [35]. Obviously, for positive values of $n$, the monotonicity is uniform, i.e. it is independent of any of the other parameters.

Finally, since the operator $G(|\mathbf{T}|) \mathbf{T}$ is continuous, bounded, monotone and coercive, the Browder-Minty theorem implies that for any $\mathbf{S} \in\left(L^{2}(\Omega)\right)^{2}$ there exists $\mathbf{T} \in\left(L^{2}(\Omega)\right)^{2}$ such that

$$
\begin{equation*}
G(|\mathbf{T}|) \mathbf{T}=\mathbf{S} \tag{3.16}
\end{equation*}
$$

as long as (3.15) holds.
Using the theoretical results above, we can prove the next result.
Proposition 3.1. Let $\lambda \in \mathbb{R}, R_{2}>0, R_{3}>0$ and $n \in(-\infty,-1 / 2]$ be such that $R_{3} \geq 2((2 n-2) /$ $(2 n+1))^{n-1}$. Then, for the variational formulation given by (3.7) there exists a unique pair $(\mathbf{T}, v) \in$ $L^{2}(\Omega)^{2} \times H_{0}^{1}(\Omega)$ such that

$$
\left.\begin{array}{l}
-\int_{\Omega} \mathbf{T} \cdot \operatorname{grad} w \mathrm{~d} x=\int_{\Omega} \lambda w \mathrm{~d} x,  \tag{3.17}\\
\int_{\Omega} \operatorname{grad} v \cdot \mathbf{S} \mathrm{~d} x=\int_{\Omega} 2 G\left(|\mathbf{T}|^{2}\right) \mathbf{T} \cdot \mathbf{S} \mathrm{d} x
\end{array}\right\}
$$

hold for any $(\mathbf{S}, w) \in L^{2}(\Omega)^{2} \times H_{0}^{1}(\Omega)$.

Proof. Let us define the function $\lambda=-\operatorname{div} \mathbf{F}_{\lambda}$. Then the problem becomes

$$
\left.\begin{array}{l}
\int_{\Omega}\left(\mathbf{T}-\mathbf{F}_{\lambda}\right) \cdot \operatorname{grad} w \mathrm{~d} x=0,  \tag{3.18}\\
\int_{\Omega} \operatorname{grad} v \cdot \mathbf{S} \mathrm{~d} x=\int_{\Omega} 2 G\left(|\mathbf{T}|^{2}\right) \mathbf{T} \cdot \mathbf{S} \mathrm{d} x,
\end{array}\right\}
$$

which holds for any $(\mathbf{S}, w) \in L^{2}(\Omega)^{2} \times H_{0}^{1}(\Omega)$.
In particular, for $\mathbf{T} \in L^{2}(\Omega)^{2}$, the second equation of (3.18) becomes

$$
\begin{equation*}
\int_{\Omega} \operatorname{grad} v \cdot \mathbf{T} \mathrm{~d} x=\int_{\Omega} 2 G\left(|\mathbf{T}|^{2}\right) \mathbf{T} \cdot \mathbf{T} \mathrm{d} x \geq R_{3}\|\mathbf{T}\|_{\left(L^{2}(\Omega)\right)^{2}}^{2} . \tag{3.19}
\end{equation*}
$$

Then, by the Cauchy-Schwartz inequality we finally establish

$$
\begin{equation*}
\|\mathbf{T}\|_{\left(L^{2}(\Omega)\right)^{2}} \leq \frac{C_{p}}{2 R_{3}}\|v\|_{H_{0}^{1}(\Omega)} . \tag{3.20}
\end{equation*}
$$

Also, by the upper boundedness of the nonlinear term, the Poincaré inequality and by virtue of the second equation of (3.5), we infer that

$$
\begin{equation*}
\|v\|_{H_{0}^{1}(\Omega)} \leq \frac{2\left(1+R_{3}\right)}{C_{p}}\|\mathbf{T}\|_{\left(L^{2}(\Omega)\right)^{2}} . \tag{3.21}
\end{equation*}
$$

Next, note that in the first equation of (3.18), we have

$$
\begin{equation*}
\int_{\Omega} \mathbf{T} \cdot \operatorname{grad} v \mathrm{~d} x=\int_{\Omega} \mathbf{F}_{\lambda} \cdot \operatorname{grad} v \mathrm{~d} x . \tag{3.22}
\end{equation*}
$$

Thus, we see that the stress has bounds given by

$$
\begin{equation*}
\|\mathbf{T}\|_{\left(L^{2}(\Omega)\right)^{2}} \leq \frac{\left(1+R_{3}\right)}{R_{3}}\left\|\mathbf{F}_{\lambda}\right\|_{\left(L^{2}(\Omega)\right)^{2}} \tag{3.23}
\end{equation*}
$$

and since the velocity is bounded by the stress in (3.21), then the velocity is also bounded by the pressure gradient. Therefore, $G\left(|\mathbf{T}|^{2}\right) \mathbf{T}$ is continuous, coercive and monotonic so, by the MintyBrowder theorem, there exists a pair $(\mathbf{T}, v) \in\left(L^{2}(\Omega)\right)^{2} \times H_{0}^{1}(\Omega)$ such that (3.17) holds.

Finally, let us assume that we have two solutions $\left(\mathbf{T}_{1}, v_{1}\right)$ and ( $\mathbf{T}_{2}, v_{2}$ ) such that they satisfy (3.5). Then subtracting one from the other we have that

$$
\left.\begin{array}{ll}
\operatorname{div}\left(\mathbf{T}_{1}-\mathbf{T}_{2}\right)=0 & \text { in } \Omega,  \tag{3.24}\\
\operatorname{grad}\left(v_{1}-v_{2}\right)=2\left(G\left(\left|\mathbf{T}_{1}\right|^{2}\right) \mathbf{T}_{1}-G\left(\left|\mathbf{T}_{2}\right|^{2}\right) \mathbf{T}_{2}\right) & \text { in } \Omega .
\end{array}\right\}
$$

Forming the inner product of the second equation with $\mathbf{T}_{1}-\mathbf{T}_{2}$ yields

$$
\begin{equation*}
\left(\operatorname{grad}\left(v_{1}-v_{2}\right), \mathbf{T}_{1}-\mathbf{T}_{2}\right)_{\left(L^{2}(\Omega)\right)^{2}}=2\left(\left(G\left(\left|\mathbf{T}_{1}\right|^{2}\right) \mathbf{T}_{1}-G\left(\left|\mathbf{T}_{2}\right|^{2}\right) \mathbf{T}_{2}\right), \mathbf{T}_{1}-\mathbf{T}_{2}\right)_{\left(L^{2}(\Omega)\right)^{2}} \tag{3.25}
\end{equation*}
$$

Then note that the left-hand side has the property

$$
\begin{equation*}
\left(\operatorname{grad}\left(v_{1}-v_{2}\right), \mathbf{T}_{1}-\mathbf{T}_{2}\right)_{\left(L^{2}(\Omega)\right)^{2}}=-\left(v_{1}-v_{2}, \operatorname{div}\left(\mathbf{T}_{1}-\mathbf{T}_{2}\right)\right)_{\left(L^{2}(\Omega)\right)^{2}}=0 . \tag{3.26}
\end{equation*}
$$

This implies by the monotonicity of the nonlinear operator that $\mathbf{T}_{1}=\mathbf{T}_{2}$, which then implies that $v_{1}=v_{2}$.

## (c) Secondary flow

Recall the system of equations for the secondary flow:

$$
\left.\begin{array}{ll}
-\frac{1}{R_{1}} \operatorname{grad}\left(\frac{1}{3} \operatorname{tr} \mathbb{T}_{1}\right)=-\left(\operatorname{grad} \mathbf{v}_{0}\right) \mathbf{v}_{1}+\frac{1}{R_{1}} \operatorname{div} \mathbb{T}_{1 \delta} & \text { in } \Omega, \\
\mathbb{D}_{1}=\left[\left(1+R_{2}\left\|\mathbb{T}_{0 \delta}\right\|^{2}\right)^{n}+R_{3}\right] \mathbb{T}_{1 \delta} & \text { in } \Omega,  \tag{3.27}\\
\operatorname{div} \mathbf{v}_{1}=0 & \text { in } \Omega .
\end{array}\right\}
$$

The terms $\mathbf{v}_{0}$ and $\mathbb{T}_{0 \delta}$ are already known, so this problem is linear in $\mathbf{v}_{1}$ and $\mathbb{T}_{1 \delta}$. Moreover, it can be written in terms of velocity and the trace of $\mathbb{T}_{1 \delta}$ as

$$
\left.\begin{array}{ll}
-\frac{1}{R_{1}} \operatorname{grad}\left(\frac{1}{3} \operatorname{tr} \mathbb{T}_{1}\right)=-\left(\operatorname{grad} \mathbf{v}_{0}\right) \mathbf{v}_{1}+\frac{1}{R_{1}} \operatorname{div}\left[\frac{1}{2 G\left(\mathbb{T}_{0}\right)}\left(\operatorname{grad} \mathbf{v}_{1}+\operatorname{grad} \mathbf{v}_{1}^{\mathrm{T}}\right)\right] & \text { in } \Omega,  \tag{3.28}\\
\operatorname{div} \mathbf{v}_{1}=0 & \text { in } \Omega,
\end{array}\right\}
$$

where $\mathbf{v}_{1}=\left(\left(\mathbf{v}_{1}\right)_{x},\left(\mathbf{v}_{1}\right)_{y}, 0\right)$ and the term $\left(\frac{1}{3} \operatorname{tr} \mathbb{T}_{1}\right)$ are unknown. Note that only the gradient on the left-hand side and the first term on the right-hand side of the first equation have components in the $z$-direction.

The two-dimensional nature of the problem and the fact that the divergence of the velocity is zero guarantees the existence of a streamfunction $\psi$ such that

$$
\left.\begin{array}{ll}
\left(\mathbf{v}_{1}\right)_{x}=\frac{\partial \psi}{\partial y} & \text { in } \Omega  \tag{3.29}\\
\left(\mathbf{v}_{1}\right)_{y}=-\frac{\partial \psi}{\partial x} & \text { in } \Omega .
\end{array}\right\}
$$

Next, on calculating the curl of the conservation of linear momentum (3.28) and substituting the streamfunction (3.29) yields

$$
\begin{array}{ll}
\Delta^{2} \psi+G\left[\left(\frac{\partial^{2}}{\partial x^{2}}-\frac{\partial^{2}}{\partial y^{2}}\right)\left(\frac{1}{G}\right)\right]\left(\frac{\partial^{2} \psi}{\partial x^{2}}-\frac{\partial^{2} \psi}{\partial y^{2}}\right)+4 G \frac{\partial^{2}}{\partial x \partial y}\left(\frac{1}{G}\right) \frac{\partial^{2} \psi}{\partial x \partial y}=0 & \text { in } \Omega, \\
\frac{\partial \psi}{\partial x}=\frac{\partial \psi}{\partial y}=0 & \text { on } \partial \Omega,  \tag{3.30}\\
\psi=\psi_{0} & \text { on } \partial \Omega,
\end{array}
$$

where the symbol $\Delta^{2}$ stands for the biharmonic operator. The boundary conditions arise naturally by virtue of the definition of $\psi$ and also from the fact that the boundary itself is also a streamline.

Note that this is a linear PDE for $\psi$ with the function $G$ given from the previous step of the axial flow. In particular, if the exponent $n=0$, the function $G$ becomes constant and the equation reduces to a biharmonic equation and the solution for $\psi$ is constant, which we can pick to be zero.

Without any loss of generality with regard to the boundary conditions, we can rewrite the problem as follows:

$$
\begin{array}{ll}
\frac{\partial^{2}}{\partial x^{2}}\left[\frac{1}{G}\left(\frac{\partial^{2} \psi}{\partial x^{2}}-\frac{\partial^{2} \psi}{\partial y^{2}}\right)\right]-\frac{\partial^{2}}{\partial y^{2}}\left[\frac{1}{G}\left(\frac{\partial^{2} \psi}{\partial x^{2}}-\frac{\partial^{2} \psi}{\partial y^{2}}\right)\right]+4 \frac{\partial^{2}}{\partial x \partial y}\left[\frac{1}{G} \frac{\partial^{2} \psi}{\partial x \partial y}\right]=0 & \text { in } \Omega, \\
\frac{\partial \psi}{\partial x}=\frac{\partial \psi}{\partial y}=0 & \text { on } \partial \Omega \\
\psi=0 & \text { on } \partial \Omega \tag{3.31}
\end{array}
$$

Next, let $\phi \in \mathcal{D}(\bar{\Omega})$ be given. After multiplying and integrating over the domain, we get the variational form given by

$$
\begin{equation*}
\int_{\Omega} \frac{1}{G}\left[\frac{\partial^{2} \psi}{\partial x^{2}} \frac{\partial^{2} \phi}{\partial x^{2}}-\frac{\partial^{2} \psi}{\partial x^{2}} \frac{\partial^{2} \phi}{\partial y^{2}}-\frac{\partial^{2} \psi}{\partial y^{2}} \frac{\partial^{2} \phi}{\partial x^{2}}+4 \frac{\partial^{2} \psi}{\partial x \partial y} \frac{\partial^{2} \phi}{\partial x \partial y}+\frac{\partial^{2} \psi}{\partial y^{2}} \frac{\partial^{2} \phi}{\partial y^{2}}\right]=0 . \tag{3.32}
\end{equation*}
$$

$$
\begin{equation*}
B(\psi, \phi)=\int_{\Omega} \frac{1}{G}\left[\Delta \psi \Delta \phi-2 \frac{\partial^{2} \psi}{\partial x^{2}} \frac{\partial^{2} \phi}{\partial y^{2}}-2 \frac{\partial^{2} \psi}{\partial y^{2}} \frac{\partial^{2} \phi}{\partial x^{2}}+4 \frac{\partial^{2} \psi}{\partial x \partial y} \frac{\partial^{2} \phi}{\partial x \partial y}\right] \mathrm{d} x \tag{3.33}
\end{equation*}
$$

From the boundedness of $G$ (upper and lower) we can get the following estimates:

$$
\left.\begin{array}{l}
|B(\psi, \phi)| \leq \frac{1}{R_{3}}\left\|\mathrm{D}^{2} \psi\right\|_{L^{2}(\Omega)}\left\|\mathrm{D}^{2} \phi\right\|_{L^{2}(\Omega)^{\prime}}  \tag{3.34}\\
B(\psi, \psi) \geq \frac{1}{1+R_{3}}\|\Delta \psi\|_{L^{2}(\Omega)^{\prime}}^{2}
\end{array}\right\}
$$

where $\mathrm{D}^{2} \phi$ is the second-order derivative operator applied in $\phi$. Note that the derivatives of $\psi$ are functions of $H_{0}^{1}(\Omega)$; therefore, we can apply the Poincaré inequality to get

$$
\begin{equation*}
\|\mathrm{D} \psi\|_{L^{2}(\Omega)} \leq c_{p}\left\|\mathrm{D}^{2} \psi\right\|_{L^{2}(\Omega)} \tag{3.35}
\end{equation*}
$$

Also, we have the property

$$
\begin{equation*}
\int_{\Omega} \frac{\partial^{2} \psi}{\partial x \partial y} \frac{\partial^{2} \psi}{\partial x \partial y} \mathrm{~d} x=-\int_{\Omega} \frac{\partial \psi}{\partial x} \frac{\partial^{3} \psi}{\partial x \partial y^{2}} \mathrm{~d} x=\int_{\Omega} \frac{\partial^{2} \psi}{\partial x^{2}} \frac{\partial^{2} \psi}{\partial y^{2}} \mathrm{~d} x \tag{3.36}
\end{equation*}
$$

and thus it is easy to see that the two previous equations imply the inequality

$$
\begin{equation*}
\|\psi\|_{L^{2}(\Omega)} \leq c_{p}^{2}\|\Delta \psi\|_{L^{2}(\Omega)} \tag{3.37}
\end{equation*}
$$

that holds for any $\psi \in \mathcal{D}(\bar{\Omega})$, and by density in $H_{0}^{2}(\Omega)$.
The Laplacian operator is continuous and bounded in $H_{0}^{2}(\Omega)$. Now, since the Laplacian is also symmetric, and positive definite, we can directly claim that the bilinear form is also an inner product. Finally, we claim that the problem has solution and that it is unique for any given righthand side element in $H^{-2}(\Omega)$.

Finally, since the boundary condition for $\psi$ is zero, and since (3.31) involves derivatives only, we have finally proved that no secondary flows are possible for the model under consideration for the specific flow field and stress field sought.

## 4. Numerical method

Recall the variational formulation for the primary flow:

$$
\left.\begin{array}{l}
-\int_{\Omega} \mathbf{T} \cdot \operatorname{grad} w \mathrm{~d} x=\int_{\Omega} \lambda w \mathrm{~d} x  \tag{4.1}\\
\int_{\Omega} \operatorname{grad} v \cdot \mathbf{S} \mathrm{~d} x=\int_{\Omega} 2 G\left(|\mathbf{T}|^{2}\right) \mathbf{T} \cdot \mathbf{S} \mathrm{d} x
\end{array}\right\}
$$

Let us take a Delaunay triangulation of the domain, and set $V_{h}$ to be defined by

$$
\begin{equation*}
V_{h}=\left\{w_{h} \in C^{0}(\Omega): w_{h} \in \mathcal{P}_{1}\left(\tau_{h}\right), \tau_{h} \in \mathcal{T}_{h}\right\} \tag{4.2}
\end{equation*}
$$

where $\mathcal{P}_{1}\left(\tau_{h}\right)$ is the space of first-degree polynomials on the triangle $\tau_{h}$ of the triangulation $\mathcal{T}_{h}$. For the stress, we can take $L_{h}$ to be

$$
\begin{equation*}
L_{h}=\left\{\mathbf{T}_{h} \in L^{2}(\Omega): \mathbf{T}_{h}=T_{x} \hat{\mathbf{e}}_{x}+T_{y} \hat{\mathbf{e}}_{y}\right\} \tag{4.3}
\end{equation*}
$$

The elements of the $L_{h}$ space are functions that are constant on each triangle. Now let us take a triangle $\tau_{h}$ of the mesh with vertices $i, j$ and $k$, and define the $i$ th nodal basis element of $V_{h}$ as

$$
\phi_{i}(x)= \begin{cases}1 & \text { if } x=P_{i}  \tag{4.4}\\ 0 & \text { if } x=P_{j} \text { or } x=P_{k} \\ \text { linear } & \text { over } \tau_{h}\end{cases}
$$



In particular, for a triangle with vertices $P_{i}, P_{j}$ and $P_{k}$ (figure 5) the $i$ th element $\phi_{i}(x, y)$ is given by

$$
\begin{equation*}
\phi_{i}(x, y)=\frac{a_{i}}{2\left|\tau_{h}\right|}+\frac{b_{i}}{2\left|\tau_{h}\right|} x+\frac{c_{i}}{2\left|\tau_{h}\right|} y, \tag{4.5}
\end{equation*}
$$

where the constants $a_{i}, b_{i}$ and $c_{i}$ are calculated in terms of coordinates of the vertices, and $\left|\tau_{h}\right|$ is the size of the triangle.

Next, we define $\mathbf{T}_{h}=\left(T_{x}, T_{y}\right), \mathbf{S}_{h}=\left(S_{x}, S_{y}\right), v_{h}=\boldsymbol{\Phi}_{\tau}^{\mathrm{T}} \mathbf{V}$ and $w_{h}=\boldsymbol{\Phi}_{\tau}^{\mathrm{T}} \mathbf{W}$, where $\boldsymbol{\Phi}_{\tau}=\left(\phi_{i}, \phi_{j}, \phi_{k}\right)^{\mathrm{T}}$, $\mathbf{V}=\left(V_{i}, V_{j}, V_{k}\right)$ and $\mathbf{W}=\left(W_{i}, W_{j}, W_{k}\right)$. This leads to

$$
\begin{align*}
\int_{\tau_{h}} \operatorname{grad} \boldsymbol{\Phi}_{\tau}^{\mathrm{T}} \mathbf{V} \cdot \mathbf{S}_{h} \mathrm{~d} x & =\int_{\tau_{h}}\left(\nabla \phi_{i} V_{i}+\nabla \phi_{j} V_{j}+\nabla \phi_{k} V_{k}\right) \cdot\left(S_{x} \hat{\mathbf{e}}_{x}+S_{y} \hat{\mathbf{e}}_{y}\right) \mathrm{d} x \\
& =\frac{1}{2}\left(\begin{array}{lll}
b_{i} & b_{j} & b_{k} \\
c_{i} & c_{j} & c_{k}
\end{array}\right)\left(\begin{array}{l}
V_{i} \\
V_{j} \\
V_{k}
\end{array}\right) \cdot\binom{S_{x}}{S_{y}},  \tag{4.6}\\
\int_{\tau_{h}} \mathbf{T}_{h} \cdot \operatorname{grad} \boldsymbol{\Phi}_{\tau}^{\mathrm{T}} \mathbf{W} \mathrm{~d} x & =\int_{\tau_{h}}\left(T_{x} \hat{\mathbf{e}}_{x}+T_{y} \hat{\mathbf{e}}_{y}\right) \cdot\left(\nabla \phi_{i} W_{i}+\nabla \phi_{j} W_{j}+\nabla \phi_{k} W_{k}\right) \mathrm{d} x \\
& =\frac{1}{2}\left(\begin{array}{ll}
b_{i} & c_{i} \\
b_{j} & c_{j} \\
b_{k} & c_{k}
\end{array}\right)\binom{T_{x}}{T_{y}} \cdot\left(\begin{array}{c}
W_{i} \\
W_{j} \\
W_{k}
\end{array}\right),  \tag{4.7}\\
& =\frac{\Delta}{2}\left(\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right)\binom{T_{x}}{T_{y}} \cdot\binom{S_{x}}{S_{y}}, \\
\int_{\tau_{h}} \mathbf{T}_{h} \cdot \mathbf{S}_{h} \mathrm{~d} x & =\int_{\tau_{h}}\left(T_{x} \hat{\mathbf{e}}_{x}+T_{y} \hat{\mathbf{e}}_{y}\right) \cdot\left(S_{x} \hat{\mathbf{e}}_{x}+S_{y} \hat{\mathbf{e}}_{y}\right) \mathrm{d} x  \tag{4.8}\\
& =\frac{\Delta}{2} G\left(T_{x}, T_{y}\right)\left(\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right)\binom{T_{x}}{T_{y}} \cdot\binom{S_{x}}{S_{y}}, \\
\int_{\tau_{h}}\left(\mathbf{T}_{h}\right) \mathbf{T}_{h} \cdot \mathbf{S}_{h} \mathrm{~d} x & =G_{\left(T_{x}, T_{y}\right) \int_{\tau_{h}}\left(T_{x} \hat{\mathbf{e}}_{x}+T_{y} \hat{\mathbf{e}}_{y}\right) \cdot\left(S_{x} \hat{\mathbf{e}}_{x}+S_{y} \hat{\mathbf{e}}_{y}\right) \mathrm{d} x}  \tag{4.9}\\
& =\int_{\tau_{h}}\left(\begin{array}{ll}
\phi_{i} \phi_{i} & \phi_{i} \phi_{j} \\
\phi_{j} \phi_{i} & \phi_{j} \phi_{j} \\
\phi_{k} \phi_{i} & \phi_{k} \\
\phi_{k} \phi_{j} & \phi_{k} \\
\phi_{k} \phi_{k}
\end{array}\right)\left(\begin{array}{l}
F_{i} \\
F_{j} \\
F_{k}
\end{array}\right) \mathrm{d} x \cdot\left(\begin{array}{l}
W_{i} \\
W_{j} \\
W_{k}
\end{array}\right)=A_{\tau}^{0} \mathbf{F} \cdot \mathbf{W},
\end{align*}
$$

where $A_{\tau}^{0}$ is the mass matrix of the element. Note that $G$ is constant over each triangle since our elements are constant too.

Unfortunately, the nonlinear nature of the problem does not allow a direct approach, and we have to appeal to a method introduced in the paper by Lions \& Mercier [39]. In our case, since one of the two operators is linear, we can use it as an initializing value in the algorithm that is explained as follows:

- Step 0: set $n=0$ in the nonlinear operator so it becomes a linear operator, and solve the associated linear problem

$$
\left.\begin{array}{l}
\int_{\Omega}\left(1+R_{3}\right) \mathbf{T}^{0} \cdot \mathbf{S} \mathrm{~d} x-\int_{\Omega} \frac{1}{2} \operatorname{grad} v^{0} \cdot \mathbf{S} \mathrm{~d} x=0 \\
-\int_{\Omega} \mathbf{T}^{0} \cdot \operatorname{grad} w \mathrm{~d} x=\int_{\Omega} \lambda w \mathrm{~d} x \tag{4.11}
\end{array}\right\}
$$

- Step 1: using the previous step $k$ and a pseudo-time derivative, calculate the linear problem for the step $k+\frac{1}{2}$ given by

$$
\left.\begin{array}{l}
\int_{\Omega}\left[\frac{1}{h}\left(\mathbf{T}^{k+\frac{1}{2}}-\mathbf{T}^{k}\right)+R_{3} \mathbf{T}^{k+\frac{1}{2}}-\nabla v^{k+\frac{1}{2}}\right] \cdot \mathbf{S} \mathrm{d} x=-\int_{\Omega}\left(1+2 R_{2}\left|\mathbf{T}^{k}\right|^{2}\right)^{n} \mathbf{T}^{k} \cdot \mathbf{S} \mathrm{~d} x  \tag{4.12}\\
-\int_{\Omega} \mathbf{T}^{k+\frac{1}{2}} \cdot \nabla w \mathrm{~d} x=\int_{\Omega} \lambda w \mathrm{~d} x
\end{array}\right\}
$$

- Step 2: using the previous step $k+\frac{1}{2}$, calculate the nonlinear $k+1$ term given by

$$
\begin{align*}
\int_{\Omega} & {\left[\frac{1}{h}\left(\mathbf{T}^{k+1}-\mathbf{T}^{k+\frac{1}{2}}\right)+\left(1+2 R_{2}\left|\mathbf{T}^{k+1}\right|^{2}\right)^{n} \mathbf{T}^{k+1}\right] \cdot \mathbf{S} \mathrm{d} x } \\
& =\int_{\Omega}\left(\nabla v^{k+\frac{1}{2}}-R_{3} \mathbf{T}^{k+\frac{1}{2}}\right) \cdot \mathbf{S} \mathrm{d} x \tag{4.13}
\end{align*}
$$

Now feed step 1 with the result of step 2 recursively until the solution converges to a steady state.

Finally, note that the finite-dimensional scheme for step 0 has the structure

$$
\left(\begin{array}{cc}
\left(1+R_{3}\right) \mathbf{D} & -\frac{1}{2} \mathbf{B}^{\mathrm{T}}  \tag{4.14}\\
\mathbf{B} & \mathbf{0}
\end{array}\right)\binom{\boldsymbol{T}_{x}}{\boldsymbol{T}_{y}}=\binom{\mathbf{0}}{\boldsymbol{V}}=\left(\begin{array}{c}
\mathbf{0}
\end{array}\right)
$$

and the scheme for step 1 has the structure

$$
\left(\begin{array}{cc}
\left(\frac{1}{h}+R_{3}\right) \mathbf{D} & -\frac{1}{2} \mathbf{B}^{\mathrm{T}}  \tag{4.15}\\
\mathbf{B} & \mathbf{0}
\end{array}\right)\binom{\binom{\boldsymbol{T}_{x}}{\boldsymbol{T}_{y}}}{\boldsymbol{V}}=\binom{\mathbf{D}\left(\boldsymbol{T}^{0}\right)\binom{\boldsymbol{T}_{x}^{0}}{\boldsymbol{T}_{y}^{0}}}{A^{0} \boldsymbol{\Lambda}}
$$

where $\mathbf{D} \in \mathbb{R}^{2 M \times 2 M}$ is a diagonal matrix, $\mathbf{B} \in \mathbb{R}^{N \times 2 M}$ is a rectangular matrix and $\mathbf{0} \in \mathbb{R}^{N \times N}$ is a null matrix.

Remark 4.1. Note that by the choice of the elements for the stress, the second step (4.13) is in reality a nonlinear algebraic system of two equations, which can be solved triangle by triangle by applying, for example, the classical Newton's method.

## (b) Numerical results

For the purposes of computation, a Matlab ${ }^{\circledR}$ algorithm was developed using a Delaunay triangular mesh obtained using Abaqus ${ }^{\circledR}$. The code had to be written since most simulation programs do not allow the computation of implicit constitutive relations.


Figure 6. Deviatoric stress norm versus $\theta$ for the aspect ratio $b / a=0.5$.


Figure 7. Deviatoric stress norm versus $\theta$ for the aspect ratio $b / a=0.1$.

## (i) Effect of the variation of the aspect ratio $(b / a)$ of the cross-section on the flow

As can be seen from the results (figures 6,7 and 8), if all the parameters of the model are kept fixed, in the case of the tube with elliptic cross-section the minor to major axis ratio has a large effect on the deviatoric stress at the boundary. From the plots, we immediately see that the minimum is attained at the vertices, whereas the maximum value is attained at the co-vertices. In addition, it shows that the smaller the aspect ratio, the steeper the change. In addition, note that the exponent $n$ also has an effect on how the stress varies across the cross-section. The effect of the aspect ratio on the velocity profile is less perceivable, the profile retaining the same ellipsoidal shape but with a different distribution of the velocity field (figure 9).

Once the points where the stress is maximum have been identified, we can proceed to calculate how it is related to the norm of the velocity gradient. From the plots (figures 10 and 11), the


Figure 8. Effect of the aspect ratio $b / a$ on the axial velocity for given values of the model's parameters. (a) $b / a=1$ and (b) $b / a=0.5$.


Figure 9. Effect of the aspect ratio $b / a$ on the shear stress for given values of the model's parameters. (a) $b / a=1$ and (b) $b / a=0.5$.
nonlinear relationship for different values of $n$ is evident: when $n$ is positive, the slope of the curve is monotonically decreasing, for $n=0$ the Newtonian (linear) solution is recovered, and for negative values of $n$ it depicts the nonlinear behaviour that was predicted by the model. Note how an increase in the value of the exponent $n$ makes the change of concavity of the curve more evident (figures 8 and 11).

As stated earlier, the numerical procedure also works for different domains (figure 12), as long as they are bounded and simply connected. In this case, the maximum stress will occur at the boundary point which is the closest to the centroid of the cross-section, and the velocity profile has a paraboloid-like structure.

## (ii) Numerical trials

To compare how well the numerical approximation works, it is important to compare it against a particular given analytical solution (table 1). It turns out that when the cross-section of the pipe is circular, the problem becomes amenable to an exact solution, and the velocity is then given analytically by the equation

$$
\begin{equation*}
v_{z}=\frac{1}{(n+1) R_{2} \lambda}\left\{\left[1+\frac{R_{2} \lambda^{2}}{2}\left(x^{2}+y^{2}\right)\right]^{n+1}-\left(1+\frac{R_{2} \lambda^{2}}{2}\right)^{n+1}\right\}+\frac{R_{3} \lambda}{2}\left(x^{2}+y^{2}-1\right) \tag{4.16}
\end{equation*}
$$



Figure 10. Deviatoric stress norm versus $D$ for the aspect ratio $b / a=0.5$.


Figure 11. Deviatoric stress norm versus $D$ for the aspect ratio $b / a=0.1$.

From the formulation of the problem, we see that when the power-law exponent is $n=-1 / 2$, the value for which monotonicity is independent of $n$ is when $R_{3} \geq 2 e^{-3 / 2}$, therefore we pick $R_{3}=1 / 2$.

## 5. Concluding remarks

In this study, we considered the flow of a stress power-law fluid and a generalization of it, in pipes of non-circular cross-sections. Unlike the classical power-law fluids that do not meet the demands of causality, stress power-law fluids do. We are interested in the flows of such fluids in pipes of non-circular cross-section (even the cross-section of pipes that are supposedly of circular cross-section are not perfectly circular). Many classes of non-Newtonian fluids develop secondary flows in pipes of non-circular cross-section, classical power-law fluids do not. However, since there is an infinite range of the power-law parameter for which stress


Figure 12. Axial velocity for different geometries. (a) L-shaped and (b) irregular.
Table 1. Experimental relative error for different pressure gradients.

| pressure gradient | 0.1 | 0.25 | 0.5 | 1 |
| :---: | :---: | :---: | :---: | :---: |
| coarse | 0.0089 | 0.0084 | 0.0079 | 0.0065 |
| refined | $8.6981 \times 10^{-4}$ | $7.8671 \times 10^{-4}$ | $7.1506 \times 10^{-4}$ | $5.7597 \times 10^{-4}$ |

power-law fluids cannot be inverted to yield classical power-law fluids, it was first necessary to determine whether secondary flows can occur in stress power-law fluids and their generalization considered herein. Having determined that secondary flows are not possible, we considered the flow of stress power-law fluids in pipes with a variety of cross-sections: elliptic, L-shaped etc. The governing equations are too complicated to afford an analytical solution and the equations have to be solved numerically. This is unlike the case of the classical power-law fluid, where the expression for the stress can be substituted into the balance of linear momentum to obtain an equation for the velocity field. In the case of a stress power-law fluid, we do not have the luxury of substituting the expression for the stress in the balance of linear momentum. We have to solve the balance of linear momentum and the constitutive equations simultaneously. This leads to a much more complicated numerical problem than that which is usually encountered. Solutions are obtained for different values of the power-law index. In the case of an elliptic cross-section, we study the problem for different values of the aspect ratio. Also, we solve the problem corresponding to an L-shaped cross-section. In addition to determining the velocity field, we determine the shear stresses that are developed and the location where the maximum and minimum shear stresses occur. We find that the maximum value of the shear on the boundary occurs at the point that is closest to the centroid of the cross-section while the minimum value occurs at the location furthest from the centroid of the cross-section.

Most viscoelastic fluids exhibit secondary flows in pipes of non-circular cross-section. Thus, one interesting extension of the current work would be to generalize the class of models considered here to include terms that are objective time derivatives of the stress, as in the Maxwell, Oldroyd and Burgers models in order to determine the character of the secondary flows that come into play.

Data accessibility. This article has no additional data.
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[^0]:    ${ }^{1}$ The following comments are not meant to diminish whatsoever the significant contributions of Rivlin to non-Newtonian fluid mechanics. It is, however, necessary to view his early efforts with the insight and hindsight that several decades of research, that have been built upon his innovative studies, offers. Rivlin-Ericksen fluids cannot stress relax, a characteristic exhibited by a very large class of non-Newtonian fluids, nor can they exhibit instantaneous elasticity. While this limits the scope of application of Rivlin-Ericksen fluids, general power-law fluids that are a special subclass of Rivlin-Ericksen fluids are useful in describing a reasonably large class of non-Newtonian fluids. Also, other than in the case of the Navier-Stokes model and power-law fluid models, that are Rivlin-Ericksen fluids of order one, one will encounter serious problems with regard to the specification of boundary conditions for the boundary-value problems for higher-order Rivlin-Ericksen fluids, as the equations governing the flow lead to partial differential equations of order $n+1$ for fluids of order $n$. This, in fact, led investigators to study very special boundary-value problems where terms of order greater than two in the equation were absent or by using formal perturbation methods to solve boundary-value problems without recognizing that they were treating singular perturbations as though they were regular perturbations. Also, while the early attempt by Greensmith \& Rivlin to determine pressure hole errors was an important investigation, they incorrectly concluded that such errors do not exist. Also, Rivlin's work on multiple integral representations with Green and others does not have much utility as it is wellnigh impossible to design an experimental programme to determine uniquely the material properties of the fluids defined using the integral representations. The equations that they engender are also very computationally challenging to use and this is why no initial-boundary-value problems that correspond to realistic practical problems have been solved within the context of such models. Rivlin's recognition of the analogous behaviour of the flows of non-Newtonian fluids and turbulent flow of a Navier-Stokes fluid, namely the similarity in the structure of the secondary flow in tubes of non-circular crosssection, while interesting, is at best a very superficial analogy and thus his conjecture that one can use this to develop models for the turbulent flow of a Navier-Stokes fluid is a gross over-simplification at best.
    ${ }^{2}$ Here $\phi$ and $\mu$ are two of the viscometric functions.
    ${ }^{3}$ Rivlin and co-workers, by virtue of their assumption for the velocity field being multiplied by a small parameter, are essentially presuming that the driving force for the problem, namely the pressure gradient, is small. Using the driving force as the perturbation parameter implies that the solution at the zeroth order is the state of rest. However, in many tube flow problems of practical relevance, one is primarily interested in considering perturbations of a non-zero basic solution, namely that which corresponds to the flow in a pipe of circular cross-section, using a measure of the departure of the cross-section from a perfectly circular geometry as the perturbation parameter. Thus, the type of approximation carried out by Rivlin and co-workers can be viewed as 'Stokes-type approximations' to the flow; the approximation considered in this work is an 'Oseen-type approximation' (see Mollica \& Rajagopal [18] for a discussion of the relevant issues; see also Baldoni et al. [19]). Truesdell \& Noll [20] provided a formal perturbation analysis under the supposition that the driving force is small.

[^1]:    They, however, recognize that such an analysis cannot be scaled up to infer anything sensible about the problem wherein the driving force is not small. While they observe that one cannot expect to obtain the solution for slow flow from those for a general flow by stating that 'It is plausible that the flow will be slow when $a$ is small. However, there is no reason to believe that the flow for a small specific driving force can be obtained from a larger specific driving force by a mere retardation', the converse that one cannot obtain the solution for a large driving force from a small driving force is even more patently obvious.

[^2]:    ${ }^{4}$ This algorithm consists in splitting the nonlinear operator into two operators which are maximal monotone. Then, by the use of a pseudo-time derivative we can define two pseudo-evolutionary equations for the forward and backward steps, respectively. Finally, the coupled equations are solved iteratively until the result stabilizes into a solution.

[^3]:    ${ }^{5}$ Unlike the assumption in all the papers by Rivlin where the primary axial flow itself is assumed to be slow as it is multiplied by the paramenter $\epsilon$ that is assumed to be small, here we only assume that the secondary flow is slow.

