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PENDULUM WITH LINEAR DAMPING AND VARIABLE LENGTH *

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Abstract: *The methods of multiple scales and approximate potential are used to study pendulums with linear damping and variable length. According to the order of the coefficient of friction compared with that of the slowly varying parameter of length, three different cases are discussed in details. Asymptotic analytical expressions of amplitude, frequency and solution are obtained. The method of approximate potential makes the results effective for large oscillations. A modified multiple scales method is used to get more accurate leading order approximations when the coefficient friction is not small. Comparisons are also made with numerical results to show the efficiency of the present method.*

Key words: pendulum; multiple scale method; approximate potential; slowly varying parameter

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1 Introduction and Problem

This paper is to study the following pendulum with slowly varying parameter:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(l^2(\tau)\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)+\mu\frac{\mathrm{d}}{\mathrm{d}t}(l(\tau)\theta)+gl(\tau)\sin\theta=0, \tag{1}$$

where θ is the angle of deviation of the pendulum from the vertical, g is the gravitational acceleration, $l(\tau)$ is the slowly varying length, $\tau=\varepsilon t$ is the slow scale and μ is the coefficient of friction. Such problem had been studied by Nayfeh[1], Bogoliubov and Mitropolsky[2]. In Ref. [3], Yuste used a generalization of the elliptic KB method[4] to obtain the amplitude of the oscillatory system governed by a first-order differential equation, which is so complicated that numerical method must be used to solve it. The expression of frequency is not given in Ref. [3]

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because of its complexity. Further examination of Eq. (1) we found that different orders of μ compared with ε will result in different expressions of frequency. In the cases of $\mu = O(\varepsilon)$, $\mu = o(\varepsilon)$ and $\varepsilon = o(\mu)$, the multiple scales method is used to deduce the asymptotic frequencies and solutions of Eq. (1), from which we can see the effect of μ and ε on frequencies clearly. Approximate potential, which was first proposed by the third author in Ref. [5], is used to make the asymptotic solutions work for large oscillations. The method is to express the potential for the oscillatory system by a polynomial of degree four such that the leading approximation is expressible in terms of elliptic functions. A modified procedure of multiple scales is also used to get more accurate leading order approximations when the coefficient of friction is not small. Examples are also given to verify the efficiency of the presented method.

2 Asymptotic Solution of Pendulum

According to the order of μ compared with ε, three cases of Eq. (1) will be discussed in details. Different orders of μ will deduce different expressions of frequencies and solutions of the oscillatory system.

2.1 Case one: $\mu = O(\varepsilon)$

Let $\mu = c\varepsilon$ and c be a constant. Then Eq. (1) becomes

$$\ddot{\theta} + \varepsilon \frac{2l'(\tau) + c}{l(\tau)} \dot{\theta} + \varepsilon^2 \frac{cl'(\tau)}{l^2(\tau)} \theta + \frac{g}{l(\tau)} \sin\theta = 0, \tag{2}$$

where $\dot{\theta} = \mathrm{d}\theta/\mathrm{d}t$, $l' = \mathrm{d}l/\mathrm{d}\tau$ and $\tau = \varepsilon t$ is the slow scale. The fast scale t^+, following Kuzmak[6], is defined as $\dfrac{\mathrm{d}t^+}{\mathrm{d}t} = \omega(\tau)$ with an unknown $\omega(\tau)$ to be determined by the periodicity of the solution of Eq. (2). Suppose that the solution of Eq. (2) can be developed into multiple scales form

$$\theta(t, \varepsilon) = \theta_0(t^+, \tau) + \varepsilon\theta_1(t^+, \tau) + \varepsilon^2\theta_2(t^+, \tau) + \cdots, \tag{3}$$

where $\theta_0, \theta_1, \theta_2, \cdots$ must be periodic functions of t^+, otherwise the expressions can not be asymptotic. Substituting Eq. (3) into Eq. (2) and equating powers of ε gives the leading order equation

$$\omega^2(\tau) \frac{\partial^2 \theta_0}{\partial t^{+2}} + \frac{g}{l(\tau)} \sin\theta_0 = 0. \tag{4}$$

Multiplying Eq. (4) by $\partial\theta_0/\partial t^+$ and integrating it with respect to t^+, we obtain the energy integral

$$\frac{\omega^2(\tau)}{2} \left(\frac{\partial \theta_0}{\partial t^+} \right)^2 + V(\theta_0) = E_0(\tau), \tag{5}$$

where

$$V(\theta_0) = \frac{g}{l(\tau)} (1 - \cos\theta_0) \tag{6}$$

is the potential and $E_0(\tau)$ is the slowly varying energy of system. We construct a fourth order polynomial to the approximate potential (6). It is denoted by

$$\bar{V}(\theta) = \frac{1}{2} a(\tau)\theta^2 + \frac{1}{4} b(\tau)\theta^4, \tag{7}$$

where the coefficients a and b are chosen such that

$$\begin{cases} \bar{V} = V \text{ at } \theta = 0, \ \theta = \theta_r \text{ and } \theta = \theta_s, \ 0 < \theta_r, \ \theta_s \leqslant \pi, \\ \bar{V}_\theta = 0 \text{ at } \theta = 0. \end{cases} \tag{8}$$

The fitting points θ_r and θ_s can be chosen according to different requirements. Substituting $\bar{V}$ for V in Eq.(5) and integrating it, we can obtain θ_0 in terms of elliptic functions (see Ref.[7] Section 3.6)

$$\theta_0 = \sqrt{-\frac{2av}{b(1+v)}}\operatorname{sn}[\mathrm{K}(v)\varphi, v], \tag{9}$$

where $\varphi = t^+ + \varphi_0$, and $\mathrm{K}(v)$ is the complete elliptic integral of the first kind associated with the modulus $\sqrt{v}$. The governing equation v is in the form (see Ref.[8] or Ref.[9])

$$\frac{L^2(v)v^2}{(1+v)^3} = \frac{D^2b^2}{4a^3}\exp\left(-2\int_0^\tau \frac{2l'(s)+c}{l(s)}\mathrm{d}s\right), \tag{10}$$

where constant D can be determined by initial values of the system, and

$$L(v) = \int_0^K \operatorname{cn}^2(u,v)\operatorname{dn}^2(u,v)\mathrm{d}u = \frac{1}{3v}[(1+v)\mathrm{E}(v) - (1-v)\mathrm{K}(v)]. \tag{11}$$

Here, $\mathrm{E}(v)$ is the complete elliptic integral of the second kind associated with the modulus $\sqrt{v}$. The frequency is

$$\omega(\tau) = -\frac{Db(1+v)}{2av\mathrm{K}(v)L(v)}\exp\left(-\int_0^\tau \frac{2l'(s)+c}{l(s)}\mathrm{d}s\right). \tag{12}$$

The details also can be found in Ref.[8] or Ref.[9].

2.2 Case two: $\mu = o(\varepsilon)$

Without loss of generality, we assume that $\mu = c\varepsilon^2$ and c is a constant. Then Eq.(1) becomes

$$\ddot{\theta} + \varepsilon\frac{2l'(\tau)}{l(\tau)}\dot{\theta} + \varepsilon^2\frac{c}{l(\tau)}\dot{\theta} + \varepsilon^3\frac{cl'(\tau)}{l^2(\tau)}\theta + \frac{g}{l(\tau)}\sin\theta = 0. \tag{13}$$

Similar to Case one, we can get the leading order approximate solution of Eq.(9). Now the governing equation v becomes

$$\frac{L^2(v)v^2}{(1+v)^3} = \frac{D^2b^2}{4a^3}\exp\left(-2\int_0^\tau \frac{2l'(s)}{l(s)}\mathrm{d}s\right) = \frac{D^2b^2}{4a^3}\frac{1}{l^4(\tau)}, \tag{14}$$

and the frequency becomes

$$\omega(\tau) = -\frac{Db(1+v)}{2av\mathrm{K}(v)L(v)}\exp\left(-\int_0^\tau \frac{2l'(s)}{l(s)}\mathrm{d}s\right) = -\frac{Db(1+v)}{2av\mathrm{K}(v)L(v)}\frac{1}{l^2(\tau)}. \tag{15}$$

2.3 Case three: $\varepsilon = o(\mu)$

Without loss of generality, we assume that $\varepsilon = c\mu^2$ and c is a constant. We define a new slow scale as $\bar{t} = \mu t$ and the fast scale t^+ as $\mathrm{d}t^+/\mathrm{d}t = \omega(\bar{t})$, then $\tau = \varepsilon t = c\mu\bar{t}$. Eq.(1) becomes

$$\ddot{\theta} + \mu\frac{1}{l(\tau)}\dot{\theta} + \mu^2\frac{2cl'(\tau)}{l(\tau)}\dot{\theta} + \mu^3\frac{cl'(\tau)}{l^2(\tau)}\theta + \frac{g}{l(\tau)}\sin\theta = 0. \tag{16}$$

Similar to Case one, we can get the leading order approximate solution of Eq.(9). The governing equation v becomes

$$\frac{L^2(v)v^2}{(1+v)^3} = \frac{D^2b^2}{4a^3}\exp\left(-2\int_0^{\bar{t}} \frac{1}{l(c\mu s)}\mathrm{d}s\right), \tag{17}$$

and the frequency becomes

$$\omega(\bar{t}) = -\frac{Db(1+v)}{2av\mathrm{K}(v)L(v)}\exp\left(-\int_0^{\bar{t}}\frac{1}{l(c\mu s)}\mathrm{d}s\right). \tag{18}$$

If μ is not small (such as $0.1 \leqslant \mu < 1$), the first-order approximation can have large errors. To avoid complicated calculation of higher order approximation, we can reserve the $O(\mu^2)$ term $\mu^2(2cl'(\tau)/l(\tau))\dot{\theta}$ in the $O(\mu)$ term $\mu[1/l(\tau)]\dot{\theta}$, that is,

$$\ddot{\theta} + \mu\left(\frac{1}{l(\tau)} + \mu\frac{2cl'(\tau)}{l(\tau)}\right)\dot{\theta} + \mu^3\frac{cl'(\tau)}{l^2(\tau)}\theta + \frac{g}{l(\tau)}\sin\theta = 0. \tag{19}$$

Then Eq.(17) becomes

$$\frac{L^2(v)v^2}{(1+v)^3} = \frac{D^2b^2}{4a^3}\exp\left(-2\int_0^{\bar{t}}\frac{1+2c\mu l'(c\mu s)}{l(c\mu s)}\mathrm{d}s\right), \tag{20}$$

and Eq.(18) becomes

$$\omega(\bar{t}) = -\frac{Db(1+v)}{2av\mathrm{K}(v)L(v)}\exp\left(-\int_0^{\bar{t}}\frac{1+2c\mu l'(c\mu s)}{l(c\mu s)}\mathrm{d}s\right). \tag{21}$$

Such modification can make the asymptotic solution much more accurate (see Example 3).

3 Examples

Exampel 1 Consider Eq.(1) of Case one with $\varepsilon = 0.01, c = 1, \mu = 0.01, l(\tau) = 1 + \tau$ and $g = 9.8$, that is,

$$\ddot{\theta} + \varepsilon\frac{3}{1+\varepsilon t}\dot{\theta} + \varepsilon^2\frac{1}{(1+\varepsilon t)^2}\theta + \frac{9.8}{1+\varepsilon t}\sin\theta = 0, \tag{22}$$

$$\theta(0) = \frac{2}{3}\pi, \quad \dot{\theta}(0) = 0. \tag{23}$$

The potential related to Eq.(6) is

$$V(\theta_0) = \frac{9.8}{1+\varepsilon t}(1-\cos\theta_0). \tag{24}$$

We seek a polynomial of the form

$$\bar{V}(\theta) = \frac{1}{2}a(\tau)\theta^2 + \frac{1}{4}b(\tau)\theta^4$$

to fit the potential V. The coefficients are chosen such that

$$\bar{V} = V \text{ at } \theta = 0,\ \theta = \frac{\pi}{3} \text{ and } \theta = \frac{2\pi}{3},$$

$$\bar{V}_\theta = 0 \text{ at } \theta = 0.$$

Then

$$\bar{V}(\theta) = \frac{1}{2}\frac{9.681\,24}{1+\tau}\theta^2 - \frac{1}{4}\frac{1.358\,19}{1+\tau}\theta^4. \tag{25}$$

The comparison of potential (24) and its approximation (25) is shown in Fig.1. The comparison of numerical solution with asymptotic solution of Eqs.(22) and (23) is shown in Fig.2. The asymptotic solution is obtained by Eqs.(9) ~ (12). In this paper, numerical solutions are obtained by using software *Mathematica*.

Example 2 Consider Eq.(1) of Case two with $\varepsilon = 0.01, c = 1, \mu = 0.000\,1, l(\tau) = 1 + \tau$ and $g = 9.8$, that is,

$$\ddot{\theta} + \varepsilon\frac{2}{1+\varepsilon t}\dot{\theta} + \varepsilon^2\frac{1}{1+\varepsilon t}\dot{\theta} + \varepsilon^3\frac{1}{(1+\varepsilon t)^2}\theta + \frac{9.8}{1+\varepsilon t}\sin\theta = 0, \tag{26}$$

$$\theta(0) = \frac{2}{3}\pi, \ \dot{\theta}(0) = 0. \tag{27}$$

The comparison of numerical solution with asymptotic solution of Eqs. (26) and (27) is shown in Fig. 3. The asymptotic solution is obtained by Eqs. (9), (14) and (15).

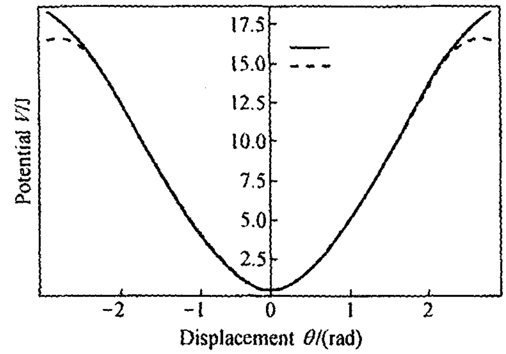

Fig. 1 Potential (24) and approximation (25)
——Potential (24), ------- Potential (25)

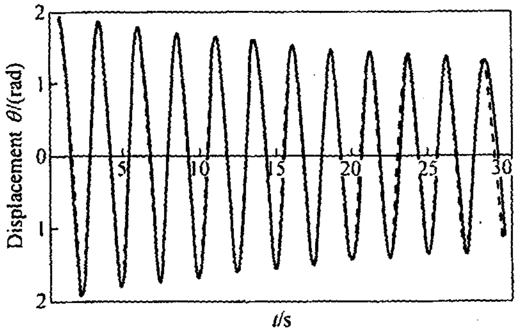

Fig. 2 Solution and Approximation of Eq. (22)
——Numerical solution, ------- Asymptotic solution

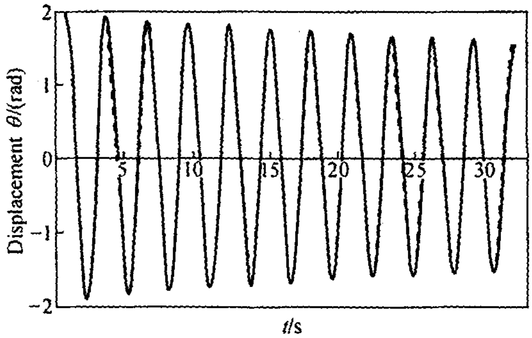

Fig. 3 Solution and approximation of Eq. (26)
——Numerical solution, ------- Asymptotic solution

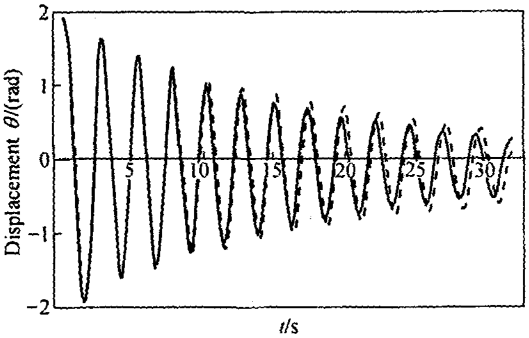

Fig. 4 Solution and approximation of Eq. (28) with (9), (17) and (18)
——Numerical solution, ------- Asymptotic solution

Example 3 Consider Eq. (1) of Case three with $\epsilon = 0.01$, $c = 1$, $\mu = 0.1$, $l(\tau) = 1 + \tau$ and $g = 9.8$, that is,

$$\ddot{\theta} + \mu \frac{1}{1 + \mu^2 t}\dot{\theta} + \mu^2 \frac{2}{1 + \mu^2 t}\dot{\theta} + \mu^3 \frac{1}{(1 + \mu^2 t)^2}\theta + \frac{9.8}{1 + \mu^2 t}\sin\theta = 0, \tag{28}$$

$$\theta(0) = \frac{2}{3}\pi, \ \dot{\theta}(0) = 0. \tag{29}$$

The comparison of numerical solution with asymptotic solution obtained by Eqs. (9), (17) and (18) is shown in Fig. 4, and the comparison

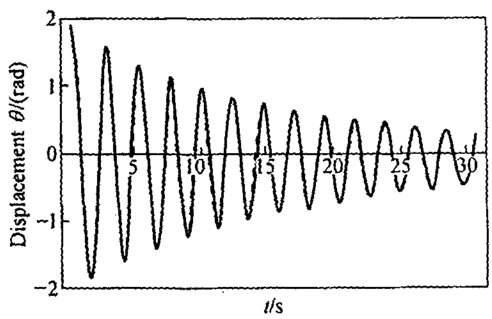

Fig. 5 Solution and approximation of Eq. (28) with (9), (20) and (21)
——Numerical solution, -------Asymptotic solution

with the asymptotic solution obtained by Eqs. (9), (20) and (21) is shown in Fig. 5, from which we can see that the leading order approximation obtained by the modified procedure of multiple scales is more accurate than that obtained by usual procedure.

It can be seen from Figs. 2, 3 and 5 that all the results obtained by our presented method are nearly identical with those numerical results even if the oscillatory amplitudes are as large as $2\pi/3$ and the parameter μ is not small.

4 Conclusions

1) The classification of the order of μ compared with ε results in different analytical expressions of the frequencies such that the effect of μ and ε on the frequencies can be seen clearly and the asymptotic solutions become more accurate.

2) The approximate potential method can make the results effective for large oscillations.

3) The modified procedure of multiple scales can get more accurate leading order approximations when the coefficient of friction is not small.

References:

[1] Nayfeh A H, Mook D T. *Nonlinear Oscillations*[M]. New York: Wiley, 1979.

[2] Bogoliubov N N, Mitropolsky Y A. *Asymptotic Methods in the Theory of Nonlinear Oscillations* [M]. Delhi: Hindustan Publishing Co, 1961.

[3] Yuste S B. On Duffing oscillators with slowly varying parameters[J]. *Internat J Non-Linear Mech*, 1991, **26**(5): 671 – 677.

[4] Coppola V T, Rand R H. MACSYMA program to implement averaging using elliptic functions[A]. In: Meyer K R, Schmidt D S Eds. *Computer Aided Proofs in Analysis*[C]. New York: Springer, 1991, 78 – 81.

[5] LI Yi-ping. Free electron lasers with variable parameter wigglers, a strictly nonlinear oscillator with slowly varying parameters[D]. Ph D Dissertation. Seattle: University of Washington, 1987.

[6] Kuzmak G Z. Asymptotic solutions of nonlinear second order differential equations with variable coefficients[J]. *Pure Math Manuscript*, 1959, **23**: 515 – 526.

[7] Kevorkian J, Cole J D. *Perturbation Methods in Applied Mathematics*[M]. New York: Springer, 1981.

[8] Kevorkian J, Li Y P. Explicit approximations for strictly nonlinear oscillators with slowly varying parameters with applications to free-electron lasers[J]. *Studies in Applied Mathematics*, 1988, **78** (2): 111 – 165.

[9] LI Yi-ping. Elapsed time of periodic motion with negative damping[J]. *Applied Mathematics and Mechanics (English Edition)*, 1992, **13**(8): 719 – 723.