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Flowing States and Vortices in the Classical XY Model in an External Field

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Uniformly flowing states and vortices in the classical XY model in an external field are studied. This is done by using a continuum approximation and by paying attention to particular solutions to nonlinear partial differential equations for two angles θ and φ of rotation of spins for which φ satisfies the Laplace equation. For these two states equations for θ have forms similar to that in the classical Ising model in a transverse field. The uniformly flowing states are therefore described by kink-type excitations identical to those in the two-dimensional Ising model. Phonon modes associated with the uniformly flowing states are also studied, which are similar to Bogoliubov phonons. Vortex solutions and vortex formation energy are studied in close similarity to the case of liquid He^4. By comparing the energies of these two states, an expression for critical velocity is obtained. By making correspondence to the case of liquid He^4, numerical values of the critical velocity and of the velocity of phonons around the uniformly flowing states are estimated. For the former the numerical value is in fair agreement with experimental data.

§ 1. Introduction

In recent years there has been much interest in soliton-like nonlinear excitations in classical continuous spin systems. In a one-dimensional case much effort has been made to obtain soliton solutions to various types of nonlinear partial differential equations for certain magnetic systems. Here one of the most interesting problems is to find completely integrable systems which admit exact multi-soliton solutions.[1)~3)] In higher dimensional cases attention has been focussed to time-independent or stationary solutions corresponding to pseudo-particle solutions such as vortex or vortex string solutions.[4)~7)] On the other hand, it is well known that several of models of magnetic systems are similar to superfluid helium in that ground states of their ordered phases exhibit broken symmetry with respect to a continuous symmetry of the Hamiltonian. Of these, the planar Heisenberg model or the XY model in an external field has received particular attention, since an equivalence of this system to a lattice gas model for Bose condensation in liquid helium has been noted by Matsubara and Matsuda.[8)] By the use of this model, Halperin and Hohenberg have developed a theory to gain insight into the foundation of two-fluid hydrodynamics.[9)] These workers[8),9)] and others[10)] have been primarily interested in static phenomena and linear excitations. It is therefore reasonable to make an attempt to deduce the properties of nonlinear excitations in this magnetic system, just as we can derive the theory of vortex excitations in superfluid helium along the line with the theories of Ginzburg and Pitaevskii[11] and of Gross.[12] In a previous paper,[13] two of the present authors (S.T. and S.H.) made a step toward this objective, obtaining vortex solutions. This work, however, is rather preliminary in this respect, since much of discussion there was given on the mathematical aspect of the problem such as soliton-like solutions in the one-dimensional case.

The purpose of the present paper, which forms a sequel to the previous one,[13] is to make a more detailed study of the properties of stationary solutions or pseudo-particle solutions to field equations and their bearing on the nonlinearity properties of the three-dimensional classical planar Heisenberg ferromagnet or the XY model in an external field. In doing this we pay attention to particular solutions to nonlinear partial differential equations for two angles θ and φ of rotation of spins for which φ satisfies the Laplace equation. Uniform-flow solutions and vortex solutions of physical interest are thereby obtained. It is shown that the equations for θ for these two states have forms similar to that for the Ising model in a transverse field. In contrast with the previous paper,[13] we are primarily concerned here with physical aspects of the problem. Namely, from these two types of solutions we study the energy of uniformly flowing state, phonon excitations around this state, the vortex formation energy, critical velocity corresponding to the case of liquid helium, and so on. We also hope that the present approach may shed some light on the problem of nonlinear excitations in superfluid helium, which is, in fact, more difficult than that of the present model magnetic system.

This paper is organized as follows. In the next section equations of motion are set up for spins in the system. Uniformly flowing states and vortex states are studied in §§ 3 and 4 from stationary solutions to the equations. In § 5 small fluctuations about the stationary solutions are studied. This is done for phonons associated with the uniformly flowing states. In § 6 a critical velocity is studied by comparing the energies of these two states. A brief discussion is given in § 7 on implications of results obtained for the model magnetic system to the problem of superfluid helium.

§ 2. Equations of motion

We consider a classical planar Heisenberg ferromagnet in an external field (CPHFF) defined by the Hamiltonian

$$H=-\epsilon \sum_{n} S_n{}^z-\sum_{nm}[J(n,\ m)(S_n{}^xS_m{}^x+S_n{}^yS_m{}^y)+J'(n,\ m)S_n{}^zS_m{}^z]. \qquad (2\cdot 1)$$

Here $\boldsymbol{S}_n=(S_n^x, S_n^y, S_n^z)$ is the spin angular momentum with magnitude S on the lattice site n, S_n^α $(\alpha=x, y, z)$ being its Cartesian α-component, and ϵ (>0) is the external field. The coupling constants $J(n, m)$ and $J'(n, m)$ are taken to be all positive and assumed to depend only on the coordinate difference between the lattice sites n and m. It is understood that for our model system the inequality

$$\sum_m J(n, m) > \sum_m J'(n, m) \tag{2·2}$$

holds.[13] The components S_n^α of $\boldsymbol{S}_n$ are parametrized by two angles of rotation

$$S_n^x = S \sin\theta_n \cos\varphi_n\ , \quad S_n^y = S \sin\theta_n \sin\varphi_n\ , \quad S_n^z = S\cos\theta_n\ . \tag{2·3}$$

Equations of motion obeyed by θ_n and φ_n are generally written as[*),7]

$$\dot\theta_n = (1/S\sin\theta_n)(\partial H/\partial\varphi_n), \qquad \dot\varphi_n = -(1/S\sin\theta_n)(\partial H/\partial\varphi_n). \tag{2·4}$$

In terms of

$$n_n = S\cos\theta_n \quad \text{and} \quad \sigma_n = S\sin\theta_n \quad \text{with} \quad n_n^2+\sigma_n^2=S^2\ , \tag{2·5}$$

explicit expressions for Eqs. (2·4) are written as

$$\dot n_n = \sigma_n \sum_m J_1(n, m)\sigma_m \sin(\varphi_m-\varphi_n), \tag{2·6a}$$

$$\sigma_n\dot\varphi_n = -\epsilon\sigma_n + \sum_m [J_1(n, m) n_n\sigma_m\cos(\varphi_n-\varphi_m) - J_2(n, m)\sigma_n n_m]. \tag{2·6b}$$

In the above equations we have put

$$J_1(n, m)=2J(n, m), \qquad J_2(n, m)=2J'(n, m). \tag{2·7}$$

We employ a continuum approximation to reduce Eqs. (2·1) and (2·6) to

$$\bar H = H/J_1(0) = \int dr\, \mathcal{H}(r) \tag{2·8}$$

with

$$\mathcal{H} = -\gamma n - (1/2)(\sigma^2+\eta n^2) + (1/2)[(\nabla\sigma)^2 + \eta(\nabla n)^2 + \sigma^2(\nabla\varphi)^2] \tag{2·9}$$

and

$$\dot n = \nabla\cdot(\sigma^2\nabla\varphi) \quad \text{or} \quad \dot n = \nabla\cdot(\rho\nabla\varphi) \quad \text{with} \quad \rho\equiv\sigma^2=\sin^2\theta\ , \tag{2·10a}$$

$$\sigma\dot\varphi = -\gamma\sigma - \eta\sigma(1+\Delta)n + n[1-(\nabla\varphi)^2+\Delta]\sigma\ , \tag{2·10b}$$

respectively. In the above equations we have taken the lattice constant of the system to be unity for the sake of simplicity and put

$$\bar t = t/J_1(0) \to t\ , \qquad \gamma = \epsilon/J_1(0), \qquad \eta = J_2(0)/J_1(0), \tag{2·11}$$

*) We use units with $\hbar=1$ from § 2 to § 6.

in which $J_i(0)$ $(i=1, 2)$ is the value of

$$J_i(q)=\sum_m J_i(n, m)\exp[iq\cdot(m-n)], \qquad (i=1, 2) \tag{2·12}$$

at $q=0$. Further we have assumed that $J_i(q)$ in the long wavelength region takes the form

$$J_i(q)=J_i(0)[1-(\alpha_{i1}q_x{}^2+\alpha_{i2}q_y{}^2+\alpha_{i3}q_z{}^2)] \tag{2·13}$$

and rescaled the coordinate variables as

$$x'=x/\alpha_{i1}^{1/2}\rightarrow x\ , \qquad y'=y/\alpha_{i2}^{1/2}\rightarrow y\ , \qquad z'=z/\alpha_{i3}^{1/2}\rightarrow z\ . \tag{2·14}$$

Equations (2·10) are basic equations to study nonlinear excitations in the CPHFF. In what follows we consider the specific case

$$J'(n, m)=J_2(n, m)=0 \quad \text{or} \quad \eta=0 \tag{2·15}$$

for the sake of simplicity. Namely, we are concerned with the classical XY model in an external field. This specification of the problem, however, does not alter the essential feature of the CPHFF, that is the axial symmetry of the system around the z-axis. Equation (2·10a) then remains unchanged, while Eqs. (2·9) and (2·10b) reduce to

$$\mathcal{H}=-\gamma(S^2-\sigma^2)^{1/2}-(1/2)\sigma^2+(1/2)[(\nabla\sigma)^2+\sigma^2(\nabla\varphi)^2] \tag{2·9$'$}$$

or

$$\mathcal{H}=-\gamma(S^2-\sigma^2)^{1/2}-(1/2)\sigma^2-(1/2)\sigma\Delta\sigma+(1/2)\sigma^2(\nabla\varphi)^2\ , \tag{2·9$''$}$$

$$\sigma\dot{\varphi}=-\gamma\sigma+n[1-(\nabla\varphi)^2+\Delta]\sigma\ . \tag{2·10b$'$}$$

In studying Eqs. (2·10a) and (2·10b′), we limit our discussion to stationary solutions, namely solutions of the following equations:

$$\nabla\cdot(\sigma^2\nabla\varphi)=0 \qquad \text{or} \qquad \nabla\cdot(\rho\nabla\varphi)=0\ , \tag{2·16a}$$

$$\Delta\sigma+[1-(\nabla\varphi)^2]\sigma-\gamma\sigma(S^2-\sigma^2)^{-1/2}=0 \tag{2·16b}$$

and small oscillations about the stationary solutions. It is seen that except the factor $(\nabla\varphi)^2$ Eq. (2·16b) is entirely identical to the stationary form of field equations for the Ising model in a transverse field.[14] Here we pay attention to particular solutions for which the quantity φ satisfies the Laplace equation

$$\Delta\varphi=0\ . \tag{2·17}$$

It is convenient to work with gradient of φ. We therefore put

$$\nabla\varphi=\boldsymbol{v}\ . \tag{2·18}$$

The quantities $\boldsymbol{v}$ and $\rho=\sigma^2$ can be considered as velocity and density of "spin fluid". Equation (2·16a) is then satisfied provided

$$\nabla\sigma\cdot\nabla\varphi=0 \qquad \text{or} \qquad \nabla\rho\cdot\boldsymbol{v}=0\,. \tag{2·19}$$

This implies that the spin fluid under consideration flows in the direction perpendicular to its density gradient. Let (x, y, z) and (r, ϕ, z) be the Cartesian and cylindrical coordinates, respectively. We are then interested in the following two types of particular solutions of physical interest:

(i) *uniform-flow solutions*

$$\boldsymbol{v}=\text{const}=\boldsymbol{v}_0=v_0\hat{\boldsymbol{z}} \quad \text{or} \quad \varphi=v_0 z \quad \text{and} \quad \theta=\theta(x, y). \tag{2·20}$$

(ii) *vortex string type solutions*

$$\varphi=q\phi \quad \text{or} \quad \boldsymbol{v}=(q/r)\hat{\boldsymbol{\phi}} \quad \text{and} \quad \theta=\theta(r),$$

$$q=\pm 1, \pm 2, \cdots\cdots \quad \text{with} \quad r=(x^2+y^2)^{1/2}\,, \quad \phi=\tan^{-1}(y/x). \tag{2·21}$$

In the above equations $\hat{\boldsymbol{z}}$ and $\hat{\boldsymbol{\phi}}$ are unit vectors in the directions of the z-axis and of the ϕ, respectively. Inserting Eqs. (2·20) and (2·21) into Eq. (2·16b), we get equations for σ or n for these two types of particular solutions. Equation (2·20) represents a uniformly flowing state in the direction of the z-axis, while Eq. (2·21) represents a vortex string localized at $r=0$. The stationary solutions corresponding to the cases (i) and (ii) will be studied in detail in §§ 3 and 4, respectively.

§ 3. Uniformly flowing states

We insert Eqs. (2·20) into Eq. (2·16b) to obtain

$$(\partial^2\sigma/\partial x^2)+(\partial^2\sigma/\partial y^2)+(1-v_0^2)\sigma-\gamma\sigma(S^2-\sigma^2)^{-1/2}=0\,. \tag{3·1}$$

We first observe that a nontrivial solution to Eq. (3·1) corresponding to a spatially uniform spin alignment is given by

$$\sigma=\pm[S^2-\{\gamma^2/(1-v_0^2)^2\}]^{1/2}\equiv\pm\sigma_0 \quad \text{with} \quad n_0=(S^2-\sigma_0^2)^{1/2}\,. \tag{3·2}$$

The symmetry-breaking state (3·2) is realizable under the condition

$$r<S \qquad \text{or} \qquad 2S\sum_m J(n, m)>\epsilon\,. \tag{3·3}$$

The maximum value $\sigma_0(\max)$ of σ_0 which exists for $v_0=0$ is given by

$$\sigma_0(\max)=(S^2-\gamma^2)^{1/2}=S[1-(\epsilon/J_s(0))^2]^{1/2}$$

with

$$J_s(0)=2SJ(0). \tag{3·4}$$

The maximum velocity $v_0(\max)$ at which the symmetry-breaking state vanishes is given by

$$v_0(\max)=[1-(\gamma/S)]^{1/2}=[1-\{\epsilon/J_s(0)\}]^{1/2}\,. \tag{3·5}$$

The energy density of the symmetry breaking state $\mathcal{H}=\mathcal{H}(\sigma=\sigma_0)\equiv E_{\rm uf}(v_0)$ is obtained by inserting Eqs. (2·20) and (3·2) into Eq. (2·9′) as follows:

$$E_{\rm uf}(v_0)=-[(1-v_0^2)/2][S^2+\{\gamma^2/(1-v_0^2)^2\}]. \tag{3·6}$$

It is seen that $E_{\rm uf}(v_0)$ increases monotonically from its minimum value $E_{\rm uf}(0)=-(1/2)(S^2+\gamma^2)$ to its maximum value $E_{\rm uf}(v_0(\max))=-\gamma S=-S\epsilon/J_1(0)$ corresponding to the state in which all the spins in the system are aligned in the positive z-direction (see Fig. 1). The energy difference $\Delta E_{\rm uf}$ of the uniformly flowing state from the ground state of the system is therefore given by

$$\Delta E_{\rm uf}=E_{\rm uf}(v_0)-E_{\rm uf}(0)=(v_0^2/2)[S^2-\{\gamma^2/(1-v_0^2)\}]. \tag{3·7}$$

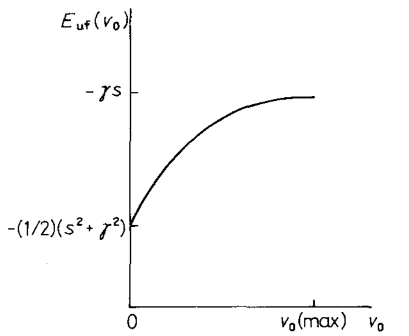

Fig. 1. Schematic feature of the energy density $E_{\rm uf}(v_0)$ of uniformly flowing states as a function of v_0.

The symmetry breaking state (3·2) with energy eigenvalue given by Eq. (3·6) can be considered as corresponding to the flowing ground state of liquid He[4]II.

We next study the effect of a wall on the uniformly flowing state. Let us assume for the sake of simplicity that the extension of the wall under consideration is infinite and that the spin fluid flows in the direction parallel to the wall. We thus take the wall as the yz-plane, where the flowing state is assumed to be uniform in the y-direction, namely σ is taken to be independent of y. We are then concerned with a specific form of Eq. (3·1):

$$(d^2\sigma/dx^2)+(1-v_0^2)\sigma-\gamma\sigma(S^2-\sigma^2)^{-1/2}=0 \tag{3·8}$$

with the boundary condition

$$\sigma=\begin{cases}0 & \text{as } x\to 0\,,\\ \sigma_0 & \text{as } x\to\infty\,.\end{cases} \tag{3·9}$$

Equation (3·8) is identical to a stationary form of a nonlinear differential equation for the one-dimensional Ising model in a transverse field. By the use of the result obtained in another previous paper,[14] the solution to Eq. (3·8) with Eq. (3·9) is obtained as follows:

$$\frac{\sigma}{S+n}=\frac{\sigma_0}{S+n_0}\tanh[(\sigma_0/2n_0)\{(1-v_0^2)^{1/2}x-\sin^{-1}(\sigma/S)\}]. \tag{3·10}$$

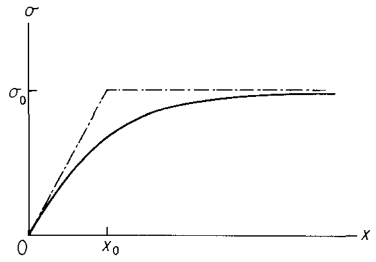

Fig. 2. Schematic feature of σ as a function of x given by Eq. (3·10). The broken line represents the approximation (3·11) to the solution of Eq. (3·10).

A schematic feature of the solution is shown in Fig. 2. Inserting Eq. (3·10) into Eqs. (2·8) and (2·9′), we obtain an increase of the energy of the symmetry breaking state due to the presence of the wall. This corresponds to the surface energy. To evaluate this, we employ an approximation to replace the curve in Fig. 2 by two pieces of straight line:

$$\sigma=\begin{cases} bx & \text{for } 0\leq x\leq x_0\,, \\ \sigma_0 & \text{for } x_0<x\,, \end{cases} \tag{3·11}$$

where

$$b=(d\sigma/dx)_{x=0}=(1-v_0^2)^{1/2}(S-n_0). \tag{3·12}$$

The quantity

$$x_0=\sigma_0/b \tag{3·13}$$

can be identified as the width of the kink obtained above, which corresponds to healing length studied by Gross.[12] By the use of this approximation procedure, the increase ΔE of the energy of the uniformly flowing state due to the presence of the wall is given by

$$\Delta E=\int_0^{x_0}dx\left[-\gamma(S^2-b^2x^2)^{1/2}-\frac{1-v_0^2}{2}b^2x^2+\gamma(S^2-\sigma_0^2)^{1/2}+\frac{1-v_0^2}{2}\sigma_0^2+\frac{1}{2}b^2\right]$$

$$=\frac{1}{b}\left[-\gamma S^2\sin^{-1}\left(\frac{\sigma_0}{S}\right)+\frac{\gamma}{2}\sigma_0(S^2-\sigma_0^2)^{1/2}+\frac{1-v_0^2}{3}\sigma_0^3+\frac{\sigma_0 b^2}{2}\right]. \tag{3·14}$$

The quantity ΔE for $v_0=0$ corresponds to the surface energy obtained by Ginzburg and Pitaevskii in their theory of superfluidity.[11]

§ 4. Vortex states

The vortex solutions given by Eqs. (2·21) are characterized by the circulation

$$\int \boldsymbol{v}\cdot d\boldsymbol{s}=2\pi q\,, \tag{4·1}$$

where ds denotes the line element. We insert Eqs. (2·21) into Eq. (2·16b) to obtain

$$\frac{d^2\sigma}{dr^2}+\frac{1}{r}\frac{d\sigma}{dr}+\left(1-\frac{q^2}{r^2}\right)\sigma-\gamma\frac{\sigma}{(S^2-\sigma^2)}=0\,. \tag{4·2}$$

We first consider the solution to Eq. (4·2) under the following boundary condition:

$$\sigma=\begin{cases}\sigma_0(\max)=(S^2-\gamma^2)^{1/2} & \text{as } r\to\infty\,, & (4·3a)\\ 0 & \text{as } r\to 0\,. & (4·3b)\end{cases}$$

Since Eq. (4·2) cannot be solved as it stands, here we content ourselves by obtaining only its asymptotic solutions:

$$\sigma=\begin{cases}\sigma_0(\max)-q^2[\gamma^2/(S^2-\gamma^2)^{1/2}](1/r^2) & \text{for large } r\,, & (4·4a)\\ J_q(r') \quad \text{with} \quad r'=[1-(\gamma/S)]r & \text{for small } r\,, & (4·4b)\end{cases}$$

where $J_q(r')$ is the Bessel function of the qth order.

We are then concerned with the energy E_v of a single vortex. In doing this, we consider only the cases $q=\pm1$, since the solutions corresponding to these two (degenerate) cases are the solutions with the lowest energy. In the evaluation of E_v we assume that the system under consideration is enclosed in a cylinder of radius R and length L. By the use of Eqs. (2·9″) and (4·2), it is written as

$$E_v=E_v^{(\text{in})}+E_v^{(\text{out})}\,, \tag{4·5}$$

where

$$E_v^{(\text{in})}=-2\pi\gamma L\int_0^{r_1}rdr[(n/2)+(S^2/2n)],$$

$$E_v^{(\text{out})}=-2\pi\gamma L\int_{r_1}^{R}rdr[(n/2)+(S^2/2n)] \tag{4·6}$$

are contributions to E_v from the regions inside and outside the vortex core with radius r_1, respectively. For the evaluation of $E_v^{(\text{out})}$ we assume that σ is a slowing varying function of r outside the vortex core region, thus neglecting the first and the second derivatives of σ in Eq. (4·2). An approximate solution of Eq. (4·2) for $q=\pm1$ is then given by

$$n=\gamma r^2/(r^2-1). \tag{4·7}$$

Inserting this into the second or Eqs. (4·6), we obtain

$$E_v^{(\text{out})}=-\pi\gamma L\left[\frac{1}{2}(R^2-r_1^2)\left(\gamma+\frac{S^2}{\gamma}\right)+\frac{\gamma}{2}\ln\frac{R^2-1}{r_1^2-1}-\frac{S^2}{\gamma}\ln\frac{R}{r_1}\right]. \tag{4·8}$$

Since $1 \leq r_1 \leq R_1$,*) we can approximate Eq. (4·8) as

$$E_v^{(\mathrm{out})} = -(\pi L/2)[\sigma_0(\max)^2 + 2\gamma^2](R^2 - r_1{}^2) + \pi L \sigma_0(\max)^2 \ln(R/r_1). \tag{4·8'}$$

The quantity $E_v^{(\mathrm{in})}$ is obtained by using Eqs. (2·5) and (4·4b). Actually, this could be done by numerical integration. Since it is independent of the system size R and its explicit expression is not required in our later discussion, we simply write it as

$$E_v^{(\mathrm{in})} = 2\pi L \epsilon_0 \quad \text{with} \quad \epsilon_0 = -\int_0^{r_1} r dr[(n/2) + (S^2/2n)], \tag{4·9}$$

omitting the evaluation of the integral. Combining Eqs. (4·8′) and (4·9), we obtain the vortex formation energy for $q = \pm 1$ as follows:

$$E_v = 2\pi L\left[\epsilon_0 - \frac{1}{4}(S^2 + \gamma^2)(R^2 - r_1{}^2) + \frac{1}{2}(S^2 - \gamma^2)\ln\frac{R}{r_1}\right]. \tag{4·10}$$

This result should be compared with that obtained by Gross.[12] Defining the excitation energy ΔE_v of a single vortex by the equation

$$\Delta E_v = E_v - \pi L R^2 E(v_0 = 0), \tag{4·11}$$

we obtain

$$\Delta E_v = 2\pi L\left[\epsilon_0 + \frac{1}{4}(S^2 + \gamma^2) r_1{}^2 + \frac{1}{2}(S^2 - \gamma^2)\ln\frac{R}{r_1}\right]. \tag{4·12}$$

This is the energy required to create a vortex with $q = \pm 1$ from the ground state. The characteristic feature of ΔE_v is the appearance of the factor $\ln(R/r_1)$. This is entirely analogous to the case of liquid He^4.[11],[12]

§ 5. Small fluctuations about uniformly flowing states

In this section we are concerned with small fluctuations around stationary states determined by Eqs. (2·16). Namely, we inquire into small oscillations about the uniformly flowing states and the vortex states which are local minima of the Hamiltonian. Let the solution to Eq. (2·16) for σ, n and φ be denoted by $\sigma(0)$, $n(0)$ and $\varphi(0)$, respectively. We treat Eqs. (2·10a) and (2·10b′) by putting

$$\sigma = \sigma(0) + \sigma', \qquad n = n(0) + n', \qquad \varphi = \varphi(0) + \varphi', \tag{5·1}$$

retaining only terms linear in σ', n' and φ'. A straightforward calculation leads to a pair of equations for n' and φ':

*) The lattice constant of the spin system has been taken to be unity.

$$\dot{n}'=-2n(0)\boldsymbol{v}_0\cdot\nabla n'+\sigma(0)^2\Delta\varphi'+\nabla\sigma(0)^2\cdot\nabla\varphi', \tag{5·2a}$$

$$\dot{\varphi}'=-2n(0)\boldsymbol{v}_0\cdot\nabla\varphi' +[1-v_0{}^2+\{S^2/\sigma(0)^3\}\Delta\sigma(0)]n'-\frac{n(0)}{\sigma(0)}\Delta\left[\frac{n(0)}{\sigma(0)}n'\right]. \tag{5·2b}$$

Here we have made use of the relation

$$\sigma'=-[n(0)/\sigma(0)]n'. \tag{5·3}$$

In studying Eqs. (5·2) three cases can be considered, in which $\sigma(0)$ or $n(0)$ and $\varphi(0)$ are given by: (i) Eqs. (3·2) and the first of Eqs. (2·20), (ii) Eq. (3·10) and the same $\varphi(0)$ and (iii) Eq. (4·2) and the first of Eqs. (2·21).

Here we limit our discussion to the case (i). Then, $\sigma(0)$, $n(0)$ and $\varphi(0)$ are spatially uniform and Eqs. (5·2) reduce to

$$\dot{n}'=-2n_0\boldsymbol{v}_0\cdot\nabla n'+\sigma_0{}^2\Delta\varphi', \tag{5·4a}$$

$$\dot{\varphi}'=-2n_0\boldsymbol{v}_0\cdot\nabla\varphi'+(1-v_0{}^2)n'-(n_0{}^2/\sigma_0{}^2)\Delta n'. \tag{5·4b}$$

The dispersion relation of linear waves is then easily obtained by putting

$$n'=A\exp[i(\omega t-\boldsymbol{k}\cdot\boldsymbol{r})],\qquad \varphi'=B\exp[i(\omega t-\boldsymbol{k}\cdot\boldsymbol{r})], \tag{5·5}$$

where ω, $\boldsymbol{k}$ and $\boldsymbol{r}$ are frequency, wave vector and the vector $\boldsymbol{r}=(x,y,z)$, respectively, and A and B are constants. A result of such a calculation is written as

$$\omega\equiv\omega(\boldsymbol{k})=2n_0\boldsymbol{v}_0\cdot\boldsymbol{k}+[(1-v_0{}^2)\sigma_0{}^2+n_0{}^2k^2]^{1/2}. \tag{5·6}$$

For $\boldsymbol{v}_0=0$ Eq. (5·6) reduces to

$$\omega(\boldsymbol{k})=[\sigma_0{}^2+n_0{}^2k^2]^{1/2}k\longrightarrow\sigma_0 k\quad\text{as}\quad k\to 0. \tag{5·6$'$}$$

Equation (5·6′) corresponds to the frequency of Bogoliubov phonons,[15] in which $\sigma_0{}^2$ has the analogy of the density of condensate as seen from the last of Eqs. (2·10a).

§ 6. Critical velocity

In §§ 3 and 4 we have obtained the energies to create the uniformly flowing state with velocity v_0 and the vortex state with vorticity $q=\pm1$. For small v_0 the energy of the uniformly flowing state is lower than that of the vortex state. With increasing v_0, however, the former state becomes less and less energetically favorable. At a certain value of v_0, the energies of these two states become equal. Such a value of v_0 is called a critical velocity in close analogy with the case of superfluid He^4 and denoted by v_{0c}. For the evaluation of v_{0c} let us consider a cylinder of radius R and length L as in the case for the vortex state. From Eqs. (3·7) and (4·12) the critical velocity v_{0c} is determined by the equation

$$\pi R^2 L\Delta E_{\mathrm{uf}}=\Delta E_v \tag{6·1}$$

or

$$\frac{v_0^2}{2}\left(S^2-\frac{\gamma^2}{1-v_0^2}\right)=\frac{2}{R^2}\left[\epsilon_1+\frac{1}{2}(S^2-\gamma^2)\ln\frac{R}{r_1}\right], \tag{6·2}$$

where

$$\epsilon_1=\epsilon_0+(1/4)(S^2+\gamma^2) \tag{6·3}$$

is the energy associated with the vortex core. Since the radius r_1 of the vortex core is of the order of atomic scale, we can neglect the quantity ϵ_1 in comparison with the second term in the square bracket on the right-hand side of Eq. (6·2). By the use of this approximation procedure Eq. (6·2) is solved as

$$v_{0c}^2=\frac{(S^2-\gamma^2)\left[1+\frac{2}{R^2}\ln\frac{R}{r_1}\right]}{2S^2}\left[1-\left\{1-\frac{(8S^2/R^2)\ln(R/r_1)}{(S^2-\gamma^2)\{1+(2/R^2)\ln(R/r_1)\}^2}\right\}^{1/2}\right]. \tag{6·4}$$

Thus, an approximate expression for v_{0c} is obtained as follows:

$$v_{0c}=[2\ln(R/r_1)]^{1/2}/R\,. \tag{6·5}$$

It is seen that the critical velocity v_{0c} depends on the system size R and the radius of the vortex core r_1. The result obtained above is similar to the critical velocity

$$v_{\mathrm{cr}}=(h/mR)\ln(R/r_1) \tag{6·6}$$

obtained by Feynman for the case of liquid He II,[16] where m is the mass of He atom. Equation (6·5) is, however, different from Eq. (6·6) in that in the former the factor $[\ln(R/r_1)]^{1/2}$ appears instead of the factor $\ln(R/r_1)$.*) Implications of Eq. (6·5) to the case of superfluid He^4 will be discussed in the next section.

§ 7. Implications of the result to liquid He II

We begin the discussion of this section by observing that a pertinent quantity of the XY model in an external field corresponding to the condensate wave function ψ in superfluid He^4 is given by[13]

$$\psi\equiv S^x+iS^y=\sin\theta\exp(i\varphi). \tag{7·1}$$

*) Here we are concerned only with the (R/r_1)-dependence of v_{0c}, since by definition (2·18) the dimension of v_{0c} given by Eq. (6·5) does not coincide with that of the critical velocity given by Eq. (6·6).

As shown in the previous paper, an equation obeyed by ψ in the continuum approximation takes the form[13)]

$$-J_s(0)\alpha a^2 \Delta\psi - J_s(0)\psi + \epsilon\psi(1-|\psi|^2)^{-1/2} = 0\,, \tag{7·2}$$

where a is the lattice constant of the spin system and we have put $\alpha_{11}=\alpha_{12}=\alpha_{13}\equiv\alpha$ in Eqs. (2·13) and (2·14) for the sake of simplicity. Here and hereafter in this section we use ordinary units to make a correspondence between two systems. Equation (7·2) is to be compared with the time-independent Gross-Pitaevskii equation[11),12)]

$$-(\hbar^2/2m)\Delta\psi - \beta\psi + g|\psi|^2\psi = 0 \tag{7·3}$$

for the condensate wave function ψ. In the above equation the quantities β and g are constants. It is seen that the interaction constant $J_s(0)$ of the XY model in a transverse field has the following correspondence:

$$\alpha a^2 J_s(0) \leftrightarrow \hbar^2/2m\,. \tag{7·4}$$

Equation (7·2) is, though similar, different from Eq. (7·3) in that it contains a nonlinearity of the form $(1-|\psi|^2)^{-1/2}\psi$ in contrast with the form $|\psi|^2\psi$ derivable from the φ^4-potential. Expanding the factor $(1-|\psi|^2)^{-1/2}$ in powers of $|\psi|^2$ and retaining only the first two terms, we get

$$-J_s(0)\alpha a^2\Delta\psi - [J_s(0)-\epsilon]\psi + (\epsilon/2)|\psi|^2\psi = 0\,. \tag{7·5}$$

Within this approximation, we can also make the following correspondence:

$$J_s(0)-\epsilon \leftrightarrow \beta\,, \qquad \epsilon/2 \leftrightarrow g\,. \tag{7·6}$$

The difference of the nonlinearity in the XY model from that in liquid He^4 is that in the former it is kinematical in origin inherent in the spin system, while in the latter it is dynamical essentially due to the existence of strong repulsive part of pair potential.

Rewriting the results obtained so far in ordinary units and using Eq. (7·4), we make a brief study of implications to the case of liquid He II. By the use of this procedure, $v_0(\max)$, $\omega(k)$ and v_{0c} are rewritten as

$$v_0(\max) = (\hbar/ma\alpha^{1/2})[1-(\epsilon/J_s(0))]^{1/2}\,, \tag{7·7}$$

$$\omega(k) = (\hbar/ma\alpha^{1/2})[1-(\epsilon^2/J_s(0)^2)]^{1/2}k \equiv ck\,, \tag{7·8}$$

$$v_{0c} = (\hbar/mR)[2\ln(R/r_1)]^{1/2}\,, \tag{7·9}$$

respectively. It is seen that $v_0(\max)$ and c, the velocity of phonons, are of the same order of magnitude. Equation (7·8) is coincident with the result obtained by Matsubara and Matsuda.[8)] Putting

$$m = 2.7\times10^{-24}\text{ g}, \qquad a = 3\times10^{-8}\text{ cm}, \qquad \alpha = 1/6,$$

$$[1-(\epsilon^2/J_s(0)^2)]^{1/2} \simeq [1-(\epsilon/J_s(0))]^{1/2}, \tag{7·10}$$

we get

$$v_0(\max) \simeq c \simeq 6\times10\text{ m/sec}. \tag{7·11}$$

The numerical value of c thus obtained here is much smaller than the experimental value c_{ex} of velocity of phonons of liquid He II:

$$c_{ex} \simeq 2.4\times10^2\text{ m/sec}. \tag{7·12}$$

A large discrepancy of c from c_{ex} is due to the fact that the Bogoliubov-phonon approximation for phonons is only applicable for very weak interactions, namely for the case $n_0 \simeq 1$, where n_0, is the fraction of condensates. On the other hand, numerical value of n_0 estimated by using various methods for liquid He II is $n_0 \simeq 0.1$. The result that $v_0(\max)$ is nearly equal to c is understood by observing that the former is the critical velocity at which the symmetry-breaking state disappears.

Our principal objective in this section is to obtain numerical value of v_{0c} given by Eq. (7·8). By putting

$$R = 10^{-5}\text{ cm}, \qquad r_1 = 4\times10^{-8}\text{ cm}, \tag{7·13}$$

which are entirely equal to the numerical values adopted by Feynman in estimating the numerical value of Eq. (6·6), we get

$$v_{0c} \simeq 50\text{ cm sec}^{-1}, \tag{7·14}$$

while the Feynman formula gives

$$v_{cr} \simeq 100\text{ cm sec}^{-1}. \tag{7·15}$$

It is seen that v_{0c} so obtained is smaller than v_{cr}. It is also seen that agreement of the R-dependence of the critical velocity and its numerical value with the experimental data[17] appears to be better than those given by Eq. (6·6).

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