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FLOWS, COALESCENCE AND NOISE

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We are interested in stationary "fluid" random evolutions with independent increments. Under some mild assumptions, we show they are solutions of a stochastic differential equation (SDE). There are situations where these evolutions are not described by flows of diffeomorphisms, but by coalescing flows or by flows of probability kernels.

In an intermediate phase, for which there exist a coalescing flow and a flow of kernels solution of the SDE, a classification is given: All solutions of the SDE can be obtained by filtering a coalescing motion with respect to a subnoise containing the Gaussian part of its noise. Thus, the coalescing motion cannot be described by a white noise.

0. Introduction. A stationary motion on the real line with independent increments is described by a Levy process, or equivalently by a convolution semigroup of probability measures. This naturally extends to "rigid" motions represented by Levy processes on Lie groups. If one assumes the continuity of the paths, a convolution semigroup on a Lie group G is determined by an element of the Lie algebra $\mathfrak{g}$ (the drift) and a scalar product on $\mathfrak{g}$ (the diffusion matrix) (see, e.g., [31]). We call them the local characteristics of the convolution semigroup.

We will be interested in stationary "fluid" random evolutions which have independent increments. Strong solutions of stochastic differential equations (SDEs) driven by smooth vector fields define such evolutions. Those are of a regular type, namely:

(a) The probability that two points thrown in the fluid at the same time and at distance ε separate at distance one in one unit of time tends to 0 as ε tends to 0.

(b) Such points will never hit each other.

Their laws can be viewed as convolution semigroups of probability measures on the group of diffeomorphisms.

On a compact manifold, let $V_0, V_1, \dots, V_n$ be vector fields and let $B^1, \dots, B^n$ be independent Brownian motions. Consider the SDE

$$dX_t = \sum_{k=1}^{n} V_k(X_t) \circ dB_t^k + V_0(X_t)\, dt, \tag{0.1}$$

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which equivalently can be written

$$df(X_t) = \sum_{k=1}^{n} V_k f(X_t)\, dB_t^k + \tfrac{1}{2} Af(X_t)\, dt \tag{0.2}$$

for every smooth function f and $Af = \sum_{k=1}^{n} V_k(V_k f) + V_0 f$. Note that $Af^2 - 2fAf = \sum_{k=1}^{n} (V_k f)^2$. Then, strong solutions (when they exist), as defined, for example, in [38], of this SDE produce a flow of maps φ_t, such that, for every x, $\varphi_t(x)$ is a strong solution of the SDE with $\varphi_0(x) = x$, which means that φ_t is a function of the Brownian paths $B^1, \dots, B^n$ up to time t. When the vector fields are smooth, strong solutions are known to exist, and to be unique. The framework can be extended to include flows of maps driven by vector field valued Brownian motions, which means essentially that $n = \infty$ (see, e.g., [3, 17, 20, 21, 27]).

In a previous work [23], this was extended again to include flows of Markovian operators S_t solutions of the SPDE

$$dS_t f = \sum_{k=1}^{\infty} S_t(V_k f)\, dB_t^k + \tfrac{1}{2} S_t(Af)\, dt, \tag{0.3}$$

assuming the covariance function $C = \sum_{k=1}^{\infty} V_k \otimes V_k$ of the Brownian vector field $\sum_{k=1}^{\infty} V_k B^k$ is compatible with A, namely that

$$Af^2 - 2fAf \le \sum_{k=1}^{\infty} (V_k f)^2. \tag{0.4}$$

Existence and uniqueness of a flow of Markovian operators S_t, which is a strong solution of the previous SPDE in the sense that S_t is a function of the Brownian paths $(B^i)_{i \ge 1}$ up to time t, holds under rather weak assumptions. However, it is assumed in [23] that A is self-adjoint with respect to a measure m and the Markovian operators act on $L^2(m)$ only. To avoid confusion with the usual notion of strong Itô solutions of SDEs, these solutions will be called Wiener solutions when they are not associated with a flow of maps.

The local characteristics of these flows are given by A and the covariance function C, and they determine the SDE or the SPDE. But it was shown in [23] that covariance functions which are not smooth on the diagonal [e.g., covariance associated with Sobolev norms of order between $d/2$ and $(d+2)/2$, d being the dimension of the space] can produce Wiener solutions, which define random evolutions of different type:

(i) turbulent evolutions where (a) is not satisfied, which means that two points thrown initially at the same place separate, though there is no pure diffusion; that is, that $Af^2 - 2fAf = \sum_{k=1}^{\infty} (V_k f)^2$.

(ii) coalescing evolutions where (b) does not hold.

In this paper, we adopt a different approach based on consistent systems of n-point Markovian Feller semigroups which can be viewed as determining the law of the motion of n indivisible points thrown into the fluid. Regular and coalescing evolutions are represented by flows of maps. Turbulent evolutions by flows of probability kernels $K_{s,t}(x, dy)$ describe how a point mass (made of a continuum of indivisible points) in x at time s is spread at time t. (Note that in that case, the motion of an indivisible point is not fully determined by the flow.)

Among turbulent evolutions, we can distinguish the intermediate ones where two points thrown in the fluid at the same place separate but can meet after, that is, where (a) and (b) are both not satisfied.

In the intermediate phase, it has been shown in [9] (for gradient fields) and (at a physical level) in [10, 11, 14] that a coalescing solution of the SDE can be defined in law, that is, in the sense of the martingale problems for the n-point motions. We present a construction of a coalescing flow in the intermediate phase. This flow obviously differs from the Wiener solution $(S_{s,t}, s \le t)$ and corresponds to an absorbing boundary condition on the diagonal for the two-point motion.

This flow generates a vector field valued white noise W and we can identify the Wiener solution to the coalescing flow $(\varphi_{s,t}, s \le t)$ filtered by the velocity field $\sigma(W)$. The noise, in Tsirelson sense (see [41]), associated to the coalescing flow, is not linearizable, that is, cannot be generated by a white noise though it contains W.

A classification of the solutions of the SDE (or of the SPDE) can be given: They are obtained by filtering a coalescing motion defined on an extended probability space with respect to a subnoise containing the Gaussian part of its noise.

Let us explain in more detail the contents of the paper. We give in Sections 1 and 2 construction results, which generalize a theorem by de Finetti on exchangeable variables (see, e.g., [18]). A stochastic flow of kernels K is associated with a general compatible family $(\mathsf{P}_t^{(n)}, n \ge 1)$ of Feller semigroups. The flow K is induced by a flow of measurable mappings when

$$\mathsf{P}_t^{(2)} f^{\otimes 2}(x, x) = \mathsf{P}_t f^2(x),$$

for all $f \in C(M)$, $x \in M$ and $t \ge 0$. The Markov process associated with $\mathsf{P}_t^{(n)}$ represents the motion of n indivisible points thrown in the fluid. The two notions are shown to be equivalent: the law of a stochastic flow of kernels is uniquely determined by the compatible system of n-point motions. This construction is related to a recent result of Ma and Xiang [28] where an associated measure valued process was constructed in a special case (the flow can actually be viewed as giving the genealogy of this process, i.e., as its "historical process") and to a result of Darling [9]. Note, however, that Darling did not get flows of measurable maps except in very special cases. See also Tsirelson [43] for an alternative approach to this construction.

In Section 3, we define the noise associated with K and introduce the notion of "filtering with respect to a subnoise."

In Section 4, coalescing flows are constructed and briefly studied. They can be obtained from any flow whose two-point motion hits the diagonal. Then the original flow is shown to be recovered by filtering the coalescing flow with respect to a subnoise.

In Section 5, we restrict our attention to diffusion generators. We define the vector field valued white noise W associated with the stochastic flow of kernels K and prove that the flow solves the SDE driven by the white noise W.

In Section 6, under some off-diagonal uniqueness assumption for the law of the n-point motion, we show there is only one Wiener solution of the SDE. In the intermediate phase described above, the classification of other solutions by filtering of the coalescing solution is established. Then we identify the linear part of the noise generated by these solutions to the noise generated by W.

The examples related to our previous work (see [23]) are presented in Section 7, with an emphasis on the verification of the Feller property for the semigroups $\mathsf{P}_t^{(n)}$, the classification of the solutions and the appearance of nonclassical noise, that is, predictable noises which cannot be generated by white noises.

1. Stochastic flow of measurable mappings.

1.1. *Compatible family of Feller semigroups.* Let M be a compact metric space and let d be a distance on M.

DEFINITION 1.1. Let $(\mathsf{P}_t^{(n)}, n \geq 1)$ be a family of Feller semigroups, respectively, defined on M^n and acting on $C(M^n)$. We say that this family is compatible as soon as, for all $k \leq n$,

$$\mathsf{P}_t^{(k)} f(x_1, \dots, x_k) = \mathsf{P}_t^{(n)} g(y_1, \dots, y_n), \tag{1.1}$$

where f and g are any continuous functions such that

$$g(y_1, \dots, y_n) = f\big(y_{i_1}, \dots, y_{i_k}\big) \tag{1.2}$$

with $\{i_1, \dots, i_k\} \subset \{1, \dots, n\}$ and $(x_1, \dots, x_k) = (y_{i_1}, \dots, y_{i_k})$.

We will denote by $\mathsf{P}_{(x_1,\dots,x_n)}^{(n)}$ the law of the Markov process associated with $\mathsf{P}_t^{(n)}$ starting from $(x_1, \dots, x_n)$. This Markov process will be called the n-point motion of this family of semigroups. It is defined on the set of càdlàg paths on M^n.

REMARK 1.1. $\mathsf{P}_t^{(n)}$ is a Feller semigroup on M^n if and only if $\mathsf{P}_t^{(n)}$ is positive (i.e., $\mathsf{P}_t^{(n)} f \geq 0$ for every $f \geq 0$), $\mathsf{P}_t^{(n)} 1 = 1$ and for every continuous function f, $\mathsf{P}_t^{(n)} f$ is continuous and $\lim_{t \to 0} \mathsf{P}_t^{(n)} f(x) = f(x)$, which implies the uniform convergence of $\mathsf{P}_t^{(n)} f$ towards f (see Theorem 9.4 in Chapter I of [7]).

1.2. *Convolution semigroups on the space of measurable mappings.* We equip M with its Borel σ-field $\mathcal{B}(M)$. Let $(F, \mathcal{F})$ be the space of measurable mappings on M equipped with the σ-field generated by the mappings $\varphi \mapsto \varphi(x)$ for every $x \in M$.

DEFINITION 1.2. A probability measure Q on $(F, \mathcal{F})$ is called regular if there exists a measurable mapping $\mathcal{J}: (F, \mathcal{F}) \to (F, \mathcal{F})$ such that

$$\big(M \times F, \mathcal{B}(M) \otimes \mathcal{F}\big) \to \big(M, \mathcal{B}(M)\big),$$
$$(x, \varphi) \mapsto \mathcal{J}(\varphi)(x),$$

is measurable and, for every $x \in M$,

$$\mathsf{Q}(d\varphi)\text{-a.s.}, \qquad \mathcal{J}(\varphi)(x) = \varphi(x), \tag{1.3}$$

that is, $\mathcal{J}$ is a measurable modification of the identity mapping on $(F, \mathcal{F}, \mathsf{Q})$. We call it a measurable presentation of Q.

PROPOSITION 1.1. *Let Q_1 and Q_2 be two probability measures on $(F, \mathcal{F})$. Assume Q_1 is regular. Let $\mathcal{J}$ be a measurable presentation of Q_1. Then the mapping*

$$(F^2, \mathcal{F}^{\otimes 2}) \to (F, \mathcal{F}),$$
$$(\varphi_1, \varphi_2) \mapsto \mathcal{J}(\varphi_1) \circ \varphi_2,$$

is measurable. Moreover, if $\mathcal{J}'$ is another measurable presentation of Q_1, then for every $x \in M$,

$$\mathsf{Q}_1(d\varphi_1) \otimes \mathsf{Q}_2(d\varphi_2)\text{-a.s.}, \qquad \mathcal{J}(\varphi_1) \circ \varphi_2(x) = \mathcal{J}'(\varphi_1) \circ \varphi_2(x). \tag{1.4}$$

REMARK 1.2. (i) $(\varphi_1, \varphi_2) \mapsto \mathcal{J}(\varphi_1) \circ \varphi_2$ is measurable but $(\varphi_1, \varphi_2) \mapsto \varphi_1 \circ \varphi_2$ is not measurable.

(ii) The law of $\mathcal{J}(\varphi_1) \circ \varphi_2$ does not depend on the chosen presentation $\mathcal{J}$.

PROOF OF PROPOSITION 1.1. Let $\mathcal{J}$ be a measurable presentation of Q_1. For every $x \in M$, the mapping $(\varphi_1, \varphi_2) \mapsto \mathcal{J}(\varphi_1) \circ \varphi_2(x)$ is measurable since it is the composition of the measurable mappings $(\varphi_1, \varphi_2) \mapsto (\varphi_1, \varphi_2(x))$ and $(\varphi_1, y) \mapsto \mathcal{J}(\varphi_1)(y)$. By definition of $\mathcal{F}$, the mapping $(\varphi_1, \varphi_2) \mapsto \mathcal{J}(\varphi_1) \circ \varphi_2$ is measurable.

For every $x \in M$, we have

$$\mathsf{Q}_1(d\varphi_1)\text{-a.s.}, \qquad \mathcal{J}(\varphi_1)(x) = \varphi_1(x).$$

Thus, for all $x \in M$ and $\varphi_2 \in F$,

$$\mathsf{Q}_1(d\varphi_1)\text{-a.s.}, \qquad \mathcal{J}(\varphi_1) \circ \varphi_2(x) = \varphi_1 \circ \varphi_2(x) = \mathcal{J}'(\varphi_1) \circ \varphi_2(x).$$

Therefore, using Fubini's theorem, for every $x \in M$,

$$\mathsf{Q}_1(d\varphi_1) \otimes \mathsf{Q}_2(d\varphi_2)\text{-a.s.}, \qquad \mathcal{J}(\varphi_1) \circ \varphi_2(x) = \mathcal{J}'(\varphi_1) \circ \varphi_2(x). \qquad \square$$

DEFINITION 1.3. We denote $\mathsf{Q}_1 * \mathsf{Q}_2$, and we call the convolution product of Q_1 and Q_2, the law of the random variable $(\varphi_1, \varphi_2) \mapsto \mathcal{J}(\varphi_1) \circ \varphi_2$ defined on the probability space $(F^2, \mathcal{F}^{\otimes 2}, \mathsf{Q}_1 \otimes \mathsf{Q}_2)$.

DEFINITION 1.4. A convolution semigroup on $(F, \mathcal{F})$ is a family $(\mathsf{Q}_t)_{t \geq 0}$ of regular probability measures on $(F, \mathcal{F})$ such that, for all nonnegative s and t, $\mathsf{Q}_{s+t} = \mathsf{Q}_s * \mathsf{Q}_t$.

DEFINITION 1.5. A convolution semigroup $(\mathsf{Q}_t)_{t\geq 0}$ on $(F, \mathcal{F})$ is called Feller if:

(i) $\forall f \in C(M)$, $\lim_{t\to 0} \sup_{x \in M} \int (f \circ \varphi(x) - f(x))^2 \mathsf{Q}_t(d\varphi) = 0$.
(ii) $\forall f \in C(M)$, $\forall t \geq 0$, $\lim_{d(x,y)\to 0} \int (f \circ \varphi(x) - f \circ \varphi(y))^2 \mathsf{Q}_t(d\varphi) = 0$.

PROPOSITION 1.2. *Let $(\mathsf{Q}_t)_{t\geq 0}$ be a Feller convolution semigroup on $(F, \mathcal{F})$. For all $n \geq 1$, $f \in C(M^n)$ and $x \in M^n$, set*

$$\mathsf{P}_t^{(n)} f(x) = \int f \circ \varphi^{\otimes n}(x) \mathsf{Q}_t(d\varphi). \tag{1.5}$$

Then $(\mathsf{P}_t^{(n)}, n \geq 1)$ is a compatible family of Feller semigroups on M satisfying

$$\mathsf{P}_t^{(2)} f^{\otimes 2}(x, x) = \mathsf{P}_t f^2(x), \tag{1.6}$$

for all $f \in C(M)$, $x \in M$ and $t \geq 0$.

PROOF. It is easy to see that this family is compatible and that, for all $n \geq 1$ and $t \geq 0$, $\mathsf{P}_t^{(n)}$ is Markovian. Let s and t be in $\mathbb{R}^+$, $f \in C(M^n)$ and $x \in M$; then

$$\begin{aligned}
\mathsf{P}_{s+t}^{(n)} f(x) &= \int f \circ \varphi^{\otimes n}(x) \mathsf{Q}_{s+t}(d\varphi) \\
&= \int f \circ \mathcal{J}(\varphi_1)^{\otimes n} \circ \varphi_2^{\otimes n}(x) \mathsf{Q}_t(d\varphi_1) \otimes \mathsf{Q}_s(d\varphi_2) \\
&= \int \mathsf{P}_t^{(n)} f \circ \varphi_2^{\otimes n}(x) \mathsf{Q}_s(d\varphi_2) \\
&= \mathsf{P}_s^{(n)} \mathsf{P}_t^{(n)} f(x),
\end{aligned}$$

where $\mathcal{J}$ is a measurable presentation of Q_s. This proves that $\mathsf{P}_t^{(n)}$ is a semigroup.

Let us now prove the Feller property. Let $h \in C(M^n)$ be in the form $f_1 \otimes \cdots \otimes f_n$, $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$. We have, for M large enough,

$$\left|\mathsf{P}_t^{(n)} h(y) - \mathsf{P}_t^{(n)} h(x)\right| \leq M \sum_{k=1}^{n} \left(\int \left(f_k \circ \varphi(y_k) - f_k \circ \varphi(x_k)\right)^2 \mathsf{Q}_t(d\varphi) \right)^{1/2}, \tag{1.7}$$

which converges toward 0 as $d(x, y)$ goes to 0 since (ii) in Definition 1.5 is satisfied. We also have

$$(1.8)\qquad \left|\mathsf{P}_t^{(n)}h(x)-h(x)\right| \le M\sum_{k=1}^{n}\left(\int \left(f_k\circ\varphi(x_k)-f_k(x_k)\right)^2\mathsf{Q}_t(d\varphi)\right)^{1/2},$$

which converges toward 0 as t goes to 0 since (i) in Definition 1.5 is satisfied. These properties extend to every function h in $C(M^n)$ by an approximation argument. This proves the Feller property of the Markovian semigroups $\mathsf{P}_t^{(n)}$.

It remains to prove (1.6). This follows from

$$\begin{aligned}\mathsf{P}_t^{(2)}f^{\otimes 2}(x,x) &= \int f^{\otimes 2}\circ\varphi^{\otimes 2}(x,x)\mathsf{Q}_t(d\varphi)\\ &= \int f^2\circ\varphi(x)\mathsf{Q}_t(d\varphi) = \mathsf{P}_t^{(1)}f^2(x).\end{aligned}$$

□

REMARK 1.3. The semigroup $(\mathsf{Q}_t)_{t\ge 0}$ is uniquely determined by $(\mathsf{P}_t^{(n)}, n\ge 1)$.

1.3. *Stochastic flows of mappings.*

DEFINITION 1.6. Let $(\Omega,\mathcal{A},\mathsf{P})$ be a probability space. A family of $(F,\mathcal{F})$-valued random variables $(\varphi_{s,t}, s\le t)$ is called a measurable stochastic flow of mappings if, for all $s\le t$, the mapping

$$\begin{aligned}\big(M\times\Omega,\mathcal{B}(M)\otimes\mathcal{A}\big) &\to \big(M,\mathcal{B}(M)\big),\\ (x,\omega) &\mapsto \varphi_{s,t}(x,\omega),\end{aligned}$$

is measurable and if it satisfies the following properties:

(a) For all $s<u<t$ and $x\in M$, P-a.s., $\varphi_{s,t}(x)=\varphi_{u,t}\circ\varphi_{s,u}(x)$ (cocycle property).

(b) For all $s\le t$, the law of $\varphi_{s,t}$ only depends on $t-s$ (stationarity).

(c) The flow has independent increments; that is, for all $t_1<t_2<\cdots<t_n$, the family $\{\varphi_{t_i,t_{i+1}}, 1\le i\le n-1\}$ is independent.

(d) For every $f\in C(M)$,

$$\lim_{(u,v)\to(s,t)}\sup_{x\in M}\mathsf{E}\left[\left(f\circ\varphi_{s,t}(x)-f\circ\varphi_{u,v}(x)\right)^2\right]=0.$$

(e) For all $f\in C(M)$ and $s\le t$,

$$\lim_{d(x,y)\to 0}\mathsf{E}\left[\left(f\circ\varphi_{s,t}(x)-f\circ\varphi_{s,t}(y)\right)^2\right]=0.$$

DEFINITION 1.7. A family of $(F,\mathcal{F})$-valued random variables $\varphi=(\varphi_{s,t}, s\le t)$ is called a stochastic flow of mappings if there exists $\varphi'=(\varphi'_{s,t}, s\le t)$, a measurable stochastic flow of mappings, such that, for all $x\in M$ and $s\le t$,

$$(1.9)\qquad \mathsf{P}\text{-a.s.},\qquad \varphi'_{s,t}(x)=\varphi_{s,t}(x).$$

The stochastic flow φ' is called a measurable modification of φ.

PROPOSITION 1.3. *Let $\varphi = (\varphi_{s,t}, s \le t)$ be a stochastic flow of mappings. For all $n \ge 1$, $f \in C(M^n)$ and $x \in M^n$, set*

$$\mathsf{P}_t^{(n)} f(x) = \mathsf{E}[f \circ \varphi_{0,t}^{\otimes n}(x)]. \tag{1.10}$$

Then $(\mathsf{P}_t^{(n)}, n \ge 1)$ is a compatible family of Feller semigroups on M satisfying (1.6).

PROOF. The proof is similar to the one of Proposition 1.2 (proving first the Feller property for f Lipschitz). □

REMARK 1.4. The law of φ is uniquely determined by $(\mathsf{P}_t^{(n)}, n \ge 1)$.

1.4. *Construction and characterization.* In this section, we present a theorem stating that to any compatible family $(\mathsf{P}_t^{(n)}, n \ge 1)$ of Feller semigroups, one can associate a Feller convolution semigroup on $(F, \mathcal{F})$ and a stochastic flow of mappings.

Let $(\Omega^0, \mathcal{A}^0)$ denote the measurable space $(\prod_{s \le t} F, \bigotimes_{s \le t} \mathcal{F})$. For $s \le t$, let $\varphi_{s,t}^0$ denote the random variable $\omega \mapsto \omega(s,t)$. Let φ^0 be the random variable $(\varphi_{s,t}^0, s \le t)$. Then $\varphi^0(\omega) = \omega$. Let $(T_h)_{h \in \mathbb{R}}$ be the one-parameter group of transformations of Ω^0 defined by $T_h(\omega)(s,t) = \omega(s+h, t+h)$, for all $s \le t$, $h \in \mathbb{R}$ and $\omega \in \Omega^0$.

DEFINITION 1.8. A probability space $(\Omega, \mathcal{A}, \mathsf{P})$ is said to be separable if the Hilbert space $L^2(\Omega, \mathcal{A}, \mathsf{P})$ is separable. [Note that this implies that, for every $1 \le p < \infty$, $L^p(\Omega, \mathcal{A}, \mathsf{P})$ is separable.]

THEOREM 1.1. (i) *Let $(\mathsf{P}_t^{(n)}, n \ge 1)$ be a compatible family of Feller semigroups on M satisfying*

$$\mathsf{P}_t^{(2)} f^{\otimes 2}(x,x) = \mathsf{P}_t f^2(x), \tag{1.11}$$

for all $f \in C(M)$, $x \in M$ and $t \ge 0$. Then there exists a unique Feller convolution semigroup $(\mathsf{Q}_t)_{t \ge 0}$ on $(F, \mathcal{F})$ such that, for all $n \ge 1$, $t \ge 0$, $f \in C(M^n)$ and $x \in M^n$,

$$\mathsf{P}_t^{(n)} f(x) = \int f \circ \varphi^{\otimes n}(x) \mathsf{Q}_t(d\varphi). \tag{1.12}$$

(ii) *For every Feller convolution semigroup $\mathsf{Q} = (\mathsf{Q}_t)_{t \ge 0}$ on $(F, \mathcal{F})$, there exists a unique $(T_h)_{h \in \mathbb{R}}$-invariant probability measure P_Q on $(\Omega^0, \mathcal{A}^0)$ such that $(\Omega^0, \mathcal{A}^0, \mathsf{P}_\mathsf{Q})$ is separable, the family of random variables $\varphi^0 = (\varphi_{s,t}^0, s \le t)$ is a stochastic flow of mappings and, for all $s \le t$, the law of $\varphi_{s,t}^0$ is Q_{t-s}. There exists a measurable modification of φ^0, φ' such that $\varphi'_{s+h,t+h} = \varphi'_{s,t} \circ T_h$.*

The flow φ^0 is called the canonical stochastic flow of mappings associated with Q [or equivalently with $(\mathsf{P}_t^{(n)}, n \ge 1)$].

REMARK 1.5. Theorem 1.1 is also satisfied when M is a locally compact separable metric space. In this case, $(\mathsf{P}_t^{(n)}, n \geq 1)$ is a compatible family of Markovian semigroups acting continuously on $C_0(M^n)$, the set of continuous functions on M^n converging toward 0 at ∞ (we call them Feller semigroups). In Definitions 1.5 and 1.6 and in the statement of Theorem 1.1, the function f has to be taken in $C_0(M)$ or in $C_0(M^n)$. Moreover (ii) of Definition 1.5 must be modified by: for all $x \in M$, $f \in C_0(M)$ and $t \geq 0$,

$$\lim_{y\to x} \int \Big(f \circ \varphi(y) - f \circ \varphi(x)\Big)^2 \mathsf{Q}_t(d\varphi) = 0 \quad \text{and} \qquad (1.13)$$
$$\lim_{y\to\infty} \int (f \circ \varphi(y))^2 \mathsf{Q}_t(d\varphi) = 0.$$

In Definition 1.6, (e) must be modified by: for all $x \in M$ and $s \leq t$,

$$\lim_{y\to x} \mathsf{E}\big[(f \circ \varphi_{s,t}(y) - f \circ \varphi_{s,t}(x))^2\big] = 0 \quad \text{and} \qquad (1.14)$$
$$\lim_{y\to\infty} \mathsf{E}\big[(f \circ \varphi_{s,t}(y))^2\big] = 0.$$

PROOF. In order to prove this remark, note that the one-point compactification of M, $\hat{M} = M \cup \{\infty\}$, is a compact metric space. On $\hat{M}$, we define the compatible family of Feller semigroups, $(\hat{\mathsf{P}}_t^{(n)}, n \geq 1)$, by the following relations:

for every $n \geq 2$ and every family of continuous functions on $\hat{M}$, $\{f_i, i \geq 1\}$,

$$\begin{aligned}\hat{\mathsf{P}}_t^{(n)} & f_1 \otimes \cdots \otimes f_n \\ &= \mathsf{P}_t^{(n)} g_1 \otimes \cdots \otimes g_n \\ &\quad + \sum_{i=1}^n f_i(\infty) \hat{\mathsf{P}}_t^{(n-1)} f_1 \otimes \cdots \otimes f_{i-1} \otimes g_{i+1} \otimes \cdots \otimes g_n\end{aligned} \qquad (1.15)$$

and

$$\hat{\mathsf{P}}_t^{(1)} f_1 = f_1(\infty) + \mathsf{P}_t^{(1)} g_1, \qquad (1.16)$$

where $g_i = f_i - f_i(\infty) \in C_0(M)$ and with the convention $\mathsf{P}_t^{(n)} g_1 \otimes \cdots \otimes g_n(x_1, \ldots, x_n) = 0$ if there exists i such that $x_i = \infty$. We apply Theorem 1.1 to $\hat{M}$ and to the family $(\hat{\mathsf{P}}_t^{(n)}, n \geq 1)$ to construct a Feller convolution semigroup $\hat{\mathsf{Q}}$ and a stochastic flow of mappings $(\hat{\varphi}_{s,t}, s \leq t)$ on $\hat{M}$. This stochastic flow of mappings satisfies:

(i) $\hat{\varphi}_{s,t}(\infty) = \infty$ for all $s \leq t$ and
(ii) $\hat{\varphi}_{s,t}(x) \neq \infty$ for all $x \in M$ and $s \leq t$.

Proof of (i). For every $f \in C(\hat{M})$,

$$\mathsf{E}\big[\big(f \circ \hat{\varphi}_{s,t}(\infty) - f(\infty)\big)^2\big] = \hat{\mathsf{P}}^{(2)}_{t-s} f^{\otimes 2}(\infty,\infty) - 2f(\infty)\hat{\mathsf{P}}^{(1)}_{t-s} f(\infty) + f(\infty)^2 = 0,$$

since $\hat{\mathsf{P}}^{(2)}_{t-s} f^{\otimes 2}(\infty,\infty) = f(\infty)^2$ and $\hat{\mathsf{P}}^{(1)}_{t-s} f(\infty) = f(\infty)$. This implies (i).

Proof of (ii). Let g_n be a sequence in $C_0(M)$ such that $g_n \in [0,1]$ and simply converging towards 1. Then $f_n = 1 - g_n \in C(\hat{M})$ is such that $f_n(\infty) = 1$ and, for every $x \in M$,

$$\mathsf{E}\big[\big(f_n \circ \hat{\varphi}_{s,t}(x)\big)^2\big] = \hat{\mathsf{P}}^{(2)}_{t-s} g_n^{\otimes 2}(x,x) + 1 - 2\hat{\mathsf{P}}^{(1)}_{t-s} g_n(x).$$

This implies that $\lim_{n\to\infty} \mathsf{E}[(f_n \circ \hat{\varphi}_{s,t}(x))^2] = 0$. Assertion (ii) follows since $\mathbb{1}_{\{\hat{\varphi}_{s,t}(x)=\infty\}} = \lim_{n\to\infty} f_n \circ \hat{\varphi}_{s,t}(x)$.

For every $x \in M$, let us denote $\hat{\varphi}_{s,t}(x)$ by $\varphi_{s,t}(x)$. Assertions (i) and (ii) imply that $\varphi_{s,t} \in F$ and that $(\varphi_{s,t}, s \le t)$ is a stochastic flow of mappings on M. In a similar way, one can show that $\hat{\mathsf{Q}}$ induces a Feller convolution semigroup on $(F, \mathcal{F})$. □

Let us explain briefly the method we use to prove Theorem 1.1. We first suppose we are given a compatible family of Feller semigroups satisfying (1.6). Then we define a convolution semigroup $(\mathsf{Q}_t, t \ge 0)$ on measurable mappings on M. For every t, to define Q_t, we define $\mathsf{P}^{(\infty)}_t$, the law of $(\varphi(z_l), l \in \mathbb{N})$, where the law of φ is Q_t, for some dense family $(z_l, l \in \mathbb{N})$ in M and get Q_t by an approximation. Hence Q_t is defined as the law of a random variable, which takes its values in the "bad" space E, but is defined on a "nice" space $M^{\mathbb{N}}$.

The approximation used to construct this convolution semigroup allows us to define a stochastic flow of mappings on M in such a way that these mappings are measurable, defining it first on the dyadic numbers. We get a measurable flow defined on a "nice" space. Note that a difficulty in getting this measurability comes from the fact that the composition of mappings from M onto M is not measurable with respect to the natural σ-field.

1.5. *Proof of the first part of Theorem* 1.1. In the following, we assume we are given $(\mathsf{P}^{(n)}_t, n \ge 1)$, a compatible family of Feller semigroups satisfying (1.6). We intend to construct a Feller convolution semigroup $(\mathsf{Q}_t)_{t\ge 0}$ on $(F, \mathcal{F})$ satisfying (1.12). The uniqueness of such a convolution semigroup is immediate since (1.12) characterizes Q_t.

1.5.1. *A measurable choice of limit points in* M. It is known that, as a compact metric space, M is homeomorphic to a closed subset of $[0,1]^{\mathbb{N}}$ (see Corollaire 1 in Section 6.1 of Chapter 9 in [8]). A point y can be represented by a sequence $(y^n)_{n\in\mathbb{N}} \in [0,1]^{\mathbb{N}}$. Let $y = (y_i)_{i\in\mathbb{N}}$ be a sequence of elements of M.

Let $y^1 = \limsup_{i\to\infty} y_i^1$. Let $i_k^1 = \inf\{i, |y^1 - y_i^1| < 1/k\}$. By induction, for every integer j, we construct y^j and $\{i_k^j, k \in \mathbb{N}\}$ by the relations

$$y^j = \limsup_{k\to\infty} y^j_{i_k^{j-1}} \quad \text{and} \quad i_k^j = \inf\bigl\{i \in \{i_k^{j-1}, k \in \mathbb{N}\}, |y^j - y_i^j| < 1/k\bigr\}.$$

We denote $(y^n)_{n\in\mathbb{N}}$ by $l(y)$. Note that $l(y)^j = \lim_{n\to\infty} y^j_{i_n^n}$. Hence $l(y)$ belongs to M. It is easy to see that l satisfies the following lemma.

LEMMA 1.1. *$l : M^{\mathbb{N}} \to M$ is a measurable mapping, M being equipped with the Borel σ-field $\mathcal{B}(M)$ and $M^{\mathbb{N}}$ with the product σ-field $\mathcal{B}(M)^{\otimes\mathbb{N}}$. Moreover, $l((y_i)_{i\in\mathbb{N}}) = y_\infty$ when y_i converges toward y_∞.*

1.5.2. *Notation and definitions.* Let $\{z_l, l \in \mathbb{N}\}$ be a dense family in M, which will be fixed in the following. We wish to define a measurable mapping $i : M^{\mathbb{N}} \to F$ such that $i((y_j)_{j\in\mathbb{N}})(z_l) = y_l$ for every integer l.

Let $(\varepsilon_k)_{k\in\mathbb{N}}$ be a positive sequence decreasing toward 0 (this sequence will be fixed later). Let $i : M^{\mathbb{N}} \to F$ be the injective mapping defined by

$$i(y)(x) = l\bigl((y_{n_k^x})_{k\in\mathbb{N}}\bigr), \tag{1.17}$$

where

$$n_k^x = \inf\{n, d(z_n, x) \le \varepsilon_k\}, \tag{1.18}$$

for $(y, x) \in M^{\mathbb{N}} \times M$. Note that $i(y)$ defined this way is a measurable mapping since l is measurable and $x \mapsto (y_{n_k^x})_{k\in\mathbb{N}}$ is measurable. Note also that the relation $i(y)(z_l) = y_l$ is satisfied for every integer l.

LEMMA 1.2. *For $n \ge 1$, the mappings $\Phi_n : (M^{\mathbb{N}})^n \to F$ and $\Psi_n : M \times (M^{\mathbb{N}})^n \to M$, defined by*

$$\Phi_n(y^1, \dots, y^n) = i(y^n) \circ i(y^{n-1}) \circ \cdots \circ i(y^1),$$
$$\Psi_n(x, y^1, \dots, y^n) = \Phi_n(y^1, \dots, y^n)(x),$$

are measurable. [*$(M^{\mathbb{N}})^n$ and $M \times (M^{\mathbb{N}})^n$ are equipped with the product σ-field.*] *In particular, i is measurable.*

PROOF. Note that Ψ_1 is the composition of the mappings l and $(x, y) \mapsto (y_{n_k^x})_{k\in\mathbb{N}}$. Since these mappings are measurable, Ψ_1 is measurable. By induction, we prove that Ψ_n is measurable since, for $n \ge 2$,

$$\Psi_n(x, y^1, \dots, y^n) = \Psi_1\bigl(\Psi_{n-1}(x, y^1, \dots, y^{n-1}), y^n\bigr).$$

For all $A \in \mathcal{B}(M)$ and $x \in M$,

$$\Phi_n^{-1}\bigl(\{\varphi \in F, \varphi(x) \in A\}\bigr) = \{y \in (M^{\mathbb{N}})^n, (x, y) \in \Psi_n^{-1}(A)\}.$$

This event belongs to $(\mathcal{B}(M)^{\otimes\mathbb{N}})^{\otimes n}$ since Ψ_n is measurable. This shows the measurability of Φ_n. □

We need to introduce Φ_n because the composition application $F^n \to F$, $(\varphi_1, \dots, \varphi_n) \mapsto \varphi_n \circ \cdots \circ \varphi_1$ is not $\mathcal{F}^{\otimes n}$-measurable in general.

Let $j : F \to M^{\mathbb{N}}$ be the mapping defined by

$$j(\varphi) = \big(\varphi(z_l)\big)_{l \in \mathbb{N}}. \tag{1.19}$$

LEMMA 1.3. *The mapping j is measurable and satisfies $j \circ i(y) = y$ for every $y \in M^{\mathbb{N}}$.*

PROOF. We have, for every $A \in \mathcal{B}(M)^{\otimes n}$,

$$j^{-1}\big(\{y \in M^{\mathbb{N}}, (y_1, \dots, y_n) \in A\}\big) = \big\{\varphi \in F, \big(\varphi(z_1), \dots, \varphi(z_n)\big) \in A\big\}.$$

This set belongs to $\mathcal{F}$. □

Note that, for all $l \in \mathbb{N}$ and $\varphi \in F$, $i \circ j(\varphi)(z_l) = \varphi(z_l)$.

1.5.3. *Constructions of probabilities on $M^{\mathbb{N}}$ and on F.* By Kolmogorov's theorem, we construct on $M^{\mathbb{N}}$ a probability measure $\mathsf{P}_t^{(\infty)}$ such that $\mathsf{P}_t^{(\infty)}(A \times M^{\mathbb{N}}) = \mathsf{P}_t^{(n)} \mathbb{1}_A(z_1, \dots, z_n)$ for any $A \in \mathcal{B}(M)^{\otimes n}$. We now prove useful lemmas satisfied by $\mathsf{P}_t^{(\infty)}$.

LEMMA 1.4. *For every positive T, there exists a positive function $\varepsilon_T(r)$ converging toward 0 as r goes to 0 such that*

$$\sup_{t \in [0,T]} \mathsf{E}_{(x,y)}^{(2)}\big[\big(d(X_t, Y_t)\big)^2\big] \le \varepsilon_T\big(d(x, y)\big). \tag{1.20}$$

PROOF. For every continuous function f, we have

$$\begin{aligned}\mathsf{E}_{(x,y)}^{(2)}\big[\big(f(X_t) - f(Y_t)\big)^2\big] &= \mathsf{P}_t f^2(x) + \mathsf{P}_t f^2(y) - 2\mathsf{P}_t^{(2)} f^{\otimes 2}(x, y) \\ &= \mathsf{P}_t^{(2)} f^{\otimes 2}(x, x) + \mathsf{P}_t^{(2)} f^{\otimes 2}(y, y) - 2\mathsf{P}_t^{(2)} f^{\otimes 2}(x, y),\end{aligned}$$

since (1.6) is satisfied. Let $(f_n)_{n \ge 1}$ be a dense sequence in $\{f \in C(M), \|f\|_\infty \le 1\}$. Then $d'(x, y) = (\sum_{n \ge 1} 2^{-n}(f_n(x) - f_n(y))^2)^{1/2}$ is a distance equivalent to d and we have

$$\mathsf{E}_{(x,y)}^{(2)}\big[\big(d'(X_t, Y_t)\big)^2\big] = \mathsf{P}_t^{(2)} h(x, x) + \mathsf{P}_t^{(2)} h(y, y) - 2\mathsf{P}_t^{(2)} h(x, y),$$

where h is the continuous function $\sum_{n \ge 1} 2^{-n} f_n \otimes f_n$. We conclude the lemma after remarking that this function is uniformly continuous in (t, x, y) on $[0, T] \times M^2$. □

From now on we fix T and define the sequence $(\varepsilon_k)_{k \in \mathbb{N}}$ [which defines the sequence $(n_k^x)_{k \in \mathbb{N}}$ for every $x \in M$ by (1.18)] such that $0 \le r \le 2\varepsilon_k$ implies $\varepsilon_T(r) \le 2^{-3k}$. The sequence $(\varepsilon_k)_{k \in \mathbb{N}}$ is well defined since $\lim_{r \to 0} \varepsilon_T(r) = 0$. Since i depends on T, we now denote i by i_T, Φ_n by Φ_n^T and Ψ_n by Ψ_n^T.

LEMMA 1.5. *For every $t \in [0, T]$ and for any independent random variables X and Y, respectively, in M and $M^{\mathbb{N}}$, such that the law of Y is $\mathsf{P}_t^{(\infty)}$, then $Y_{n_k^X}$ converges almost surely towards $l((Y_{n_k^X})_{k\in\mathbb{N}}) = i_T(Y)(X)$ as k goes to ∞.*

PROOF. Note that $(Y_{n_k^X})_{k\in\mathbb{N}}$ is a random variable [the mapping $(x, y) \mapsto (y_{n_k^x})_{k\in\mathbb{N}}$ is measurable]. For every integer k, $d(z_{n_k^x}, z_{n_{k+1}^x}) \leq 2\varepsilon_k$ and

$$\mathsf{P}\big[d\big(Y_{n_k^X}, Y_{n_{k+1}^X}\big) > 2^{-k}\big] \leq 2^{2k}\mathsf{E}\big[\varepsilon_T\big(d\big(z_{n_k^X}, z_{n_{k+1}^X}\big)\big)\big] \leq 2^{-k}. \tag{1.21}$$

Using the Borel–Cantelli lemma, we prove that a.s., $(Y_{n_k^X})_{k\in\mathbb{N}}$ is a Cauchy sequence and therefore converges. Its limit can only be $l((Y_{n_k^X})_{k\in\mathbb{N}})$. □

LEMMA 1.6. *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of random variables in M converging in probability toward a random variable X. Let Y be a random variable in $M^{\mathbb{N}}$ of law $\mathsf{P}_t^{(\infty)}$ independent of $(X_n)_{n\in\mathbb{N}}$. Then $i_T(Y)(X_n) = l((Y_{n_k^{X_n}})_{k\in\mathbb{N}})$ converges in probability towards $i_T(Y)(X) = l((Y_{n_k^X})_{k\in\mathbb{N}})$ as n tends to ∞.*

PROOF. Let $Z_n = l((Y_{n_k^{X_n}})_{k\in\mathbb{N}})$ and $Z = l((Y_{n_k^X})_{k\in\mathbb{N}})$. For every integer k, we have

$$\mathsf{P}[d(Z_n, Z) > \varepsilon] \leq \mathsf{P}\big[d\big(Z_n, Y_{n_k^{X_n}}\big) > \varepsilon/3\big] + \mathsf{P}\big[d\big(Y_{n_k^{X_n}}, Y_{n_k^X}\big) > \varepsilon/3\big] + \mathsf{P}\big[d\big(Y_{n_k^X}, Z\big) > \varepsilon/3\big].$$

Lemma 1.5 implies that the first and last terms of the right-hand side of the preceding equation converge toward 0 as k goes to ∞. The second term is lower than $(9/\varepsilon^2)\mathsf{E}[\varepsilon_T(d(z_{n_k^{X_n}}, z_{n_k^X}))]$. Since for every positive α, there exists a positive η such that $|r| < \eta$ implies $(9/\varepsilon^2)|\varepsilon_T(r)| < \alpha$, we get

$$\begin{aligned}\mathsf{P}\big[d\big(Y_{n_k^{X_n}}, Y_{n_k^X}\big) > \varepsilon/3\big] &\leq \alpha + C\mathsf{P}\big[d\big(z_{n_k^{X_n}}, z_{n_k^X}\big) > \eta\big] \\ &\leq \alpha + C\mathsf{P}[d(X_n, X) > \eta - 2\varepsilon_k],\end{aligned}$$

where $C = 9D^2/\varepsilon^2$ and D is the diameter of M (one can choose ε_T such that $\varepsilon_T(r) \leq D^2$ for every r). Therefore, we get $\mathsf{P}[d(Z_n, Z) > \varepsilon] \leq \alpha + C\mathsf{P}[d(X_n, X) \geq \eta]$ and for every positive α, $\limsup_{n\to\infty} \mathsf{P}[d(Z_n, Z) > \varepsilon] \leq \alpha$. Thus we prove that Z_n converges in probability toward Z. □

For every $t \in [0, T]$, set $\mathsf{Q}_t = i_T^*(\mathsf{P}_t^{(\infty)})$. It is a probability measure on $(F, \mathcal{F})$ and it satisfies the following proposition.

PROPOSITION 1.4. *Q_t is the unique probability measure on $(F, \mathcal{F})$ such that, for any continuous function f on M^n and any $x \in M^n$,*

$$\int_F f \circ \varphi^{\otimes n}(x)\mathsf{Q}_t(d\varphi) = \mathsf{P}_t^{(n)} f(x). \tag{1.22}$$

Moreover, $j^*(\mathsf{Q}_t) = \mathsf{P}_t^{(\infty)}$ *and* $(i_T \circ j)^*(\mathsf{Q}_t) = i_T^*(\mathsf{P}_t^{(\infty)}) = \mathsf{Q}_t$.

PROOF. The unicity is obvious since (1.22) characterizes Q_t. Let us check that $\mathsf{Q}_t = i_T^*(\mathsf{P}_t^{(\infty)})$ satisfies (1.22). Let Y be a random variable of law $\mathsf{P}_t^{(\infty)}$. Then, for every $f \in C(M^n)$ and every $x \in M^n$,

$$\begin{aligned}\int_F f \circ \varphi^{\otimes n}(x)\mathsf{Q}_t(d\varphi) &= \mathsf{E}\big[f\big(i_T(Y)(x_1), \dots, i_T(Y)(x_n)\big)\big]\\ &= \lim_{k\to\infty} \mathsf{E}\big[f\big(Y_{n_k^{x_1}}, \dots, Y_{n_k^{x_n}}\big)\big]\\ &= \lim_{k\to\infty} \mathsf{P}_t^{(n)} f\big(z_{n_k^{x_1}}, \dots, z_{n_k^{x_n}}\big) = \mathsf{P}_t^{(n)} f(x),\end{aligned}$$

using first dominated convergence theorem and Lemma 1.5, then the definition of $\mathsf{P}_t^{(\infty)}$ and the fact that $\mathsf{P}_t^{(n)}$ is Feller. □

REMARK 1.6. Since T can be taken arbitrarily large, we can define Q_t for every positive t and the definition of Q_t is independent of the chosen T, since Q_t satisfies Proposition 1.4.

1.5.4. *A convolution semigroup on* $(F, \mathcal{F})$.

LEMMA 1.7. *For every* $t \geq 0$, Q_t *is regular. And for every* $T \geq t$, $i_T \circ j$ *is a measurable presentation of* Q_t.

PROOF. Let $0 \leq t \leq T$. For all $x \in M$ and $\varphi \in F$, $i_T \circ j(\varphi)(x) = \Psi_1^T(x, j(\varphi))$. Since Ψ_1^T and j are measurable, the mapping $(x, \varphi) \mapsto i_T \circ j(\varphi)(x)$ is measurable.

Let $x \in M$. Since $\mathsf{Q}_t = i_T^*(\mathsf{P}_t^{(\infty)})$, if Y is a random variable of law $\mathsf{P}_t^{(\infty)}$,

$$\begin{aligned}\mathsf{Q}_t\big[d\big(\varphi(z_{n_k^x}), \varphi(x)\big) > 2^{-k}\big] &= \mathsf{P}\big[d\big(Y_{n_k^x}, i_T(Y)(x)\big) \geq 2^{-k}\big]\\ &= \lim_{l\to\infty} \mathsf{P}\big[d\big(Y_{n_k^x}, Y_{n_l^x}\big) \geq 2^{-k}\big] \leq 2^{-k},\end{aligned}$$

since for all $l \geq k$, $d(z_{n_k^x}, z_{n_l^x}) \leq 2\varepsilon_k$ [see (1.21)]. Using the Borel–Cantelli lemma, we prove that $\varphi(z_{n_k^x})$ converges a.s. toward $\varphi(x)$. Therefore,

$$\mathsf{Q}_t(d\varphi)\text{-a.s.,} \qquad i_T \circ j(\varphi)(x) = \varphi(x).$$

This proves the lemma. □

REMARK 1.7. Let φ and X be independent random variables, respectively, F-valued and M-valued. Then, if the law of φ is Q_t and if $M \times \Omega \ni (x, \omega) \mapsto \varphi(x, \omega) \in M$ is measurable, Fubini's theorem implies that, for every $T \geq t$,

$$\mathsf{P}\text{-a.s.,} \qquad i_T \circ j(\varphi)(X) = \varphi(X). \tag{1.23}$$

LEMMA 1.8. *For all* $t_1, \dots, t_n$ *in* $[0, T]$,

$$(\Phi_n^T)^*(\mathsf{P}_{t_1}^{(\infty)} \otimes \cdots \otimes \mathsf{P}_{t_n}^{(\infty)}) = \mathsf{Q}_{t_1+\cdots+t_n}. \tag{1.24}$$

PROOF. Let us prove that $(\Phi_n^T)^*(\mathsf{P}_{t_1}^{(\infty)} \otimes \cdots \otimes \mathsf{P}_{t_n}^{(\infty)})$ satisfies (1.22) for all $f \in C(M^k)$, $x \in M^k$ and $t = t_1 + \cdots + t_n$. To simplify, we prove this for $k = 1$. Let $f \in C(M)$ and $x \in M$; then, applying Fubini's theorem,

$$\begin{aligned}
&\int_F f(\varphi(x))(\Phi_n^T)^*(\mathsf{P}_{t_1}^{(\infty)} \otimes \cdots \otimes \mathsf{P}_{t_n}^{(\infty)})(d\varphi) \\
&\qquad = \int f\big(i_T(y^n) \circ i_T(y^{n-1}) \circ \cdots \circ i_T(y^1)(x)\big)\mathsf{P}_{t_1}^{(\infty)}(dy^1) \otimes \cdots \otimes \mathsf{P}_{t_n}^{(\infty)}(dy^n) \\
&\qquad = \int \mathsf{P}_{t_n}^{(1)} f\big(i(y^{n-1}) \circ \cdots \circ i_T(y^1)(x)\big)\mathsf{P}_{t_1}^{(\infty)}(dy^1) \otimes \cdots \otimes \mathsf{P}_{t_{n-1}}^{(\infty)}(dy^{n-1}) \\
&\qquad = \cdots = \mathsf{P}_{t_1+\cdots+t_n}^{(1)} f(x).
\end{aligned}$$

The proof is similar for $f \in C(M)^k$ and $x \in M^k$. We conclude using Proposition 1.4. □

PROPOSITION 1.5. $(\mathsf{Q}_t)_{t \geq 0}$ *is a Feller convolution semigroup on* $(F, \mathcal{F})$.

PROOF. For all nonnegative s and t, $\Phi_2^T \circ j^{\otimes 2}$ is measurable. Proposition 1.4 and Lemma 1.8 imply that $(\Phi_2^T \circ j^{\otimes 2})^*(\mathsf{Q}_s \otimes \mathsf{Q}_t) = \mathsf{Q}_{s+t}$. Since $(\Phi_2^T \circ j^{\otimes 2})(\varphi_1, \varphi_2) = (i_T \circ j)(\varphi_1) \circ (i_T \circ j)(\varphi_2)$, we have easily that $\mathsf{Q}_s * \mathsf{Q}_t = \mathsf{Q}_{s+t}$. The Feller property for Q is easy to prove. □

This proves the first part of Theorem 1.1.

1.6. *Proof of the second part of Theorem* 1.1. We now assume we are given a Feller convolution semigroup $\mathsf{Q} = (\mathsf{Q}_t)_{t \geq 0}$. With Q, we associate a compatible family of Feller semigroups $(\mathsf{P}_t^{(n)}, n \geq 1)$ and construct $\mathsf{P}_t^{(\infty)}$ as in Section 1.5.3.

1.6.1. *Construction of a probability space.* For every $n \in \mathbb{N}$, let $D_n = \{j2^{-n}, j \in \mathbb{Z}\}$ and let $D = \bigcup_{n \in \mathbb{N}} D_n$ be the set of the dyadic numbers. We take $T = 1$ and set $i = i_1$ and $\Phi_n = \Phi_n^1$.

For every integer $n \geq 1$, let $(S_n, \mathcal{S}_n, \mathsf{P}_n)$ denote the probability space $(M^{\mathbb{N}}, \mathcal{B}(M)^{\otimes \mathbb{N}}, \mathsf{P}_{2^{-n}}^{(\infty)})^{\otimes \mathbb{Z}}$. Let $\pi_{n-1,n} : S_n \to S_{n-1}$, $\omega^n \mapsto \omega^{n-1}$, where

$$\omega_{i/(2^{n-1})}^{n-1} = j \circ \Phi_2\big(\omega_{(2i-1)/2^n}^n, \omega_{2i/2^n}^n\big) = j\big(i\big(\omega_{2i/2^n}^n\big) \circ i\big(\omega_{(2i-1)/2^n}^n\big)\big). \tag{1.25}$$

From Lemma 1.8, $\pi_{n-1,n}^*(\mathsf{P}_n) = \mathsf{P}_{n-1}$.

Let $\Omega = \{(\omega^n)_{n \in \mathbb{N}} \in \prod S_n, \pi_{n-1,n}(\omega^n) = \omega^{n-1}\}$ and let $\mathcal{A}$ be the σ-field on Ω generated by the mappings $\pi_n : \Omega \to S_n$, with $\pi_n((\omega^k)_{k \in \mathbb{N}}) = \omega^n$. Let P be the unique probability on $(\Omega, \mathcal{A})$ such that $\pi_n^*(\mathsf{P}) = \mathsf{P}_n$ (see Theorem 3.2 in [34]).

For all dyadic numbers $s < t$, let $\mathcal{F}_{s,t}$ be the σ-field generated by the mappings $(\omega^k)_{k\in\mathbb{N}} \mapsto \omega_u^n$ for all $n \in \mathbb{N}$ and $u \in D_n \cap [s, t]$.

1.6.2. *A measurable stochastic flow of mappings on M.*

DEFINITION 1.9. On $(\Omega, \mathcal{A}, \mathsf{P})$, we define the following random variables:

1. For all $s < t \in D_n$, let $\varphi_{s,t}^n((\omega^k)_{k\in\mathbb{N}}) = \Phi_{(t-s)2^n}(\omega_s^n, \ldots, \omega_{t-2^{-n}}^n)$.
2. For all $s < t \in D$, let $\varphi_{s,t} = \varphi_{s,t}^n$ where $n = \inf\{k, (s,t) \in D_k^2\}$.

Then, for every $s \in D_n$, $\varphi_{s,s+2^{-n}}(\omega) = i(\omega_s^n)$. Let us remark that, for all $s < t \in D_n$, the law of $\varphi_{s,t}$ and of $\varphi_{s,t}^n$ is Q_{t-s} (this is a consequence of Lemma 1.8) and that $M \times \Omega \ni (x, \omega) \mapsto \varphi_{s,t}(x, \omega) \in M$ is measurable. Note also that, for all $s < u < t \in D_n$, we have $\varphi_{s,t}^n = \varphi_{u,t}^n \circ \varphi_{s,u}^n$.

PROPOSITION 1.6. *For all $s < t \in D_n$ and every M-valued random variable X independent of $\mathcal{F}_{s,t}$,*

$$\varphi_{s,t}^n(X) = \varphi_{s,t}(X) \qquad \mathsf{P}\text{-}a.s.$$

PROOF. It is enough to prove that, for all $s < t \in D_n$, $\varphi_{s,t}^n(X) = \varphi_{s,t}^{n+1}(X)$ a.s.. This holds since

$$\begin{aligned}\varphi_{s,t}^n(X) &= i(\omega_{t-2^{-n}}^n) \circ \cdots \circ i(\omega_s^n)(X) \\ &= (i \circ j)(\varphi_{t-2^{-n},t}^{n+1}) \circ \cdots \circ (i \circ j)(\varphi_{s,s+2^{-n}}^{n+1})(X).\end{aligned}$$

Using Remark 1.7 and the independence of the family of random variables $\{\omega_u^{n+1}, u \in D_{n+1}\}$, we prove that the last term is a.s. equal to $\varphi_{t-2^{-n},t}^{n+1} \circ \cdots \circ \varphi_{s,s+2^{-n}}^{n+1}(X) = \varphi_{s,t}^{n+1}(X)$. $\square$

REMARK 1.8. The preceding proposition implies that, for all $s < u < t \in D$ and every M-valued random variable X independent of $\mathcal{F}_{s,t}$,

$$\varphi_{s,t}(X) = \varphi_{u,t} \circ \varphi_{s,u}(X), \qquad \mathsf{P}\text{-a.s.} \tag{1.26}$$

We now intend to define by approximation, for all $s < t$ in $\mathbb{R}$, an $(F, \mathcal{F})$-valued random variable $\varphi_{s,t}$ of law Q_{t-s}. In order to do this, we prove the following lemma.

LEMMA 1.9. *For every continuous function f on M^2, the mapping*

$$(s, t, u, v, x, y) \mapsto \mathsf{E}\big[f\big(\varphi_{s,t}(x), \varphi_{u,v}(y)\big)\big] \tag{1.27}$$

is continuous on $\{(s,t) \in D^2, s \le t\}^2 \times M^2$ (and therefore uniformly continuous on every compact).

PROOF. For all $s \le u \le t \le v$ in D, using the cocycle property, we have

$$\begin{aligned} \mathsf{E}\big[f\big(\varphi_{s,t}(x), \varphi_{u,v}(y)\big)\big] &= \mathsf{E}\big[f\big(\varphi_{u,t} \circ \varphi_{s,u}(x), \varphi_{t,v} \circ \varphi_{v,t}(y)\big)\big] \\ &= \big(\mathsf{P}^{(1)}_{u-s} \otimes I\big)\mathsf{P}^{(2)}_{t-u}\big(I \otimes \mathsf{P}^{(1)}_{v-t}\big) f(x, y). \end{aligned}$$

For all $s \le u \le v \le t$ in D, using the cocycle property, we have

$$\begin{aligned} \mathsf{E}\big[f\big(\varphi_{s,t}(x), \varphi_{u,v}(y)\big)\big] &= \mathsf{E}\big[f\big(\varphi_{v,t} \circ \varphi_{u,v} \circ \varphi_{s,u}(x), \varphi_{u,v}(y)\big)\big] \\ &= \big(\mathsf{P}^{(1)}_{u-s} \otimes I\big)\mathsf{P}^{(2)}_{v-u}\big(\mathsf{P}^{(1)}_{t-v} \otimes I\big) f(x, y). \end{aligned}$$

For all $s \le t \le u \le v$ in D,

$$\mathsf{E}\big[f\big(\varphi_{s,t}(x), \varphi_{u,v}(y)\big)\big] = \big(\mathsf{P}^{(1)}_{t-s} \otimes \mathsf{P}^{(1)}_{v-u}\big) f(x, y).$$

All these functions are continuous and they join. This implies the lemma. □

For every real t and every integer n, let $t_n = \sup\{u \in D_n, u \le t\}$. For all $s < t \in \mathbb{R}$, we define the increasing sequences $(s_n)_{n\in\mathbb{N}}$ and $(t_n)_{n\in\mathbb{N}}$. Using Lemma 1.9 for $f(x, y) = d(x, y)$ and the Markov inequality, for every positive ε, we have

$$\lim_{n\to\infty} \sup_{k>n} \sup_{x\in M} \mathsf{P}\big[d\big(\varphi_{s_n,t_n}(x), \varphi_{s_k,t_k}(x)\big) \ge \varepsilon\big] = 0. \tag{1.28}$$

Set $\varphi_{s,t}(x) = l((\varphi_{s_n,t_n}(x)))$. Then $M \times \Omega \ni (x, \omega) \mapsto \varphi_{s,t}(x, \omega) \in M$ is measurable and $\varphi_{s,t}$ is an $(F, \mathcal{F})$-valued random variable.

LEMMA 1.10. *For every positive ε and all $s \le t$,*

$$\lim_{n\to\infty} \sup_{x\in M} \mathsf{P}\big[d\big(\varphi_{s_n,t_n}(x), \varphi_{s,t}(x)\big) \ge \varepsilon\big] = 0. \tag{1.29}$$

PROOF. Equation (1.28) implies that $\varphi_{s_n,t_n}(x)$ converges in probability towards $\varphi_{s,t}(x)$. Thus, for every positive ε,

$$\mathsf{P}\big[d\big(\varphi_{s_n,t_n}(x), \varphi_{s,t}(x)\big) \ge \varepsilon\big] = \lim_{k\to\infty} \mathsf{P}\big[d\big(\varphi_{s_n,t_n}(x), \varphi_{s_k,t_k}(x)\big) \ge \varepsilon\big].$$

Therefore,

$$\sup_{x\in M} \mathsf{P}\big[d\big(\varphi_{s_n,t_n}(x), \varphi_{s,t}(x)\big) \ge \varepsilon\big] \le \sup_{k>n} \sup_{x\in M} \mathsf{P}\big[d\big(\varphi_{s_n,t_n}(x), \varphi_{s_k,t_k}(x)\big) \ge \varepsilon\big],$$

which implies the lemma. □

PROPOSITION 1.7. *For all $s < t \in \mathbb{R}$, the law of $\varphi_{s,t}$ is Q_{t-s}.*

PROOF. For all $k \geq 1$, $f \in C(M^k)$ and $x \in M^k$, Lemma 1.10 and the dominated convergence theorem imply that

$$\begin{aligned} \mathsf{E}[f \circ \varphi_{s,t}^{\otimes k}(x)] &= \lim_{n\to\infty} \mathsf{E}\big[f \circ \varphi_{s_n,t_n}^{\otimes k}(x)\big] \\ &= \lim_{n\to\infty} \mathsf{P}_{t_n-s_n}^{(k)} f(x) = \mathsf{P}_{t-s}^{(k)} f(x) \end{aligned}$$

since $\mathsf{P}_t^{(k)}$ is Feller. This implies that the law of $\varphi_{s,t}$ is Q_{t-s}. □

Let us now prove the cocycle property.

PROPOSITION 1.8. *For all $x \in M$ and $s < u < t$, P-a.s.,*

$$\varphi_{s,t}(x) = \varphi_{u,t} \circ \varphi_{s,u}(x). \tag{1.30}$$

PROOF. Almost surely we have $\varphi_{s_n,t_n}(x) = \varphi_{u_n,t_n} \circ \varphi_{s_n,u_n}(x)$ since $s_n < u_n < t_n$ belong to D. On one hand, $\varphi_{s_n,t_n}(x)$ converges in probability towards $\varphi_{s,t}(x)$. On the other hand,

$$\begin{aligned} &\mathsf{P}\big[d\big(\varphi_{u_n,t_n} \circ \varphi_{s_n,u_n}(x), \varphi_{u,t} \circ \varphi_{s,u}(x)\big) \geq \varepsilon\big] \\ &\quad\leq \mathsf{P}\big[d\big(\varphi_{u_n,t_n} \circ \varphi_{s_n,u_n}(x), \varphi_{u,t} \circ \varphi_{s_n,u_n}(x)\big) \geq \varepsilon/2\big] \\ &\quad\quad + \mathsf{P}\big[d\big(\varphi_{u,t} \circ \varphi_{s_n,u_n}(x), \varphi_{u,t} \circ \varphi_{s,u}(x)\big) \geq \varepsilon/2\big]. \end{aligned}$$

Lemma 1.10 shows that the first term converges towards 0 and Lemma 1.6 shows that the second term converges towards 0 [with $X_n = \varphi_{s_n,u_n}(x)$, $X = \varphi_{s,u}(x)$, $Y = j(\varphi_{u,t})$ and using the fact that $i \circ j(\varphi_{u,t})(x) = \varphi_{u,t}(x)$ a.s. for every $x \in M$]. □

Thus we have constructed a stochastic flow of measurable mappings on M associated with the compatible family of Feller semigroups $(\mathsf{P}_t^{(k)}, k \geq 1)$ and with the Feller convolution semigroup $(\mathsf{Q}_t, t \geq 0)$.

Let φ be the $(\Omega^0, \mathcal{A}^0)$-valued random variable defined by $\varphi = (\varphi_{s,t}, s \leq t)$. Let $\mathsf{P}_\mathsf{Q} = \varphi^*(\mathsf{P})$ be the law of φ. Then by a monotone class argument, we show that $T_h^*(\mathsf{P}_\mathsf{Q}) = \mathsf{P}_\mathsf{Q}$ for every $h \in \mathbb{R}$.

Let us now prove that on $(\Omega^0, \mathcal{A}^0, \mathsf{P}_\mathsf{Q})$ the canonical random variable $\varphi^0(\omega) = \omega$ is a stochastic flow. For every $t \geq 0$, there exists $\mathcal{J}_t$ a measurable presentation of Q_t (one can take $\mathcal{J}_t = i_t \circ j$). For all $s \leq t$, set $\varphi'_{s,t} = \mathcal{J}_{t-s}(\varphi^0_{s,t})$. Then, for all $x \in M$ and $s \leq t$, P_Q-a.s. $\varphi'_{s,t}(x) = \varphi^0_{s,t}(x)$. Then, $\varphi' = (\varphi'_{s,t}, s \leq t)$ is a measurable stochastic flow of mappings. Indeed, to prove (a) in Definition 1.6, we remark that, for $s \leq u \leq t$ and $x \in M$, the mapping

$$\begin{aligned} &G : (F^3, \mathcal{F}^{\otimes 3}) \to \big(M^2, \mathcal{B}(M)^{\otimes 2}\big), \\ &\quad(\varphi_1, \varphi_2, \varphi_3) \mapsto \big(\mathcal{J}_{t-u}(\varphi_1) \circ \mathcal{J}_{u-s}(\varphi_2)(x), \mathcal{J}_{t-s}(\varphi_3)(x)\big), \end{aligned}$$

is measurable. Thus $G(\varphi_{u,t}, \varphi_{s,u}, \varphi_{s,t})$ and $G(\varphi^0_{u,t}, \varphi^0_{s,u}, \varphi^0_{s,t})$ have the same law. Since

$$\mathsf{P}\text{-a.s.,} \qquad \mathcal{J}_{t-s}(\varphi_{s,t})(x) = \mathcal{J}_{t-u}(\varphi_{u,t}) \circ \mathcal{J}_{s-u}(\varphi_{s,u})(x),$$

we have

$$\mathsf{P}_{\mathsf{Q}}\text{-a.s.,} \qquad \mathcal{J}_{t-s}(\varphi^0_{s,t})(x) = \mathcal{J}_{t-u}(\varphi^0_{u,t}) \circ \mathcal{J}_{s-u}(\varphi^0_{s,u})(x).$$

This proves (a) and φ' is a measurable stochastic flow of mappings, proving also that φ is a stochastic flow of mappings. Finally, for $s \le t$ and $h \in \mathbb{R}$, we have

$$\begin{aligned}\varphi'_{s+h,t+h} &= \mathcal{J}_{t-s}(\varphi_{s+h,t+h}) \\ &= \mathcal{J}_{t-s}(\varphi_{s,t} \circ \theta_h) = \varphi'_{s,t} \circ \theta_h.\end{aligned}$$

Thus, we have constructed the canonical stochastic flow of mappings on M associated with the Feller convolution semigroup Q. Note that P_{Q} is uniquely determined by Q and is associated to a unique compatible family of Feller semigroups. The fact that $(\Omega^0, \mathcal{A}^0, \mathsf{P}_{\mathsf{Q}})$ is separable is a consequence of the construction of φ. The proof of Theorem 1.1 is finished.

1.7. *The example of Lipschitz SDEs.* We first show a sufficient condition for a compatible family of Markovian kernel semigroups to be constituted of Feller semigroups.

LEMMA 1.11. *A compatible family* $(\mathsf{P}^{(n)}_t, n \ge 1)$ *of semigroups of Markovian kernels is constituted of Feller semigroups when the following condition is satisfied*:

(F) *For all* $f \in C(M)$ *and* $x \in M$, $\lim_{t\to 0} \mathsf{P}^{(1)}_t f(x) = f(x)$ *and for all* $x \in M$, $\varepsilon > 0$ *and* $t > 0$, $\lim_{y\to x} \mathsf{P}^{(2)}_t f_\varepsilon(x, y) = 0$, *where* $f_\varepsilon(x, y) = 1_{d(x,y)>\varepsilon}$.

PROOF. Let $h \in C(M^n)$ be in the form $f_1 \otimes \cdots \otimes f_n$ and $x = (x_1, \dots, x_n)$ in M^n. We have, for M large enough,

$$\big|\mathsf{P}^{(n)}_t h(x) - h(x)\big| \le M \sum_{k=1}^n \big(\mathsf{P}^{(1)}_t f_k^2 + f_k^2 - 2 f_k \mathsf{P}^{(1)}_t f_k\big)^{1/2}(x_k), \tag{1.31}$$

which converges toward 0 as t goes to 0 since, for every $f \in C(M)$ and every $x \in M$, $\lim_{t\to 0} \mathsf{P}^{(1)}_t f(x) = f(x)$. We also have, for $y = (y_1, \dots, y_n)$ in M^n,

$$\big|\mathsf{P}^{(n)}_t h(y) - \mathsf{P}^{(n)}_t h(x)\big| \le M \sum_{k=1}^n \mathsf{P}^{(2)}_t (|1 \otimes f_k - f_k \otimes 1|)(y_k, x_k), \tag{1.32}$$

which converges toward 0 as y tends to x since, for all $f \in C(M)$ and $x \in M$, $\lim_{y\to x} \mathsf{P}^{(2)}_t(|1 \otimes f - f \otimes 1|)(y, x) = 0$. Indeed, $\forall \alpha > 0$, $\exists \varepsilon > 0$ such that

$d(x, y) < \varepsilon$ implies $|f(y) - f(x)| < \alpha$. This implies

$$\mathsf{P}_t^{(2)}(|1 \otimes f - f \otimes 1|)(y, x) \le \alpha + 2\|f\|_\infty \mathsf{P}_t^{(2)} f_\varepsilon(x, y). \tag{1.33}$$

This implies $\limsup_{y \to x} \mathsf{P}_t^{(2)}(|1 \otimes f - f \otimes 1|)(y, x) \le \alpha$ for every $\alpha > 0$. □

REMARK 1.9. (i) The previous result extends to the locally compact case [using the fact that $C_0(M)$ is constituted of uniformly continuous functions].

(ii) When (F) is satisfied, for all positive t, $f \in C_0(M)$ and $x \in M$, $\mathsf{P}_t^{(2)} f^{\otimes 2}(x, x) = \mathsf{P}_t^{(1)} f^2(x)$. This implies that (F) is not a necessary condition. Theorem 1.1 shows that a stochastic flow of mappings is associated with this family of semigroups.

DEFINITION 1.10. A two parameter family $(W_{s,t}, s \le t)$ of real random variables is called a real white noise if:

(i) for all $s < t$, $W_{s,t}$ is a centered Gaussian variable with variance $t - s$,
(ii) for all $((s_i, t_i), 1 \le i \le n)$ with $s_i \le t_i \le s_{i+1}$, the random variables $(W_{s_i,t_i}, 1 \le i \le n)$ are independent, and
(iii) for all $s \le t \le u$, $W_{s,u} = W_{s,t} + W_{t,u}$.

Let $V, V_1, \dots, V_k$ be bounded Lipschitz vector fields on a smooth locally compact manifold M. We also assume that $V_1, \dots, V_k$ are C^1. Let $W^1, \dots, W^k$ be k independent real white noises. We consider the SDE on M

$$dX_t = \sum_{i=1}^k V_i(X_t) \circ dW_t^i + V(X_t)\, dt, \qquad t \in \mathbb{R}. \tag{1.34}$$

From the usual theory of strong solutions of SDEs (see, e.g., [20]), it is possible to construct a stochastic flow of diffeomorphisms $(\varphi_{s,t}, s \le t)$ such that, for every $x \in M$, $\varphi_{s,t}(x)$ is a strong solution of the SDE (1.34) with $\varphi_{s,s}(x) = x$.

Using this stochastic flow, it is possible to construct a compatible family of Markovian semigroups $(\mathsf{P}_t^{(n)}, n \ge 1)$ with

$$\mathsf{P}_t^{(n)} h(x_1, \dots, x_n) = \mathsf{E}\big[h\big(\varphi_{0,t}(x_1), \dots, \varphi_{0,t}(x_n)\big)\big] \tag{1.35}$$

for $h \in C(M^n)$ and $x_1, \dots, x_n$ in M. Using Lemma 1.11, it is easy to check that these semigroups are Feller (these properties were previously observed in [3]).

It can easily be shown that the canonical stochastic flow of maps associated with this family of semigroups is equal in law to $(\varphi_{s,t}, s \le t)$.

2. Stochastic flow of kernels.

2.1. *Notation and definitions.* We denote by $\mathcal{P}(M)$ the space of probability measures on M, equipped with the weak convergence topology. Let $(f_n)_{n \in \mathbb{N}}$ be a sequence of functions dense in $\{f \in C(M), \|f\|_\infty \le 1\}$. We will equip $\mathcal{P}(M)$ with the distance $\rho(\mu,\nu) = (\sum_n 2^{-n}(\int f_n\,d\mu - \int f_n\,d\nu)^2)^{1/2}$ for all μ and ν in $\mathcal{P}(M)$. Thus $\mathcal{P}(M)$ is a compact metric space.

Let us recall that a kernel K on M is a measurable mapping from M into $\mathcal{P}(M)$, M and $\mathcal{P}(M)$ being equipped with their Borel σ-fields. For all $f \in C(M)$ and $x \in M$, $Kf(x)$ denotes $\int f(y)K(x,dy)$. For every $\mu \in \mathcal{P}(M)$, μK denotes the probability measure defined by $\int f(y)\mu K(dy) = \int Kf(x)\mu(dx)$. We denote by E the space of all kernels on M and we equip E with the σ-field generated by the mappings $K \mapsto \mu K$, for every $\mu \in \mathcal{P}(M)$ [$\mathcal{P}(M)$ is equipped with its Borel σ-field $\mathcal{B}(\mathcal{P}(M))$]. We denote this σ-field by $\mathcal{E}$.

Let Γ denote the space of measurable mappings on $\mathcal{P}(M)$. We equip Γ with the σ-field generated by the mappings $\Phi \mapsto \Phi(\mu)$ for every $\mu \in \mathcal{P}(M)$. Note that $(\Gamma, \mathcal{G}) = (F, \mathcal{F})$ once we have replaced M by $\mathcal{P}(M)$.

2.2. *Convolution semigroups on the space of kernels.* Let $\mathcal{I}$ denote the measurable mapping from $(E, \mathcal{E})$ on $(\Gamma, \mathcal{G})$ defined by $\mathcal{I}(K)(\mu) = \mu K$. Note that $\mathcal{I}(E)$ is not measurable in Γ but $\mathcal{I}$ is measurable.

DEFINITION 2.1. (i) A probability measure ν on $(E, \mathcal{E})$ is called regular if $\mathcal{I}^*(\nu)$ is a regular probability measure on $(\Gamma, \mathcal{G})$.

(ii) A convolution semigroup on $(E, \mathcal{E})$ is a family $(\nu_t)_{t\ge 0}$ of regular probability measures on $(E, \mathcal{E})$ such that $(\mathcal{I}^*(\nu_t))_{t\ge 0}$ is a convolution semigroup on $(\Gamma, \mathcal{G})$.

Let $\delta : \Gamma \to E$ be the mapping defined by $\delta(\Phi)(x) = \Phi(\delta_x)$. Note that δ is not measurable in general.

PROPOSITION 2.1. *Let* Q *be a regular probability measure on* $(\Gamma, \mathcal{G})$ *and let* $\mathcal{J}$ *be a measurable presentation of* Q. *Then* $\delta \circ \mathcal{J}$ *is measurable and the probability measure* $\nu = (\delta \circ \mathcal{J})^*(\mathsf{Q})$ *is a regular probability measure on* $(E, \mathcal{E})$ *if* $\mathcal{I}^*(\nu) = \mathsf{Q}$.

PROOF. Let Q be a regular probability measure on $(\Gamma, \mathcal{G})$ and let $\mathcal{J}$ be a measurable presentation of Q. The mappings $\mathcal{P}(M) \times \Gamma \ni (\mu, \Phi) \mapsto \mathcal{J}(\Phi)(\mu) \in \mathcal{P}(M)$ and $M \ni x \mapsto \delta_x \in \mathcal{P}(M)$ are measurable. Thus $M \times \Gamma \ni (x, \Phi) \mapsto \delta \circ \mathcal{J}(\Phi)(x) = \mathcal{J}(\Phi)(\delta_x) \in \mathcal{P}(M)$ is measurable, which implies that $\delta \circ \mathcal{J}$ is measurable. $\square$

REMARK 2.1. The probability measure ν defined in Proposition 2.1 depends only on Q. Indeed, if $\mathcal{J}'$ is another measurable presentation of Q, for every $x \in M$, $\mathsf{Q}(d\Phi)$-a.s., $\delta \circ \mathcal{J}(\Phi)(x) = \delta \circ \mathcal{J}'(\Phi)(x)$, which implies by the Fubini theorem that, for every $\mu \in \mathcal{P}(M)$, $\mathsf{Q}(d\Phi)$-a.s., $\mu(\delta \circ \mathcal{J}(\Phi)) = \mu(\delta \circ \mathcal{J}'(\Phi))$ and then that $(\delta \circ \mathcal{J})^*(\mathsf{Q}) = (\delta \circ \mathcal{J}')^*(\mathsf{Q})$.

DEFINITION 2.2. A convolution semigroup $(\nu_t)_{t\ge 0}$ on $(E, \mathcal{E})$ is called Feller if:

(i) for every $f \in C(M)$, $\lim_{t\to 0} \sup_{x\in M} \int (Kf(x) - f(x))^2 \nu_t(dK) = 0$,

(ii) for every $f \in C(M)$ and every $t \geq 0$, $\lim_{d(x,y)\to 0} \int (Kf(x) - Kf(y))^2 \times \nu_t(dK) = 0$.

PROPOSITION 2.2. *Let $(\nu_t)_{t\geq 0}$ be a Feller convolution semigroup on $(E, \mathcal{E})$. For all $n \geq 1$, $f \in C(M^n)$ and $x \in M^n$, set*

$$\mathsf{P}_t^{(n)} f(x) = \int K^{\otimes n} f(x) \nu_t(dK). \tag{2.1}$$

Then $(\mathsf{P}_t^{(n)}, n \geq 1)$ is a compatible family of Feller semigroups on M.

PROOF. This is the same proof as the one of Proposition 1.2. □

PROPOSITION 2.3. *Let $(\mathsf{Q}_t)_{t\geq 0}$ be a convolution semigroup on $(\Gamma, \mathcal{G})$. Let $\mathcal{J}_t$ be a measurable presentation of Q_t and $\nu_t = (\delta \circ \mathcal{J}_t)^*(\mathsf{Q}_t)$. If $\mathsf{Q}_t = \mathcal{I}^*(\nu_t)$, $(\nu_t)_{t\geq 0}$ is a convolution semigroup on $(E, \mathcal{E})$. Then, $(\mathsf{Q}_t)_{t\geq 0}$ is Feller if and only if $(\nu_t)_{t\geq 0}$ is Feller.*

PROOF. The fact that $(\nu_t)_{t\geq 0}$ is a convolution semigroup follows from Definition 2.1.

Note that $(\mathsf{Q}_t)_{t\geq 0}$ is Feller if and only if, for every $f \in C(M)$,

$$\lim_{t\to 0} \sup_{\mu\in\mathcal{P}(M)} \int \big(\Phi(\mu)f - \mu f\big)^2 \mathsf{Q}_t(d\Phi) = 0, \tag{2.2}$$

$$\lim_{\rho(\mu,\nu)\to 0} \int \big(\Phi(\mu)f - \Phi(\nu)f\big)^2 \mathsf{Q}_t(d\Phi) = 0. \tag{2.3}$$

We first prove (2.2) and (i) in Definition 2.2 are equivalent. Equation (2.2) implies (i) since $\int (Kf(x) - f(x))^2 \nu_t(dK) = \int (\Phi(\delta_x)f - \delta_x f)^2 \mathsf{Q}_t(d\Phi)$. And (i) implies (2.2) since

$$\begin{aligned}\int \big(\Phi(\mu)f - \mu f\big)^2 \mathsf{Q}_t(d\Phi) &= \int (\mu K f - \mu f)^2 \nu_t(dK) \\ &\leq \int \bigg(\int \big(Kf(x) - f(x)\big)^2 \nu_t(dK)\bigg) \mu(dx).\end{aligned}$$

We now prove (2.3) and (ii) in Definition 2.2 are equivalent. Equation (2.3) implies (ii) since $\int (Kf(x) - Kf(y))^2 \nu_t(dK) = \int (\Phi(\delta_x)f - \Phi(\delta_y)f)^2 \mathsf{Q}_t(d\Phi)$ and $\lim_{d(x,y)\to 0} \rho(\delta_x, \delta_y) = 0$. Assume (ii) holds. For μ and ν in $\mathcal{P}(M)$, we have

$$\begin{aligned}\int \big(\Phi(\mu)f - \Phi(\nu)f\big)^2 \mathsf{Q}_t(d\Phi) &= \int (\mu K f - \nu K f)^2 \nu_t(dK) \\ &= (\mu - \nu)^{\otimes 2} \int K^{\otimes 2} f^{\otimes 2} \nu_t(dK).\end{aligned}$$

We conclude since $\int K^{\otimes 2} f^{\otimes 2} \nu_t(dK)$ is a continuous function (see Proposition 2.2). □

2.3. *Stochastic flows of kernels.*

DEFINITION 2.3. Let $(\Omega, \mathcal{A}, \mathsf{P})$ be a probability space. Then a family of $(E, \mathcal{E})$-valued random variables $(K_{s,t}, s \le t)$ is called a measurable stochastic flow of kernels if, for all $s \le t$,

$$(x, \omega) \mapsto K_{s,t}(x, \omega) \tag{2.4}$$

is a measurable mapping from $(M \times \Omega, \mathcal{B}(M) \otimes \mathcal{A})$ onto $(\mathcal{P}(M), \mathcal{B}(\mathcal{P}(M)))$ and if it satisfies the following properties:

(a) For all $s < u < t$ and $x \in M$, P-a.s., for every $f \in C(M)$, $K_{s,t}f(x) = K_{s,u}(K_{u,t}f)(x)$ (cocycle property).

(b) For all $s \le t$, the law of $K_{s,t}$ only depends on $t - s$ (stationarity).

(c) The flow has independent increments; that is, for all $t_1 < t_2 < \cdots < t_n$, the family $\{K_{t_i,t_{i+1}}, 1 \le i \le n-1\}$ is independent.

(d) For every $f \in C(M)$,

$$\lim_{(u,v)\to(s,t)} \sup_{x \in M} \mathsf{E}\big[\big(K_{s,t}f(x) - K_{u,v}f(x)\big)^2\big] = 0. \tag{2.5}$$

(e) For all $f \in C(M)$ and $s < t$,

$$\lim_{d(x,y)\to 0} \mathsf{E}\big[\big(K_{s,t}f(x) - K_{s,t}f(y)\big)^2\big] = 0. \tag{2.6}$$

DEFINITION 2.4. A family of $(E, \mathcal{E})$-valued random variables $K = (K_{s,t}, s \le t)$ is called a stochastic flow of kernels if there exists $K' = (K'_{s,t}, s \le t)$, a measurable stochastic flow of kernels, such that, for all $s \le t$ and $\mu \in \mathcal{P}(M)$,

$$\mathsf{P}\text{-a.s.,} \qquad \mu K'_{s,t} = \mu K_{s,t}. \tag{2.7}$$

The stochastic flow K' is called a measurable modification of K.

PROPOSITION 2.4. *Let $(K_{s,t}, s \le t)$ be a stochastic flow of kernels. For all $n \ge 1$, $f \in C(M^n)$ and $x \in M^n$, set*

$$\mathsf{P}_t^{(n)} f(x) = \mathsf{E}[K_{0,t}^{\otimes n} f(x)]. \tag{2.8}$$

Then $(\mathsf{P}_t^{(n)}, n \ge 1)$ is a compatible family of Feller semigroups on M.

PROOF. This is the same proof as the one to prove Proposition 1.2. □

2.4. *Construction and characterization.* Let $(\Omega^0, \mathcal{A}^0)$ denote the measurable space $(\prod_{s \le t} E, \bigotimes_{s \le t} \mathcal{E})$. For $s \le t$, let $K_{s,t}^0$ denote the random variable $\omega \mapsto \omega(s,t)$. Let also K^0 be the random variable $(K_{s,t}^0, s \le t)$. Then $K^0(\omega) = \omega$. Let $(T_h)_{h \in \mathbb{R}}$ be the one-parameter group of transformations of Ω^0 defined by $T_h(\omega)(s,t) = \omega(s+h, t+h)$, for all $s \le t$, $h \in \mathbb{R}$ and $\omega \in \Omega^0$.

THEOREM 2.1. (i) *For every compatible family* $(\mathsf{P}_t^{(n)}, n \geq 1)$ *of Feller semigroups on* M, *there exists a unique Feller convolution semigroup* $(\nu_t)_{t\geq 0}$ *on* $(E, \mathcal{E})$ *such that, for all* $n \geq 1$, $t \geq 0$, $f \in C(M^n)$ *and* $x \in M^n$,

$$\mathsf{P}_t^{(n)} f(x) = \int K^{\otimes n} f(x) \nu_t(dK). \tag{2.9}$$

(ii) *For every Feller convolution semigroup* $\nu = (\nu_t)_{t\geq 0}$ *on* $(E, \mathcal{E})$, *there exists a unique* $(T_h)_{h\in\mathbb{R}}$-*invariant probability measure* P_ν *on* $(\Omega^0, \mathcal{A}^0)$ *such that* $(\Omega^0, \mathcal{A}^0, \mathsf{P}_\nu)$ *is separable, the family of random variables* $(K_{s,t}^0, s \leq t)$ *is a stochastic flow of kernels and, for all* $s \leq t$, *the law of* $K_{s,t}^0$ *is* ν_{t-s}. *There exists a measurable modification of* K^0, K' *such that* $K'_{s+h,t+h} = K'_{s,t} \circ T_h$.

The flow K^0 *is called the canonical stochastic flow of kernels associated with* ν [*or equivalently with* $(\mathsf{P}_t^{(n)}, n \geq 1)$].

REMARK 2.2. In the case (1.6) is satisfied, the stochastic flow of kernels K is induced by a stochastic flow of mappings φ. More precisely, there exists a measurable modification of K in the form $(\delta_{\varphi_{s,t}}, s \leq t)$, where φ is a measurable flow of mappings.

2.5. *Proof of Theorem* 2.1. Let $(\mathsf{P}_t^{(n)}, n \geq 1)$ be a compatible family of Feller semigroups on M. Starting with this family of semigroups, we intend to construct a Feller convolution semigroup $\nu = (\nu_t)_{t\geq 0}$ on $(E, \mathcal{E})$. The idea is to construct a compatible family of Feller semigroups on $\mathcal{P}(M)$, then to apply Theorem 1.1 to construct a Feller convolution semigroup $\mathsf{Q} = (\mathsf{Q}_t)_{t\geq 0}$ on $(\Gamma, \mathcal{G})$ and to construct ν using the mappings $\delta \circ \mathcal{J}_t$, where $\mathcal{J}_t$ is a measurable presentation of Q_t.

2.5.1. *Construction of a compatible family of Feller semigroups on* $\mathcal{P}(M)$. For every integer k, we define a Feller semigroup $\Pi_t^{(k)}$ acting on the continuous functions on $\mathcal{P}(M)^k$ (see [28] for a similar construction when $k = 1$).

Let $\mathcal{A}_k$ denote the algebra of functions $g : \mathcal{P}(M)^k \to \mathbb{R}$ such that

$$g(\mu_1, \ldots, \mu_k) = \langle f, \mu_1^{\otimes n_1} \otimes \cdots \otimes \mu_k^{\otimes n_k} \rangle \tag{2.10}$$

[here and in the following, for all measure μ and $f \in L^1(\mu)$, we denote $\int f\, d\mu$ by $\langle f, \mu \rangle$, $\langle \mu, f \rangle$ or μf] for $f \in C(M^n)$ and $n_1, \ldots, n_k$ integers such that $n = n_1 + \cdots + n_k$ ($\mathcal{A}_k$ is the union of an increasing family of algebras $\mathcal{A}_{n_1,\ldots,n_k}$). For every $g \in \mathcal{A}_k$, given by (2.10), let

$$\Pi_t^{(k)} g(\mu) = \langle \mathsf{P}_t^{(n)} f, \mu_1^{\otimes n_1} \otimes \cdots \otimes \mu_k^{\otimes n_k} \rangle, \tag{2.11}$$

with $\mu = (\mu_1, \ldots, \mu_k) \in \mathcal{P}(M)^k$. Since the family of semigroups $(\mathsf{P}_t^{(n)}, n \geq 1)$ is compatible, (2.11) is independent of the expression of g in (2.10).

Let us notice that $\Pi_t^{(k)}$ acts on $\mathcal{A}_k$ and that, by the theorem of Stone and Weierstrass, the algebra $\mathcal{A}_k$ is dense in $C(\mathcal{P}(M)^k)$.

LEMMA 2.1. *$\Pi_t^{(k)}$ is a Markovian operator acting on $\mathcal{A}_k$.*

PROOF. The only thing to be proved is the positivity property (it is obvious that $\Pi_t^{(k)} 1 = 1$).

For every integer N, let $(X^{j,i}, 1 \le i \le k, 1 \le j \le N)$ be a Markov process associated with the Markovian semigroup $\mathsf{P}_t^{(Nk)}$ such that the random variables $(X_0^{j,i}, 1 \le i \le k, 1 \le j \le N)$ are independent and the law of $X_0^{j,i}$ is μ_i, where $(\mu_1, \dots, \mu_k) \in \mathcal{P}(M)^k$. Let us introduce the following Markov process on $\mathcal{P}(M)^k$, $\mu_t^N = (\mu_t^{N,1}, \dots, \mu_t^{N,k})$, where

$$\mu_t^{N,i} = \frac{1}{N} \sum_{j=1}^{N} \delta_{X_t^{j,i}} \qquad \text{for } 1 \le i \le k. \tag{2.12}$$

For $g(\mu_1, \dots, \mu_k) = \langle f, \mu_1^{\otimes n_1} \otimes \cdots \otimes \mu_k^{n_k} \rangle$, we have

$$\begin{aligned}
\mathsf{E}[g(\mu_t^N)] &= \mathsf{E}[\langle f, (\mu_t^{N,1})^{\otimes n_1} \otimes \cdots \otimes (\mu_t^{N,k})^{\otimes n_k} \rangle] \\
&= \frac{1}{N^n} \sum_{i=1}^{k} \sum_{l=1}^{n_k} \sum_{j_i^l = 1}^{N} \mathsf{E}\big[f\big(X_t^{j_1^1,1}, X_t^{j_1^2,1}, \dots, X_t^{j_1^{n_1},1}, X_t^{j_2^1,2}, \dots, X_t^{j_k^{n_k},n_k}\big)\big] \\
&= \big\langle \mathsf{P}_t^{(n)} f, \mu_1^{\otimes n_1} \otimes \cdots \otimes \mu_k^{n_k} \big\rangle + R_N.
\end{aligned}$$

The remainder term R_N comes from terms in which $j_i^a = j_i^b$ for some $a \ne b$ and some i and is therefore dominated by $2\|f\|_\infty (1 - \prod_{i=1}^k (N(N-1) \cdots (n - n_i + 1)/N^{n_i}))$. Thus

$$\lim_{N \to \infty} E[g(\mu_t^N)] = \big\langle \mathsf{P}_t^{(n)} f, \mu_1^{\otimes n_1} \otimes \cdots \otimes \mu_k^{n_k} \big\rangle \tag{2.13}$$

$$= \Pi_t^{(k)} g(\mu_1, \dots, \mu_k). \tag{2.14}$$

This shows that $\Pi_t^{(k)}$ is positive. □

Using this lemma, it is easy to define $\Pi_t^{(k)} g$ for every continuous function g and to show that $\Pi_t^{(k)}$ is a Markovian semigroup acting on $C(M^n)$.

LEMMA 2.2. *$(\Pi_t^{(k)}, k \ge 1)$ is a compatible family of Feller semigroups on $\mathcal{P}(M)$ satisfying* (1.6).

PROOF. Since the semigroups $\mathsf{P}_t^{(n)}$ are Feller, the semigroups $\Pi_t^{(k)}$ are also Feller: for every g in $\mathcal{A}_k$, $\Pi_t^{(k)} g$ is continuous and $\lim_{t \to 0} \Pi_t^{(k)} g = g$ and these properties extend to every continuous function.

It is clear that the family of semigroups $(\Pi_t^{(k)}, k \ge 1)$ is compatible (in the sense given in Section 1.1). Thus $(\Pi_t^{(k)}, k \ge 1)$ is a compatible family of Feller semigroups on $\mathcal{P}(M)$. We denote $\Pi^{(2)}_{(\mu,\nu)}$ the law of the Markov process associated with $\Pi^{(2)}_t$ starting from (μ,ν) and we denote this process by (μ_t,ν_t).

For $g \in \mathcal{A}_1$ in the form (2.10), $t \geq 0$ and $\mu \in \mathcal{P}(M)$, we have

$$\Pi^{(2)}_t g^{\otimes 2}(\mu,\mu) = \langle \mathsf{P}^{(2n)}_t f^{\otimes 2}, \mu^{\otimes 2n}\rangle = \Pi^{(1)}_t g^2(\mu).$$

Thus (1.6) is satisfied for $g \in \mathcal{A}_1$ and this extends to $C(\mathcal{P}(M))$. □

2.5.2. *Proof of the first part of Theorem* 2.1. Using Theorem 1.1, we construct $(\mathsf{Q}_t)_{t\geq 0}$ a Feller convolution semigroup on $(\Gamma, \mathcal{G})$. Let $\mathcal{J}_t$ be a measurable presentation of Q_t. Set $\nu_t = (\delta \circ \mathcal{J}_t)^* \mathsf{Q}_t$.

LEMMA 2.3. *For all $\mu \in \mathcal{P}(M)$ and $t \geq 0$,*

$$\mathsf{Q}_t(d\Phi)\text{-a.s.}, \qquad \Phi(\mu) = \mu\big(\delta \circ \mathcal{J}_t(\Phi)\big). \tag{2.15}$$

And for every $t \geq 0$, $\mathcal{I}^*(\nu_t) = \mathsf{Q}_t$.

PROOF. For every $f \in C(M)$, set $g(\mu) = \mu f$; then

$$\begin{aligned}
&\mathsf{E}\big[\big(\mu\big(\delta \circ \mathcal{J}_t(\Phi)\big) f - \Phi(\mu) f\big)^2\big] \\
&\quad = \mathsf{E}\bigg[\bigg(\int g\big(\Phi(\delta_x)\big)\mu(dx) - g\big(\Phi(\mu)\big)\bigg)^2\bigg] \\
&\quad = \int \Pi^{(2)}_t g^{\otimes 2}(\delta_x,\delta_y)\mu(dx)\mu(dy) + \Pi^{(2)}_t g^{\otimes 2}(\mu,\mu) \\
&\qquad - 2\int \Pi^{(2)}_t g^{\otimes 2}(\delta_x,\mu)\mu(dx).
\end{aligned}$$

Since for all μ and ν in $\mathcal{P}(M)$,

$$\Pi^{(2)}_t g^{\otimes 2}(\mu,\nu) = \int \mathsf{P}^{(2)}_t f^{\otimes 2}(x,y)\mu(dx)\nu(dy),$$

we get $\mathsf{E}[(\mu(\delta \circ \mathcal{J}_t(\Phi)) f - \Phi(\mu) f)^2] = 0$. This proves the lemma. □

Lemma 2.3 implies that $\nu = (\nu_t)_{t\geq 0}$ is a Feller convolution semigroup on $(E, \mathcal{E})$ (we apply Proposition 2.3) and (2.9) holds. This proves the first part of Theorem 2.1.

2.5.3. *Proof of the second part of Theorem* 2.1. Suppose now we are given $\nu = (\nu_t)_{t\geq 0}$ a Feller convolution semigroup on $(E, \mathcal{E})$. For $t \geq 0$, set $\mathsf{Q}_t = \mathcal{I}^*(\nu_t)$. Then $\mathsf{Q} = (\mathsf{Q}_t)_{t\geq 0}$ is a Feller convolution semigroup on $(\Gamma, \mathcal{G})$. Using Theorem 1.1, we construct P_{Q} the law of a stochastic flow of mappings on $\mathcal{P}(M)$ associated with Q. Let $(\Phi_{s,t}, s \leq t)$ be a measurable stochastic flow of mappings of law P_{Q}. For $s \leq t$, set $K_{s,t} = \delta \circ \mathcal{J}_{t-s}(\Phi_{s,t})$, where $\mathcal{J}_{t-s}$ is a measurable presentation of Q_{t-s}.

We now show that $K=(K_{s,t}, s\le t)$ is a stochastic flow of kernels. Note that the law of $K_{s,t}$ is ν_{t-s}. Thus it is easy to check that K satisfies (b)–(e). In order to show (a), we use the following lemma.

LEMMA 2.4. *For all $\mu\in\mathcal{P}(M)$ and $s\le t$,*

$$\mathsf{P}\text{-}a.s.,\qquad \mu K_{s,t}=\Phi_{s,t}(\mu). \tag{2.16}$$

PROOF. For every $f\in M$, set $g(\mu)=\mu f$; then as in the proof of Lemma 2.3,

$$\begin{aligned}
&\mathsf{E}\big[\big(\mu K_{s,t}f-\Phi_{s,t}(\mu)f\big)^2\big]\\
&\quad=\mathsf{E}\bigg[\bigg(\int g\big(\Phi_{s,t}(\delta_x)\big)\mu(dx)-g\big(\Phi_{s,t}(\mu)\big)\bigg)^2\bigg]\\
&\quad=\int \Pi_{t-s}^{(2)}g^{\otimes 2}(\delta_x,\delta_y)\mu(dx)\mu(dy)+\Pi_{t-s}^{(2)}g^{\otimes 2}(\mu,\mu),\\
&\qquad-2\int \Pi_{t-s}^{(2)}g^{\otimes 2}(\delta_x,\mu)\mu(dx)\\
&\quad=0.
\end{aligned}$$

This proves the lemma. □

Let $s\le u\le t$ and $\mu\in\mathcal{P}(M)$. Lemma 2.4 and the cocycle property of Φ imply that a.s.,

$$\mu K_{s,t}=\Phi_{s,t}(\mu)=\Phi_{u,t}\circ\Phi_{s,u}(\mu).$$

Lemma 2.4 implies that a.s., $\Phi_{u,t}\circ\Phi_{s,u}(\mu)=\Phi_{u,t}(\mu K_{s,u})$. Fubini's theorem, Lemma 2.4 and the fact that $\mu K_{s,u}$ and $\Phi_{u,t}$ are independent imply that a.s.,

$$\Phi_{u,t}(\mu K_{s,u})=\mu K_{s,u}K_{u,t}.$$

This proves (a), that is, a.s. $\mu K_{s,t}=\mu K_{s,u}K_{u,t}$. We let P_ν be the law of K. Then $T_h^*(\mathsf{P}_\nu)=\mathsf{P}_\nu$. The rest of the proof is similar to the end of the proof of Theorem 1.1.

2.6. *Sampling the flow.* Let $(K_{s,t}, s\le t)$ be a stochastic flow of kernels defined on a probability space $(\Omega,\mathcal{A},\mathsf{P})$ and let $(T_h)_{h\in\mathbb{R}}$ be a one-parameter group of transformations of Ω preserving P and such that $K_{s,t}\circ T_h=K_{s+h,t+h}$. In this section, we construct on an extension of $(\Omega,\mathcal{A},\mathsf{P})$ a random path X_t starting at x such that, for every positive t,

$$K_{0,t}f(x)=\mathsf{E}[f(X_t)|\mathcal{A}]. \tag{2.17}$$

For $x\in M$ and $\omega\in\Omega$, by Kolmogorov's theorem, we define on $M^{\mathbb{R}^+}$, a probability $\mathsf{P}_{x,\omega}^0$ such that

$$\mathsf{E}_{x,\omega}^0\Bigg[\prod_{i=1}^n f_i(X_{t_i}^0)\Bigg]=K_{0,t_1}\big(f_1\big(K_{t_1,t_2}f_2\big(\cdots\big(f_{n-1}K_{t_{n-1},t_n}f_n\big)\big)\big)\big)(x), \tag{2.18}$$

for all $f_1, \dots, f_n$ in $C(M)$, $0 < t_1 < t_2 < \cdots < t_n$.

With P and $\mathsf{P}^0_{x,\omega}$, we construct a probability $\mathsf{P}^0_x(d\omega, d\omega') = \mathsf{P}(d\omega) \otimes \mathsf{P}^0_{x,\omega}(d\omega')$ on $\Omega \times M^{\mathbb{R}^+}$. Then, on the probability space $(\Omega \times M^{\mathbb{R}^+}, \mathcal{A} \otimes \mathcal{B}(M)^{\otimes \mathbb{R}^+}, \mathsf{P}^0_x)$, the random process $(X^0_t, t \geq 0)$, defined by $X^0_t(\omega, \omega') = \omega'(t)$, is a Markov process starting at x with semigroup $\mathsf{P}^{(1)}_t$ since

$$\mathsf{E}^0_x\left[\prod_{i=1}^n f_i(X^0_{t_i})\right] = \mathsf{P}^{(1)}_{t_1}\big(f_1\big(\mathsf{P}^{(1)}_{t_2-t_1} f_2\big(\cdots\big(f_{n-1}\mathsf{P}^{(1)}_{t_n-t_{n-1}} f_n\big)\big)\big)\big)(x), \tag{2.19}$$

for all $f_1, \dots, f_n$ in $C(M)$, $0 < t_1 < t_2 < \cdots < t_n$.

Therefore, there is a càdlàg (or continuous when $\mathsf{P}^{(1)}_t$ is the semigroup of a continuous Markov process) modification $X = (X_t, t \geq 0)$ of $(X^0_t, t \geq 0)$. Let now $\mathsf{P}_{x,\omega}$ be the law of X knowing $\mathcal{A}$. It is a law on $D(\mathbb{R}^+, M)$, the space of càdlàg functions [or $C(\mathbb{R}^+, M)$ when $\mathsf{P}^{(1)}_t$ is the semigroup of a continuous Markov process]. Equipped with the Skorohod topology (see [29] or [5]), $D(\mathbb{R}^+, M)$ becomes a Polish space [resp. $C(\mathbb{R}^+, M)$ is equipped with the topology of uniform convergence on every compact on $\mathbb{R}^+$].

On the probability space $(\Omega \times D(\mathbb{R}^+, M), \mathcal{A} \otimes \mathcal{B}(D(\mathbb{R}^+, M)), \mathsf{P}_x)$ [resp. on $(\Omega \times C(\mathbb{R}^+, M), \mathcal{A} \otimes \mathcal{B}(C(\mathbb{R}^+, M)), \mathsf{P}_x)$], where $\mathsf{P}_x(d\omega, d\omega') = \mathsf{P}(d\omega) \otimes \mathsf{P}_{x,\omega}(d\omega')$, let X be the random process $X(\omega, \omega') = \omega'$. Then X is a càdlàg (resp. continuous) process and

$$\begin{aligned}\mathsf{E}_x\left[\prod_{i=1}^n f_i(X_{t_i})\middle|\mathcal{A}\right] &= \mathsf{E}_{x,\omega}\left[\prod_{i=1}^n f_i(X_{t_i})\right]\\ &= K_{0,t_1}\big(f_1\big(K_{t_1,t_2} f_2\big(\cdots\big(f_{n-1}K_{t_{n-1},t_n} f_n\big)\big)\big)\big)(x),\end{aligned} \tag{2.20}$$

ar where E_x denotes the expectation with respect to P_x.

Let $(K'_{s,t}, s \leq t)$ be the stochastic flow of kernels defined on $(\Omega, \mathcal{A}, \mathsf{P})$ by

$$K'_{s,t} f(x, \omega) = K'_{0,t-s} f(x, T_s\omega), \tag{2.21}$$

where

$$K'_{0,t} f(x) = \mathsf{E}_x[f(X_t)|\mathcal{A}] = \int f\big(X_t(\omega, \omega')\big)\mathsf{P}_{x,\omega}(d\omega') \tag{2.22}$$

for $f \in C(M)$, $x \in M$. Then $(K'_{s,t}, s \leq t)$ is a càdlàg in t (resp. continuous in t) modification of $(K_{s,t}, s \leq t)$.

REMARK 2.3. The concept of sampling will be used in Section 5.4.

Replacing $K_{0,t}$ by $K^{\otimes n}_{0,t}$ and $\mathsf{P}^{(1)}_t$ by $\mathsf{P}^{(n)}_t$ in the above, we obtain a random process $X^{(n)}$ in M^n which represents an n-sampling of the flow. The coordinates of $X^{(n)}$ are independent given the flow K.

Let $(x_i)_{i\geq 1}$ be a sequence in M. For $\omega \in \Omega$, let $\mathsf{P}_{x_1,\dots,x_n,\omega} = \bigotimes_{i=1}^n \mathsf{P}_{x_i,\omega}$, $\mathsf{P}_{(x_i)_{i\geq 1},\omega} = \bigotimes_{i\geq 1}^n \mathsf{P}_{x_i,\omega}$, $\mathsf{P}_{x_1,\dots,x_n}(d\omega, d\omega'_1,\dots,d\omega'_n) = \mathsf{P}(d\omega)\otimes \mathsf{P}_{x_1,\dots,x_n,\omega}(d\omega'_1, \dots, d\omega'_n)$ and $\mathsf{P}_{(x_i)_{i\geq 1}}(d\omega, d\omega') = \mathsf{P}(d\omega)\otimes \mathsf{P}_{(x_i)_{i\geq 1},\omega}(d\omega')$. Then the process $X^{(n)}(\omega,\omega') = (\omega'_1,\dots,\omega'_n)$ defines an n-sampling of the flow (under $P_{x_1,\dots,x_n}$ or $\mathsf{P}_{(x_i)_{i\geq 1}}$). Let $X^i(\omega,\omega') = \omega'_i$. Then, under $\mathsf{P}_{(x_i)_{i\geq 1}}$, the sequence $(X^i)_{i\geq 1}$ is independent conditionally to $\mathcal{A}$. Moreover, if for every $i\geq 1$, $x_i = x$, this sequence is identically distributed and the law of large numbers implies that, for every $f\in C_0(M)$, $\frac{1}{n}\sum_{i=1}^n f(X_t^i)$ converges a.s. toward $\mathsf{E}_x[f(X_t^1)|\mathcal{A}] = K_{0,t}f(x)$.

Since, under $\mathsf{P}_{(x_i)_{i\geq 1}}$, $X^{(n)}$ is equal in law to the n-point motion of K starting from $(x_1,\dots,x_n)$, if for every $n\geq 1$, we let $X^{(n)}$ denote the n-point motion starting from $(x,\dots,x)$, we have that $\frac{1}{n}\sum_{i=1}^n f(X_t^i)$ converges in law toward $K_{0,t}f(x)$ for every $f\in C^0(M^n)$. This gives an intuitive way to recover $K_{0,t}(x)$ out of the n-point motions.

3. Noise and stochastic flows.

3.1. *Noise generated by a stochastic flow of kernels.* The definition of a noise we give here is very close to the one given by Tsirelson in [41].

DEFINITION 3.1. A noise consists of a separable probability space $(\Omega,\mathcal{A},\mathsf{P})$, a one-parameter group $(T_h)_{h\in\mathbb{R}}$ of P-preserving L^2-continuous transformations of Ω and a family $\{\mathcal{F}_{s,t}, -\infty\leq s\leq t\leq\infty\}$ of sub-σ-fields of $\mathcal{A}$ such that:

(a) T_h sends $\mathcal{F}_{s,t}$ onto $\mathcal{F}_{s+h,t+h}$ for all $h\in\mathbb{R}$ and $s\leq t$,
(b) $\mathcal{F}_{s,t}$ and $\mathcal{F}_{t,u}$ are independent for all $s\leq t\leq u$,
(c) $\mathcal{F}_{s,t}\vee\mathcal{F}_{t,u} = \mathcal{F}_{s,u}$ for all $s\leq t\leq u$.

Moreover, we will assume that, for all $s\leq t$, $\mathcal{F}_{s,t}$ contains all P-negligible sets of $\mathcal{F}_{-\infty,\infty}$, denoted $\mathcal{F}$.

In the following, $(\Omega^0,\mathcal{A}^0,\mathsf{P}_\nu)$ denotes the canonical probability space of a stochastic flow of kernels associated with a Feller convolution semigroup ν. $K^0 = (K^0_{s,t}, s\leq t)$ denotes this canonical flow. When this stochastic flow is induced by a flow of maps, one can take, for $(\Omega^0,\mathcal{A}^0,\mathsf{P}_\nu)$, the canonical probability space associated to this stochastic flow of mappings.

For all $-\infty\leq s\leq t\leq\infty$, let $\mathcal{F}^\nu_{s,t}$ be the sub-σ-field of $\mathcal{A}^0$ generated by the random variables $K^0_{u,v}$ for all $s\leq u\leq v\leq t$ completed by all P_ν-negligible sets of $\mathcal{A}^0$. Then the cocycle property of K^0 implies that $N_\nu := (\Omega^0,\mathcal{A}^0,(\mathcal{F}^\nu_{s,t})_{s\leq t},\mathsf{P}_\nu, (T_h)_{h\in\mathbb{R}})$ is a noise (T_h is L^2-continuous because of the Feller property). We call it the noise generated by the canonical flow K^0.

DEFINITION 3.2. Let ν be a Feller convolution semigroup, let $N = (\Omega,\mathcal{A}, (\mathcal{F}_{s,t})_{s\leq t},\mathsf{P},(T_h)_{h\in\mathbb{R}})$ be a noise and let K be a measurable stochastic flow of kernels of law P_ν defined on $(\Omega, \mathcal{A}, \mathsf{P})$ such that, for all $s < t$, $K_{s,t}$ is $\mathcal{F}_{s,t}$-measurable and, for every $h \in \mathbb{R}$,

$$K_{s+h,t+h} = K_{s,t} \circ T_h \qquad \text{a.s.} \tag{3.1}$$

We will call (N, K) an extension of the noise N_ν.

Let (N_1, K_1) and (N_2, K_2) be two extensions of the noise N_ν. Let $\Omega = \Omega_1 \times \Omega_2$, $\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2$ and let P be the probability measure on $(\Omega, \mathcal{A})$ defined by

$$\mathsf{E}[Z] = \int \mathsf{E}_1[Z_1|K_1 = K]\mathsf{E}_2[Z_2|K_2 = K]\mathsf{P}_\nu(dK), \tag{3.2}$$

for any bounded random variable $Z(\omega_1, \omega_2) = Z_1(\omega_1)Z_2(\omega_2)$. Let $(T_h)_{h\in\mathbb{R}}$ be the one-parameter group of P-preserving transformations of Ω defined by $T_h(\omega_1, \omega_2) = (T_h^1(\omega_1), T_h^2(\omega_2))$. For all $s < t$, let $\mathcal{F}_{s,t} = \mathcal{F}_{s,t}^1 \otimes \mathcal{F}_{s,t}^2$. Then $N := (\Omega, \mathcal{A}, (\mathcal{F}_{s,t})_{s\le t}, \mathsf{P}, (T_h)_{h\in\mathbb{R}})$ is a noise. And if K denotes the random variable $K(\omega_1, \omega_2) = K_1(\omega_1)(= K_2(\omega_2)$ P-a.s.), then (N, K) is an extension of N_ν. We will call (N, K) the product of the extensions (N_1, K_1) and (N_2, K_2). Note that N_1 and N_2 are isomorphic to subnoises of N.

3.2. *Filtering by a subnoise.* Let $\bar{N}$ be a subnoise of an extension (N, K) of N_ν; that is, $\bar{N}$ is a noise $(\Omega, \mathcal{A}, (\bar{\mathcal{F}}_{s,t})_{s\le t}, \mathsf{P}, (T_h)_{h\in\mathbb{R}})$ such that $\bar{\mathcal{F}}_{s,t} \subset \mathcal{F}_{s,t}$ for all $s \le t$.

REMARK 3.1. A subnoise is characterized by $\bar{\mathcal{F}}_{-\infty,\infty}$, denoted $\bar{\mathcal{F}}$. This σ-field has to be stable under T_h, to contain all P-negligible sets of $\mathcal{F}$, and be such that $\bar{\mathcal{F}} = (\bar{\mathcal{F}} \cap \mathcal{F}_{-\infty,0}) \vee (\bar{\mathcal{F}} \cap \mathcal{F}_{0,\infty})$.

For every $n \ge 1$, let $\bar{\mathsf{P}}_t^{(n)}$ be the operator acting on $C(M^n)$ defined by

$$\bar{\mathsf{P}}_t^{(n)}(f_1 \otimes \cdots \otimes f_n)(x_1, \dots, x_n) = \mathsf{E}\left[\prod_{i=1}^n \mathsf{E}[K_{0,t} f_i(x_i)|\bar{\mathcal{F}}_{0,t}]\right], \tag{3.3}$$

for all $x_1, \dots, x_n$ in M and all $f_1, \dots, f_n$ in $C(M)$.

LEMMA 3.1. *The family* $(\bar{\mathsf{P}}_t^{(n)}, n \ge 1)$ *is a compatible family of Feller semigroups.*

PROOF. The semigroup property of $\bar{\mathsf{P}}_t^{(n)}$ follows directly from the independence of the increments of the flow. The Markovian property and in particular the positivity property hold since, for every $h \in C(M^n)$,

$$\bar{\mathsf{P}}_t^{(n)} h(x_1, \dots, x_n) = \mathsf{E}\left[\left\langle h, \bigotimes_{i=1}^n \mathsf{E}[K_{0,t}(x_i)|\bar{\mathcal{F}}_{0,t}]\right\rangle\right]. \tag{3.4}$$

From this, it is clear that $(\bar{\mathsf{P}}_t^{(n)}, n \geq 1)$ is a compatible family of Markovian semigroups, respectively, acting on $C(M^n)$.

It remains to prove the Feller property. For all continuous functions $f_1, \dots, f_n$, $h = f_1 \otimes \cdots \otimes f_n$, $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ in M^n, for M large enough,

$$\begin{aligned}|\bar{\mathsf{P}}_t^{(n)} h(x) - \bar{\mathsf{P}}_t^{(n)} h(y)| &\leq M \sum_{i=1}^n \mathsf{E}\big[\big(\mathsf{E}[K_{0,t} f_i(x_i) - K_{0,t} f_i(y_i) | \bar{\mathcal{F}}_{0,t}]\big)^2\big]^{1/2} \\ &\leq M \sum_{i=1}^n \mathsf{E}\big[\big(K_{0,t} f_i(x_i) - K_{0,t} f_i(y_i)\big)^2\big]^{1/2},\end{aligned} \tag{3.5}$$

which converges toward 0 as y tends to x since (e) in Definition 2.3 is satisfied.

We also have, for all $h = f_1 \otimes \cdots \otimes f_n$ and $x = (x_1, \dots, x_n)$ in M^n, for M large enough,

$$\begin{aligned}|\bar{\mathsf{P}}_t^{(n)} h(x) - h(x)| &\leq M \sum_{i=1}^n \mathsf{E}\big[\big(\mathsf{E}[K_{0,t} f_i(x_i) - f_i(x_i) | \bar{\mathcal{F}}_{0,t}]\big)^2\big]^{1/2} \\ &\leq M \sum_{i=1}^n \mathsf{E}\big[\big(K_{0,t} f_i(x_i) - f_i(x_i)\big)^2\big]^{1/2},\end{aligned} \tag{3.6}$$

which converges toward 0 as t tends to 0 since (d) in Definition 2.3 is satisfied. Hence, for every function $h \in C(M^n)$ such that h is a linear combination of functions of the type $f_1 \otimes \cdots \otimes f_n$, we have $\bar{\mathsf{P}}_t^{(n)} h$ is continuous and $\lim_{t \to 0} \bar{\mathsf{P}}_t^{(n)} h(x) = h(x)$ for every $x \in M^n$. This extends to all functions $h \in C(M^n)$. $\square$

Let us denote by $\bar{\nu} = (\bar{\nu}_t)_{t \geq 0}$ the Feller convolution semigroup on $(E, \mathcal{E})$ associated with $(\bar{\mathsf{P}}_t^{(n)}, n \geq 1)$. Note that the one-point motion of ν and $\bar{\nu}$ is the same, that is, $\bar{\mathsf{P}}_t^{(1)} = \mathsf{P}_t^{(1)}$.

LEMMA 3.2. (i) *Let K be an $(E, \mathcal{E})$-valued random variable defined on a probability space $(\Omega, \mathcal{A}, \mathsf{P})$. Assume that*

$$\lim_{d(x,y) \to 0} \mathsf{E}\big[\rho\big(K(x), K(y)\big)^2\big] = 0. \tag{3.7}$$

Let $\mathcal{G}$ be a sub-σ-field of $\mathcal{A}$. Then there exists an $(E, \mathcal{E})$-valued random variable $K^{\mathcal{G}}$ which is $\mathcal{G}$-measurable and such that $(x, \omega) \mapsto K^{\mathcal{G}}(x, \omega)$ is measurable and that

$$K^{\mathcal{G}} f(x) = \mathsf{E}[K f(x) | \mathcal{G}], \qquad \mathsf{P}\text{-}a.s. \tag{3.8}$$

for all $f \in C(M)$ and $x \in M$. Thus $K^{\mathcal{G}} = \mathsf{E}[K | \mathcal{G}]$. Note that $K^{\mathcal{G}} = K^{\tilde{\mathcal{G}}}$, where $\tilde{\mathcal{G}} = \sigma(K^{\mathcal{G}})$.

(ii) *Let* (N, K) *be an extension of* N_ν *and let* $\bar{N}$ *be a subnoise of* N. *Then there exists* $\bar{K} = (\bar{K}_{s,t}, s \le t)$ *a stochastic flow of kernels of law* $\mathsf{P}_{\bar{\nu}}$ *such that* $(\bar{N}, \bar{K})$ *is an extension of* $N_{\bar{\nu}}$ *and*

$$\bar{K}_{s,t} f(x) = \mathsf{E}[K_{s,t} f(x) | \bar{\mathcal{F}}_{s,t}] = \mathsf{E}[K_{s,t} f(x) | \bar{\mathcal{F}}], \qquad \mathsf{P}\text{-}a.s. \tag{3.9}$$

for all $s \le t$, $x \in M$ *and* $f \in C(M)$. *We say* $\bar{K}$ *is obtained by filtering* K *with respect to* $\bar{N}$.

PROOF. (i) Let $(x_i)_{i \in \mathbb{N}}$ be a dense sequence in M. Equation (3.7) implies the existence of a sequence $(\varepsilon_k)_{k \in \mathbb{N}}$ such that if $d(x, y) \le \varepsilon_k$, then

$$\mathsf{E}\big[\rho\big(K(x), K(y)\big)^2\big] \le 2^{-3k}. \tag{3.10}$$

For every $x \in M$, let $(n_k^x)_{k \in \mathbb{N}}$ be defined by $n_k^x = \inf\{n \in \mathbb{N}, d(x_n, x) \le \varepsilon_k\}$. Then $\lim_{k \to \infty} x_{n_k^x} = x$ and the Borel–Cantelli lemma shows that

$$\lim_{k \to \infty} K\big(x_{n_k^x}\big) = K(x), \qquad \mathsf{P}\text{-a.s.} \tag{3.11}$$

(since $\mathsf{P}[\rho(K(x_{n_k^x}), K(x)) \ge 2^{-k}] \le 2^{-k}$). Then by dominated convergence,

$$\lim_{k \to \infty} \mathsf{E}\big[K\big(x_{n_k^x}\big) | \mathcal{G}\big] = \mathsf{E}[K(x) | \mathcal{G}], \qquad \mathsf{P}\text{-a.s.} \tag{3.12}$$

Let us choose an everywhere defined $\mathcal{G}$-measurable version of $\mathsf{E}[K(x_i) | \mathcal{G}]$ for every $i \in \mathbb{N}$.

Let $K^{\mathcal{G}}$ be defined by $K^{\mathcal{G}}(x) = l((\mathsf{E}[K(x_{n_k^x}) | \mathcal{G}])_{k \in \mathbb{N}})$. Then $K^{\mathcal{G}}$ is an $(E, \mathcal{E})$-valued $\mathcal{G}$-measurable random variable, $(x, \omega) \mapsto K^{\mathcal{G}}(x, \omega)$ is measurable and, for every $x \in M$,

$$K^{\mathcal{G}}(x) = \lim_{k \to \infty} \mathsf{E}\big[K\big(x_{n_k^x}\big) | \mathcal{G}\big] = \mathsf{E}[K(x) | \mathcal{G}], \qquad \mathsf{P}\text{-a.s.} \tag{3.13}$$

(ii) Since for every $t \ge 0$, ν_t is the law of a random variable satisfying (3.7), (i) shows that, for all $s \le t$, there exists $\bar{K}_{s,t}$ an $(E, \mathcal{E})$-valued $\bar{\mathcal{F}}_{s,t}$-measurable random variable such that $(x, \omega) \mapsto \bar{K}_{s,t}(x, \omega)$ is measurable and

$$\bar{K}_{s,t} f(x) = \mathsf{E}[K_{s,t} f(x) | \bar{\mathcal{F}}_{s,t}], \qquad \mathsf{P}\text{-a.s.} \tag{3.14}$$

for all $s \le t$, $x \in M$ and $f \in C(M)$.

It is easy to see that $\bar{K} = (\bar{K}_{s,t}, s \le t)$ is a measurable stochastic flow of kernels of law $\mathsf{P}_{\bar{\nu}}$ and that $(\bar{N}, \bar{K})$ is an extension of $N_{\bar{\nu}}$. Let us just show the cocycle property. For all $s \le u \le t$, $x \in M$ and $f \in C(M)$, P_ν-a.s.,

$$\begin{aligned}
\mathsf{E}[K_{s,t} f(x) | \bar{\mathcal{F}}_{s,t}] &= \mathsf{E}[K_{s,u} K_{u,t} f(x) | \bar{\mathcal{F}}_{s,t}] \\
&= \mathsf{E}\big[\mathsf{E}[K_{s,u} K_{u,t} f(x) | \mathcal{F}_{s,u} \vee \bar{\mathcal{F}}_{u,t}] | \bar{\mathcal{F}}_{s,t}\big] \\
&= \bar{K}_{s,u} \bar{K}_{u,t} f(x).
\end{aligned}$$

Thus the lemma is proved. □

DEFINITION 3.3. Given two Feller convolution semigroups on $(E, \mathcal{E})$, ν^1 and ν^2, we say that ν^1 dominates (resp. weakly dominates) ν^2, denoted $\nu^1 \succeq \nu^2$ (resp. $\nu^1 \overset{w}{\succeq} \nu^2$), if there exists a subnoise of N_{ν^1} [resp. of an extension (N^1, K^1) of N_{ν^1}] such that P_{ν^2} is the law of the flow obtained by filtering the canonical flow of law P_{ν^1} (resp. by filtering K^1) with respect to this subnoise.

Notice that in Lemma 3.2, ν weakly dominates $\bar{\nu}$ and ν dominates $\bar{\nu}$ if $\bar{N}$ is a subnoise of N_ν. Note that the domination relation is in fact an extension of the notion of barycenter.

LEMMA 3.3. *Let ν and $\bar{\nu}$ be two Feller convolution semigroups such that ν dominates $\bar{\nu}$. Let (N, K) be an extension of N_ν. Let $\tilde{N}_\nu$ be the subnoise (isomorphic to N_ν) of N generated by K. Then there exists a subnoise $\bar{N}$ of $\tilde{N}_\nu$ such that $\mathsf{P}_{\bar{\nu}}$ is the law of the flow obtained by filtering K with respect to $\bar{N}$.*

PROOF. Let $N_\nu := (\Omega^0, \mathcal{A}^0, (\mathcal{F}^\nu_{s,t})_{s\le t}, \mathsf{P}_\nu, (T_h)_{h\in\mathbb{R}})$ be the noise generated by the canonical flow associated with ν. Notice that $\nu \succeq \bar{\nu}$ means the existence of $\bar{N}^0$ a subnoise of N_ν such that $\mathsf{P}_{\bar{\nu}}$ is the law of $\bar{K}^0$, the flow obtained by filtering the canonical flow of law P_ν with respect to $\bar{N}^0$.

Note that the mapping $K : (\Omega, \mathcal{A}) \to (\Omega^0, \mathcal{A}^0)$ is measurable. Let $\bar{\mathcal{F}}$ be the completion of $K^{-1}(\bar{\mathcal{F}}^0)$ by all P-negligible sets of $\mathcal{A}$ and, for all $s \le t$, set $\bar{\mathcal{F}}_{s,t} = \bar{\mathcal{F}} \cap \mathcal{F}_{s,t}$. Then $\bar{N} = (\Omega, \mathcal{A}, (\bar{\mathcal{F}}_{s,t})_{s\le t}, \mathsf{P}, (T_h)_{h\in\mathbb{R}})$ is a subnoise of N. Lemma 3.2 allows us to define $\bar{K}$ the flow obtained by filtering K with respect to $\bar{N}$. One can check that $\bar{K} = \bar{K}^0(K)$. This implies that the law of $\bar{K}$ is $\mathsf{P}_{\bar{\nu}}$. Thus the proposition is proved. □

PROPOSITION 3.1. *The domination relation and the weak domination relation are partial orders on the class of Feller convolution semigroups.*

PROOF. (i) The transitivity of the domination relation follows from Lemma 3.3 by the chain rule for conditional expectations.

Let us observe that if $\nu^1 \preceq \nu^2$ and $\nu^2 \preceq \nu^1$, then $\nu^1 = \nu^2$. Indeed, if $\nu^1 \succeq \nu^2$, Jensen's inequality shows that, for all $f_1, \dots, f_n$ in $C(M)$, $x_1, \dots, x_n$ in M and $t \ge 0$,

$$\mathsf{E}_{\nu^1}\left[\exp\left(\sum_{i=1}^n K_{0,t} f_i(x_i)\right)\right] \ge \mathsf{E}_{\nu^2}\left[\exp\left(\sum_{i=1}^n K_{0,t} f_i(x_i)\right)\right]. \tag{3.15}$$

Therefore, if moreover $\nu^1 \preceq \nu^2$, the preceding inequality becomes an equality. This proves $\nu^1 = \nu^2$.

(ii) For the weak domination relation, the proof is similar. We prove the transitivity using the product of extensions. Indeed, if $\bar{\nu} \overset{w}{\preceq} \nu$, given any extension (N^1, K^1) of N_ν, there exist a larger extension (N, K) and a subnoise $\bar{N}$ of N such that $\bar{K}$ has law $\mathsf{P}_{\bar{\nu}}$. Let $\bar{N}^2$ be a subnoise of an extension (N^2, K^2) of N_ν such that $\bar{K}^2$ has law $\mathsf{P}_{\bar{\nu}}$; then (N, K) is taken as the product of the extensions (N^1, K^1) and (N^2, K^2), and $\bar{N}$ is induced by $\bar{N}^2$. $\square$

REMARK 3.2. The concept of filtering will be used in Sections 4.3, 5.5 and 6.2 and an example is given in the following section.

3.3. *An example of filtering.* Let $M = \{0, 1\}$. Then F, the set of maps from $\{0, 1\}$ on $\{0, 1\}$ is constituted of the maps σ, I, f_0 and f_1, with I the identity, $\sigma(0) = 1$, $\sigma(1) = 0$, $f_0 = 0$ and $f_1 = 1$. Let (N_t) be a Poisson process on $\mathbb{R}$ and let $(\varphi_n)_{n\in\mathbb{Z}}$ be a sequence, independent of the Poisson process, of independent random variables taking their values in F with law

$$\tfrac{1}{4}\big(\delta_{f_0} + \delta_{f_1} + \delta_I + \delta_\sigma\big).$$

We then define a stochastic flow of mappings on $\{0, 1\}$ by

$$\begin{aligned} \varphi_{s,t} &= I, && \text{if } N_t - N_s = 0,\\ \varphi_{s,t} &= \varphi_{N_t-1} \circ \cdots \circ \varphi_{N_s}, && \text{if } N_t - N_s > 0, \end{aligned}$$

for all $s \le t$. Note that φ is a coalescing flow since for every s, there is a.s. a finite time T such that, for all $t \ge T$, $\varphi_{s,t}(0) = \varphi_{s,t}(1)$. The one-point motion of this flow is given by the symmetric random walk with generator $A^{(1)}$ given by

$$A^{(1)} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}.$$

Note also that, since $\{0, 1\}$ has only two points, the n-point motions associated with this stochastic flow of mappings are determined by the two-point motion. The generator $A^{(2)}$ of the two-point motion is [the state space is $\{(0, 0), (1, 1), (0, 1), (1, 0)\}$]

$$A^{(2)} = \begin{pmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 \end{pmatrix}.$$

With the stochastic flow φ and an independent sequence of random variables $(Y_n)_{n\in\mathbb{Z}}$ with $\mathsf{P}[Y_n = 1] = p = 1 - \mathsf{P}[Y_n = 0]$, we define a stochastic flow of kernels K, by

$$\begin{aligned} K_{s,t}(i) &= \delta_i, && \text{if } N_t - N_s = 0,\\ K_{s,t} &= K_{N_s} \cdots K_{N_t-1}, && \text{if } N_t - N_s > 0, \end{aligned}$$

where $K_n = Y_n\delta_{\varphi_n} + (1 - Y_n)\frac{1}{2}(\delta_0 + \delta_1)$.

Denote by N^c the noise of φ, by N the noise of K and by $\hat{N}$ the noise of (φ, Y). Then N^c is the noise of (N_t, φ_{N_t}), $\hat{N}$ is the noise of $(N_t, \varphi_{N_t}, Y_{N_t})$ and N is the noise of (N_t, K_{N_t}). The noises N^c and N are subnoises of $\hat{N}$. And N cannot be isomorphic to a subnoise of N^c. Indeed, for ε small, $\mathcal{F}_{0,\varepsilon}^{N^c}$ has one atom of probability $e^{-\varepsilon}$ and four atoms of probability $\frac{1}{4}\varepsilon e^{-\varepsilon}$, and $\mathcal{F}_{0,\varepsilon}^{N}$ has one atom of probability $e^{-\varepsilon}$ as well but one atom of probability $(1-p)\varepsilon e^{-\varepsilon}$ and four atoms of probability $\frac{p}{4}\varepsilon e^{-\varepsilon}$.

The flow K coincides with the flow obtained by filtering φ with respect to N. Thus the law of K is weakly dominated by the law of φ but is not dominated.

3.4. *Continuous martingales.* Let $(K_{s,t}, s \le t)$ be a stochastic flow of kernels. For all $s \le t$, set $\mathcal{F}_{s,t} = \sigma(K_{u,v}, s \le u \le v \le t)$. Let $\mathcal{F}$ be the filtration $(\mathcal{F}_{0,t})_{t\ge 0}$. Let $\mathcal{M}(\mathcal{F})$ be the space of locally square integrable $\mathcal{F}$-martingales.

PROPOSITION 3.2. *Suppose that $\mathsf{P}_t^{(1)}$ is the semigroup of a Markov process with continuous paths. Then all martingales of $\mathcal{M}(\mathcal{F})$ are continuous.*

PROOF. Let $M \in \mathcal{M}(\mathcal{F})$ be a martingale in the form $\mathsf{E}[F|\mathcal{F}_{0,t}]$, where $F = \prod_{i=1}^n K_{s_i,t_i} f_i(x_i)$, with $f_1, \dots, f_n$ in $C(M)$, $x_1, \dots, x_n$ in M and $0 \le s_i < t_i$ (we take here the continuous modification in t of the stochastic flow of kernels). By definition of the filtration, functions in this form are dense in $L^2(\mathcal{F}_{0,\infty})$. This implies that martingales of this form are dense in $\mathcal{M}(\mathcal{F})$. Since the space of continuous martingales is closed in $\mathcal{M}(\mathcal{F})$, it is enough to prove the continuity of these martingales.

For every t, let $\tilde{K}_t$ be the kernel defined on $\mathbb{R}^+ \times M$ by

$$\tilde{K}_t(s,x) = \begin{cases} \delta_{s-t} \otimes \delta_x, & \text{for } s \ge t, \\ \delta_0 \otimes K_{s,t}(x), & \text{for } s \le t. \end{cases} \tag{3.16}$$

Then we can rewrite F in the form $\prod_{i=1}^n \tilde{K}_{t_i} \tilde{f}_i(s_i, x_i)$, where $\tilde{f}_i(s,x) = f_i(x)$.

Note that $(\tilde{K}_{t_i}(s_i, x_i), 1 \le i \le n)$ is a Markov process on $(\mathcal{B}(\mathbb{R}^+) \otimes \mathcal{P}(M))^n$. This Markov process is continuous and Feller [the Feller property follows from the Feller property of the semigroups $(\Pi_t^{(k)}, k \ge 1)$]. It is well known that the martingales relative to the filtration denoted here $(\mathcal{F}_t^{\{s_i,x_i\}_{1\le i\le n}}, t \ge 0)$ generated by such a process are continuous (see [38], tome II).

This proves that $\mathsf{E}[F|\mathcal{F}_t^{\{s_i,x_i\}_{1\le i\le n}}]$ is a continuous martingale. We conclude after remarking that $M_t = \mathsf{E}[F|\mathcal{F}_t^{\{s_i,x_i\}_{1\le i\le n}}]$, which holds since the σ-field $\mathcal{F}_t^{\{s_i,x_i\}_{1\le i\le n}}$ is a sub-σ-field of $\mathcal{F}_t$ and M_t is easily seen to be $\mathcal{F}_t^{\{s_i,x_i\}_{1\le i\le n}}$-measurable. □

4. Stochastic coalescing flows. In this section, we study stochastic coalescing flows, we denote by $(\varphi_{s,t}, s \le t)$. In Section 4.2, it is shown that, for all $s < t$, $\varphi_{s,t}^*(\lambda)$ is atomic (where λ denotes any positive Radon measure on M).

We study this point measure valued process which provides a description of the coalescing flow.

In Section 4.3, starting from a compatible family of Feller semigroups, under the hypothesis that starting close to the diagonal the two-point motion hits the diagonal with a probability close to 1, we construct another compatible family of Feller semigroups to which is associated a stochastic coalescing flow. We then show that the stochastic flow of kernels associated with the first family of semigroups can be defined by filtering the stochastic coalescing flow with respect to a subnoise of an extension of its canonical noise.

Finally, we give three examples. The first one, due to Arratia [2], describes the flow of independent Brownian motions sticking together when they meet. The second one is due to Propp and Wilson [35] in the context of perfect simulation of the invariant distribution of a finite-state irreducible Markov chain, their stochastic flows being indexed by the integers. The third one is the construction of a stochastic coalescing flow solution of Tanaka's SDE

$$dX_t = \operatorname{sgn}(X_t)\, dW_t, \tag{4.1}$$

where W is a real white noise. This coalescing flow was constructed by Watanabe in [45] and Warren in [44]. In [23], a stochastic flow of kernels solution of this SDE was constructed as the only Wiener solution of this SDE.

4.1. *Definition.* Let M be a locally compact separable metric space.

DEFINITION 4.1. A stochastic flow of mappings on M, $(\varphi_{s,t}, s \le t)$, is called a stochastic coalescing flow if, for some $(x, y) \in M^2$, $T_{x,y} = \inf\{t \ge 0, \varphi_{0,t}(x) = \varphi_{0,t}(y)\}$ is finite with a positive probability and, for every $t \ge T_{x,y}$, $\varphi_{0,t}(x) = \varphi_{0,t}(y)$. In other words, a pair of points stick together after a finite time with a positive probability.

REMARK 4.1. This definition depends only on the two-point motion.

Let $(\mathsf{P}_t^{(n)}, n \ge 1)$ be a compatible family of Feller semigroups. We denote by $\mathsf{P}_{(x,y)}^{(2)}$ the law of the Markov process associated with $\mathsf{P}_t^{(2)}$ starting from (x, y) and we denote this process (X_t, Y_t) or $X_t^{(2)}$. Let $T_\Delta = \inf\{t \ge 0, X_t = Y_t\}$.

REMARK 4.2. A compatible family $(\mathsf{P}_t^{(n)}, n \ge 1)$ of Feller semigroups defines a stochastic coalescing flow if and only if, for every $(x, y) \in M^2$, for every $t \ge T_\Delta$, $X_t = Y_t$, $\mathsf{P}_{(x,y)}^{(2)}$-a.s., and for some $(x, y) \in M^2$, $\mathsf{P}_{(x,y)}^{(2)}[T_\Delta < \infty] > 0$.

4.2. *A point measure valued process associated with a stochastic coalescing flow.* In this section, we suppose we are given a compatible family of Feller semigroups $(\mathsf{P}_t^{(n)}, n \geq 1)$ such that

$$\forall x \in M, \forall t > 0, \qquad \lim_{y \to x} \mathsf{P}^{(2)}_{(x,y)}[X_t \neq Y_t] = 0,$$
$$\forall (x,y) \in M^2, \qquad \mathsf{P}^{(2)}_{(x,y)}[T_\Delta < \infty] > 0. \tag{4.2}$$

REMARK 4.3. Assumption (4.2) implies that the associated stochastic flow is a stochastic coalescing flow and is verified in all the examples of coalescing flows we will study except for the example presented in Section 4.4.3, where $\mathsf{P}^{(2)}_{(x,y)}[X_t \neq Y_t]$ does not converge toward 0 as y tends to x when $x \neq 0$.

Let $\varphi = (\varphi_{s,t}, s \leq t)$ be a measurable stochastic coalescing flow associated with $(\mathsf{P}_t^{(n)}, n \geq 1)$. For all $s < t \in \mathbb{R}$, let $\mu_{s,t} = \varphi^*_{s,t}(\lambda)$, where λ is any positive Radon measure on M.

PROPOSITION 4.1. (a) *For all $s < t \in \mathbb{R}$, a.s., $\mu_{s,t}$ is atomic.*
(b) *For all $s < u < t \in \mathbb{R}$, a.s., $\mu_{s,t}$ is absolutely continuous with respect to $\mu_{u,t}$.*

PROOF. Fix $s < t \in \mathbb{R}$. For all $\varepsilon > 0$ and $x \in M$, let

$$m^x_\varepsilon = \int_{B(x,\varepsilon)} 1_{\varphi_{s,t}(x) = \varphi_{s,t}(y)} \lambda(dy)$$

[m^x_ε is well defined since $(x,\omega) \mapsto \varphi_{s,t}(x,\omega)$ is measurable]. For all $\alpha \in]0,1[$ and $x \in M$, let

$$A_n^{\alpha,x} = \{m^x_{\varepsilon^x_n} < (1-\alpha)\lambda(B(x,\varepsilon^x_n))\}, \tag{4.3}$$

where ε^x_n is a positive sequence such that $d(x,y) \leq \varepsilon^x_n$ implies

$$\mathsf{P}^{(2)}_{(x,y)}[X_{t-s} \neq Y_{t-s}] \leq 2^{-n}.$$

LEMMA 4.1. *For all positive α, $x \in M$ and $n \in \mathbb{N}$, $\mathsf{P}(A^{\alpha,x}_n) \leq \frac{1}{\alpha 2^n}$.*

PROOF. For every integer n, we have

$$\begin{aligned}\mathsf{E}[m^x_{\varepsilon^x_n}] &= \int_{B(x,\varepsilon^x_n)} \mathsf{P}^{(2)}_{(x,y)}[X_{t-s} = Y_{t-s}]\lambda(dy) \\ &= \int_{B(x,\varepsilon^x_n)} (1 - \mathsf{P}^{(2)}_{(x,y)}[X_{t-s} \neq Y_{t-s}])\lambda(dy) \geq (1-2^{-n})\lambda(B(x,\varepsilon^x_n)).\end{aligned}$$

And we conclude since

$$\mathsf{E}[m^x_{\varepsilon^x_n}] \leq \mathsf{P}(A^{\alpha,x}_n)(1-\alpha)\lambda(B(x,\varepsilon^x_n)) + (1 - \mathsf{P}(A^{\alpha,x}_n))\lambda(B(x,\varepsilon^x_n))$$

[we use the fact that $m^x_{\varepsilon^x_n} \leq \lambda(B(x,\varepsilon^x_n))$]. $\square$

LEMMA 4.2. *For every $x \in M$, a.s., $m^x_{\varepsilon^x_n} \sim \lambda(B(x, \varepsilon^x_n))$ as $n \to \infty$.*

PROOF. Using the Borel–Cantelli lemma, for every $\alpha \in]0, 1[$,

$$1 - \alpha \le \liminf_{n\to\infty} \frac{m^x_{\varepsilon^x_n}}{\lambda(B(x, \varepsilon^x_n))} \le \limsup_{n\to\infty} \frac{m^x_{\varepsilon^x_n}}{\lambda(B(x, \varepsilon^x_n))} \le 1 \qquad \text{a.s.}$$

This implies $\lim_{n\to\infty} \frac{m^x_{\varepsilon^x_n}}{\lambda(B(x,\varepsilon^x_n))} = 1$ a.s. □

Since for every $(x, \omega) \in M \times \Omega$,

$$\begin{aligned}\mu_{s,t}\big(\{\varphi_{s,t}(x)\}\big) &= \lambda\big(\{y, \varphi_{s,t}(y) = \varphi_{s,t}(x)\}\big)\\ &\ge \lambda\big(\{y \in B(x, \varepsilon^x_n), \varphi_{s,t}(y) = \varphi_{s,t}(x)\}\big),\end{aligned}$$

Lemma 4.2 implies that, for every $x \in M$,

$$\mu_{s,t}\big(\{\varphi_{s,t}(x)\}\big) > 0 \qquad \text{a.s.} \tag{4.4}$$

Since $(x, \omega) \mapsto \mu_{s,t}(\{\varphi_{s,t}(x)\})$ is measurable,

$$\lambda(dx) \otimes \mathsf{P}(d\omega)\text{-a.e.,} \qquad \mu_{s,t}\big(\{\varphi_{s,t}(x)\}\big) > 0. \tag{4.5}$$

This equation implies [since $\mu_{s,t} = \varphi^*_{s,t}(\lambda)$]

$$\mu_{s,t}(dy)\text{-a.e.,} \qquad \mu_{s,t}(\{y\}) > 0 \qquad \text{a.s.} \tag{4.6}$$

This last equation is one characterization of the atomic nature of $\mu_{s,t}$ and (a) is proved.

To prove (b), note first that $\lambda(dx) \otimes \mathsf{P}(d\omega)$-a.e., $\varphi^*_{u,t}(\delta_x) = \delta_{\varphi_{u,t}(x)}$ is absolutely continuous with respect to $\varphi^*_{u,t}(\lambda) = \mu_{u,t}$ since (4.4) holds. Note also that $\lambda(dx) \otimes \mathsf{P}(d\omega)$-a.e., $\varphi_{s,t}(x) = \varphi_{u,t} \circ \varphi_{s,u}(x)$. This implies

$$\mu_{s,t} = \varphi^*_{u,t}(\mu_{s,u}) \qquad \text{a.s.} \tag{4.7}$$

Since $\mu_{s,u}$ is atomic, independent of $\varphi_{u,t}$ and $\mathsf{E}[\mu_{s,u}] = \lambda$, it follows that $\mu_{s,t}$ is absolutely continuous with respect to $\mu_{u,t}$. This proves (b). □

REMARK 4.4. (i) $(\mu_{s,t}, s \le t)$ is Markovian in t.

(ii) Since $\mu_{s,t}$ is atomic for $t > s$, there exist a point process $\xi_{s,t} = \{\xi^i_{s,t}\}$ and weights $\{\alpha^i_{s,t}\} \in \mathbb{R}^{\mathbb{N}}$ such that $\mu_{s,t} = \sum_i \alpha^i_{s,t}\delta_{\xi^i_{s,t}}$. The point process $\xi_{s,t}$ and the marked point process $(\xi_{s,t}, \alpha_{s,t})$ are Markovian in t since, for all $s < u < t$, $\xi_{s,t} = \varphi_{u,t}(\xi_{s,u})$ and $\alpha^i_{s,t} = \sum_{\{j, \xi^i_{s,t} = \varphi_{u,t}(\xi^j_{s,u})\}} \alpha^j_{s,u}$.

(iii) Let $A^i_{s,t} = \varphi^{-1}_{s,t}(\xi^i_{s,t})$ and let $\Pi_{s,t}$ be the collection of the sets $A^i_{s,t}$. Note that $\bigcup_i A^i_{s,t} = M$ λ-a.e, the union being disjoint. Note also that $\xi_{s,t}$ and $\Pi_{s,t}$ determine $\varphi_{s,t}$ λ-a.e. Note finally that $\Pi_{s,t}$ is Markovian in s when s decreases, since for all $s < u < t$, $\Pi_{s,t} = \{\varphi^{-1}_{s,u}(A^i_{u,t})\}$. This Markov process has also a coalescence property: one can have, for $i \ne j$, $\varphi^{-1}_{s,u}(A^i_{u,t}) = \varphi^{-1}_{s,u}(A^j_{u,t})$. When s decreases, the partition $\Pi_{s,t}$ becomes coarser.

4.3. *Construction of a family of coalescent semigroups.* Let $(\mathsf{P}_t^{(n)}, n \geq 1)$ be a compatible family of Feller semigroups on a locally compact separable metric space M and let $\nu = (\nu_t)_{t\in\mathbb{R}}$ be the associated Feller convolution semigroup on $(E, \mathcal{E})$. Let $\Delta_n = \{x \in M^n, \exists\, i \neq j, x_i = x_j\}$ and $T_{\Delta_n} = \inf\{t \geq 0, X_t^{(n)} \in \Delta_n\}$, where $X_t^{(n)}$ denotes the n-point motion, that is, the Markov process on M^n associated with the semigroup $\mathsf{P}_t^{(n)}$. We will denote Δ_2 by Δ.

THEOREM 4.1. *There exists a unique compatible family* $(\mathsf{P}_t^{(n),c}, n \geq 1)$ *of Markovian semigroups on* M *such that if* $X^{(n),c}$ *is the associated* n*-point motion and* $T^c_{\Delta_n} = \inf\{t \geq 0, X_t^{(n),c} \in \Delta_n\}$, *then*:

(i) $(X_t^{(n),c}, t \leq T^c_{\Delta_n})$ *is equal in law to* $(X_t^{(n)}, t \leq T_{\Delta_n})$,

(ii) *for* $t \geq T^c_{\Delta_n}$, $X_t^{(n),c} \in \Delta_n$.

Moreover, this family is constituted of Feller semigroups if condition (C) *is satisfied*:

(C) *For all* $t > 0$, $\varepsilon > 0$ *and* $x \in M$,

$$\lim_{y\to x} \mathsf{P}^{(2)}_{(x,y)}[\{T_\Delta > t\} \cap \{d(X_t, Y_t) > \varepsilon\}] = 0,$$

where $(X_t, Y_t) = X_t^{(2)}$. *And for some* x *and* y *in* M, $\mathsf{P}^{(2)}_{(x,y)}[T_\Delta < \infty] > 0$.

In this case, $(\mathsf{P}_t^{(n),c}, n \geq 1)$ *satisfies* (1.6) *and is associated with a coalescing flow.*

PROOF. For every $n \geq 1$, let $\mathcal{P}_n$ be the set of all partitions of $\{1, \dots, n\}$. The number of elements of $\pi \in \mathcal{P}_n$ is denoted $|\pi|$. For every $\pi \in \mathcal{P}_n$, we define the equivalent relation $i \overset{\pi}{\sim} j$ if i and j belong to the same element of π. We define a partial order on $\mathcal{P}_n$ by $\pi' \leq \pi$ if $i \overset{\pi}{\sim} j$ implies $i \overset{\pi'}{\sim} j$ (π is finer than π').

For every $\pi \in \mathcal{P}_n$, we let E_π be the set of elements $x \in M^n$ such that $x_i = x_j$ if $i \overset{\pi}{\sim} j$ and $\partial E_\pi = \bigcup_{\pi' < \pi} E_{\pi'}$, the set of elements $x \in E_\pi$ such that there exists i and j with $i \overset{\pi}{\not\sim} j$ and $x_i = x_j$. Let j_π be an isometry between $M^{|\pi|}$ and E_π.

By induction on $k = |\pi|$, we define a Markov process X^π on E_π. For $k = 1$, we let $X^\pi = j_\pi(X^{(1)})$. Assume now we have defined a Markov process on E_π for every π such that $|\pi| \leq k$. Let $\pi \in \mathcal{P}_n$ with $|\pi| = k + 1$; we define X^π concatenating the process $j_\pi(X^{(k+1)})$ stopped at the entrance time T in ∂E_π with the process $X^{\pi'}$ starting from the corresponding point and where π' is the finest partition such that $j_\pi(X_T^{(k+1)}) \in E_{\pi'}$. This way, we construct a Markov process on M^n, $X^{(n),c} = X^\pi$ for $\pi = \{\{1\}, \dots, \{n\}\}$.

For every integer n, let $\mathsf{P}_t^{(n),c}$ be the Markovian semigroup associated with the Markov process $X^{(n),c}$. From the above construction, it is clear that the family $(\mathsf{P}_t^{(n),c}, n \geq 1)$ of Markovian semigroups is compatible.

It remains to prove that when (C) is satisfied, this family of Markovian semigroups is constituted of Feller semigroups. This holds since (C) implies (F) in Lemma 1.11: for every positive ε, $\mathsf{P}^{(2),c}_{(x,y)}[d(X_t, Y_t) > \varepsilon] \le \mathsf{P}^{(2)}_{(x,y)}[\{T_\Delta > t\} \cap \{d(X_t, Y_t) > \varepsilon\}]$, which converges toward 0 as $y \to x$. Note that when (C) holds, it is easy to see that the canonical flow is a coalescing flow. □

We now suppose that $(\mathsf{P}^{(n),c}_t, n \ge 1)$ is constituted of Feller semigroups [which is true when (C) holds]. We denote by ν^c the associated Feller convolution semigroup.

THEOREM 4.2. *The convolution semigroup ν^c weakly dominates ν.*

PROOF. The idea of the proof is to define a coupling between the flows of kernels K and K^c, respectively, of law P_ν and P_{ν^c}. [Since we did not assume (C) holds, it is not clear that K^c is a flow of mappings.]

In a way similar to the construction of the Markov process $X^{(n),c}$ in the proof of Theorem 4.1, for every integer $n \ge 1$, we construct a Markov process $\hat{X}^{(n)}$ on $(M \times M)^n$ identified with $M^n \times M^n$ such that:

(i) $(\hat{X}^{(n)}_1, \dots, \hat{X}^{(n)}_n)$ is the n-point motion of ν^c,
(ii) $(\hat{X}^{(n)}_{n+1}, \dots, \hat{X}^{(n)}_{2n})$ is the n-point motion of ν,
(iii) between the coalescing times, $\hat{X}^{(n)}$ is described by the $(k+n)$-point motion of ν [when $(\hat{X}^{(n)}_1, \dots, \hat{X}^{(n)}_n)$ belongs to E_π, with $|\pi| = k$].

Let $\hat{\mathsf{P}}^{(n)}_t$ be the Markovian semigroup associated with $\hat{X}^{(n)}$. One easily checks that this semigroup is Feller using the fact that $\mathsf{P}^{(n)}_t$ and $\mathsf{P}^{(n),c}_t$ are Feller. Then $(\hat{\mathsf{P}}^{(n)}_t, n \ge 1)$ is a compatible family of Feller semigroups, associated with a Feller convolution semigroup $\hat{\nu}$.

Let $\hat{K}$ be the canonical stochastic flow associated with this family of semigroups. Straightforward computations show that, for all $s < t$, $(f, g) \in C(M)^2$ and $(x, y) \in M^2$,

$$\mathsf{E}\big[\big(\hat{K}_{s,t}(f \otimes g)(x, y)\big)^2\big] = \mathsf{P}^{(3)}_{t-s} f^2 \otimes g \otimes g(x, y, y),$$

$$\mathsf{E}\big[\big(\hat{K}_{s,t}(f \otimes 1)\hat{K}_{s,t}(1 \otimes g)\big)^2(x, y)\big] = \mathsf{P}^{(3)}_{t-s} f^2 \otimes g \otimes g(x, y, y),$$

$$\mathsf{E}\big[\big(\hat{K}_{s,t}(f \otimes g)\hat{K}_{s,t}(f \otimes 1)\hat{K}_{s,t}(1 \otimes g)\big)(x, y)\big] = \mathsf{P}^{(3)}_{t-s} f^2 \otimes g \otimes g(x, y, y).$$

This implies that

$$\mathsf{E}\big[\big(\hat{K}_{s,t}(f \otimes g) - \hat{K}_{s,t}(f \otimes 1)\hat{K}_{s,t}(1 \otimes g)\big)^2(x, y)\big] = 0. \tag{4.8}$$

Thus we have $\hat{K}_{s,t}(x, y) = K^c_{s,t}(x) \otimes K_{s,t}(y)$ and it is easy to check that the laws of K^c and of K are, respectively, P_{ν^c} and P_ν. Thus $(N_{\hat{\nu}}, K^c)$ is an extension of N_{ν^c}. Let $\tilde{N}_\nu$ be the subnoise of $N_{\hat{\nu}}$ generated by K.

Let us notice now that, for all $g, f_1, \dots, f_n$ in $C_0(M)$, all $y, x_1, \dots, x_n$ in M and all $s < t$, we have (setting $y_i = x_{n+1} = y$ and for $i \le n$, $h_i = f_i \otimes 1$ and $h_{n+1} = 1 \otimes g$)

$$\mathsf{E}\left[K^c_{s,t} g(y) \prod_{i=1}^n K_{s,t} f_i(x_i)\right] = \mathsf{E}\left[\prod_{i=1}^{n+1} \hat{K}_{s,t} h_i(x_i, y_i)\right]$$
$$= \mathsf{P}^{(n+1)}_{t-s} f_1 \otimes \cdots \otimes f_n \otimes g(x_1, \dots, x_n, y).$$

More generally, one can prove in a similar way, for all g, $f_1, \dots, f_n$ in $C_0(M)$, all $y, x_1, \dots, x_n$ in M, all $s < t$ and all $(s_i, t_i)_{1 \le i \le n}$ with $s_i \le t_i$, that

$$\mathsf{E}\left[K^c_{s,t} g(y) \prod_{i=1}^n K_{s_i,t_i} f_i(x_i)\right] = \mathsf{E}\left[K_{s,t} g(y) \prod_{i=1}^n K_{s_i,t_i} f_i(x_i)\right]. \tag{4.9}$$

This implies that $K_{s,t} g(y) = \mathsf{E}[K^c_{s,t}(y) | \sigma(K)]$ and therefore that $\nu^c \overset{w}{\succeq} \nu$. $\square$

REMARK 4.5. Let $(X^{(n)}, n \ge 1)$ be a family of strong Markov processes, respectively, taking their values in M^n. We suppose that the associated family of Markovian semigroups $(\mathsf{P}^{(n)}_t, n \ge 1)$ is compatible and that, for every $x \in M$,

$$\lim_{y \to x} \mathsf{P}^{(2)}_{(x,y)}[\{T_\Delta > t\} \cap \{d(X_t, Y_t) > \varepsilon\}] = 0 \tag{4.10}$$

for all $\varepsilon > 0$ and $t > 0$. Then $(\mathsf{P}^{(n)}_t, n \ge 1)$ [and $(\mathsf{P}^{(n),c}_t, n \ge 1)$] are Feller semigroups.

One can prove this with a coupling similar to the coupling given in the proof of the previous theorem: the idea is to construct on the same probability space two Markov processes $X^{(n)}$ and $Y^{(n)}$ associated to $\mathsf{P}^{(n)}_t$ and such that $X^{(n)}_i(t) = Y^{(n)}_i(t)$ if $t \ge \inf\{s, X^{(n)}_i(s) = Y^{(n)}_i(s)\}$.

REMARK 4.6. The example given in Section 3.3 gives an illustration of the two theorems of this section, first with $\mathsf{P}^{(n)}_t = \mathsf{P}^{\otimes n}_t$, then with $\mathsf{P}^{(n)}_t$ the semigroup of the n-point motion of K. This example shows in particular that one can have $\nu \overset{w}{\preceq} \nu^c$ and $\nu \not\preceq \nu^c$.

4.4. *Examples.*

4.4.1. *Arratia's coalescing flow of independent Brownian motions.* The first example of coalescing flows was given by Arratia [2]. On $\mathbb{R}$, let P_t be the semigroup of a Brownian motion. With this semigroup we define the compatible family $(\mathsf{P}^{\otimes n}_t, n \ge 1)$ of Feller semigroups. Note that the n-point motion of this family of semigroups is given by n independent Brownian motions. Let us also remark that the canonical stochastic flow of kernels associated with this family of semigroups is not random and is given by $(\mathsf{P}_{t-s}, s \le t)$.

Let $(\mathsf{P}_t^{(n)}, n \ge 1)$ be the compatible family of Markovian coalescent semigroups associated with $(\mathsf{P}_t^{\otimes n}, n \ge 1)$ (see Section 4.3). Note that the n-point motion of this family of semigroups is given by n independent Brownian motions who stick together when they meet.

PROPOSITION 4.2. *The family* $(\mathsf{P}_t^{(n)}, n \ge 1)$ *is constituted of Feller semigroups and is associated with a coalescing flow.*

PROOF. It is obvious after remarking that two real independent Brownian motions meet each other a.s. [condition (C) is verified]. □

4.4.2. *Propp–Wilson algorithm.* Similarly to Arratia's coalescing flow, let P_t be the semigroup of an irreducible aperiodic Markov process on a finite set M, with invariant probability measure m. Let $(\mathsf{P}_t^{(n)}, n \ge 1)$ be the compatible family of Markovian coalescent semigroups associated with $(\mathsf{P}_t^{\otimes n}, n \ge 1)$. The coalescing flow in Section 3.3 is of this type.

PROPOSITION 4.3. *The family* $(\mathsf{P}_t^{(n)}, n \ge 1)$ *is constituted of Feller semigroups and is associated with a coalescing flow.*

PROOF. It is obvious since the two-point motion defined by $\mathsf{P}_t^{\otimes 2}$ hits the diagonal almost surely. □

Let $\varphi = (\varphi_{s,t}, s \le t)$ denote this coalescing flow. Then a.s., for all x, y in M, $\tau_{x,y} = \inf\{t > 0, \varphi_{0,t}(x) = \varphi_{0,t}(y)\}$ is finite. Therefore, after a finite time, $\mathrm{Card}\{\varphi_{0,t}(x), x \in M\} = 1$.

In [35], an algorithm to exactly simulate a random variable distributed according to the invariant probability measure of a Markov chain with finite state space is given. The method consists in constructing a stochastic coalescing flow. We explain this in our context.

Let $\tau = \inf\{t > 0, \varphi_{-t,0}(x) = \varphi_{-t,0}(y) \text{ for all } (x, y) \in M^2\}$.

PROPOSITION 4.4. τ *is a.s. finite and the law of* X_τ, *the random variable* $\varphi_{-\tau,0}(x)$ (*independent of* $x \in M$), *is* m.

PROOF. Let us remark that, for $t > \tau$ and every $x \in M$, the cocycle property implies that $\varphi_{-t,0}(x) = X_\tau$.

For every positive t,

$$\begin{aligned} \mathsf{P}[\tau \ge t] &= \mathsf{P}[\exists\, x, y, \varphi_{-t,0}(x) \ne \varphi_{-t,0}(y)] \\ &\le \sum_{(x,y)\in M^2} \mathsf{P}[\tau_{x,y} \ge t], \end{aligned} \tag{4.11}$$

which converges toward 0 as t goes to ∞. Thus $\tau < \infty$ a.s.

For every function f on M and every $x \in M$, $\lim_{t\to\infty} \mathsf{P}_t f(x) = \sum_{y\in M} f(y) \times m(y)$ and

$$\mathsf{P}_t f(x) = \mathsf{E}\big[f\big(\varphi_{-t,0}(x)\big)\big] = \mathsf{E}\big[f\big(\varphi_{-t,0}(x)\big)1\!\!1_{t\le\tau}\big] + \mathsf{E}[f(X_\tau)1\!\!1_{\tau<t}]. \tag{4.12}$$

Since τ is a.s. finite, as t goes to ∞, the first term of the right-hand side of the preceding equation converges toward 0 and the second term converges toward $\mathsf{E}[f(X_\tau)]$. Therefore we prove that $\mathsf{E}[f(X_\tau)] = \sum_{y\in M} f(y)m(y)$. □

4.4.3. *Tanaka's SDE.* In [23], starting from a real Brownian motion B, we constructed a family of random operators $(S_t, t \ge 0)$, Wiener solution of the SDE

$$dX_t = \operatorname{sgn}(X_t)\,dB_t, \qquad t \ge 0. \tag{4.13}$$

For f continuous,

$$S_t f(x) = f(R_t^x)1\!\!1_{t<T_x} + \tfrac{1}{2}\big(f(R_t^x) + f(-R_t^x)\big)1\!\!1_{t\ge T_x}, \tag{4.14}$$

where R_t^x is the Brownian motion $x + B_t$ reflected at 0 and T_x is the first time it hits 0. For all continuous functions $f_1, \dots, f_n$, let

$$\mathsf{P}_t^{(n)}(f_1 \otimes \cdots \otimes f_n)(x_1, \dots, x_n) = \mathsf{E}\left[\prod_{i=1}^n S_t f_i(x_i)\right]. \tag{4.15}$$

Then it is easy to see that $(\mathsf{P}_t^{(n)}, n \ge 1)$ is a compatible family of Feller semigroups. Let $(\mathsf{P}_t^{(n),c}, n \ge 1)$ be the family of semigroups constructed by Theorem 4.1.

Let us describe the n-point motion associated with $(\mathsf{P}_t^{(n),c}, n \ge 1)$. Let $(X_t, t \ge 0)$ be a Brownian motion starting at 0. Let $B_t = \int_0^t \operatorname{sgn}(X_s)\,dX_s$; $(B_t, t \ge 0)$ is also a Brownian motion starting at 0. For every $x \in \mathbb{R}$, let $\tau_x = \inf\{t \ge 0, |x| + B_t = 0\}$. Note that $X_{\tau_x} = 0$. For every $x \in \mathbb{R}$, let

$$X_t^x = \begin{cases} x + \operatorname{sgn}(x)B_t, & \text{if } t < \tau_x, \\ X_t, & \text{if } t \ge \tau_x. \end{cases} \tag{4.16}$$

Then $B_t = \int_0^t \operatorname{sgn}(X_t^x)\,dX_t^x$ and X^x is a solution of the SDE

$$dX_t^x = \operatorname{sgn}(X_t^x)\,dB_t, \qquad t \ge 0, X_0^x = x. \tag{4.17}$$

Thus, for all $x_1, \dots, x_n$ in M, $((X_t^{x_1}, \dots, X_t^{x_n}), t \ge 0)$ is the n-point motion of the family of semigroups $(\mathsf{P}_t^{(n),c}, n \ge 1)$.

PROPOSITION 4.5. *The family* $(\mathsf{P}_t^{(n),c}, n \ge 1)$ *is constituted of Feller semigroups and is associated with a coalescing flow.*

PROOF. It is easy to see that $(\mathsf{P}_t^{(n),c}, n \geq 1)$ is constituted of Feller semigroups since, for all t and x_0, $x \mapsto X_t^x$ is a.s. continuous at x_0 [it implies that (F) in Lemma 1.11 is satisfied]. This also implies that (1.6) is satisfied. Thus, the associated stochastic flow is a flow of mappings. And it is a coalescing flow since a.s., every pair of points meets after a finite time. Note that condition (C) is verified. □

5. Stochastic flows of kernels and SDEs.

5.1. *Hypotheses.* In this section, M is a smooth locally compact manifold and we suppose we are given $(\mathsf{P}_t^{(n)}, n \geq 1)$, a compatible family of Feller semigroups, or equivalently a Feller convolution semigroup $\nu = (\nu_t)_{t \geq 0}$ on $(E, \mathcal{E})$. For every positive integer n, we will denote by $X_t^{(n)}$ the n-point motion, that is, the Markov process associated with the semigroup $\mathsf{P}_t^{(n)}$. We denote by $A^{(n)}$ the infinitesimal generator of $\mathsf{P}_t^{(n)}$ and by $\mathcal{D}(A^{(n)})$ its domain. (f is in the domain of the infinitesimal generator A of a Feller semigroup P_t if and only if $\frac{\mathsf{P}_t f - f}{t}$ converges uniformly as t goes toward 0. Its limit is denoted Af.) We assume that:

(i) The space $C_K^2(M) \otimes C_K^2(M)$ of functions of the form $f(x)g(y)$, with f, g in $C_K^2(M)$ and x, y in M, is included in $\mathcal{D}(A^{(2)})$.

(ii) The one-point motion $X_t^{(1)}$ has continuous paths.

[$C_K(M)$, resp. $C_K^2(M)$, denotes the set of continuous, resp. C^2, functions with compact support.] In that case, we say that ν is a *diffusion convolution semigroup* on $(E, \mathcal{E})$ and that the $\mathsf{P}_t^{(n)}$ are diffusion semigroups.

5.2. *Local characteristics of a diffusion convolution semigroup.* Let us denote by A the restriction of $A^{(1)}$ to $C_K^2(M)$. Note that it follows easily from (i) and (ii) that, for every $f \in C_K^2(M)$,

$$M_t^f = f(X_t^{(1)}) - f(X_0^{(1)}) - \int_0^t Af(X_s^{(1)})\, ds \tag{5.1}$$

is a martingale. Since f^2 also belongs to $C_K^2(M)$, using the martingale M^{f^2}, it is easy to see that

$$\langle M^f \rangle_t = \int_0^t \Gamma(f)(X_s^{(1)})\, ds, \tag{5.2}$$

where

$$\Gamma(f) = Af^2 - 2fAf. \tag{5.3}$$

In the following $\Gamma(f, g)$ will denote $A(fg) - fAg - gAf$, for f and g in $C_K^2(M)$.

LEMMA 5.1. *On a smooth local chart on an open set $U \subset M$, there exist continuous functions on U, $a^{i,j}$ and b^i such that, for every $f \in C_K^2(M)$,*

$$Af = \frac{1}{2} a^{i,j} \frac{\partial^2 f}{\partial x^i \, \partial x^j} + b^i \frac{\partial f}{\partial x^i}. \tag{5.4}$$

PROOF. For every $x \in U$, let $\varphi^i(x) = x^i$ denote the coordinate functions of the local chart. We can extend φ^i into an element of $C_K^2(M)$. For $f \in C_K^2(M)$, using Itô's formula, for $t < T_U$, the exit time of U,

$$f(X_t^{(1)}) - f(X_0^{(1)}) - \int_0^t \left(\frac{1}{2} a^{i,j}(X_s^{(1)}) \frac{\partial^2 f}{\partial x^i \, \partial x^j}(X_s^{(1)}) + b^i(X_s^{(1)}) \frac{\partial f}{\partial x^i}(X_s^{(1)}) \right) ds,$$

is a martingale, where $b^i(x) = A\varphi^i(x)$ and $a^{i,j}(x) = \Gamma(\varphi^i, \varphi^j)(x)$. And we get (5.4) since for every $x \in U$, $Af(x) = \lim_{t \to 0} \frac{\mathsf{P}_t^{(1)} f(x) - f(x)}{t}$. $\square$

Note that the two-point motion $X_t^{(2)}$ has also continuous trajectories and these results also apply to functions in $C_K^2(M) \otimes C_K^2(M)$. For all f, g in $C_K^2(M)$, let

$$C(f, g) = A^{(2)}(f \otimes g) - f \otimes Ag - Af \otimes g. \tag{5.5}$$

It is clear that on a local chart on $U \times V \subset M \times M$,

$$C(f, g)(x, y) = c^{i,j}(x, y) \frac{\partial f}{\partial x^i}(x) \frac{\partial g}{\partial y^j}(y), \tag{5.6}$$

where $c^{i,j} \in C(U \times V)$. Then we can shortly write $A^{(2)} = A \otimes I + I \otimes A + C$. On a local chart on $U \times V$, for every $h \in C_K^2(M) \otimes C_K^2(M)$,

$$\begin{aligned} A^{(2)} h(x, y) &= \frac{1}{2} a^{i,j}(x) \frac{\partial^2}{\partial x^i \, \partial x^j} h(x, y) + b^i(x) \frac{\partial}{\partial x^i} h(x, y) \\ &\quad + \frac{1}{2} a^{i,j}(y) \frac{\partial^2}{\partial y^i \, \partial y^j} h(x, y) + b^i(y) \frac{\partial}{\partial y^i} h(x, y) \\ &\quad + c^{i,j}(x, y) \frac{\partial^2}{\partial x^i \, \partial y^j} h(x, y). \end{aligned} \tag{5.7}$$

We will call $\Gamma(f, g)(x) - C(f, g)(x, x) = \frac{1}{2} A^{(2)}(1 \otimes f - g \otimes 1)^2(x, x) - (1 \otimes f - g \otimes 1)(1 \otimes Af - Ag \otimes 1)(x, x)$ the *pure diffusion form*. It can easily be seen that it is nonnegative and it vanishes if the associated canonical flow is a flow of maps. Indeed,

$$\begin{aligned} \Gamma(f, f)(x) &= \lim_{t \to 0} \frac{1}{t} \big(\mathsf{P}_t^{(1)} f^2(x) - \mathsf{P}_t^{(2)} f^{\otimes 2}(x, x) \big) \\ &= \lim_{t \to 0} \frac{1}{2t} \big(\mathsf{P}_t^{(2)} (1 \otimes f - f \otimes 1)^2(x, x) \big). \end{aligned}$$

The converse is not true (see examples in Section 7). Diffusive flows for which the pure diffusion form vanishes may be called *turbulent*.

The two-point motion $X_t^{(2)} = (X_t, Y_t)$ solves the following martingale problem associated with $A^{(2)}$:

$$M_t^{f\otimes g} := f(X_t)g(Y_t) - f(X_0)g(Y_0) - \int_0^t A^{(2)}(f\otimes g)(X_s, Y_s)\,ds \tag{5.8}$$

is a martingale for all f and g in $C_K^2(M)$.

Note that for all functions h_1 and h_2 in $C_K^2(M)\otimes C_K^2(M)$, the martingale bracket $\langle h_1(X^{(2)}), h_2(X^{(2)})\rangle_t$ is equal to

$$\int_0^t (A^{(2)}(h_1h_2) - h_1A^{(2)}h_2 - h_2A^{(2)}h_1)\big(X_s^{(2)}\big)\,ds, \tag{5.9}$$

and for all functions f and g in $C_K^2(M)$,

$$\langle f(X), g(Y)\rangle_t = \int_0^t C(f,g)(X_s, Y_s)\,ds. \tag{5.10}$$

DEFINITION 5.1. (i) A covariance function on the space of vector fields is a symmetric map C from T^*M^2 in $\mathbb{R}$ such that its restriction to $T_x^*M \times T_y^*M$ is bilinear and, for any n-uples $(\xi_1,\dots,\xi_n)$ of T^*M^2, $\sum_{1\le i,j\le n} C(\xi_i,\xi_j) \ge 0$ (see [23]). For f and g in $C_K^1(M)$, we denote $C(df(x), dg(y))$ by $C(f,g)(x,y)$.

(ii) We say the covariance function is continuous if $C(f,g)$ is continuous for all f and g in $C_K^1(M)$.

PROPOSITION 5.1. (i) *C is a continuous covariance function on the space of vector fields.*

(ii) *For all $f_1,\dots,f_n$ in $C_K^2(M)$, $g = f_1\otimes\cdots\otimes f_n \in \mathcal{D}(A^{(n)})$, and for $x = (x_1,\dots,x_n)\in M^n$,*

$$A^{(n)}g(x) = \sum_i \prod_{j\ne i} f_j(x_i)Af_i(x_i) + \sum_{i<j} C(f_i,f_j)(x_i,x_j)\prod_{k\ne i,j} f_k(x_k). \tag{5.11}$$

PROOF. For all f and g in $C_K^2(M)$, $C(f,g)(x,y)$ is a function of $df(x)$ and of $dg(y)$ we denote $C(df(x), dg(y))$. Hence C is a symmetric map from T^*M^2 in $\mathbb{R}$ and its restriction to $T_x^*M\times T_y^*M$ is bilinear. To prove (i), it remains to prove $\sum_{i,j} C(\xi_i,\xi_j)\ge 0$ for all $\xi_1,\dots,\xi_n$ in T^*M^2. This holds since, for all $f_1,\dots,f_n$ in $C^2(M)$ and all $x_1,\dots,x_n$ in M,

$$\sum_{i,j} C(f_i,f_j)(x_i,x_j) = \big(A^{(n)}g^2 - 2gA^{(n)}g\big)(x_1,\dots,x_n), \tag{5.12}$$

where $g(x_1,\dots,x_n) = \sum_{i=1}^n f_i(x_i) \in \mathcal{D}(A^{(n)})$. This expression is nonnegative since $A^{(n)}g^2 - 2gA^{(n)}g = \lim_{t\to 0}\frac{1}{t}(\mathsf{P}_t^{(n)}g^2 - (\mathsf{P}_t^{(n)}g)^2 + (\mathsf{P}_t^{(n)}g - g)^2)$.

The proof of (ii) is an application of Itô's formula. □

DEFINITION 5.2. The diffusion generator A and the covariance function C are called the local characteristics of the family $(\mathsf{P}_t^{(n)}, n \geq 1)$ or of the diffusion convolution semigroup.

When there is no pure diffusion, to give the local characteristics (A, C) in a system of local charts is equivalent to giving a drift b and C (this corresponds to the usual definition of the local characteristics of a stochastic flow) since in this case $a^{i,j}(x) = c^{i,j}(x, x)$.

REMARK 5.1. When $(\mathsf{P}_t^{(n)}, n \geq 1)$ satisfies (C), (i) and (ii) of Theorem 4.1, then $(\mathsf{P}_t^{(n),c}, n \geq 1)$ also satisfies (i) if and only if, for every x in M and all f, g in $C_K^2(M)$, $C(f, g)(x, x) = \Gamma(f, g)(x)$ [this holds since we have $C(f, g)(x, x) - \Gamma(f, g)(x) = \lim_{t \to 0} \frac{1}{t}(\mathsf{P}_t^{(2),c}(f \otimes g)(x, x) - \mathsf{P}_t^{(1)}(fg)(x))$], that is, when there is no pure diffusion. So the results of this section also apply to $(\mathsf{P}_t^{(n),c}, n \geq 1)$.

Then in this case $(\mathsf{P}_t^{(n)}, n \geq 1)$ and $(\mathsf{P}_t^{(n),c}, n \geq 1)$ have the same local characteristics.

Let $K = (K_{s,t}, s \leq t)$ be a measurable stochastic flow of kernels associated with $(\mathsf{P}_t^{(n)}, n \geq 1)$ defined on a probability space $(\Omega, \mathcal{A}, \mathsf{P})$. We here consider the modification of K, which is continuous in t (see Section 2.6).

DEFINITION 5.3. Let C be a covariance function on the space of vector fields. A two-parameter family $W = (W_{s,t}, s \leq t)$ of random variables taking their values in the space of vector fields on M is called a vector field valued white noise of covariance C if:

(i) for all $s_i \leq t_i \leq s_{i+1}$, the random variables $(W_{s_i,t_i}, 1 \leq i \leq n)$ are independent,

(ii) for all $s \leq u \leq t$, $W_{s,t} = W_{s,u} + W_{u,t}$ a.s. and

(iii) for all $s \leq t$, $\{\langle W_{s,t}, \xi\rangle, \xi \in T^*M\}$ [when $\xi = (x, u)$, $\langle W_{s,t}, \xi\rangle = \langle W_{s,t}(x), u\rangle$] is a centered Gaussian process of covariance given by

$$\mathsf{E}[\langle W_{s,t}, \xi\rangle\langle W_{s,t}, \xi'\rangle] = (t - s)C(\xi, \xi'), \tag{5.13}$$

for ξ and ξ' in T^*M.

In this section, we intend to define on $(\Omega, \mathcal{A}, \mathsf{P})$ a vector field valued white noise W of covariance C such that K solves a SDE driven by W.

In Section 6, under an additional assumption, we will prove that the linear (or Gaussian) part of the noise generated by K (in the case it is the canonical flow) is the noise generated by the vector field valued white noise W.

5.3. *The velocity field* W. For all $s \le t$, $f \in C_K^2(M)$ and $x \in M$, let

$$M_{s,t} f(x) = K_{s,t} f(x) - f(x) - \int_s^t K_{s,u}(Af)(x)\,du. \tag{5.14}$$

LEMMA 5.2. *For all $s \in \mathbb{R}$, $f \in C_K^2(M)$ and $x \in M$, $M_s^{f,x} = (M_{s,t} f(x), t \ge s)$ is a martingale with respect to the filtration $\mathcal{F}^s = (\mathcal{F}_{s,t}, t \ge s)$ and*

$$\frac{d}{dt}\langle M_s^{f,x}, M_s^{g,y}\rangle_t = K_{s,t}^{\otimes 2} C(f,g)(x,y), \tag{5.15}$$

for all f, g in $C_K^2(M)$ and all x, y in M.

PROOF. Since K is a measurable stochastic flow of kernels and since, for every positive h and every f in $C_K^2(M)$, a.s.

$$M_{s,t+h} f(x) - M_{s,t} f(x) = K_{s,t}(M_{t,t+h} f)(x), \tag{5.16}$$

$M_s^{f,x}$ is a martingale. Note that (5.16) also implies that, for every positive h, all f, g in $C_K^2(M)$ and all x, y in M,

$$\begin{aligned}\mathsf{E}\big[\big(M_{s,t+h} f(x) - M_{s,t} f(x)\big)\big(M_{s,t+h} g(y) - M_{s,t} g(y)\big) | \mathcal{F}_{s,t}\big] \\ = K_{s,t}^{\otimes 2}(\mathsf{E}[M_{t,t+h} f \otimes M_{t,t+h} g])(x,y).\end{aligned}$$

The stationarity implies that $\mathsf{E}[M_{t,t+h} f(x) M_{t,t+h} g(y)] = \mathsf{E}[M_{0,h} f(x) M_{0,h} g(y)]$. Elementary computations using the fact that $\mathsf{P}_t^{(1)} f - f = \int_0^t \mathsf{P}_s^{(1)} Af\,ds$ and $\mathsf{P}_t^{(2)}(f \otimes g) - f \otimes g = \int_0^t \mathsf{P}_s^{(2)} A^{(2)}(f \otimes g)\,ds$ give

$$\mathsf{E}[M_{0,h} f(x) M_{0,h} g(y)] = \int_0^h \mathsf{P}_s^{(2)}\big(C(f,g)\big)(x,y)\,ds. \tag{5.17}$$

Since $\mathsf{P}_t^{(2)}$ is Feller and $C(f,g)$ is continuous with compact support,

$$\mathsf{E}[M_{0,h} f(x) M_{0,h} g(y)] = hC(f,g)(x,y) + o(h), \tag{5.18}$$

uniformly in $(x,y) \in M^2$.

Therefore $\mathsf{E}[(M_{s,t+h} f(x) - M_{s,t} f(x))(M_{s,t+h} g(y) - M_{s,t} g(y)) | \mathcal{F}_{s,t}]$ is equivalent as h tends to 0 to $hK_{s,t}^{\otimes 2} C(f,g)(x,y)$. This proves the lemma. □

REMARK 5.2. In the case of Arratia's coalescing flow $(\varphi_{s,t}, s \le t)$, $C = 0$ but $\frac{d}{dt}\langle M_s^{f,x}, M_s^{g,y}\rangle_t = 1\!\!1_{\{\varphi_{s,t}(x) = \varphi_{s,t}(y)\}}$. In this case, $C_K^2(M) \otimes C_K^2(M)$ is not included in $\mathcal{D}(A^{(2)})$. This property also fails for the coalescing flow associated with Tanaka's SDE.

For all $s < t$, $n \geq 1$ and $0 \leq k \leq 2^n - 1$, let $t_k^n = s + k2^{-n}(t-s)$ and

$$W_{s,t}^n f = \sum_{k=0}^{2^n-1} M_{t_k^n, t_{k+1}^n} f, \tag{5.19}$$

where $f \in C_K^2(M)$. Note that $(M_{t_k^n, t_{k+1}^n})_{0 \leq k \leq 2^n - 1}$ are independent equidistributed random variables.

5.3.1. *Convergence in law.*

LEMMA 5.3. *For all $s < t$ and $((x_i, f_i), 1 \leq i \leq m) \in (M \times C_K^2(M))^m$, we have $\sum_{i=1}^m W_{s,t}^n f_i(x_i)$ converges in law toward $\sum_{i=1}^m W_{s,t} f_i(x_i)$ as n tends to ∞, where W is a vector field valued white noise of covariance C.*

PROOF. Using Lemma 5.2, we have, for all f, g in $C_K^2(M)$ and all x, y in M,

$$\begin{aligned} \mathsf{E}\big[M_{t_k^n, t_{k+1}^n} f(x) M_{t_k^n, t_{k+1}^n} g(y)\big] &= \int_0^{2^{-n}(t-s)} \mathsf{P}_u^{(2)} C(f,g)(x,y)\, du \\ &= 2^{-n}(t-s) C(f,g)(x,y) + o(2^{-n}), \end{aligned} \tag{5.20}$$

and this development is uniform in x and y in M.

We will only prove the proposition when $m = 1$ (the proof being the same for $m > 1$). The proposition is just an application of the central limit theorem for arrays (see [6]), which we can apply since (5.20) is satisfied provided the Lyapounov condition

$$\lim_{n \to \infty} \sum_{k=0}^{2^n-1} \mathsf{E}\big[\big|M_{t_k^n, t_{k+1}^n} f(x)\big|^{2+\delta}\big] = 0, \tag{5.21}$$

for some positive δ, is satisfied.

Using the Burkholder–Davies–Gundy inequality and Lemma 5.2,

$$\begin{aligned} \mathsf{E}\big[\big|M_{t_k^n, t_{k+1}^n} f(x)\big|^{2+\delta}\big] &\leq C\mathsf{E}\bigg[\bigg(\int_0^{2^{-n}(t-s)} K_{0,u}^{\otimes 2}\big(C(f,f)\big)(x,x)\, du\bigg)^{(2+\delta)/2}\bigg] \\ &\leq C 2^{-((2+\delta)n)/2}, \end{aligned}$$

where C is a constant (changing every line) depending only on f, $(t-s)$ and δ. This implies

$$\sum_{k=0}^{2^n-1} \mathsf{E}\big[\big|M_{t_k^n, t_{k+1}^n} f(x)\big|^{2+\delta}\big] \leq C 2^n 2^{-((2+\delta)n)/2} \leq C 2^{-n\delta/2}. \tag{5.22}$$

□

REMARK 5.3. For Arratia's coalescing flow, one can show the convergence in law as n goes to ∞ of $(W_{s,t}^n(x_1), \dots, W_{s,t}^n(x_k))$ toward $(B_{s,t}^1, \dots, B_{s,t}^k)$, where $(B^1, \dots, B^k)$ is a k-dimensional white noise.

5.3.2. *Convergence in $L^2(\mathsf{P})$.* In the preceding section, we proved the convergence in law of W^n toward a vector field valued white noise of covariance C. In this section, we prove that this convergence holds in $L^2(\mathsf{P})$.

LEMMA 5.4. *For all $s < t$, $x \in M$ and $f \in C_K^2(M)$, $W_{s,t}^n f(x)$ converges in $L^2(\mathsf{P})$.*

PROOF. For all $f \in C_K^2(M)$, $x \in M$ and $s < t$,

$$\begin{aligned} &\mathsf{E}\big[\big(W_{s,t}^n f(x) - W_{s,t}^{n+k} f(x)\big)^2\big] \\ &\qquad = \mathsf{E}\big[\big(W_{s,t}^n f(x)\big)^2\big] + \mathsf{E}\big[\big(W_{s,t}^{n+k} f(x)\big)^2\big] - 2\mathsf{E}[W_{s,t}^n f(x) W_{s,t}^{n+k} f(x)]. \end{aligned} \tag{5.23}$$

Elementary computations using (5.18) imply

$$\mathsf{E}\big[\big(W_{s,t}^n f(x)\big)^2\big] = (t-s)\, C(f,f)(x,x) + o(1), \tag{5.24}$$

$$\mathsf{E}\big[\big(W_{s,t}^{n+k} f(x)\big)^2\big] = (t-s)\, C(f,f)(x,x) + o(1), \tag{5.25}$$

as n goes to ∞ and this uniformly in $k \in \mathbb{N}$. Using the independence of the increments, the last term in (5.23) can be rewritten as

$$\mathsf{E}[W_{s,t}^n f(x) W_{s,t}^{n+k} f(x)] = \sum_{i=0}^{2^n-1} \sum_{j=i2^k}^{(i+1)2^k-1} \mathsf{E}\big[M_{t_i^n, t_{i+1}^n} f(x) M_{t_j^{n+k}, t_{j+1}^{n+k}} f(x)\big]. \tag{5.26}$$

Note that for $s \le u \le v \le t$, using first the martingale property, then (5.18) and the uniform continuity of $C(f,f)$, we have

$$\begin{aligned} \mathsf{E}[M_{s,t} f(x) M_{u,v} f(x)] &= \mathsf{E}[M_{s,v} f(x) M_{u,v} f(x)] \\ &= \mathsf{E}[(K_{s,u} \otimes I)(M_{u,v} f \otimes M_{u,v} f)(x,x)] \\ &= \mathsf{E}[(K_{s,u} \otimes I)(\mathsf{E}[M_{u,v} f \otimes M_{u,v} f])(x,x)] \\ &= (v-u) C(f,f)(x,x) + o(v-u), \end{aligned} \tag{5.27}$$

uniformly in $x \in M$. This implies

$$\mathsf{E}[W_{s,t}^n f(x) W_{s,t}^{n+k} f(x)] = (t-s) C(f,f)(x,x) + o(1) \tag{5.28}$$

as n tends to ∞ and uniformly in $k \in \mathbb{N}$. We therefore have

$$\lim_{n\to\infty} \sup_{k\in\mathbb{N}} \mathsf{E}\big[\big(W_{s,t}^n f(x) - W_{s,t}^{n+k} f(x)\big)^2\big] = 0, \tag{5.29}$$

that is, $(W_{s,t}^n f(x), n \in \mathbb{N})$ is a Cauchy sequence in $L^2(\mathsf{P})$. This proves the lemma. □

REMARK 5.4. For Arratia's coalescing flow, this lemma is not satisfied since $(W_{s,t}^n f(x), n \in \mathbb{N})$ fails to be a Cauchy sequence in $L^2(\mathsf{P})$.

Thus, for all $s < t$, we have defined the vector field valued random variable $W_{s,t}$ such that $W_{s,t}f(x)$ is the $L^2(\mathsf{P})$-limit of $W^n_{s,t}f(x)$ for all $x \in M$ and $f \in C(M)$. Then, using Lemma 5.3, it is easy to see that $W = (W_{s,t}, s \le t)$ is a vector field valued white noise of covariance C.

5.4. *The stochastic flow of kernels solves a SDE.* In [23], it is shown that a vector field valued white noise W of covariance C can be constructed with a sequence of independent real white noises $(W^\alpha)_\alpha$ by the formula $W = \sum_\alpha V_\alpha W^\alpha$, where $(V^\alpha)_\alpha$ is an orthonormal basis of H_C, the self-reproducing space associated with C.

For every predictable [with respect to the filtration $(\mathcal{F}_{-\infty,t}, t \in \mathbb{R})$] process $(H_t)_{t\in\mathbb{R}}$ taking its values in the dual of H_C, we define the stochastic integral of H with respect to W by the formula

$$\int_s^t H_u\big(W(du)\big) = \sum_\alpha \int_s^t \langle H_u, V_\alpha\rangle W^\alpha(du), \tag{5.30}$$

for $s < t$. Note that the above definition is independent of the choice of the orthonormal basis $(V^\alpha)_\alpha$.

In particular, this applies to $H_u(V) = K_{s,u}(Vf)(x)1\!\!1_{s\le u<t}$ for $f \in C_K(M)$ and $x \in M$. Then the stochastic integral $\sum_\alpha \int_s^t K_{s,u}(V^\alpha f)W^\alpha(du)$ is denoted

$$\int_s^t K_{s,u}\big(Wf(du)\big)(x). \tag{5.31}$$

REMARK 5.5. The stochastic integral (5.31) is equal to the limit in $L^2(\mathsf{P})$ of

$$\sum_{k=0}^{2^n-1} K_{s,t_k^n}\big(W_{t_k^n t_{k+1}^n} f\big)(x)$$

as n tends to ∞, where $t_k^n = s + k2^{-n}(t-s)$. Indeed,

$$\begin{aligned}\mathsf{E}\left[\left(\int_s^t K_{s,u}\big(Wf(du)\big)(x) - \sum_{k=0}^{2^n-1} K_{s,t_k^n}\big(W_{t_k^n t_{k+1}^n} f\big)(x)\right)^2\right]&\\ = \sum_{k=0}^{2^n-1}\int_{t_k^n}^{t_{k+1}^n} \mathsf{P}^{(2)}_{t_k^n - s}\big(I + \mathsf{P}^{(2)}_{u-t_k^n} - 2I\otimes \mathsf{P}^{(1)}_{u-t_k^n}\big)C(f,f)(x,x)\,du,&\end{aligned}$$

which tends to 0 as n tends to ∞.

PROPOSITION 5.2. *W is the unique vector field valued white noise such that, for all $s<t$, $x\in M$ and $f \in C_K^2(M)$, P-a.s.,*

$$K_{s,t}f(x) = f(x) + \int_s^t K_{s,u}\big(Wf(du)\big)(x) + \int_s^t K_{s,u}(Af)(x)\,du. \tag{5.32}$$

Note that giving the local characteristics of the flow is equivalent to giving this SDE. This SDE will be called the (A, C)-SDE.

PROOF. For all $s < t$, from Remark 5.5,

$$\int_s^t K_{s,u}\big(Wf(du)\big)(x) = \lim_{n\to\infty} \sum_{k=0}^{2^n-1} K_{s,t_k^n}\big(W_{t_k^n,t_{k+1}^n} f\big)(x) \tag{5.33}$$

in $L^2(\mathsf{P})$, where $t_k^n = s + k2^{-n}(t-s)$.

For all integers i, l, k and n such that $l \geq n$ and $k2^{l-n} \leq i \leq (k+1)2^{l-n} - 1$, the development (5.27) implies

$$\mathsf{E}\big[M_{t_i^l,t_{i+1}^l} f(x) M_{t_k^n,t_{k+1}^n} f(x)\big] = 2^{-l}(t-s)C(f,f)(x,x) + o(2^{-l}), \tag{5.34}$$

uniformly in $x \in M$. This implies that, for $l \geq n$,

$$\mathsf{E}\left[\left(\sum_{i=k2^{l-n}}^{(k+1)2^{l-n}-1} M_{t_i^l,t_{i+1}^l} f(x) - M_{t_k^n,t_{k+1}^n} f(x)\right)^2\right] = o(2^{-n}), \tag{5.35}$$

uniformly in $x \in M$. Taking the limit as l goes to ∞, we get

$$\mathsf{E}\big[\big(W_{t_k^n,t_{k+1}^n} f(x) - M_{t_k^n,t_{k+1}^n} f(x)\big)^2\big] = o(2^{-n}), \tag{5.36}$$

uniformly in $x \in M$. We use this estimate to prove that

$$\int_s^t K_{s,u}\big(W(du)f\big)(x) = \lim_{n\to\infty} \sum_{k=0}^{2^n-1} K_{s,t_k^n}\big(M_{t_k^n,t_{k+1}^n} f\big)(x) \tag{5.37}$$

in $L^2(\mathsf{P})$. This holds since

$$\begin{aligned}
&\mathsf{E}\left[\left(\sum_{k=0}^{2^n-1} K_{s,t_k^n}\big(W_{t_k^n,t_{k+1}^n} f\big) - \sum_{k=0}^{2^n-1} K_{s,t_k^n}\big(M_{t_k^n,t_{k+1}^n} f\big)\right)^2(x)\right] \\
&\quad = \sum_{k=0}^{2^n-1} \mathsf{E}\big[\big(K_{s,t_k^n}\big(W_{t_k^n,t_{k+1}^n} f - M_{t_k^n,t_{k+1}^n} f\big)\big)^2(x)\big] \\
&\quad \leq \sum_{k=0}^{2^n-1} \mathsf{P}^{(1)}_{t_k^n - s}\big(\mathsf{E}\big[\big(W_{t_k^n,t_{k+1}^n} f - M_{t_k^n,t_{k+1}^n} f\big)^2\big]\big)(x) \\
&\quad \leq 2^n o(2^{-n}) = o(1).
\end{aligned}$$

Note now that

$$\begin{aligned}
\sum_{k=0}^{2^n-1} K_{s,t_k^n}\big(M_{t_k^n,t_{k+1}^n} f\big)(x) &= \sum_{k=0}^{2^n-1} K_{s,t_k^n}\left(K_{t_k^n,t_{k+1}^n} f - f - \int_{t_k^n}^{t_{k+1}^n} K_{t_k^n,u}(Af)\,du\right)(x) \\
&= K_{s,t} f(x) - f(x) - \int_s^t K_{s,u}(Af)(x)\,du.
\end{aligned}$$

This proves that K solves the (A, C)-SDE driven by W. Finally, note that if K solves the (A, C)-SDE driven by a vector field valued white noise W', then we must have $W' = W$. $\square$

Let $X = (X_t, t \geq 0)$ be the Markov process defined in Section 2.6 on $(\Omega \times C(\mathbb{R}^+, M), \mathcal{A} \otimes \mathcal{B}(C(\mathbb{R}^+, M)), \mathsf{P}(d\omega) \otimes \mathsf{P}_{x,\omega}(d\omega'))$ by $X(\omega, \omega') = \omega'$.

PROPOSITION 5.3. *Assume there is no pure diffusion* [*i.e., for all* $f \in C_K^2(M)$ *and* $x \in M$, $\Gamma(f)(x) = C(f, f)(x, x)$]. *Then, for all* $t \geq 0$, $x \in M$ *and* $f \in C_K^2(M)$, $\mathsf{P}(d\omega) \otimes \mathsf{P}_{x,\omega}(d\omega')$*-a.s.*,

$$f(X_t) = f(x) + \int_0^t W(du) f(X_u) + \int_0^t Af(X_u)\, du, \tag{5.38}$$

that is, X *is a weak solution of this SDE* (*in the sense given in* [38]).

PROOF. As in the proof of (5.37) in Proposition 5.2, we show that

$$\int_0^t W(du) f(X_u) = \lim_{n\to\infty} \sum_{k=0}^{2^n-1} M_{t_k^n, t_{k+1}^n} f\big(X_{t_k^n}\big) \tag{5.39}$$

in $L^2(\mathsf{P}_x)$, with $\mathsf{P}_x = \mathsf{P}(d\omega) \otimes \mathsf{P}_{x,\omega}(d\omega'))$. Let

$$M_t^f = f(X_t) - f(x) - \int_0^t Af(X_u)\, du;$$

then $(M_t^f, t \geq 0)$ is a martingale relative to the filtration $(\mathcal{F}_t^X, t \geq 0)$ generated by the Markov process X. We now prove that $\mathsf{E}_x[(M_t^f - \int_0^t W(du) f(X_u))^2] = 0$, where E_x denotes the expectation with respect to P_x. It is easy to see that, since there is no pure diffusion,

$$\mathsf{E}_x[(M_t^f)^2] = \mathsf{E}_x\left[\left(\int_0^t W(du) f(X_u)\right)^2\right] = \mathsf{E}_x\left[\int_0^t C(f, f)(X_u, X_u)\, du\right]. \tag{5.40}$$

Equation (5.39) and the martingale property of M_t^f imply that

$$\begin{aligned} &\mathsf{E}_x\left[M_t^f \int_0^t W(du) f(X_u)\right] \\ &\quad = \lim_{n\to\infty} \sum_{k=0}^{2^n-1} \mathsf{E}_x\big[M_{t_{k+1}^n}^f \times M_{t_k^n, t_{k+1}^n} f\big(X_{t_k^n}\big)\big]. \\ &\quad = \lim_{n\to\infty} \sum_{k=0}^{2^n-1} \mathsf{E}_x\big[\big(M_{t_{k+1}^n}^f - M_{t_k^n}^f\big) \times M_{t_k^n, t_{k+1}^n} f\big(X_{t_k^n}\big)\big]. \end{aligned} \tag{5.41}$$

Since, for all $0 \le s < t$, $\mathsf{E}_x[M_t^f - M_s^f | \mathcal{A} \vee \mathcal{F}_s^X] = M_{s,t} f(X_s)$, we get

$$\begin{aligned}\mathsf{E}_x\left[M_t^f \int_0^t W(du) f(X_u)\right] &= \lim_{n\to\infty} \sum_{k=0}^{2^n-1} \mathsf{E}_x\big[\big(M_{t_k^n, t_{k+1}^n} f\big(X_{t_k^n}\big)\big)^2\big] \\ &= \mathsf{E}_x\left[\left(\int_0^t W(du) f(X_u)\right)^2\right].\end{aligned} \tag{5.42}$$

Therefore $\mathsf{E}_x[(M_t^f - \int_0^t W(du) f(X_u))^2] = 0$. □

5.5. *The* (A, C)*-SDE.* In this section and the following, we let A be a second-order differential operator mapping $C_K^2(M)$ in $C_K(M)$ and let C be a continuous covariance on vector fields.

DEFINITION 5.4. Let K be a stochastic flow of kernels and let W be a vector field valued white noise, defined on a probability space $(\Omega, \mathcal{A}, \mathsf{P})$.

(i) (K, W) is a solution of the (A, C)-SDE if the covariance of W is C and (K, W) satisfies (5.32) for all $s < t$, $x \in M$ and $f \in C_K^2(M)$.

(ii) (K, W) is called a Wiener solution of the (A, C)-SDE if moreover, for all $s \le t$, $K_{s,t}$ is $\mathcal{F}_{s,t}^W$-measurable, where $\mathcal{F}_{s,t}^W$ is the completion by all P-negligible sets of $\mathcal{A}$ of the σ-field $\sigma(W_{u,v}, s \le u \le v \le t)$.

(iii) When a solution (K, W) of the (A, C)-SDE is not a Wiener solution, we say it is a weak solution.

REMARK 5.6. The Wiener solution is the usual strong Itô solution of the SDE when the solution is a flow of mappings, which is the case for the SDE (1.34), or when C satisfies condition (8.2) in [23] and when there is no pure diffusion.

REMARK 5.7. Let (K, W) be a solution of the (A, C)-SDE and let ν be the Feller convolution semigroup associated with K. Then ν is a diffusion convolution semigroup with local characteristics (A, C).

The proof of this remark is left to the reader.

REMARK 5.8. The fact that (K, W) is a Wiener (resp. a weak) solution of the (A, C)-SDE only depends on the law of K. So that we can say shortly that K is a Wiener (resp. a weak) solution of the (A, C)-SDE.

DEFINITION 5.5. We will say that $(\mathsf{P}_t^{(n)}, n \ge 1)$, a compatible family of Feller semigroup, or $\nu = (\nu_t)$, a Feller convolution semigroup, defines a Wiener (resp. a weak) solution of the (A, C)-SDE if P_ν is the law of a stochastic flow of kernels, which is a Wiener (resp. a weak) solution of the (A, C)-SDE.

Under some additional assumptions, we will give in Section 6 a representation of all solutions of the (A, C)-SDE.

DEFINITION 5.6. We say that (Wiener) uniqueness holds for the (A, C)-SDE when there is only one diffusion convolution semigroup with local characteristics (A, C) defining a (Wiener) solution.

5.6. *Wiener solution and filtering.* Let us now consider the canonical flow associated with ν, a diffusion convolution semigroup, with local characteristics (A, C). Let $N_\nu^W := (\Omega^0, \mathcal{A}^0, (\mathcal{F}_{s,t}^W)_{s\le t}, \mathsf{P}_\nu, (T_h)_{h\in\mathbb{R}})$ be the noise generated by the vector field valued white noise W. Note that N_ν^W is a linear or Gaussian subnoise of N_ν, the noise generated by the canonical flow. [The noise $(\mathcal{G}_{s,t})_{s\le t}$ is Gaussian if and only if there exists a countable family of independent real white noises $\{W^\alpha\}$ such that, up to negligible sets, $\mathcal{G}_{s,t}$ is generated by the random variables $W_{u,v}^\alpha$ for all $s \le u \le v \le t$ and every α.]

Let $\bar{K} = (\bar{K}_{s,t}, s \le t)$ be the stochastic flow of kernels obtained by filtering the canonical flow with respect to the subnoise N^W (see Section 3.2). It is easy to see that $\bar{K}$ also solves the (A, C)-SDE (see the proof of Lemma 3.9 in [23]) and has the same local characteristics as the canonical flow. Since, for all $s \le t$, $\bar{K}_{s,t}$ is $\mathcal{F}_{s,t}^W$-measurable, $(\bar{K}, W)$ is a Wiener solution of the (A, C)-SDE. Let ν^s denote the associated diffusion convolution semigroup.

For any $f \in C_0(M)$ and $x \in M$, $\bar{K}_{s,t} f(x)$ can be expanded into a sum of Wiener chaos elements, that is, iterated Wiener integrals of the form

$$\sum_{\alpha_1,\dots,\alpha_n} \int C^{\alpha_1,\dots,\alpha_n}(s_1, \dots, s_n)\, dW_{s_n}^{\alpha_n} \cdots dW_{s_1}^{\alpha_1}. \tag{5.43}$$

Since W was constructed from the flow, it is clear that the functions $C^{\alpha_1,\dots,\alpha_n}$ are determined by the law of the flow. (We will give, under some additional assumptions, an explicit form of them in the following section.)

5.7. *The Krylov–Veretennikov expansion.* We still assume we are given $\nu = (\nu_t)_{t\ge 0}$ a diffusion convolution semigroup, in the sense of Section 5.1, associated with a set of local characteristics (A, C).

We suppose in this section the existence of a Radon measure m on M such that A is symmetric with respect to m.

Moreover, we assume that $\mathrm{Im}(I - A)$ is dense in $C_0(M)$ (it implies that $\mathsf{P}_t^{(1)}$ is symmetric with respect to m and is the unique Feller semigroup whose generator extends A).

Following [23], starting from the vector field valued white noise W, one can define $(S_{s,t}, s \le t)$ a stochastic flow of Markovian operators [acting on $L^2(m)$] such that, for all $s \le t$, $S_{s,t}$ is $\sigma(W)$-measurable and, for $f \in L^2(m)$ and $s \le u \le t$,

$$S_{s,t} f = S_{s,u} S_{u,t} f,$$

$$S_{s,t} f = \mathsf{P}_{t-s}^{(1)} f + \int_s^t S_{s,u} W(du) \mathsf{P}_{t-u}^{(1)} f,$$

where both equalities hold in $L^2(m\otimes \mathsf{P})$. These operators are given by the Wiener chaos expansion (called the Krylov–Veretennikov expansion)

$$S_{s,t}f = \mathsf{P}^{(1)}_{t-s}f + \sum_{n\geq 1} J^n_{s,t}f, \tag{5.44}$$

with

$$J^n_{s,t}f = \int_{s\leq s_1\leq\cdots\leq s_n\leq t} \mathsf{P}^{(1)}_{s_1-s}W(ds_1)\mathsf{P}^{(1)}_{s_2-s_1}\cdots\mathsf{P}^{(1)}_{s_n-s_{n-1}}W(ds_n)\mathsf{P}^{(1)}_{t-s_n}f. \tag{5.45}$$

They can be characterized (Theorem 3-2 in [23]) as the unique flow of random operators on $L^2(m)$, $\sigma(W)$-measurable, such that $\mathsf{E}[(S_{s,t}f)^2]\leq \mathsf{P}^{(1)}_{t-s}f^2$ and

$$S_{s,t}f - f = \int_s^t S_{s,u}W(du)f + \frac{1}{2}\int_s^t S_{s,u}\bar{A}f\,du \qquad \text{in } L^2(m\otimes\mathsf{P}) \tag{5.46}$$

for every f in the domain of the L^2-generator $\bar{A}$, denoted $\mathcal{D}(\bar{A})$. It implies the following.

PROPOSITION 5.4. (i) *If ν defines a Wiener solution (K,W) of the (A,C)-SDE, then for all $s\leq t$, $m\otimes\mathsf{P}$-a.e., for every $f\in C_K(M)$,*

$$K_{s,t}f = S_{s,t}f. \tag{5.47}$$

(ii) *Wiener uniqueness holds.*

PROOF. (i) It is clear that K induces a flow of Markovian operators on $L^2(m)$ which verifies (5.46) for $f\in C^2_K(m)$. Then (5.46) extends to functions in the domain of the Feller generator and finally to $\mathcal{D}(\bar{A})$.

(ii) From (i), it is clear that $m^{\otimes n}$-a.e., $\mathsf{P}^{(n)}_t = \mathsf{E}[S^{\otimes n}_{0,t}]$. Since it is a Feller semigroup, it is uniquely determined. □

6. Noise and classification.

6.1. *Assumptions.* In this section, as before, M denotes a smooth locally compact manifold. We fix a pair of local characteristics (A,C) on M. A is a second-order differential operator mapping $C^2_K(M)$ in $C_K(M)$ and C is a continuous covariance on vector fields. The associated differential operators $A^{(n)}$ on $C^2_K(M)^{\otimes n}$ are defined by (5.11).

Let $\mathcal{M}(n,x)$ be the following martingale problem associated with $A^{(n)}$ and $x\in M^n$: There exists a probability space on which is constructed an M^n-valued stochastic process $X^{(n)}=(X^{(n)}_t, t\geq 0)$ such that

$$f\big(X^{(n)}_t\big) - f(x) - \int_0^t A^{(n)}f\big(X^{(n)}_s\big)\,ds \tag{6.1}$$

is a martingale for every test function f in $C^2_K(M)\otimes\cdots\otimes C^2_K(M)$.

We suppose that the local characteristics (A,C) verify the following assumption.

(U) For every $n \geq 1$, the martingale problem $\mathcal{M}(n, x)$ has a unique solution in law on the set of continuous trajectories stopped at Δ_n.

REMARK 6.1. Condition (U) is satisfied when the coefficients of the local characteristics are C^2 outside of Δ_n (see Theorem 12.12 and Section V.19 in [38]) or when $A^{(n)}$ is elliptic outside of Δ_n (see Section V.24 in [38]).

Our purpose is to classify Feller convolution semigroups associated with these local characteristics. We will treat two cases:

(A) The noncoalescing case where the solution of the martingale problem $\mathcal{M}(2, x)$ does not hit the diagonal when $x = (x_1, x_2)$ with $x_1 \neq x_2$.
(B) The coalescing case where there is no pure diffusion [i.e., $(\frac{1}{2}Af^2 - fAf)(x) = C(f, f)(x, x)$ for all $f \in C_K^2(M)$ and $x \in M$], and where assumption (C) of Theorem 4.1 holds for $X_t^{(2)} = (X_t, Y_t)$ a solution of $\mathcal{M}(2, x)$.

When the local characteristics are noncoalescing [case (A)], these local characteristics are associated with at most a unique convolution semigroup and a unique canonical flow (which is not always a flow of maps). From Section 5.5, we know the latter has to be a Wiener solution of the SDE (otherwise uniqueness would be violated). Assumption (F) (see Section 1.7) is a sufficient (but not necessary) condition for existence. The family of semigroups given in the example of Lipschitz SDEs (see Section 1.7) satisfies these assumptions.

In Sections 6.2–6.4, we assume (B) is satisfied.

6.2. *The coalescing case*: *classification.* Following [15], $\mathcal{M}(n, x)$ has a unique solution in law on the set of coalescing trajectories; that is, $X^{(n)}(\omega) \in C^{(n)}$, where $C^{(n)}$ is the set of continuous functions $f : \mathbb{R}^+ \to M^n$ such that if $f_i(s) = f_j(s)$ for $1 \leq i, j \leq n$ and $s \geq 0$, then for all $t \geq s$, $f_i(t) = f_j(t)$. (In [15], this martingale problem is solved when $M = \mathbb{R}$, but the proof can obviously be adapted to our framework.) Since assumption (C) holds, Remark 4.5 implies that the associated semigroups are Feller.

Hence all coalescing flows with these local characteristics have the same law P_{ν^c}. They induce the same family of semigroups $(\mathsf{P}_t^{(n),c}, n \geq 1)$ and the same convolution semigroup ν^c. This convolution semigroup is a diffusion convolution semigroup with local characteristics (A, C) since, for all f and g in $C_K^2(M)$ and all x, y in M,

$$f(X_t)g(Y_t) - f(x)g(y) - \int_0^t A^{(2)}(f \otimes g)(X_s, Y_s)\, ds$$

is a martingale, where (X_t, Y_t) denotes the two-point motion of ν^c started at (x, y).

Let N_{ν^c} be the noise generated by the canonical coalescing flow asociated with the local characteristics (A, C).

Let W be the vector field valued white noise defined on $(\Omega^0, \mathcal{A}^0, \mathsf{P}_{\nu^c})$ in Section 5 and let $N_{\nu^c}^W$ be the subnoise of N_{ν^c} generated by W. Then $N_{\nu^c}^W$ is a Gaussian subnoise of N and it is possible to represent it by a countable family of independent real white noises $\{W^\alpha\}$ such that $W = \sum_\alpha V_\alpha W^\alpha$, where $\{V_\alpha\}$ is a countable family of vector fields on M.

We denote by ν^s the diffusion convolution semigroup associated with the flow obtained by filtering the canonical coalescing flow of law P_{ν^c} with respect to $N_{\nu^c}^W$.

The following theorem gives a representation of all flows with the same local characteristics. They lie "between" the Wiener solution and the coalescing solution of the SDE, which are distinct when the coalescing solution is not a Wiener solution of the SDE.

THEOREM 6.1. *Suppose we are given a set of local characteristics* (A, C) *and that assumption* (B) *is verified. Then*:

(a) ν^c *is the unique diffusion convolution semigroup associated with* (A, C) *and defining a flow of maps* (*which is coalescing*).

(b) ν^s *is the unique diffusion convolution semigroup associated with* (A, C) *and defining a Wiener solution of the* (A, C)*-SDE.*

(c) *The diffusion convolution semigroups associated with* (A, C) *are all the Feller convolution semigroups weakly dominated by* ν^c *and dominating* ν^s.

Note that ν^c and ν^s are not necessarily distinct.

PROOF OF THEOREM 6.1. We have already proved (a) at the begining of this section. Theorem 4.2 implies that every diffusion convolution semigroup $\bar{\nu}$ with local characteristics (A, C) is weakly dominated by ν^c so that a stochastic flow $\bar{K}$ of law $\mathsf{P}_{\bar{\nu}}$ can be obtained by filtering on an extension (N, φ) of N_{ν^c} the coalescing flow φ with respect to a subnoise $\bar{N}$ of N.

Let $\bar{W}$ be the velocity field associated with $\bar{K}$. Proposition 5.2 shows that $(\bar{K}, \bar{W})$ solves the (A, C)-SDE. Notice that $\bar{W}$ can be obtained by filtering W with respect to $\bar{N}$. Indeed, Section 5.3 shows that $\bar{W}_{s,t}^n$ (defined from $\bar{K}$) converges (in L^2) toward $\bar{W}_{s,t}$ and we have that, for all $s \le t$, $f \in C_K^2(M)$ and $x \in M$, $\bar{W}_{s,t}^n f(x) = \mathsf{E}[W_{s,t}^n f(x) | \bar{\mathcal{F}}_{s,t}]$ a.s. and therefore that $\bar{W}_{s,t} f(x) = \mathsf{E}[W_{s,t} f(x) | \bar{\mathcal{F}}_{s,t}]$ a.s. Since $\bar{W}$ and W have the same law, we must have $W_{s,t} = \bar{W}_{s,t}$ a.s. This proves that $\bar{\nu}$ dominates ν^s.

Let us now suppose that $(\bar{K}, \bar{W})$ is a Wiener solution of the (A, C)-SDE. Then, since $\bar{W} = W$, we must have $N_{\nu^c}^W = \bar{N}$ (since $\bar{K}_{s,t}$ is $\mathcal{F}_{s,t}^W$-measurable) and thus $\nu^s = \bar{\nu}$. This proves the Wiener uniqueness for the (A, C)-SDE.

Finally let $\bar{\nu}$ be a Feller convolution semigroup weakly dominated by ν^c and dominating ν^s. The fact that $\bar{\nu} \overset{w}{\preceq} \nu^c$ implies that a stochastic flow $\bar{K}$ of law $\mathsf{P}_{\bar{\nu}}$ can be obtained by filtering on an extension (N, φ) of N_{ν^c} the coalescing flow φ with respect to a subnoise $\bar{N}$ of N. Then Section 5.3 shows that $\bar{W}_{s,t}^n$ (defined from $\bar{K}$) converges (in L^2) toward $\bar{W}_{s,t} = \mathsf{E}[W_{s,t}|\bar{\mathcal{F}}_{s,t}]$. Now, since $\bar{\nu} \succeq \nu^s$, there exists (see Lemma 3.3) a subnoise $\bar{\bar{N}}$ of $\bar{N}$ such that the flow obtained by filtering $\bar{K}$ or equivalently, the coalescing flow, with respect to $\bar{\bar{N}}$ has law P_{ν^s}. The associated white noise $\bar{\bar{W}}$ verifies, for all $s \le t$, $x \in M$ and $f \in C_K^2(M)$,

$$\bar{\bar{W}}_{s,t} f(x) = \mathsf{E}[\bar{W}_{s,t} f(x)|\bar{\bar{\mathcal{F}}}_{s,t}] = \mathsf{E}[W_{s,t} f(x)|\bar{\bar{\mathcal{F}}}_{s,t}]. \tag{6.2}$$

Since $\bar{\bar{W}}$ has covariance C, it has to coincide with W and $\bar{W} = W$.

Thus, $(\bar{K}, \bar{W})$ solves the (A, C)-SDE so that $\bar{\nu}$ is a diffusion convolution semigroup whose local characteristics are (A, C). □

6.3. *The coalescing case*: *martingale representation.* On the probability space $(\Omega^0, \mathcal{A}^0, \mathsf{P}_{\nu^c})$, let $\mathcal{F}^{\nu^c}$ be the filtration $(\mathcal{F}_{0,t}^{\nu^c})_{t\ge0}$ and let $\mathcal{M}(\mathcal{F}^{\nu^c})$ be the space of locally square integrable $\mathcal{F}^{\nu^c}$-martingales.

PROPOSITION 6.1. *For every $\mathcal{F}^{\nu^c}$-martingale $M = (M_t)_{t\in\mathbb{R}^+}$, there exist predictable processes $\Phi^\alpha = (\Phi_s^\alpha)_{s\ge0}$ such that*

$$M_t = \sum_\alpha \int_0^t \Phi_s^\alpha W^\alpha(ds). \tag{6.3}$$

REMARK 6.2. Of course, this does not imply that $\mathcal{F}^{\nu^c}$ is generated by W.

PROOF OF PROPOSITION 6.1. We follow an argument by Dellacherie (see [38], (V-25)). Suppose there exists $F \in L^2(\mathcal{F}_{0,\infty}^{\nu^c})$ orthogonal in $L^2(\mathcal{F}_{0,\infty}^{\nu^c})$ to all stochastic integrals of $(W^\alpha)_\alpha$ of the form (6.3). Then $M_t = \mathsf{E}[F|\mathcal{F}_{0,t}^{\nu^c}]$ is orthogonal to W^α for every α; that is, $\langle M, W_{0,\cdot}^\alpha\rangle_t = 0$.

Let $\tau = \inf\{t, |M_t| = 1/2\}$ and $\hat{\mathsf{P}}_{\nu^c} = (1 + M_\tau)\cdot \mathsf{P}_{\nu^c}$. Since M is a uniformly integrable martingale and τ is a stopping time (with $1 + M_\tau \ge 1/2$), $\hat{\mathsf{P}}_{\nu^c}$ is a probability measure on $(\Omega^0, \mathcal{A}^0)$. Since $\langle M, W_{0,\cdot}^\alpha\rangle_t = 0$, we get that under $\hat{\mathsf{P}}_{\nu^c}$, $(W_{0,t}^\alpha)_\alpha$ is a family of independent Brownian motions.

We are now going to prove that since (U) is satisfied, we must have $\mathsf{P}_{\nu^c} = \hat{\mathsf{P}}_{\nu^c}$, which implies $M_t = 0$ and a contradiction.

Let $F = \prod_{i=1}^n f_i(\varphi_{0,t_i}(x_i))$, for $f_1, \dots, f_n$ in $C_K^2(M)$, $t_1, \dots, t_n$ in $\mathbb{R}^+$ and $x_1, \dots, x_n$ in M. We know that under P_{ν^c}, for all $1 \le i \le n$, $(\varphi_{0,t}(x_i), t \ge 0)$ is a solution of the SDE

$$dg_i(\varphi_{0,t}(x_i)) = \sum_\alpha V_\alpha g_i(\varphi_{0,t}(x_i)) W^\alpha(dt) + Af(\varphi_{0,t}(x_i))\, dt, \tag{6.4}$$

for all $g_1, \dots, g_n$ in $C_K^2(M)$. Note that under $\hat{\mathsf{P}}_{\nu^c}$, these SDEs are also satisfied. Since under $\hat{\mathsf{P}}_{\nu^c}$, $(W^\alpha)_\alpha$ is a family of independent Brownian motions, $((\varphi_{0,t}(x_i), t \ge 0), 1 \le i \le n)$ is a coalescing solution of the martingale problem associated with $A^{(n)}$ and (U) implies that the law of $((\varphi_{0,t}(x_i), t \geq 0), 1 \leq i \leq n)$ is the same under P_{ν^c} and under $\hat{\mathsf{P}}_{\nu^c}$. Therefore $\hat{\mathsf{E}}[F] = \mathsf{E}[F]$, where $\hat{\mathsf{E}}$ denotes the expectation with respect to $\hat{\mathsf{P}}_{\nu^c}$.

To conclude that $\hat{\mathsf{P}}_{\nu^c} = \mathsf{P}_{\nu^c}$, we need to prove $\hat{\mathsf{E}}[F] = \mathsf{E}[F]$ with $F = \prod_{i=1}^n f_i(\varphi_{s_i,t_i}(x_i))$ for all $f_1, \dots, f_n$ in $C_K^2(M)$, $0 \leq s_i < t_i$ in $\mathbb{R}^+$ and $x_1, \dots, x_n$ in M. This can be proved the same way but using the kernel $\tilde{K}_t$ introduced in Section 2.6. In this case $\tilde{K}_t = \delta_{\tilde{\varphi}_t}$, where $\tilde{\varphi}_t : \mathbb{R}^+ \times M \to \mathbb{R}^+ \times M$ is measurable. Then $F = \prod_{i=1}^n \tilde{f}_i(\tilde{\varphi}_{t_i}(s_i, x_i))$ and $(\tilde{\varphi}_t(s_i, x_i), t \geq 0)$ is a solution of an SDE on $\mathbb{R}^+ \times M$. $\square$

6.4. *The coalescing case*: *the linear noise.* Let us remark that if ν is a diffusion convolution semigroup, then N_ν is a predictable noise (see Proposition 3.2); that is, $\mathcal{M}(\mathcal{F}^\nu)$ is formed of continuous martingales (in particular, a Gaussian noise is predictable). Following [41], a linear representation of a predictable noise $N = (\Omega, \mathcal{A}, (\mathcal{F}_{s,t})_{s \leq t}, \mathsf{P}, (T_h)_{h \in \mathbb{R}})$ is a family of real random variables $X = (X_{s,t}, s \leq t)$ such that:

(a) $X_{s,t} \circ T_h = X_{s+h,t+h}$ for all $s \leq t$ and $h \in \mathbb{R}$,
(b) $X_{s,t}$ is $\mathcal{F}_{s,t}$-measurable for all $s \leq t$,
(c) $X_{r,s} + X_{s,t} = X_{r,t}$ a.s., for all $r \leq s \leq t$.

The space of linear representations is a vector space. Equipped with the norm $\|X\| = (\mathsf{E}[|X_{0,1}|^2])^{1/2}$, it is a Hilbert space we denote by H_{lin}. Let H_{lin}^0 be the orthogonal in H_{lin} of the one-dimensional vector space consisting of the representation $X_{s,t} = v(t-s)$ for $v \in \mathbb{R}$; then H_{lin}^0 is constituted with the centered linear representations. Note that if $X \in H_{\text{lin}}^0$ with $\|X\| = 1$, then $(X_{0,t})_{t \geq 0}$ is a standard Brownian motion. The Hilbert space H_{lin}^0 is a Gaussian system and every $X \in H_{\text{lin}}^0$ is a real white noise.

Note that if X and Y are orthogonal linear representations, then X and Y are independent.

For all $-\infty \leq s \leq t \leq \infty$, let $\mathcal{F}_{s,t}^{\text{lin}}$ be the σ-field generated by the random variables $X_{u,v}$ for all $X \in H_{\text{lin}}^0$ and $s \leq u \leq v \leq t$, and completed by all P-negligible sets of $\mathcal{F}_{-\infty,+\infty}$. Then $N_{\text{lin}} := (\Omega, \mathcal{A}, (\mathcal{F}_{s,t}^{\text{lin}})_{s \leq t}, \mathsf{P}, (T_h)_{h \in \mathbb{R}})$ is a noise. It is called the linearizable part of the noise N. The noise N_{lin} is a maximal Gaussian subnoise of N, hence N is Gaussian if and only if $N_{\text{lin}} = N$. When N_{lin} is trivial (i.e., consisting of trivial σ-fields), one says that N is a black noise (when N is not trivial).

THEOREM 6.2. $N_{\nu^c}^W = N_{\nu^c}^{\text{lin}}$.

PROOF. Let H^W be the space of centered linear representations of the noise $N_{\nu^c}^W$. Then H^W is an Hilbert space [an orthonormal basis of H^W is given by

$\{(W^{\alpha}_{s,t})_{s\le t}\}]$ and we have $H^W \subset H^0_{\text{lin}}$. This implies that $N^W_{\nu^c}$ is a Gaussian subnoise of $N^{\text{lin}}_{\nu^c}$.

If $N^W_{\nu^c} \neq N^{\text{lin}}_{\nu^c}$, then there exists a linear representation $X \neq 0 \in H^0_{\text{lin}}$ orthogonal to H^W and therefore independent of $\{W^{\alpha}\}$. Since $(X_{0,t})_{t\ge 0} \in \mathcal{M}(\mathcal{F})$, Proposition 6.1 implies that the martingale bracket of $X_{0,t}$ equals 0. This is a contradiction. □

In Section 7, we give an example of a stochastic coalescing flow whose noise is predictable but not Gaussian. It is an example of nonuniqueness of the diffusion convolution semigroup associated with a set of local characteristics.

REMARK 6.3. In Section 4.4.3, although the covariance function C is not continuous, it is still possible to construct a white noise W from the coalescing flow $(\varphi_{s,t}, s \le t)$. For all $s < t$, we set $W_{s,t} = \int_s^t \operatorname{sgn}(\varphi_{s,u}(0))\, d\varphi_{s,u}(0)$. Then we have $W_{s,t} = \int_s^t \operatorname{sgn}(\varphi_{s,u}(x))\, d\varphi_{s,u}(x)$ for every $x \in \mathbb{R}$. Therefore one can check that $W = (W_{s,t}, s \le t)$ is a real white noise.

The coalescing flow $(\varphi_{s,t}, s \le t)$ solves the SDE

$$\varphi_{s,t}(x) = \int_s^t \operatorname{sgn}(\varphi_{s,u}(x))\, dW_u \qquad \text{for } s < t \text{ and } x \in \mathbb{R}. \tag{6.5}$$

The results of this section apply since Proposition 6.1 is also satisfied if we only assume the uniqueness in law of the coalescing solutions [i.e. such that if $(X^1, \dots, X^n)$ solves the SDE, then if, for $i \neq j$ and $s \ge 0$, $X^i_s = X^j_s$, then $X^i_t = X^j_t$ for all $t \ge s$] of the SDE satisfied by the n-point motion [i.e., the SDE (6.4)], which here is almost obvious. Therefore, the linear part of the noise generated by the coalescing flow is given by the noise generated by W. But since the Wiener solution of the SDE (6.5) is not a flow of mappings, the coalescing flow is not a strong solution. Therefore, we recover the result of [44] and [45] that the noise of this stochastic coalescing flow is predictable but not Gaussian.

The Wiener solution given in Section 4.4.3 can be recovered by filtering the coalescing solution with respect to the noise generated by W.

7. Isotropic Brownian flows. In this section, we give examples of compatible families of Feller semigroups. They are constructed on M, a two-point symmetric space, with C an isotropic covariance function on the space of vector fields and the semigroup of a Brownian motion on M.

7.1. *Isotropic covariance functions.* Let $M = G/K$ be a two-point symmetric space. This class of spaces includes euclidean spaces, hyperbolic spaces and spheres; see [16], Chapter III. G is the group of isometries on M. A covariance function C is said to be isotropic if

$$C(g\cdot\xi, g\cdot\xi') = C(\xi, \xi') \tag{7.1}$$

for all $g \in G$ and $(\xi, \xi') \in (T^*M)^2$ and where $g \cdot \xi = Tg(\xi)$ [or $g \cdot (x, u) = (gx, Tg_x u)$ for $(x, u) \in T^*M$].

Examples of isotropic covariances are given in [32] on $\mathbb{R}^d$ and in [37, 36] on the sphere and on the hyperbolic plane. In these examples, the group G of isometries on $\mathbb{R}^d$ (making $\mathbb{R}^d$ homogeneous) is generated by $O(d)$ and by the translations. For the sphere $\mathbb{S}^d$, this group is $O(d+1)$, and for the hyperbolic space, it is $O(d, 1)$.

7.2. *A compatible family of Markovian semigroups.* Let C be an isotropic covariance on $\mathcal{X}(M)$, the space of vector fields on the two-point symmetric space $M = G/K$. To this isotropic covariance function is associated a Brownian vector field on M [i.e., a $\mathcal{X}(M)$-valued Brownian motion W such that $\mathsf{E}[\langle W_t, \xi\rangle\langle W_s, \xi'\rangle] = t \wedge s C(\xi, \xi')$]. Let P be the associated Wiener measure, constructed on the canonical space $\Omega = \{\omega : \mathbb{R}^+ \to \mathcal{X}(M)\}$, equipped with the σ-field $\mathcal{A}$ generated by the coordinate functions.

We denote by W the random variable $W(\omega) = \omega$. W is a Brownian vector field of covariance C which is isotropic in the sense that, for every $g \in G$, $(Tg_x^{-1} W_t(gx), t \in \mathbb{R}^+, x \in M)$ is a Brownian vector field of covariance C.

Let P_t be the heat semigroup on M, let m be the volume element and let Δ be the Laplacian.

Let $(S_t, t \geq 0)$ be the family of random operators defined in [23], associated with W and to the heat semigroup P_t. Following [23], we define the associated semigroups of the n-point motion, $\mathsf{P}_t^{(n)} = \mathsf{E}[S_t^{\otimes n}]$ (with $\mathsf{P}_t^{(1)} = \mathsf{P}_t$). Then, it is obvious that $(\mathsf{P}_t^{(n)}, n \geq 1)$ is a compatible family of Markovian semigroups of operators acting on $L^2(m^{\otimes n})$. We now prove that these semigroups are induced by Feller semigroups (the question was raised in [28]).

One can extend $(W_t)_{t\geq 0}$ into a vector field valued white noise $(W_{s,t}, s \leq t)$ of covariance C such that $W_t = W_{0,t}$ for $t \geq 0$ and associate to it a stationary cocycle of random operators $(S_{s,t}, s \leq t)$ such that $S_{0,t} = S_t$ for $t \geq 0$.

7.3. *Verification of the Feller property.* For every $g \in G$, let $L_g : \Omega \to \Omega$ defined by $L_g\omega_t(\cdot) = Tg^{-1}(\omega_t(g\cdot))$, for all $t \in \mathbb{R}$ and $x \in M$. Then L_g is linear and, for all g_1 and g_2 in G, $L_{g_1 g_2} = L_{g_1} L_{g_2}$ (i.e., $g \mapsto L_g$ is a representation of G). It is easy to check that, for every $g \in G$, $(L_g)^*\mathsf{P} = \mathsf{P}$. Note that this last condition is also a characterization that C is isotropic.

For every $g \in G$, L_g induces a linear transformation on $L^2(\Omega, \mathcal{A}, \mathsf{P})$ we will also denote by L_g. Then for every $f \in L^2(\Omega, \mathcal{A}, \mathsf{P})$, we have $L_g f(\omega) = f(L_g\omega)$. This transformation is unitary since

$$\|L_g f\|^2 = \int f^2(L_g\omega)\, \mathsf{P}(d\omega) = \int f^2(\omega)\big((L_g)^*\mathsf{P}\big)(d\omega) = \|f\|^2$$

[where $\|\cdot\|$ denotes the $L^2(\mathsf{P})$-norm].

PROPOSITION 7.1. *For every $v \in L^2(\Omega, \mathcal{A}, \mathsf{P})$, the mapping $g \mapsto L_g v$ is continuous.*

PROOF. Note that, since L is a representation, it is enough to prove the continuity at e, the identity element in G.

REMARK 7.1. Let $(v_n, n \in \mathbb{N})$ be a sequence in $L^2(\Omega, \mathcal{A}, \mathsf{P})$ converging toward $v \in L^2(\Omega, \mathcal{A}, \mathsf{P})$ as $n \to \infty$ such that $\lim_{g\to e} L_g v_n = v_n$ for every integer n; then $\lim_{g\to e} L_g v = v$. Indeed, since for every $g \in G$, L_g is unitary, $\|L_g v - v\| \le 2\|v_n - v\| + \|L_g v_n - v_n\|$. Hence $\limsup_{g\to e} \|L_g v - v\| \le 2\|v_n - v\|$ for every integer n.

We first prove that $\lim_{g\to e} L_g v = v$ for every v of the form $\sum_i W_{t_i}(\xi_i)$ [with $W_t(x,u) = \langle W_t(x), u\rangle$, where $\langle\cdot,\cdot\rangle$ denotes the Riemannian metric]:

$$\left\| L_g\left(\sum_i W_{t_i}(\xi_i)\right) - \sum_i W_{t_i}(\xi_i)\right\|^2 = 2\sum_{i,j} t_i \wedge t_j \big(C(\xi_i, \xi_j) - C(g\cdot\xi_i, \xi_j)\big),$$

which converges toward 0 as g tends to e.

Let H denote the closure (in $L^2(\Omega, \mathcal{A}, \mathsf{P})$) of the class of every v of the form $\sum_i W_{t_i}(\xi_i)$. Remark 7.1 implies that $\lim_{g\to e} L_g v = v$ holds for every $v \in H$.

It is well known that $L^2(\Omega, \mathcal{A}, \mathsf{P})$ is the orthogonal sum of the Wick powers H^n of H (see [39]), also called the nth Wiener chaos (see [33]); H^0 is constituted by the constants. The space H^n is isometric to the symmetric tensor product Hilbert space $H^{\otimes^s n}$. We now prove that $\lim_{g\to e} L_g v = v$ holds for every $v \in H^n$. For every $v = v_1 \otimes^s \cdots \otimes^s v_n \in H^n$ (or $: v_1 v_2 \cdots v_n :$ in wick notation), with $v_1, \dots, v_n$ in H,

$$\begin{aligned}\|L_g v - v\| &\le \sum_j \|L_g v_1 \otimes^s \cdots \otimes^s L_g v_{j-1} \otimes^s (L_g v_j - v_j) \otimes^s v_{j+1} \otimes^s \cdots \otimes^s v_n\| \\ &\le \sqrt{n!}\sum_j \|L_g v_j - v_j\| \times \prod_{i\ne j} \|v_i\|,\end{aligned}$$

which converges toward 0 as g tends to e. Since the class of linear combinations of elements of the form $v_1 \otimes^s \cdots \otimes^s v_n$ is dense in H^n, we have $\lim_{g\to e} L_g v = v$ for every v in H^n. And we conclude since $L^2(\Omega, \mathcal{A}, \mathsf{P}) = \bigoplus_{n\ge 0} H^n$. $\square$

For all $x \in M$, $s \le t$ and $f \in C_0(M)$, since $\mathsf{P}_\varepsilon^{(1)}$ is absolutely continuous with respect to m, we have

$$\mathsf{P}_\varepsilon^{(1)} S_{s+\varepsilon,t} f(x) = \mathsf{E}\big[\mathsf{P}_{\varepsilon'}^{(1)} S_{s+\varepsilon',t} f(x) | \mathcal{F}_{s+\varepsilon',t}\big], \tag{7.2}$$

for $0 < \varepsilon' \le \varepsilon$. Thus, for all $s < t$, $\mathsf{P}_\varepsilon^{(1)} S_{s+\varepsilon,t} f(x)$ is a martingale as ε decreases. This martingale converges and we denote its limit by $K_{s,t} f(x)$. Then $S_{s,t} f = K_{s,t} f$ in $L^2(m \otimes \mathsf{P})$ and $\mathsf{P}_t^{(n)} = \tilde{P}_t^{(n)}$ $m^{\otimes n}$-a.e., where $\tilde{P}_t^{(n)}$ denotes $\mathsf{E}[K_{s,t}^{\otimes n}]$.

LEMMA 7.1. *The mapping $x \mapsto K_{s,t} f(x)$ is continuous for every Lipschitz function f and all $s \le t$.*

PROOF. Note that for all $g \in G$ and $x \in M$,

$$L_g K_{s,t} f(x) = K_{s,t} f^{g^{-1}}(gx), \tag{7.3}$$

where $f^{g^{-1}}(x) = f(g^{-1}x)$. We then have

$$\begin{aligned}&\|K_{s,t} f(gx) - K_{s,t} f(x)\| \\ &\quad \le \|K_{s,t} f(gx) - K_{s,t} f^{g^{-1}}(gx)\| + \|L_g K_{s,t} f(x) - K_{s,t} f(x)\|.\end{aligned}$$

Hence $\lim_{g \to e} K_{s,t} f(gx) = K_{s,t} f(x)$ since $\lim_{g \to e} L_g K_{s,t} f(x) = K_{s,t} f(x)$ and $\|K_{s,t} f(gx) - K_{s,t} f^{g^{-1}}(gx)\| \le \|f - f^{g^{-1}}\|_\infty$, which converges toward 0 [since $|f(x) - f^{g^{-1}}(x)| \le C d(x, g^{-1}x)$, which converges toward 0 as $g \to e$]. This implies the lemma. □

PROPOSITION 7.2. (i) *$(\tilde{\mathsf{P}}_t^{(n)}, n \ge 1)$ is a compatible family of Feller semigroups.*

(ii) *The associated convolution semigroup $\nu^s = (\nu_t^s)_{t \ge 0}$ is a diffusion convolution semigroup with local characteristics $(\frac{1}{2}\Delta, C)$.*

PROOF. For all bounded Lipschitz functions $f_1, \dots, f_n$, Lemma 7.1 implies that $(x_1, \dots, x_n) \mapsto \tilde{\mathsf{P}}_t^{(n)} f_1 \otimes \cdots \otimes f_n(x_1, \dots, x_n) = \mathsf{E}[\prod_{i=1}^n K_{s,t} f_i(x_i)]$ is continuous. This suffices to prove (i). [The proof that $\lim_{t \to 0} \mathsf{P}_t^{(n)} h(x) = h(x)$ for every $h \in C(M^n)$ is the same as in Lemma 1.11.]

To prove (ii), notice that Itô's formula for $(S_{s,t}, s \le t)$ (see Theorem 3.2 in [23]) implies that, for all $f \in C_K^2(M)$ and $s \le t$,

$$K_{s,t} f(x) = f(x) + \int_s^t K_{s,u}\big(W f(du)\big)(x) + \frac{1}{2} \int_s^t K_{s,u}(\Delta f)(x)\, du, \tag{7.4}$$

that is, (K, W) solves the $(\frac{1}{2}\Delta, C)$-SDE. □

7.4. *Classification.* Let ν^s be the diffusion convolution semigroup constructed above. It defines a Wiener solution of the $(\frac{1}{2}\Delta, C)$-SDE. Note that there is no pure diffusion.

Let $(d_t)_{t \ge 0}$ denote the distance process induced by the two-point motion $X_t^{(2)} = (X_t, Y_t)$ [then $d_t = d(X_t, Y_t)$]. The isotropy condition and the fact that in two-point homogeneous spaces, pairs of equidistant points can be exchanged by an isometry imply that d_t is a real diffusion. We denote in the following the law of this diffusion starting from $r \ge 0$ by P_r. Let $H_r = \inf\{t > 0, d_t = r\}$.

PROPOSITION 7.3. (i) ν^s *defines a noncoalescing flow of maps* (*i.e., such that the two-point motion starting outside of the diagonal never hits the diagonal*) *if and only if* 0 *is a natural boundary point, that is, if*

$$\forall r > 0 \qquad \mathsf{P}_r[H_0 < \infty] = 0 \quad \textit{and} \quad \mathsf{P}_0[H_r < \infty] = 0. \tag{7.5}$$

(ii) ν^s *defines a coalescing flow of maps if and only if* 0 *is a closed exit boundary point, that is, if*

$$\exists r > 0 \qquad \mathsf{P}_r[H_0 < \infty] > 0 \quad \textit{and} \quad \forall r > 0 \qquad \mathsf{P}_0[H_r < \infty] = 0. \tag{7.6}$$

(iii) ν^s *defines a turbulent flow without hitting* (*i.e., such that the two-point motion starting outside of the diagonal never hits the diagonal*) *if and only if* 0 *is an open entrance boundary point, that is, if*

$$\forall r > 0 \qquad \mathsf{P}_r[H_0 < \infty] = 0 \quad \textit{and} \quad \exists r > 0 \qquad \mathsf{P}_0[H_r < \infty] > 0. \tag{7.7}$$

[*We recall that a turbulent flow was defined as a stochastic flow of kernels which is not a flow of maps and without pure diffusion*.]

(iv) ν^s *defines a turbulent flow with hitting* (*i.e., such that the two-point motion starting outside of the diagonal hits the diagonal with a positive probability*) *if and only if* 0 *is a reflecting regular boundary point, that is, if*

$$\exists r > 0 \qquad \mathsf{P}_r[H_0 < \infty] > 0 \quad \textit{and} \quad \exists r > 0 \qquad \mathsf{P}_0[H_r < \infty] > 0. \tag{7.8}$$

In all cases except (iv), ν^s *is the unique diffusion convolution semigroup with local characteristics* $(\frac{1}{2}\Delta, C)$.

In case (iv), *called the intermediate phase*, $\nu^c \neq \nu^s$ *and Theorems* 6.1 *and* 6.2 *apply. Thus* N_{ν^c} *is a predictable non-Gaussian noise.*

PROOF. The proof of (i)–(iv) is straightforward. Notice that the local characteristics satisfy (U). In all cases, ν^s defines a Wiener solution of the $(\frac{1}{2}\Delta, C)$-SDE. This with Theorem 6.1 implies that in the coalescing case (ii), since $\nu^s = \nu^c$, ν^s is the unique diffusion convolution semigroup whose local characteristics are $(\frac{1}{2}\Delta, C)$.

In the noncoalescing case (i) and in the turbulent case without hitting (iii), the fact that ν^s is the unique diffusion convolution semigroup whose local characteristics are $(\frac{1}{2}\Delta, C)$ follows directly from (U).

In the intermediate phase (iv), we must have $\nu^c \neq \nu^s$ since ν^s defines a turbulent flow and ν^c a flow of maps. Moreover, condition (B) holds so that we can conclude using Theorems 6.1 and 6.2. □

REMARK 7.2. The $(\frac{1}{2}\Delta, C)$-SDE has a solution, unique in law except in the intermediate phase, in which case all solutions are obtained by filtering, on an extension (N, φ) of the noise of the coalescing solution, this coalescing solution φ with respect to a subnoise of N containing W.

REMARK 7.3. The conditions involving the distance process can be verified using the speed and scale measures of this process which are explicitly determined by the spectral measures of the isotropic fields (cf. [23] for $\mathbb{R}^d$ and for $\mathbb{S}^d$).

7.5. *Sobolev flows.* In [23], Sobolev flows $(S_{s,t}, s \le t)$ on $\mathbb{R}^d$ and on $\mathbb{S}^d$ are studied. The Sobolev covariances are described with two parameters $\alpha > 0$ and $\eta \in [0,1]$. The associated self-reproducing spaces are Sobolev spaces of vector fields of order $(d+\alpha)/2$. The incompressible and gradient subspaces are orthogonal and, respectively, weighted by factors η and $1-\eta$.

Let us apply the results obtained in [23]. We will call the stochastic flow associated with $(S_{s,t}, s \le t)$ (see Sections 5.7 and 7.3) Sobolev flow as well. When $\alpha > 2$, we are in case (i) and Sobolev flows are flows of diffeomorphisms. More interestingly, when $0 < \alpha < 2$ then:

(i) if $d \in \{2,3\}$ and $\eta < 1 - \frac{d}{\alpha^2}$, we are in case (ii) of Proposition 7.3 and the Sobolev flow is a coalescing flow,

(ii) if $d \ge 4$ or if $d \in \{2,3\}$ and $\eta > \frac{1}{2} - \frac{(d-2)}{2\alpha}$, we are in case (iii) and the Sobolev flow is turbulent without hitting,

(iii) if $d \in \{2,3\}$ and $1 - \frac{d}{\alpha^2} < \eta < \frac{1}{2} - \frac{(d-2)}{2\alpha}$, we are in case (iv) (i.e., the intermediate phase) and the Sobolev flow is turbulent with hitting.

In dimension 1, the parameter η vanishes. The critical case was studied in [1, 12, 30]. There is a strong coalescing solution for $\alpha \in [1,2[$ and an intermediate phase for $\alpha \in]0,1[$.

By construction, in all these cases, the noises generated by the Sobolev flows are Gaussian noises. For the intermediate phase, in which there exist two different solutions to the $(\frac{1}{2}\Delta, C)$-SDE (namely the coalescing one and the turbulent one), the noise of the associated coalescing flow is predictable but not Gaussian.

These different cases are represented by the phase diagram (Figure 1) for the homogeneous space $\mathbb{S}^3$. Recall that a flow of diffeomorphisms is called stable

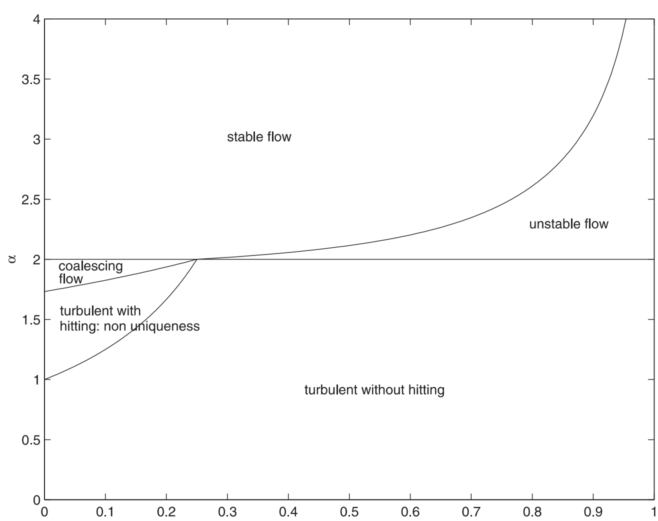

FIG. 1.

(resp. unstable) when the first Lyapounov exponent is negative (resp. positive). These exponents actually converge actually toward $-\infty$ or to $+\infty$ as α approaches the critical value 2.

8. Conclusion. Looking at the phase diagram in Figure 1, it looks as if this case has been fully analyzed.

The three different types of motion which can be defined by a consistent system of Feller semigroups appear in Figure 1. Flows of noncoalescing maps occur when, for the two-point motion, the diagonal and the complement of the diagonal are absorbing.

When the first condition fails, that is, when the diagonal is not absorbing, we get a diffusive flow, that is, a flow of nontrivial Markov kernels. We see in this example that this can happen without pure diffusion, that is, when the evolution equation has no dissipative term. In that case we say that the flow is turbulent. It can be viewed as an effect of extreme instability due to the importance of very high frequency divergence-free components in the velocity field near the diagonal.

When the second condition fails, that is, when the complement of the diagonal is not absorbing, we get flows of coalescing maps. We see, in the intermediate phase, that a turbulent and a coalescing flow can have the same local characteristics. This happens when both conditions fail for the two-point motion associated with the turbulent flow.

Moreover, it is likely that at least in the other isotropic situations, a very similar picture will occur, the parameters being the singularity of the covariance on the diagonal and the balance between gradient and incompressible velocity fields.

Yet there is still some important work to do about the intermediate phase. We know there exist two remarkable distinct solutions in that case for the SDE: the coalescing flow, the noise of which is not linear but for which the linear part has been identified as the velocity white noise W, and the unique Wiener solution which is a flow of nontrivial kernels obtained by averaging the coalescing flow with respect to W. Other solutions do exist and we have shown that their associated convolution semigroups are weakly dominated by the "coalescing" convolution semigroup and dominate the "Wiener" or "linear" one. But this classification should be made analytically precise and one can conjecture it involves a "gluing" parameter on the diagonal (see Section 3.3, [24–26] for first steps in this direction.) Moreover, the nonlinear part of the relevant noises remains to be fully analyzed. Finally, one can expect that more complex phenomena occur for SDEs in which a multiplicity of weak solutions with different one-point motions do exist. Hence this paper can only be a step in the understanding of the multiplicity of flows with given velocity field, or given local characteristics.

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