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FLOWS OF NEWTONIAN AND POWER-LAW FLUIDS IN SYMMETRICALLY CORRUGATED CAPPILARY FISSURES AND TUBES

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In this paper, an analytical method for deriving the relationships between the pressure drop and the volumetric flow rate in laminar flow regimes of Newtonian and power-law fluids through symmetrically corrugated capillary fissures and tubes is presented. This method, which is general with regard to fluid and capillary shape, can also be used as a foundation for different fluids, fissures and tubes. It can also be a good base for numerical integration when analytical expressions are hard to obtain due to mathematical complexities.

Five converging-diverging or diverging-converging geometrics, viz. wedge and cone, parabolic, hyperbolic, hyperbolic cosine and cosine curve, are used as examples to illustrate the application of this method. For the wedge and cone geometry the present results for the power-law fluid were compared with the results obtained by another method; this comparison indicates a good compatibility between both the results.

Key words: power-law fluid, laminar flow, symmetrically corrugated capillaries, capillary fissures or tubes.

1. Introduction

Modelling the flow in tapered or corrugated channels and tubes is required for a number of scientific technological, medical and industrial applications [1]. In the literature on fluid dynamics, there are numerous studies on the flow through channels or tubes of tapered-expanded or corrugated nature. Many of these studies use numerical techniques; cf. the papers by Lahbabi and Chang [2], Burdette *et al*. [3], James *et al*. [4], Momemi- Masuleh and Phillips [5], Wang *et al.* [6], Hayat *et al*. [7, 8], Mekheimer and Kot [9], Nadeem *et al*. [10]. Some others adopt analytical approaches based on simplified assumptions and normally deal with very special cases (Williams and Javadpour [11], Walicki *et al*. [12], Walicki and Walicka [13÷15], Walicka and Walicki [16], Sochi [17÷20]. Note that most of these studies concern the flows in conical (or similar) geometry, namely in the converging-diverging tubes.

This paper presents an analytical method for deriving mathematical relations between the volumetric flow rate and pressure drop or pressure gradient in tapered-expanded or corrugated capillary fissures or tubes, such as those shown schematically in Fig.1.

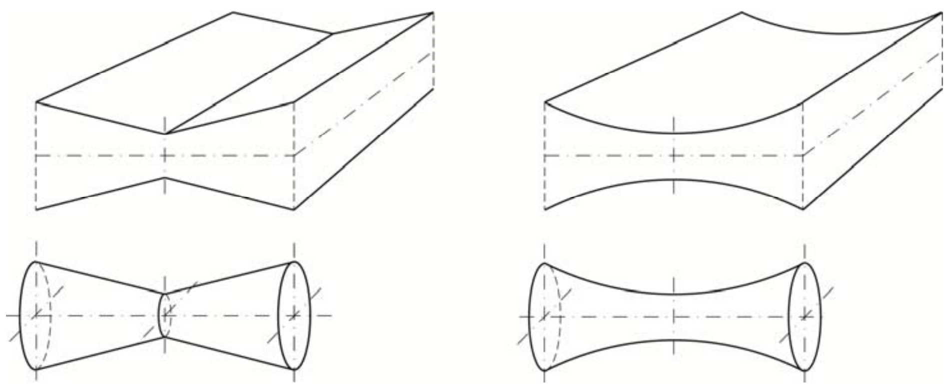

Fig.1. Profiles of converging-diverging capillary fissures or tubes.

Developing some results obtained by Sochi [17], concerning the flows in capillary tubes, we also present five examples of flows both in capillary fissures and tubes for Newtonian and power-law fluids. Both these flows may be used to model the flows through porous media [21].

2. Flows through rectilinear converging-diverging capillary fissures or tubes

Frequently, to model the flow through porous media, rectilinear fissures or tubes of constant cross-sections are used (Fig.2).

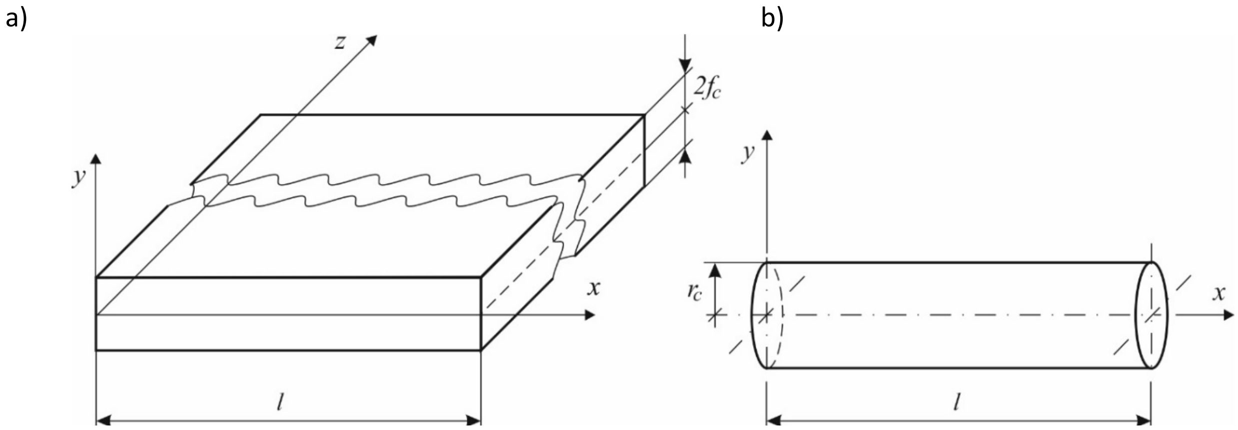

Fig.2. Geometry of a rectilinear capillary fissure (a) and a capillary tube (b) of a constant cross-section.

The velocity of the power-law fluid is, respectively [21]:

– for a capillary fissure

$$\upsilon_f = \frac{f_c^{m+1}}{(m+2)\mu}\left(-\frac{dp}{dx}\right)^m, \tag{2.1}$$

– for a capillary tube

$$\upsilon_t = \frac{r_c^{m+1}}{2^m (m+3)\mu}\left(-\frac{dp}{dx}\right)^m, \tag{2.2}$$

whereas the volumetric flow rate Q is equal:

– for a capillary fissure

$$Q_f = 2f_c\upsilon_f = \frac{2f_c^{m+2}}{(m+2)\mu}\left(-\frac{dp}{dx}\right)^m, \tag{2.3}$$

here Q_f is counted on the unit of a fissure width;

– for a capillary tube

$$Q_t = \pi r_c^2 \upsilon_t = \frac{\pi r_c^{m+3}}{2^m (m+3)\mu}\left(-\frac{dp}{dx}\right)^m . \tag{2.4}$$

For capillaries of variable cross-sections we have, respectively (see Fig.3):

– for a capillary fissure

$$\Delta p = \left[\frac{\mu(m+2)Q_f}{2}\right]^{\frac{1}{m}} \int_{-l/2}^{l/2} \frac{dx}{f_c^{\frac{m+2}{m}}}, \tag{2.5}$$

– while for a capillary tube

$$\Delta p = 2\left[\frac{\mu(m+3)Q_t}{\pi}\right]^{\frac{1}{m}} \int_{-l/2}^{l/2} \frac{dx}{r_c^{\frac{m+3}{m}}}, \tag{2.6}$$

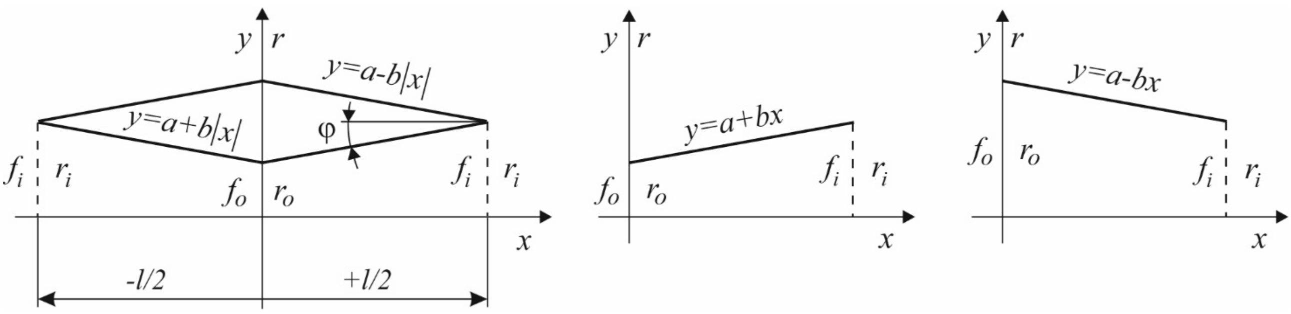

Fig.3. Scheme of half of converging-diverging and diverging-converging capillaries with rectilinear generatrices.

The current thickness of the capillary fissure or the radius of the capillary tube are given, respectively, by

$$\left.\begin{array}{l} f_c(x) \\ r_c(x) \end{array}\right\} = \left\{\begin{array}{l} y = a + b|x| \\ y = a - b|x| \end{array}\right. \quad \text{where} \quad -\frac{l}{2} \le x \le +\frac{l}{2} \tag{2.7}$$

and there are, respectively

$$a = \begin{cases} f_o \\ r_o \end{cases}, \qquad b = \begin{cases} \pm\dfrac{2(f_i - f_o)}{l} \\ \pm\dfrac{2(r_i - r_o)}{l} \end{cases} . \tag{2.8}$$

Introducing formulae (2.7) and (2.8) into Eq.(2.5) or Eq.(2.6) we will obtain – after integration – the following expressions (see formulae (A.3)÷(A.4) in the Appendix):

– for the capillary fissure

$$\Delta p=\left[\frac{(m+2)\mu Q_f}{2}\right]^{\frac{1}{m}}\frac{ml\left[\pm\left(f_o^{-\frac{2}{m}}-f_i^{-\frac{2}{m}}\right)\right]}{2\left[\pm\left(f_i-f_o\right)\right]} \tag{2.9}$$

and

$$Q_f=\frac{2^{m+1}\left[\pm\left(f_i-f_o\right)^m\right]}{m^m\left(m+2\right)\mu\left[\pm\left(f_o^{-\frac{2}{m}}-f_i^{-\frac{2}{m}}\right)\right]^m}\left(\frac{\Delta p}{l}\right)^m; \tag{2.10}$$

– for the capillary tube

$$\Delta p=\left[\frac{(m+3)\mu Q_t}{\pi}\right]^{\frac{1}{m}}\frac{2ml\left[\pm\left(r_o^{-\frac{3}{m}}-r_i^{-\frac{3}{m}}\right)\right]}{3\left[\pm\left(r_i-r_o\right)\right]} \tag{2.11}$$

and

$$Q_t=\frac{3^m\pi\left[\pm\left(r_i-r_o\right)^m\right]}{\left(2m\right)^m\left(m+3\right)\mu\left[\pm\left(r_o^{-\frac{3}{m}}-r_i^{-\frac{3}{m}}\right)\right]^m}\left(\frac{\Delta p}{l}\right)^m. \tag{2.12}$$

The flow velocities through a thin porous layer, composed of convergent-divergent capillaries, will be given, respectively, as

$$\upsilon_f=\frac{Q_f}{2f_i}=\frac{2^m\left(f_i-f_o\right)^m\varphi_p}{m^m\left(m+2\right)\mu\left(f_o^{-\frac{2}{m}}-f_i^{-\frac{2}{m}}\right)^m f_i}\left(-\frac{dp}{dx}\right)^m, \tag{2.13}$$

$$\upsilon_t=\frac{Q_t}{\pi r_i^2}=\frac{3^m\left(r_i-r_o\right)^m\varphi_p}{\left(2m\right)^m\left(m+3\right)\mu\left(r_o^{-\frac{3}{m}}-r_i^{-\frac{3}{m}}\right)^m r_i^2}\left(-\frac{dp}{dx}\right)^m \tag{2.14}$$

where φ_p is the porosity of the porous layer. Note that it will be similar for a thin porous layer composed of divergent-convergent capillaries.

Let us refer to the papers [12-16]. The formulae for the pressure losses in a divergent wedge flow and in a divergent conical flow presented there are as follows:

– for the wedge flow

$$\Delta p = \frac{m}{2}\left[\frac{(m+2)\mu Q_f}{2}\right]^{\frac{1}{m}}\left(f_o^{-\frac{2}{m}} - f_i^{-\frac{2}{m}}\right)\left(\cot\varphi + \frac{1}{12}\tan\varphi\right), \tag{2.15}$$

– for the conical flow

$$\Delta p = \frac{2m}{3}\left[\frac{(m+3)\mu Q_t}{\pi}\right]^{\frac{1}{m}}\left(r_o^{-\frac{3}{m}} - r_i^{-\frac{3}{m}}\right)\left(\cot\varphi + \frac{1}{2}\tan\varphi\right). \tag{2.16}$$

Taking into account that

$$\cot\varphi = \frac{1}{\tan\varphi} = \begin{cases} \dfrac{l}{2(f_i - f_o)} & \text{for the wedge} \\ \dfrac{l}{2(r_i - r_o)} & \text{for the cone} \end{cases}$$

and that $\cot\varphi >> \tan\varphi$ for small values of φ , then the second terms in braces of Eqs (2.15) and (2.16) can be neglected. Assuming that the pressure loss in a double wedge or double cone should been taken doubly, it is easy to see that the present results are consistent with the results obtained in the earlier papers [12-16] by another method. Note that the first terms in braces of Eqs (2.15) and (2.16) are connected with the pressure drop due to simple shear deformation of the fluid while the second terms are connected with the pressure drop due to simple tension of the fluid in the wedge or conical die.

It easy to see that in the case when $m = 1$, all the above formulae describe the flows of Newtonian fluids.

Let us introduce the following notation

$$\beta = \frac{f_o}{f_i} \quad \text{or} \quad \beta = \frac{r_o}{r_i} \tag{2.17}$$

then it can be assumed that

$$f_c = \frac{1}{2}(f_i + f_o) = \frac{f_i}{2}(1+\beta) \tag{2.18}$$

or

$$r_c = \frac{1}{2}(r_i + r_o) = \frac{r_i}{2}(1+\beta).$$

Note that

$$f_i = \frac{2}{1+\beta} f_c \quad \text{or} \quad r_i = \frac{2}{1+\beta} r_c \,. \tag{2.19}$$

Introducing these expressions into Eqs (2.13) and (2.14) we have

$$\upsilon_f = \frac{f_c^{m+1}\varphi_p}{(m+2)\mu} F_{cor} \left(-\frac{dp}{dx}\right)^m, \tag{2.20}$$

where

$$F_{cor} = \frac{2\beta^2}{1+\beta}\left[\frac{4(1-\beta)}{m(1+\beta)\left(1-\beta^{2/m}\right)}\right]^m \tag{2.21}$$

and

$$\upsilon_t = \frac{r_c^{m+1}\varphi_p}{2^m (m+3)\mu} T_{cor} \left(-\frac{dp}{dx}\right)^m \tag{2.22}$$

where

$$T_{cor} = \frac{2\beta^3}{1+\beta}\left[\frac{6(1-\beta)}{m(1+\beta)\left(1-\beta^{3/m}\right)}\right]^m \tag{2.23}$$

here F_{cor} and T_{cor} are the correction factors which fulfill the condition: $F_{cor}, T_{cor} < 1$. This result indicates that the flow velocity in porous media with corrugated capillaries is always less than the flow velocity in porous media with rectilinear capillaries of constant cross-sections.

3. Flows through parabolic capillaries

Parabolic capillaries, depicted in Fig.4, are described by the formulae

$$\left.\begin{matrix} f_c(x) \\ r_c(x) \end{matrix}\right\} = a \pm bx^2 \qquad \text{where} \qquad -\frac{l}{2} \le x \le +\frac{l}{2} \tag{3.1}$$

and

$$a = \begin{cases} f_o \\ r_o \end{cases}, \qquad b = \begin{cases} \pm\left(\dfrac{2}{l}\right)^2 (f_i - f_o) \\ \pm\left(\dfrac{2}{l}\right)^2 (r_i - r_o) \end{cases}. \tag{3.2}$$

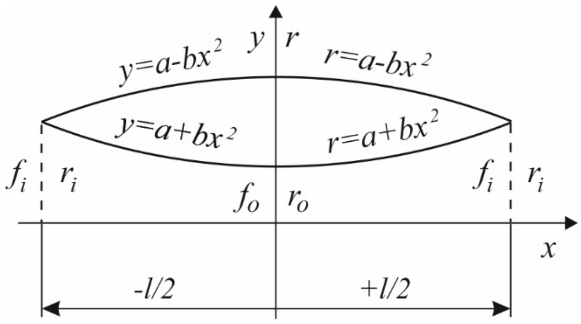

Fig.4. Schematic representation of half of converging-diverging and diverging-converging capillaries with parabolic profiles.

Introducing formulae (3.1) and (3.2) into Eqs (2.5) or (2.6) we will obtain – after integration – the following expressions (see formula (A.8) in the Appendix):

– for the capillary fissure

$$\Delta p = \left[\frac{(m+2)\mu Q_f}{2}\right]^{\frac{1}{m}} \frac{l}{f_o^{\frac{m+2}{m}}} F_{fp}, \tag{3.3}$$

or

$$Q_f = \frac{2 f_o^{m+2}}{(m+2)\mu} \frac{1}{\left(F_{fp}\right)^m} \left(\frac{\Delta p}{l}\right)^m \tag{3.4}$$

where

$$F_{fp} = F\left[\frac{1}{2}, \frac{m+2}{m}; \frac{3}{2}; 1 - \frac{f_i}{f_o}\right]; \tag{3.5}$$

– for the capillary tube

$$\Delta p = \left[\frac{(m+3)\mu Q_t}{\pi}\right]^{\frac{1}{m}} \frac{2l}{r_o^{\frac{m+3}{m}}} F_{tp}, \tag{3.6}$$

or

$$Q_t = \frac{\pi r_o^{m+3}}{2^m (m+3)\mu} \frac{1}{\left(F_{tp}\right)^m} \left(\frac{\Delta p}{l}\right)^m \tag{3.7}$$

where

$$F_{tp} = F\left[\frac{1}{2}, \frac{m+3}{m}; \frac{3}{2}; 1 - \frac{r_i}{r_o}\right]. \tag{3.8}$$

The minimal flow velocities through the thin porous layer will be, respectively,

$$\upsilon_f = \frac{f_o^{m+2}}{(m+2)\mu f_i} \frac{\varphi_p}{(F_{fp})^m} \left(-\frac{dp}{dx} \right)^m , \tag{3.9}$$

$$\upsilon_t = \frac{r_o^{m+3}}{2^m (m+3)\mu r_i^2} \frac{\varphi_p}{(F_{tp})^m} \left(-\frac{dp}{dx} \right)^m . \tag{3.10}$$

The functions $F[...]$ are so called "hypergeometric functions" and they are defined in the Appendix.

For the flows of Newtonian fluids $(m=1)$ we will have, respectively (see formulae (A.9)-(A.13) in the Appendix):

- for the capillary fissure

$$\Delta p = \frac{3\mu}{2} Q_f l F_p , \tag{3.11}$$

$$Q_f = \frac{2l}{3\mu F_p} \left(\frac{\Delta p}{l} \right) \tag{3.12}$$

where

$$F_p = \frac{1}{4 f_i^2 f_o} + \frac{3}{8 f_i f_o^2} + \frac{3}{8 f_o^2} J(\pm 1) \tag{3.13}$$

and

$$J(-1) = \frac{2}{\left[f_o (f_o - f_i) \right]^{1/2}} \operatorname{arctanh} \left(\frac{f_o - f_i}{f_o} \right)^{1/2} ,$$

$$J(+1) = \frac{2}{\left[f_o (f_i - f_o) \right]^{1/2}} \arctan \left(\frac{f_i - f_o}{f_o} \right)^{1/2} ; \tag{3.14}$$

- for the capillary tube

$$\Delta p = \frac{8\mu Q_t}{\pi} l T_p , \tag{3.15}$$

$$Q_t = \frac{\pi}{8\mu T_p} \left(\frac{\Delta p}{l} \right) \tag{3.16}$$

where

$$T_p = \frac{1}{6 r_i^3 r_o} + \frac{5}{24 r_i^2 r_o^2} + \frac{5}{16 r_i r_o^3} + \frac{5}{16 r_o^3} J(\pm 1) \tag{3.17}$$

and

$$J(-l) = \frac{2}{\left[r_o\left(r_o - r_i\right)\right]^{1/2}} \operatorname{arctanh}\left(\frac{r_o - r_i}{r_o}\right)^{1/2},$$

$$J(+l) = \frac{2}{\left[r_o\left(r_i - r_o\right)\right]^{1/2}} \arctan\left(\frac{r_i - r_o}{r_o}\right)^{1/2}. \tag{3.18}$$

The flow velocities of the Newtonian fluid through a thin porous layer will be given, respectively

$$\upsilon_f = \frac{\varphi_p}{3\mu f_i F_p}\left(-\frac{dp}{dx}\right), \tag{3.19}$$

$$\upsilon_t = \frac{\varphi_p}{8\mu r_i^2 T_p}\left(-\frac{dp}{dx}\right). \tag{3.20}$$

4. Flows through hyperbolic capillaries

For capillaries of hyperbolic profiles, similar to the profiles shown in Fig.4, the geometric description is as follows

$$\left.\begin{matrix} f_c(x) \\ r_c(x) \end{matrix}\right\} = \left(a \pm bx^2\right)^{1/2} \qquad \text{where} \qquad -\frac{l}{2} \le x \le +\frac{l}{2}, \tag{4.1}$$

and

$$a = \begin{cases} f_o^2 \\ r_o^2 \end{cases}, \qquad b = \begin{cases} \pm\left(\dfrac{2}{l}\right)^2\left(f_i^2 - f_o^2\right) \\ \pm\left(\dfrac{2}{l}\right)^2\left(r_i^2 - r_o^2\right) \end{cases}. \tag{4.2}$$

Introducing formulae (4.1) and (4.2) into Eqs (2.5) or (2.6) we will obtain – the expressions (see formula (A.15) in the Appendix):

- for the capillary fissure

$$\Delta p = \left[\frac{(m+2)\mu Q_f}{2}\right]^{\frac{1}{m}} \frac{l}{f_o^{\frac{m+2}{m}}} F_{fh}, \tag{4.3}$$

or

$$Q_f = \frac{2 f_o^{m+2}}{(m+2)\mu} \frac{1}{\left(F_{fh}\right)^m}\left(\frac{\Delta p}{l}\right)^m, \tag{4.4}$$

where

$$F_{fh} = F\left[\frac{1}{2}, \frac{m+2}{2m}; \frac{3}{2}; 1 - \frac{f_i^2}{f_o^2}\right]; \tag{4.5}$$

– for the capillary tube

$$\Delta p = \left[\frac{(m+3)\mu Q_t}{\pi}\right]^{\frac{1}{m}} \frac{2l}{r_o^{\frac{m+3}{m}}} F_{th}, \tag{4.6}$$

or

$$Q_t = \frac{\pi r_o^{m+3}}{2^m (m+3)\mu} \frac{1}{(F_{th})^m} \left(\frac{\Delta p}{l}\right)^m \tag{4.7}$$

where

$$F_{th} = F\left[\frac{1}{2}, \frac{m+3}{2m}; \frac{3}{2}; 1 - \frac{r_i^2}{r_o^2}\right]. \tag{4.8}$$

The flow velocities through the thin porous layer will be as follows

$$\upsilon_f = \frac{f_o^{m+2}}{(m+2)\mu f_i} \frac{\varphi_p}{(F_{fh})^m} \left(-\frac{dp}{dx}\right)^m, \tag{4.9}$$

$$\upsilon_t = \frac{r_o^{m+3}}{2^m (m+3)\mu r_i^2} \frac{\varphi_p}{(F_{th})^m} \left(-\frac{dp}{dx}\right)^m. \tag{4.10}$$

For the Newtonian flows $(m=1)$ we will have, respectively (see formulae (A.16)-(A.19) in the Appendix):

– for the capillary fissure

$$\Delta p = \frac{3\mu Q_f l}{2 f_i^2 f_o^2}, \tag{4.11}$$

$$Q_f = \frac{2 f_i^2 f_o^2}{3\mu}\left(\frac{\Delta p}{l}\right), \tag{4.12}$$

$$\upsilon_f = \frac{f_i f_o^2 \varphi_p}{3\mu}\left(-\frac{dp}{dx}\right), \tag{4.13}$$

– for the capillary tube

$$\Delta p = \frac{8\mu Q_t l}{\pi} T_p, \tag{4.14}$$

$$Q_t = \frac{\pi}{8\mu T_p}\left(\frac{\Delta p}{l}\right), \tag{4.15}$$

$$\upsilon_t = \frac{\varphi_p}{8\mu r_i^2 T_p}\left(-\frac{dp}{dx}\right) \tag{4.16}$$

where

$$T_p = \frac{1}{2r_i^2 r_o^2} + \frac{1}{2r_o^2} J(\pm 1) \tag{4.17}$$

where now

$$\begin{aligned} J(-1) &= \frac{2}{\left[r_o\left(r_o^2 - r_i^2\right)\right]^{1/2}} \operatorname{arctanh}\left(\frac{r_o - r_i^2}{r_o^2}\right)^{1/2}, \\ J(+1) &= \frac{2}{\left[r_o\left(r_i^2 - r_o^2\right)\right]^{1/2}} \arctan\left(\frac{r_i^2 - r_o^2}{r_o^2}\right)^{1/2}. \end{aligned} \tag{4.18}$$

5. Flows through hyperbolic cosine capillaries

For capillaries of hyperbolic cosine profiles, similar to the profiles shown in Fig.4, the geometric description is given as follows

$$\left.\begin{matrix} f_c(x) \\ r_c(x) \end{matrix}\right\} = \left[a\cosh(bx)\right]^{\pm 1} \qquad \text{where} \qquad -\frac{l}{2} \le x \le +\frac{l}{2} \tag{5.1}$$

and

$$a = \begin{cases} f_o^{\pm 1} \\ r_o^{\pm 1} \end{cases}, \qquad b = \begin{cases} \left(\dfrac{2}{l}\right)\operatorname{arccosh}\left(\dfrac{f_i}{f_o}\right)^{\pm 1} \\ \left(\dfrac{2}{l}\right)\operatorname{arccosh}\left(\dfrac{r_i}{r_o}\right)^{\pm 1} \end{cases}. \tag{5.2}$$

Note that the exponent value equal $+1$ is adequate to the lower profile curve on Fig.4, whereas the exponent value -1 is adequate to the upper profile curve.

Hence, Eqs (2.5) or (2.6) become, respectively (see formulae (A.20) and (A.21) in the Appendix) for the exponent value equal $+1$:

- for the capillary fissure

$$\Delta p = \left[\frac{(m+2)\mu Q_f}{2}\right]^{\frac{1}{m}} \frac{ml}{2 f_o f_i^{2/m} \operatorname{arccosh}\left(\dfrac{f_i}{f_o}\right)} \operatorname{Im}\left(F_{fh}\right), \tag{5.3}$$

or

$$Q_f = \frac{2^{m+1} f_o^m f_i^2 \left[\operatorname{arccosh}\left(\frac{f_i}{f_o} \right) \right]^m}{m^m (m+2)\mu} \frac{1}{\left[\operatorname{Im}\left(F_{fh} \right) \right]^m} \left(\frac{\Delta p}{l} \right)^m \tag{5.4}$$

where

$$\operatorname{Im}\left(F_{fh} \right) = \operatorname{Im}\left\{ F\left[\frac{1}{2}, -\frac{1}{m}; 1 - \frac{1}{m}; \frac{f_i^2}{f_o^2} \right] \right\}; \tag{5.5}$$

– for the capillary tube

$$\Delta p = \left[\frac{(m+3)\mu Q_t}{\pi} \right]^{\frac{1}{m}} \frac{2ml}{3 r_o r_i^{3/m} \operatorname{arccosh}\left(\frac{r_i}{r_o} \right)} \operatorname{Im}\left(F_{th} \right), \tag{5.6}$$

or

$$Q_t = \frac{3^m \pi r_o^m r_i^3 \left[\operatorname{arccosh}\left(\frac{r_i}{r_o} \right) \right]^m}{2^m m^m (m+3)\mu} \frac{1}{\left[\operatorname{Im}\left(F_{th} \right) \right]^m} \left(\frac{\Delta p}{l} \right)^m \tag{5.7}$$

where

$$\operatorname{Im}\left(F_{th} \right) = \operatorname{Im}\left\{ F\left[\frac{1}{2}, -\frac{3}{2m}; 1 - \frac{3}{2m}; \frac{r_i^2}{r_o^2} \right] \right\}. \tag{5.8}$$

Here, $\operatorname{Im}(F)$ is the imaginary part of the hypergeometric function.
The flow velocities through the thin porous layer will be

$$\upsilon_f = \frac{2^{2m} f_o^m f_i \left[\operatorname{arccosh}\left(\frac{f_i}{f_o} \right) \right]^m}{m^m (m+2)\mu} \frac{\varphi_p}{\left[\operatorname{Im}\left(F_{fh} \right) \right]^m} \left(-\frac{dp}{dx} \right)^m, \tag{5.9}$$

$$\upsilon_t = \frac{3^m r_o^m r_i \left[\operatorname{arccosh}\left(\frac{r_i}{r_o} \right) \right]^m}{2^m m^m (m+3)\mu} \frac{\varphi_p}{\left[\operatorname{Im}\left(F_{th} \right) \right]^m} \left(-\frac{dp}{dx} \right)^m. \tag{5.10}$$

For the flows of Newtonian fluids $(m = 1)$ we have (see formulae (A.22)÷(A.25) in the Appendix):

– for the capillary fissure

$$\Delta p=\frac{3\mu Q_f l}{4f_o^3\operatorname{arccosh}\left(\frac{f_i}{f_o}\right)}F_h, \tag{5.11}$$

$$Q_f=\frac{4f_o^3\left[\operatorname{arccosh}\left(\frac{f_i}{f_o}\right)\right]}{3\mu}\frac{1}{F_h}\left(\frac{\Delta p}{l}\right), \tag{5.12}$$

$$\upsilon_f=\frac{2f_o^3\left[\operatorname{arccosh}\left(\frac{f_i}{f_o}\right)\right]}{3\mu f_i}\frac{\varphi_p}{F_h}\left(-\frac{dp}{dx}\right) \tag{5.13}$$

where

$$F_h=\frac{f_o^2\sinh\left[\operatorname{arccosh}\left(\frac{f_i}{f_o}\right)\right]}{f_i^2}+\arctan\left\{\sinh\left[\operatorname{arccosh}\left(\frac{f_i}{f_o}\right)\right]\right\}; \tag{5.14}$$

– for the capillary tube

$$\Delta p=\frac{8\mu Q_t l}{\pi r_o^4\left[\operatorname{arccosh}\left(\frac{r_i}{r_o}\right)\right]}T_h, \tag{5.15}$$

$$Q_t=\frac{\pi r_o^4\left[\operatorname{arccosh}\left(\frac{r_i}{r_o}\right)\right]}{8\mu}\frac{1}{T_h}\left(\frac{\Delta p}{l}\right), \tag{5.16}$$

$$\upsilon_t=\frac{r_o^4\left[\operatorname{arccosh}\left(\frac{r_i}{r_o}\right)\right]}{8\mu r_i^2}\frac{1}{T_h}\left(-\frac{dp}{dx}\right) \tag{5.17}$$

where

$$T_p=\left\{1-\frac{1}{3}\left(\tanh\left[\operatorname{arccosh}\left(\frac{r_i}{r_o}\right)\right]\right)^2\right\}\tanh\left[\operatorname{arccosh}\left(\frac{r_i}{r_o}\right)\right]. \tag{5.18}$$

Equations (2.5) and (2.6), for the exponent value equal -1 will become (see formulae (A.26) and (A.27) in the Appendix):

– for the capillary fissure

$$\Delta p = \left[\frac{(m+2)\mu Q_f}{2}\right]^{\frac{1}{m}} \frac{ml}{2(m+1) f_o^{-1} f_i^{\frac{2m+2}{m}} \operatorname{arc}\cosh\left(\frac{f_o}{f_i}\right)} I_m\left(F_{fh}\right), \tag{5.19}$$

or

$$Q_f = \frac{2^{m+1}(m+1)^m f_o^{-m} f_i^{2m+2}\left[\operatorname{arc}\cosh\left(\frac{f_o}{f_i}\right)\right]^m}{m^m (m+2)\mu\left[\operatorname{Im}\left(F_{fh}\right)\right]^m}\left(\frac{\Delta p}{l}\right)^m \tag{5.20}$$

where now

$$\operatorname{Im}\left(F_{fh}\right) = \operatorname{Im}\left\{F\left[\frac{1}{2}, \frac{m+1}{m}; \frac{2m+1}{m}; \left(\frac{f_o}{f_i}\right)^2\right]\right\}; \tag{5.21}$$

– for the capillary tube

$$\Delta p = \left[\frac{(m+3)\mu Q_t}{\pi}\right]^{\frac{1}{m}} \frac{2ml}{(2m+3) r_o^{-1} r_i^{\frac{2m+3}{m}} \operatorname{arccosh}\left(\frac{r_i}{r_o}\right)} \operatorname{Im}\left(F_{th}\right), \tag{5.22}$$

or

$$Q_t = \frac{(2m+3)^m \pi r_o^{-m} r_i^{2m+3}\left[\operatorname{arccosh}\left(\frac{r_o}{r_i}\right)\right]^m}{m^m (m+3)\mu\left[\operatorname{Im}\left(F_{th}\right)\right]^m} \tag{5.23}$$

where now

$$\operatorname{Im}\left(F_{th}\right) = \operatorname{Im}\left\{F\left[\frac{1}{2}, \frac{2m+3}{2m}; \frac{4m+3}{2m}; \left(\frac{r_o}{r_i}\right)^2\right]\right\}. \tag{5.24}$$

The flow velocity through the thin porous layer will be:

$$\upsilon_f = \frac{2^m (m+1)^m f_i^{2m+1}\left[\operatorname{arccosh}\left(\frac{f_o}{f_i}\right)\right]^m \varphi_p}{m^m (m+2)\mu f_o^m\left[\operatorname{Im}\left(F_{fh}\right)\right]^m}\left(-\frac{dp}{dx}\right)^m, \tag{5.25}$$

$$\upsilon_t = \frac{(2m+3) r_i^{2m+1}\left[\operatorname{arccosh}\left(\frac{r_o}{r_i}\right)\right]^m \varphi_p}{m^m (m+3)\mu r_o^m\left[\operatorname{Im}\left(F_{th}\right)\right]^m}\left(-\frac{dp}{dx}\right). \tag{5.26}$$

For the flow of Newtonian fluids $(m = 1)$ we have (se formulae (A.28)÷(A.31) in the Appendix:

– for the capillary fissure

$$\Delta p = \frac{9\mu Q_f l}{8 f_o^3 \operatorname{arccosh}\left(\frac{f_o}{f_i}\right)} F_h , \tag{5.27}$$

$$Q_f = \frac{8 f_o^3 \left[\operatorname{arccosh}\left(\frac{f_o}{f_i}\right)\right]}{9\mu} \frac{1}{F_h}\left(\frac{\Delta p}{l}\right), \tag{5.28}$$

$$\upsilon_f = \frac{4 f_o^3 \left[\operatorname{arccos} h\left(\frac{f_o}{f_i}\right)\right]}{9 f_i} \frac{\varphi_p}{F_h}\left(-\frac{dp}{dx}\right) \tag{5.29}$$

where

$$F_h = \sinh\left[\operatorname{arccosh}\left(\frac{f_o}{f_i}\right)\right] + \frac{1}{9}\sinh\left[3\operatorname{arccosh}\left(\frac{f_o}{f_i}\right)\right]; \tag{5.30}$$

– for the capillary tube

$$\Delta p = \frac{\mu Q_t l}{\pi r_o^4 \operatorname{arccosh}\left(\frac{r_o}{r_i}\right)} F_t , \tag{5.31}$$

$$Q_t = \frac{\pi r_o^4 \left[\operatorname{arccos}\left(\frac{r_o}{r_i}\right)\right]}{\mu} \frac{1}{F_t}\left(\frac{\Delta p}{l}\right), \tag{5.32}$$

$$\upsilon_t = \frac{r_o^4 \left[\operatorname{arccosh}\left(\frac{r_o}{r_i}\right)\right]}{\mu r_i^2} \frac{\varphi_p}{F_t}\left(-\frac{dp}{dt}\right) \tag{5.33}$$

where

$$F_t = 3\operatorname{arccosh}\left(\frac{r_o}{r_i}\right) + 2\sinh\left[2\operatorname{arccosh}\left(\frac{r_o}{r_i}\right)\right] + \frac{1}{4}\sinh\left[4\operatorname{arccosh}\left(\frac{r_o}{r_i}\right)\right]. \tag{5.34}$$

6. Flows through cosine curve capillaries

For capillaries of cosine curve profile, shown in Fig.5, where the capillary length l spans one complete wavelength, the current capillary thickness or radius are given by

$$\left.\begin{matrix} f_c(x) \\ r_c(x) \end{matrix}\right\} = a \mp b\cos(kx) \qquad \text{where} \qquad -\frac{l}{2} \le x \le +\frac{l}{2} \tag{6.1}$$

where

$$a = \begin{cases} \dfrac{f_i + f_o}{2} \\ \dfrac{r_i + r_o}{2} \end{cases}, \qquad b = \begin{cases} \pm\dfrac{f_i - f_o}{2} \\ \pm\dfrac{r_i - r_o}{2} \end{cases} \qquad \text{and} \qquad k = \frac{2\pi}{l}. \tag{6.2}$$

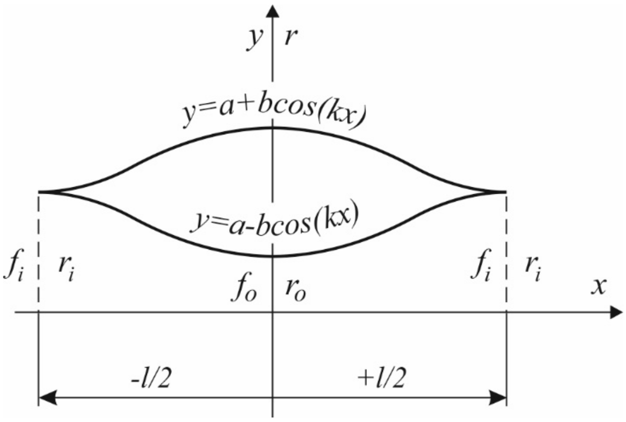

Fig.5. Schematic representation of the thickness or radius of converging-diverging and diverging-converging capillaries with a cosine curve profile.

Introducing formulae (6.1) and (6.2) into Eqs (2.5) or (2.6) we will obtain (see formulae (A.32)÷(A.36) in the Appendix):

– for the capillary fissure

$$\Delta p = \left[\frac{(m+2)\mu Q_f}{2}\right]^{\frac{1}{m}} \frac{ml}{2\pi f_i^{2/m}(f_i f_o)^{1/2}} \mathrm{Im}\left(F_{1fh}\right), \tag{6.3}$$

or

$$Q_f = \frac{2^{m+1}\pi^m f_i^2 (f_i f_o)^{m/2}}{m^m (m+2)\mu} \frac{1}{\left[\mathrm{Im}\left(F_{1fh}\right)\right]^m} \left(\frac{\Delta p}{l}\right)^m \tag{6.4}$$

where

$$\mathrm{Im}\left(F_{1fh}\right) = \mathrm{Im}\left\{F_1\left[-\frac{2}{m}, \frac{1}{2}; \frac{1}{2}; 1 - \frac{2}{m}; 1, \frac{f_i}{f_o}\right]\right\}; \tag{6.5}$$

– for the capillary tube

$$\Delta p = \left[\frac{(m+3)\mu Q_t}{\pi}\right]^{\frac{1}{m}} \frac{ml}{3\pi r_i^{3/m}\left(r_i r_o\right)^{1/2}} \operatorname{Im}\left(F_{1th}\right), \tag{6.6}$$

or

$$Q_t = \frac{3^m \pi^{m+1} r_i^3 \left(r_i r_o\right)^{m/2}}{m^m (m+3)\mu} \frac{1}{\left[\operatorname{Im}\left(F_{1th}\right)\right]^m} \left(\frac{\Delta p}{l}\right)^m \tag{6.7}$$

where

$$\operatorname{Im}\left(F_{1th}\right) = \operatorname{Im}\left\{F_1\left[-\frac{3}{m};\frac{1}{2};\frac{1}{2};1-\frac{3}{m};1,\frac{r_i}{r_o}\right]\right\}. \tag{6.8}$$

Here , $F_1[...]$ is the Appell hypergeometric function and $\operatorname{Im}\left(F_1\right)$ is the imaginary part of this function. The minimal flow velocities through the thin porous layer will be

$$\upsilon_f = \frac{2^m \pi^m f_i \left(f_i f_o\right)^{m/2}}{m^m (m+2)\mu} \frac{\varphi_p}{\left\{\operatorname{Im}\left(F_{1fh}\right)\right\}^m} \left(-\frac{dp}{dx}\right)^m, \tag{6.9}$$

$$\upsilon_t = \frac{3^m \pi^m r_i \left(r_i r_o\right)^{m/2}}{m^m (m+3)\mu} \frac{\varphi_p}{\left\{\operatorname{Im}\left(F_{1th}\right)\right\}^m} \left(-\frac{dp}{dx}\right)^m. \tag{6.10}$$

For the flows of Newtonian fluids $(m=1)$ we have (see formulae (A.37)÷(A.40) in the Appendix):

– for the capillary fissure

$$\Delta p = \frac{3\mu Q_f l}{4} F_c, \tag{6.11}$$

$$Q_f = \frac{4}{3\mu}\frac{1}{F_c}\left(\frac{\Delta p}{l}\right), \tag{6.12}$$

$$\upsilon_f = \frac{2}{3\mu f_i}\frac{\varphi_p}{F_c}\left(-\frac{dp}{dx}\right) \tag{6.13}$$

where

$$F_c = \frac{3 f_i^2 + 2 f_i f_o + 3 f_o^2}{\left(f_i f_o\right)^{5/2}}; \tag{6.14}$$

– for the capillary tube

$$\Delta p = \frac{2\mu Q_t l}{\pi} T_c, \tag{6.15}$$

$$Q_t = \frac{\pi}{2\mu}\frac{1}{T_c}\left(\frac{\Delta p}{l}\right), \tag{6.16}$$

$$\upsilon_t = \frac{1}{2\mu r_i^2}\frac{\varphi_p}{T_c}\left(-\frac{dp}{dx}\right) \tag{6.17}$$

where

$$T_c = \frac{\left(r_i + r_o\right)\left(5r_i^2 - 2r_i r_o + 5r_o^2\right)}{\left(r_i r_o\right)^{7/2}}. \tag{6.18}$$

7. Conclusions

In this paper, an approximate mathematical method for obtaining analytical relations between the pressure drop and the volumetric flow rate in symmetrically corrugated fissures and tubes is presented and applied to the flow of Newtonian and power-law fluids.

The method is illustrated by five examples of capillary fissures or tubes with converging-diverging or diverging-converging shape. The results presented for the flows in the wedge or cone geometries were compared with the results of an earlier study yielded by another method; this comparison indicates a good agreement between both the results for the geometry of small convergence or divergence.

For the flow velocities (in the thin layers) it may be concluded that any corrugation or complexity of the capillary geometry leads to the diminution of these velocities with respect to the flow velocities in the simple capillaries of constant cross-section.

Appendix

In this Appendix we will derive analytical expressions for the integrals appearing in the previous sections of the present paper.

The first of them, for rectilinear capillaries of variable cross-sections, is:

- for the convergent-divergent capillary

$$J_n\Big|_{-l/2}^{+l/2} = \int_{-l/2}^{+l/2} \frac{dx}{\left(a + b|x|\right)^n} = J_{n1}\Big|_{-l/2}^{0} + J_{n2}\Big|_{0}^{+l/2} \tag{A.1}$$

where

$$J_{n1}\Big|_{-l/2}^{0} = \int_{-l/2}^{0} \frac{dx}{\left(a - bx\right)^n} = \frac{1}{(n-1)b}\frac{1}{\left(a - bx\right)^{n-1}}\Bigg|_{-l/2}^{0} = \frac{1}{(n-1)b}\left[a^{1-n} - \left(a + \frac{bl}{2}\right)^{1-n}\right],$$

$$J_{n2}\Big|_{0}^{+l/2} = \int_{0}^{+l/2} \frac{dx}{\left(a + bx\right)^n} = -\frac{1}{(n-1)b}\frac{1}{\left(a + bx\right)^{n-1}}\Bigg|_{0}^{l/2} = \frac{1}{(n-1)b}\left[a^{1-n} - \left(a + \frac{bl}{2}\right)^{1-n}\right], \tag{A.2}$$

and

$$J_n\Big|_{-l/2}^{+l/2} = \frac{2}{(n-1)b}\left[a^{1-n} - \left(a + \frac{bl}{2}\right)^{1-n}\right]; \tag{A.3}$$

– for the divergent-convergent capillary

$$J_n\Big|_{-l/2}^{+l/2} = \int\limits_{-l/2}^{+l/2} \frac{dx}{\left(a - b|x|\right)^n} = \frac{2}{(n-1)b}\left[\left(a - \frac{bl}{2}\right)^{1-n} - a^{1-n}\right]; \tag{A.4}$$

here

$$n = \begin{cases} \dfrac{m+2}{m} & \text{for a fissure} \\ \dfrac{m+3}{m} & \text{for a tube} \end{cases}; \quad \text{if} \quad m = 1 \quad \text{then} \quad n = \begin{cases} 3 \\ 4 \end{cases}. \tag{A.5}$$

The second one, for parabolic capillaries, is as follows

$$J_n = \int \frac{dx}{\left(a \pm bx^2\right)^n} = \frac{x}{a^n} F\left[\frac{1}{2}, n; \frac{3}{2}; \mp\frac{bx^2}{a}\right] \tag{A.6}$$

where $F[...]$ is a hypergeometric function [22,23] defined by the Gauss series [24]

$$F[a,b;c;z] = \sum_{s=0}^{\infty} \frac{(a)_s (b)_s}{(c)_s\, s!} z^s = 1 + \frac{ab}{c} z + \frac{a(a+1)b(b+1)}{c(c+1)2!} z^2 + ... \tag{A.7}$$

here, for convenience, we used the Pochhammer symbol notation for the shifted factorial

$$(a)_s := \begin{cases} a(a+1)...(a+s-1) \\ 1 \end{cases} \quad \text{if} \quad s = \begin{cases} 1,2,..., \\ 0. \end{cases}$$

Accordingly, we have

$$(a)_s = \frac{\Gamma(a+s)}{\Gamma(a)}$$

which is used as a definition for the shifted factorial in the case when s is not necessarily a nonnegative integer.

Introducing in (A.6) the limits of integration we will obtain

$$J_n\Big|_{-l/2}^{+l/2} = \frac{l}{a^n} F\left[\frac{1}{2}, n; \frac{3}{2}; \mp\frac{b}{a}\left(\frac{l}{2}\right)^2\right]. \tag{A.8}$$

For the Newtonian flows we have, respectively

$$J_3 = \int \frac{dx}{\left(a \pm bx^2\right)^3} = \frac{x}{4a\left(a \pm bx^2\right)^2} + \frac{3x}{8a^2\left(a \pm bx^2\right)} + \frac{3}{8a^2} I(\pm) \tag{A.9}$$

and

$$J_3\Big|_{-l/2}^{+l/2} = \frac{l}{4a\left(a \pm \frac{bl^2}{4}\right)^2} + \frac{3l}{8a^2\left(a \pm \frac{bl^2}{4}\right)} + \frac{3}{8a^2} I(\pm)\Big|_{-l/2}^{+l/2}, \tag{A.10}$$

$$J_4 = \int \frac{dx}{\left(a \pm bx^2\right)^4} = \frac{x}{6a\left(a \pm bx^2\right)^3} + \frac{5x}{24a^2\left(a \pm bx^2\right)^2} + \frac{5x}{16a^3\left(a \pm bx^2\right)} + \frac{5}{16a^3} I(\pm) \tag{A.11}$$

and

$$J_4\Big|_{-l/2}^{+l/2} = \frac{l}{6a\left(a \pm \frac{bl^2}{4}\right)^3} + \frac{5l}{24a^2\left(a \pm \frac{bl^2}{4}\right)^2} + \frac{5l}{16a^3\left(a \pm \frac{bl^2}{4}\right)} + \frac{5}{16a^3} I(\pm)\Big|_{-l/2}^{+l/2} \tag{A.12}$$

$$I(\pm) = \int \frac{dx}{\left(a \pm bx^2\right)} = \begin{cases} \dfrac{1}{\sqrt{ab}} \arctan \dfrac{bx}{\sqrt{ab}} \\ \dfrac{1}{\sqrt{ab}} \operatorname{arctanh} \dfrac{bx}{\sqrt{ab}} \end{cases}, \qquad I(\pm)\Big|_{-l/2}^{+l/2} = \begin{cases} \dfrac{2}{\sqrt{ab}} \arctan \dfrac{bl}{2\sqrt{ab}} \\ \dfrac{2}{\sqrt{ab}} \operatorname{arctanh} \dfrac{bl}{2\sqrt{ab}} \end{cases}. \tag{A.13}$$

The third integral, for hyperbolic capillaries, is given by the following expression

$$J_n = \int \frac{dx}{\left(a \pm bx^2\right)^{n/2}} = \frac{x}{a^{n/2}} F\left[\frac{1}{2}, \frac{n}{2}; \frac{3}{2}; \mp \frac{bx^2}{a}\right] \tag{A.14}$$

and

$$J_n\Big|_{-l/2}^{+l/2} = \frac{l}{a^{n/2}} F\left[\frac{1}{2}, \frac{n}{2}; \frac{3}{2}; \mp \frac{b}{a}\left(\frac{l}{2}\right)^2\right]. \tag{A.15}$$

For the Newtonian flows we have

$$J_3 = \int \frac{dx}{\left(a \pm bx^2\right)^{3/2}} = \frac{x}{a\left(a \pm bx^2\right)^{1/2}} \tag{A.16}$$

and

$$J_3\Big|_{-l/2}^{+l/2} = \frac{l}{a\left(a \pm \frac{bl^2}{4}\right)^{1/2}} \tag{A.17}$$

or

$$J_4 = \int \frac{dx}{\left(a \pm bx^2\right)^2} = \frac{x}{2a\left(a \pm bx^2\right)} + \frac{1}{2a} I(\pm) \tag{A.18}$$

and

$$J_4\Big|_{-l/2}^{+l/2} = \frac{l}{2a\left(a \pm \dfrac{bl^2}{4}\right)} + \frac{1}{2a} I(\pm)\Big|_{-l/2}^{+l/2}. \tag{A.19}$$

The fourth and fifth integrals, for hyperbolic cosine capillaries, are as follows:

– the forth one

$$J_n = \int \frac{dx}{\left[a\cosh(bx)\right]^n} = \frac{-i\,\mathrm{sech}^{n-1}(bx)}{(n-1)a^n b} F\left[\frac{1}{2}, \frac{1-n}{2}; \frac{3-n}{2}; \cosh^2(bx)\right], \tag{A.20}$$

where: $i = \sqrt{-1}$, and the real value of the definite integral $J_n\Big|_{-l/2}^{+l/2}$ is equal to

$$J_n\Big|_{-l/2}^{+l/2} = \frac{2\,\mathrm{sech}^{n-1}\left(\dfrac{bl}{2}\right)}{(n-1)a^n b} \mathrm{Im}\left(F\left[\frac{1}{2}, \frac{1-n}{2}; \frac{3-n}{2}; \cosh^2\left(\frac{bl}{2}\right)\right]\right); \tag{A.21}$$

here $\mathrm{Im}\left(F[...]\right)$ is the real value of the imaginary part of the hypergeometric function $F[...]$; for the Newtonian flows we have

$$J_3 = \int \frac{dx}{\left[a\cosh(bx)\right]^3} = \frac{1}{a^3 b}\left\{\frac{\sinh(bx)}{2\left[\cosh(bx)\right]^2} + \frac{1}{2}\arctan\left[\sinh(bx)\right]\right\} \tag{A.22}$$

and

$$J_3\Big|_{-l/2}^{+l/2} = \frac{1}{a^3 b}\left\{\frac{\sinh\left(\dfrac{bl}{2}\right)}{\left[\cosh\left(\dfrac{bl}{2}\right)\right]^2} + \arctan\left[\sinh\left(\frac{bl}{2}\right)\right]\right\} \tag{A.23}$$

or

$$J_4 = \int \frac{dx}{\left[a\cosh(bx)\right]^4} = \frac{1}{a^4 b}\left\{1 - \frac{\left[\tanh(bx)\right]^2}{3}\right\}\tanh(bx) \tag{A.24}$$

and

$$J_4\Big|_{-l/2}^{+l/2} = \frac{2}{a^4 b}\left\{1 - \frac{\left[\tanh\left(\frac{bl}{2}\right)\right]^2}{3}\right\}\tanh\left(\frac{bl}{2}\right) \; ; \tag{A.25}$$

– the fifth one

$$J_n = \int \frac{dx}{\left[a\cosh(bx)\right]^{-n}} = \frac{i\cosh^{n+1}(bx)}{(n-1)a^{-n}b} F\left[\frac{1}{2},\frac{1+n}{2};\frac{3+n}{2};\cosh^2(bx)\right], \tag{A.26}$$

and the real value of the definite integral $J_n\big|_{-l/2}^{+l/2}$ is equal to

$$J_n\Big|_{-l/2}^{+l/2} = \frac{2\cosh^{n+1}\left(\frac{bl}{2}\right)}{(n+1)a^{-n}b}\mathrm{Im}\left(F\left[\frac{1}{2},\frac{1+n}{2};\frac{3+n}{2};\cosh^2\left(\frac{bl}{2}\right)\right]\right). \tag{A.27}$$

For the Newtonian flows we have

$$J_3 = \int\left[a\cosh(bx)\right]^3 dx = \frac{3a^3}{4b}\left[\sinh(bx) + \frac{1}{9}\sinh(3bx)\right] \tag{A.28}$$

and

$$J_3\Big|_{-l/2}^{+l/2} = \frac{3a^3}{2b}\left[\sinh\left(\frac{bl}{2}\right) + \frac{1}{9}\sinh\left(\frac{3bl}{2}\right)\right] \tag{A.29}$$

or

$$J_4 = \int\left[a\cosh(bx)\right]^4 dx = \frac{a^4}{8b}\left[3bx + 2\sinh(2bx) + \frac{1}{4}\sinh(4bx)\right] \tag{A.30}$$

and

$$J_4\Big|_{-l/2}^{+l/2} = \frac{2}{4^4 b}\left[\frac{3bl}{2} + 2\sinh(2bl) + \frac{1}{4}\sinh(2bl)\right] . \tag{A.31}$$

The last integral, for cosine curve capillaries, is given by the following expression [22, 23]

$$J_n = \int \frac{dx}{\left[a \mp b\cos(kx)\right]^n} = \frac{-i}{(n-1)k\sqrt{a^2-b^2}\left[a \mp b\cos(kx)\right]^{n-1}} F_1[...] \tag{A.32}$$

where $F_1[...]$ is the Appell hypergeometric function described here by the formula

$$F_1[...] = F_1\left[1-n;\frac{1}{2},\frac{1}{2};2-n;\frac{a \mp b\cos(kx)}{a \pm b},\frac{a \mp b\cos(kx)}{a \mp b}\right]. \tag{A.33}$$

Generally, the Appell hypergeometric function $F_1(x,y)$ is defined by the following double hypergeometric series [24]

$$F_1\left[a;b_1,b_2;c;x,y\right]=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a)_{m+n}(b_1)_m(b_2)_n}{(c)_{m+n}\,m!n!}x^m y^n. \tag{A.34}$$

It is easy to see that it is a bivariate generalization of the Gauss hypergeometric series defined by formula (A.7). Introducing in (A.20) the limits of integration we will obtain

$$J_n\Big|_{-l/2}^{+l/2}=\frac{2}{(n-1)k\sqrt{a^2-b^2}\left[a\mp b\cos\left(\frac{kl}{2}\right)\right]^{n-1}}\operatorname{Im}\left(F_1[...]\right) \tag{A.35}$$

where

$$\operatorname{Im}\left(F_1[...]\right)=\operatorname{Im}\left(F_1\left[1-n;\frac{1}{2},\frac{1}{2};2-n;\frac{a\mp b\cos\left(\frac{kl}{2}\right)}{a\pm b},\frac{a\mp b\cos\left(\frac{kl}{2}\right)}{a\mp b}\right]\right); \tag{A.36}$$

Here, $\operatorname{Im}\left(F_1[...]\right)$ is the imaginary part of the Appell hypergeometric function $F_1[...]$.
For the Newtonian flows we have

$$J_3=\int\frac{dx}{\left[a\mp b\cos(kx)\right]^3}=\frac{1}{2k\left(a^2-b^2\right)}\left\{\frac{\pm b\sin(kx)}{\left[a\mp b\cos(kx)\right]^2}+\right.$$
$$\left.+\frac{3a}{\left(a^2-b^2\right)}\frac{\pm b\sin(kx)}{\left[a\mp b\cos(kx)\right]}+\frac{2a^2+b^2}{\left(a^2-b^2\right)}\frac{2}{\sqrt{a^2-b^2}}\arctan\frac{\sqrt{a^2-b^2}\,\tan\left(\frac{kx}{2}\right)}{a\mp b}\right\} \tag{A.37}$$

and

$$J_3\Big|_{-l/2}^{+l/2}=\frac{1}{k\left(a^2-b^2\right)}\left\{\frac{\pm b\sin\left(\frac{kl}{2}\right)}{\left[a\mp b\cos\left(\frac{kl}{2}\right)\right]^2}+\frac{3a}{\left(a^2-b^2\right)}\frac{\pm b\sin\left(\frac{kl}{2}\right)}{\left[a\mp b\cos\left(\frac{kl}{2}\right)\right]}+\right.$$
$$\left.+\frac{2a^2+b^2}{\left(a^2-b^2\right)}\frac{2}{\sqrt{a^2-b^2}}\arctan\frac{\sqrt{a^2-b^2}\,\tan\left(\frac{kl}{4}\right)}{a\mp b}\right\}; \tag{A.38}$$

or

$$J_4=\int\frac{dx}{\left[a\mp b\cos(kx)\right]^4}=\frac{1}{3k\left(a^2-b^2\right)}\left\{\frac{\pm b\sin(kx)}{\left[a\mp b\cos(kx)\right]^3}+\frac{5a}{\left(a^2-b^2\right)}\frac{\pm b\sin(kx)}{\left[a\mp b\cos(kx)\right]^2}+\right.$$
$$\left.+\frac{11a^2+4b^2}{2\left(a^2-b^2\right)^2}\frac{\pm b\sin(kx)}{\left[a+b\cos(kx)\right]}+\frac{6a^3+9ab^2}{\left(a^2-b^2\right)^2}\frac{1}{\sqrt{a^2-b^2}}\arctan\frac{\sqrt{a^2-b^2}\,\tan\left(\frac{kx}{2}\right)}{a\mp b}\right\} \tag{A.39}$$

and

$$J_4\Big|_{-l/2}^{+l/2}=\frac{2}{3k\left(a^2-b^2\right)}\left\{\frac{\pm b\sin\left(\frac{kl}{2}\right)}{\left[a\mp b\cos\left(\frac{kl}{2}\right)\right]^3}+\frac{5a}{\left(a^2-b^2\right)}\frac{\pm b\sin\left(\frac{kl}{2}\right)}{\left[a\mp b\cos\left(\frac{kl}{2}\right)\right]^2}+\right.$$
$$\left.+\frac{11a^2+4b^2}{2\left(a^2-b^2\right)^2}\frac{\pm b\sin\left(\frac{kl}{2}\right)}{\left[a\mp b\cos\left(\frac{kl}{2}\right)\right]}+\frac{6a^3+9ab^2}{\left(a^2-b^2\right)^2}\frac{1}{\sqrt{a^2-b^2}}\arctan\frac{\sqrt{a^2-b^2}\tan\left(\frac{kl}{4}\right)}{a\mp b}\right\}. \tag{A.40}$$

Nomenclature

a, b – auxiliary constants in the formula describing a converging-diverging capillary
F – hypergeometric function
F_1 – Appell hypergeometric function
F_{cor} – correction factor for a capillary fissure
f_c – half thickness of a capillary fissure
f_i – inlet half thickness of a capillary fissure
f_o – middle half thickness of a capillary fissure
$i=\sqrt{-1}$
l – capillary length
m – flow behaviour index for power-law fluid
n – auxiliary in integrals and hypergeometric functions
p – pressure
Δp – pressure drop
Q – volumetric flow rate
Q_f – volumetric flow rate through the unity width of a capillary fissure
Q_t – volumetric flow rate through a capillary tube
r_c – radius of a capillary tube
r_i – inlet radius of a capillary tube
r_o – middle radius of a capillary tube
T_{cor} – correction factor for capillary tube
υ_f – flow velocity through a thin porous layer modelled by capillary fissures
υ_t – flow velocity through a thin porous layer modelled by capillary tubes
φ_p – porosity of a porous layer
μ – fluid viscosity

References

[1] Bird R.B., Steward W.F. and Lightfoot E.N. (1960): *Transport Phenomena*. – New York: Wiley.

[2] Lahbabi A. and Chang H.-C. (1986): *Flow in periodically constricted tubes: transition to inertial and nonsteady flows*. – Chem. Eng. Sci., vol.41, No.10, pp. 2487-2505.

[3] Burdette S.R., Coates P.J, Armstrong R.C. and Brown R.A. (1989): *Calculations of viscoelastic flow through an axisymmetric corrugated tube using the explicity elliptic momentum equation formulation* (*EEME*). – J. Non-Newtonian Fluid Mech., vol.33, No.1, pp.1-23.

[4] James D.F., Phan-Thien N., Khan M.M.K., Beris A.N. and Pilitsis S. (1990): *Flow of test fluid M1 in corrugated tubes*. – J. Non-Newtonian Fluid Mech., vol.35, No.2-3, pp.405-412.

[5] Momeni-Masuleh S.H. and Phillips T.N. (2004): *Viscoelastic flow in an undulating tube using spectral methods*. – Comput. Fluids, vol.33, No.8, pp.1075-1095.

[6] Wang Y., Hayat T., Ali N. and Oberlack M. (2008): *Magnetohydrodynamic peristaltic motion of a Sisko fluid in a symmetric or asymmetric channel*. – Physica A, vol.387, pp.347-362.

[7] Hayat T., Maqbool K. and Asghar S. (2009): *Hall and heat transfer effects on the steady flow of a Sisko fluid*. – Z. Naturforsch. vol.64a, No.769-782.

[8] Hayat T., Moitsheki R.J. and Abelman S. (2010): *Stokes' first problem for Sisko fluid over a porous wall*. – Appl. Math. Comp., vol.217, No.2, pp.622-628.

[9] Mekheimer Kh.S. and El Kot M.A. (2012): *Mathematical modelling of unsteady flow of a Sisko fluid through an anisotropically tapered elastic arteries with time-variant overlapping stenosis.* – Appl. Math. Modelling. vol.36, No.11, pp.5393-5407.

[10] Nadeem S., Sadaf H. and Akbar N.S. (2015): *Effects of nanoparticles on the peristaltic motion of tangent hyperbolic fluid model in an annulus*. – Alexandria Eng. J., vol.54, pp.843-851.

[11] Williams E.W. and Javadpour S.H. (1980): *The flow of an elastico-viscous liquid in an axisymmetric pipe of slowly varying cross-section*. – J. Non-Newtonian Fluid Mech., vol.7, No.2-3, pp.171-188.

[12] Walicki E., Walicka A. Michalski D. and Ratajczak P. (1998): *Approximate analysis for conical flow of generalized second grade fluid.* – Les Cahiers de Rhéologie, vol.16, No.1, pp.309-316.

[13] Walicki E. and Walicka A. (2000*): Pressure drops in a wedge flow of generalized second grade fluids of a power-law type and a Bingham type*. – Les Cahiers de Rhéologie, vol.17, No.1, pp.541-550.

[14] Walicki E. and Walicka A. (2000): *Conical flow of generalized second grade fluids.* – Chem. Proc. Eng., vol.21, No.1, pp.75-85.

[15] Walicki E. and Walicka A. (2002): *Convergent flows of molten polymers modeled by generalized second-grade fluids of a power type.* – Mech. Comp. Mat., vol.38, No.1, pp.89-94.

[16] Walicka A. and Walicki E. (2010): *Pressure drops in convergent flows of polymer melts.* – Int. J. Appl. Mech. Eng., vol.15, No.4, pp.1273-1285.

[17] Sochi T. (2010): *The flow of Newtonian fluids in axisymmetric corrugated tubes*. – arXiv:1006.1515v1.

[18] Sochi T. (2011): *Newtonian flow in converging-diverging capillares*. – arXiv:1108.0163v1.

[19] Sochi T. (2011): *The flow of power-law fluids in axisymmetric corrugated tubes*. – J. Petrol. Sci. Eng., vol.78, pp.582-585.

[20] Sochi T. (2015): *Navier-Stokes flow in converging-diverging distensible tubes*. – Alexandria Eng. J., vol.54, pp.713-723.

[21] Walicka A. (2017): *Rheology of Fluids in Mechanical Engineering*. – Zielona Gora: University Press.

[22] Gradshteyn I.S and Ryzhik I.M. (2014): *Tables of Integrals, Series and Products, 8th ed*. – Orlando, Florida: Academic Press.

[23] Wolfram: *Mathematica 10*. – Wolfram Res. Ing.

[24] Olver F.W.J., Lozier D.W., Boisvert R.F. and Clark C.W. (2010): *NIST Handbook of Mathematical Functions*. – NIST & Cambridge University Press.

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