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Fluctuations for a conservative interface model on a wall

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Abstract. We consider an effective interface model on a hard wall in (1+1) dimensions, with conservation of the area between the interface and the wall. We prove that the equilibrium fluctuations of the height variable converge in law to the solution of a SPDE with reflection and conservation of the space average. The proof is based on recent results obtained with L. Ambrosio and G. Savaré on stability properties of Markov processes with log-concave invariant measures.

1. Introduction

This paper concerns fluctuations of a $\nabla\phi$ interface model on a hard wall with conservation of the area between the interface and the wall. The system is defined on the one-dimensional lattice $\Gamma_N := \{1, 2, \dots, N\}$ and the location of the interface at time t is represented by the height variables $\phi_t = \{\phi_t(x), x \in \Gamma_N\} \in \Omega_N^+ := [0,\infty)^{\Gamma_N}$ measured from the wall Γ_N.

In order to describe the dynamics of ϕ_t we need some notation. Let $\{(w_t(x))_{t\geq 0} : x = 1, \dots, N\}$ be independent standard Brownian motions and define the $N \times N$ matrices

$$\sigma := \begin{pmatrix} -1 & & & & \\ 1 & \cdot & & & \\ & \cdot & \cdot & & \\ & & 1 & -1 & \\ & & & 1 & 0 \end{pmatrix}, \qquad \sigma^T := \begin{pmatrix} -1 & 1 & & & \\ & \cdot & \cdot & & \\ & & \cdot & \cdot & \\ & & & -1 & 1 \\ & & & 0 & 0 \end{pmatrix}$$

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Then the dynamics of $(\phi_t(x) : x \in \Gamma_N)_{t\geq 0}$, height from the wall of the reflected interface, is governed by the stochastic differential equation of the Skorohod type

$$d\phi_t \;=\; -\,\sigma\sigma^T\{\sigma V'(\sigma^T\phi_t)\,dt + \,dl_t\} + \sqrt{2}\,\sigma\,dw_t \tag{1.1}$$

for all $x \in \Gamma_N$, subject to the conditions

$$\begin{aligned} &\phi_t(x) \,\geq\, 0, \qquad t \mapsto l_t(x) \text{ continuous and non} - \text{decreasing}, \qquad l_0(x) = 0, \\ &\int_0^\infty \phi_t(x)\,dl_t(x) \;=\; 0, \qquad x \in \Gamma_N. \end{aligned} \tag{1.2}$$

We refer to Funaki (2005) for an introduction to interface models.

Throughout the paper the potential V satisfies the following conditions

(V1) (convexity) $V \in C^2(\mathbb{R})$ is convex and $\lim_{|r|\to\infty} V(r) = +\infty$.

Notice that for a convex V

$$\lim_{|r|\to\infty} V(r) = +\infty \iff \int \exp(-V)\,dr < \infty \iff V(r) \geq a + b|r| \quad \forall\, r \in \mathbb{R},$$

for some $a \in \mathbb{R}$ and $b > 0$. In particular we have

$$q \;:=\; \int_{\mathbb{R}} r^2\, \exp(-V(r))\,dr \;<\; \infty \tag{1.3}$$

(V2) (normalization), $\displaystyle\int_{\mathbb{R}} \exp(-V(r))\,dr \;=\; 1.$

(V3) (0 mean), $\displaystyle\int_{\mathbb{R}} r\, \exp(-V(r))\,dr \;=\; 0.$

The normalization (V2) does not affect equation (1.1), where only V' appears.

If X is a real random variable with distribution $e^{-V}dx$, then X has zero mean by (V3) and variance equal to q by (1.3). Notice that q is the only trace of the potential V which survives in the limit fluctuation process, see (1.5) and Theorem 1.1 below. See also subsection 1.3 below for a discussion of related random walk models.

We shall prove in the following sections existence and uniqueness of solutions of (1.1) and other properties.

1.1. *The main result.* For any $N \in \mathbb{N}$ we set $\Lambda_N : \mathbb{R}^N \mapsto L^2(0,1)$,

$$\Lambda_N(\phi)(\theta) \;:=\; \frac{1}{\sqrt{N}}\,\phi(\lfloor N\theta \rfloor + 1), \quad \theta \in [0,1), \tag{1.4}$$

where $\lfloor\cdot\rfloor$ denotes the integer part, and we define the spaces

$$H_N = \Lambda_N(\mathbb{R}^N) \subset L^2(0,1), \qquad \Omega_N^+ := (\mathbb{R}_+)^N, \qquad K_N := \Lambda_N(\Omega_N^+).$$

Notice that K_N can be identified with the space of non-negative functions on $[0,1)$ being constant on $I(x) = [(x-1)/N, x/N)$ for all $x \in \Gamma_N$.

For all $k \in K_N$ and $t \geq 0$ we define now the rescaled interface Φ^N

$$\Phi_t^N \;:=\; \Lambda_N\left(\phi_{N^4 t}\right), \qquad \Phi_0^N := \Lambda_N\left(\phi_0\right).$$

In other words

$$\Phi_t^N(\theta) \;=\; \frac{1}{\sqrt{N}}\,\phi_{N^4 t}(\lfloor N\theta \rfloor + 1), \quad \theta \in [0,1).$$

In the main result of this paper, i.e. Theorem 1.1 below, we state the weak convergence of Φ^N to the unique solution u of the following stochastic Cahn-Hilliard equation on $[0,1]$ with homogeneous Neumann boundary condition and reflection at $u=0$

$$\begin{cases} \dfrac{\partial u}{\partial t} = -\dfrac{\partial^2}{\partial\theta^2}\left(\dfrac{1}{q}\dfrac{\partial^2 u}{\partial\theta^2}+\eta\right)+\sqrt{2}\,\dfrac{\partial}{\partial\theta}\dot{W}, \\ \dfrac{\partial u}{\partial\theta}(t,0)=\dfrac{\partial u}{\partial\theta}(t,1)=\dfrac{\partial^3 u}{\partial\theta^3}(t,0)=\dfrac{\partial^3 u}{\partial\theta^3}(t,1)=0, \\ u(0,\theta)=u_0(\theta), \quad \theta\in[0,1] \end{cases} \tag{1.5}$$

where $\dot{W}$ is a space-time white noise on $[0,+\infty)\times[0,1]$, u is a continuous function of $(t,\theta)\in[0,+\infty)\times[0,1]$, η is a locally finite positive measure on $(0,+\infty)\times[0,1]$, subject to the constraint

$$u\geq 0, \qquad \int_{(0,+\infty)\times[0,1]} u\,d\eta \,=\, 0. \tag{1.6}$$

Such equation has been studied by Debussche and Zambotti (2007), see Proposition 6.2 below.

With an abuse of notation, we say that a sequence of measures $(\mathbf{P}_n)$ on the space $C([a,b];L^2(0,1))$ converges weakly in $C([a,b];L^2_w(0,1))$ if, for all $m\in\mathbb{N}$ and $h_1,\ldots,h_m\in C^1([0,1])$, the process $(\langle X_\cdot,h_i\rangle_{L^2(0,1)},\ i=1,\ldots,m)$ under $(\mathbf{P}_n)$ converges weakly in $C([a,b];\mathbb{R}^m)$ as $n\to\infty$.

Then we can state the main result of this paper.

Theorem 1.1. *If $\Phi_0^N\to u_0$ in $L^2(0,1)$ as $N\to\infty$ with*

$$\Phi_0^N\geq 0, \qquad \int_0^1 \Phi_0^N(\theta)\,d\theta = c>0 \qquad \forall\ N\in\mathbb{N},$$

then, for any $0<\varepsilon\leq T<\infty$, the law of $(\Phi_t^N, t\in[\varepsilon,T])$ converges to the law of the unique solution u of (1.5)*, weakly in $C([\varepsilon,T];L^2_w(0,1))$.*

Notice that the technique used in this paper to prove Theorem 1.1 could also be applied to other situations, see e.g. the discussion below. However, not all situations can be covered: for instance, if the macroscopic (hydrodynamical) limit is not constant, then the fluctuations process is not in general a time-homogeneous Markov process and our technique can not be applied (in its present formulation).

1.2. *A conservative dynamics.* The starting point of this work is the paper by Funaki and Olla (2001). In that paper, the following $\nabla\phi$ interface model on a hard wall is considered

$$d\phi_t(x) = -\,\sigma V'(\sigma^T\phi_t)\,dt + \,dl_t(x)\, +\, \sqrt{2}\,dw_t(x), \qquad x\in\Gamma_N, \tag{1.7}$$

with constraints analogous to (1.2) and Dirichlet boundary condition $\phi_t(0)=\phi_t(N+1)=0$. Using the definition (1.4), it is then proven that in the stationary case, the process $(\Lambda_N(\phi_{N^2t}), t\geq 0)$ converges in law as $N\to\infty$ to the law of the unique stationary solution of the second order equation

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{1}{q}\dfrac{\partial^2 u}{\partial \theta^2} + \eta + \sqrt{2}\,\dfrac{\partial^2 W}{\partial t \partial \theta} \\ u(t,0) = u(t,1) = 0, \quad t \geq 0 \\ u \geq 0, \ d\eta \geq 0, \ \int u \, d\eta = 0 \end{cases} \tag{1.8}$$

At the end of the introduction of Funaki and Olla (2001), it is remarked that it would be more natural to consider a stochastic dynamics conserving the area between the interface and the wall, namely $\sum_x \phi(x)$. Such conservative dynamics, but without the hard wall constraint, has indeed been studied in Nishikawa (2002) and Nishikawa (2007), where respectively hydrodynamic limit and large deviations are considered; the hydrodynamic scaling limit of the interface is the solution of a fourth-order equation, as predicted by Spohn (1993).

The SDE (1.1) combines the hard wall and the conservation of volume constraints; indeed, $\sigma^T \mathbf{1} = 0$, where $\mathbf{1} = (1, \ldots, 1) \in \mathbb{R}^N$, and it is easy to see that

$$d\left[\sum_{x=1}^{N} \phi_t(x)\right] = \sum_{x=1}^{N} \left[\sigma^T \mathbf{1}\right](x) \left\{ - \left[\sigma^T \{\sigma V'(\sigma^T \phi_t)\, dt + dl_t\}\right](x) + \sqrt{2}\, dw_t(x) \right\} = 0.$$

The main novelty of this paper is the use of a technique recently developed by Ambrosio et al. (2008) for the convergence in law of stochastic processes associated with symmetric Dirichlet forms of gradient type and with log-concave invariant measures; see section 2 below. The general principle is in fact very simple: this class of reversible dynamics is parametrized by two objects, the invariant measure and the scalar product of the Hilbert space which defines the gradient. If such objects converge (in a sense to be made precise), it is natural to conjecture that the associated processes converge; the results of Ambrosio et al. (2008) confirm this conjecture in the case of log-concave reference measures: see section 2 below.

The solutions of equations (1.1), (1.5), (1.7) and (1.8) are all in this class and the techniques of Ambrosio et al. (2008) give a general framework to prove results like Theorem 1.1 or the convergence result of Funaki and Olla (2001). We recall that the results by Funaki and Olla (2001) are based on monotonicity properties, i.e. on rather special properties of (1.7)-(1.8), not shared by (1.1)-(1.5). One can notice that, given the general results of Ambrosio et al. (2008), the proof of convergence of equilibrium fluctuations as in Funaki and Olla (2001) and in this paper becomes much easier.

We also notice that Theorem 1.1 is comparatively stronger than the analogous statement in Funaki and Olla (2001). Indeed, we consider a convex microscopic interaction potential V, instead of a strictly convex and symmetric one. Moreover the convergence is proven not only in the stationary case, but for any sequence of initial conditions which converge under the rescaling (1.4). Using the techniques of this paper, one could improve correspondingly the results of Funaki and Olla (2001).

1.3. *Boundary conditions and random walk models.* Finally, we notice that the boundary conditions we consider are of Neumann type, like in Collet et al. (1995), while many other papers consider the Dirichlet case, see e.g. Funaki and Olla

(2001), or the periodic case, see e.g. Nishikawa (2002). The case of periodic boundary condition could be proven with no additional difficulty with the techniques of this paper. Indeed, like in the Neumann case, the invariant measure of the limit SPDE is absolutely continuous w.r.t. the Gaussian invariant probability measure of the linear SPDE (i.e. without reflection). The weak convergence of the rescaled stationary measures is then a simple consequence of a standard invariance principle: see the proof of Proposition 6.3.

For the case of homogeneous Dirichlet boundary conditions, on the contrary, the invariant measure of the limit SPDE is singular w.r.t. the Gaussian invariant probability measure of the linear SPDE, due to the interplay of the homogeneous boundary conditions and the non-negativity constraint. This makes the convergence of the rescaled invariant measures more delicate.

In fact, we could prove the results of this paper for Dirichlet boundary condition, if we could prove the following invariance principle: we consider a random walk $S_n = X_1 + \cdots + X_n$, $n = 1, \dots, N$, with step distribution $X_i \sim e^{-V} dx$, conditioned to be non-negative (i.e. $S_1, \dots, S_N \geq 0$), to be 0 at time N (i.e. $S_N = 0$) and to have a fixed sum (i.e. $\sum_{n=1}^{N} S_n = cN^{3/2}$, $c > 0$); then we would like to prove that such processes converge under Brownian rescaling as $N \to \infty$ to a Brownian excursion e conditioned to have integral c (i.e. $\int_0^1 e_x \, dx = c$). Since we have not found a proof for this invariance principle, we restrict to the Neumann case, for which we can prove convergence of the stationary measures. In the Dirichlet boundary condition case the limit SPDE would be an analog of (1.5), with boundary conditions

$$u(t,0) = u(t,1) = \frac{\partial^3 u}{\partial \theta^3}(t,0) = \frac{\partial^3 u}{\partial \theta^3}(t,1) = 0,$$

i.e. Dirichlet for u and Neumann for $\frac{\partial^2 u}{\partial \theta^2}$. Such equation is studied in Zambotti (2008).

The invariance principle for S_n, conditioned to be 0 at time N (i.e. $S_N = 0$) and to have a fixed sum (i.e. $\sum_{n=1}^{N} S_n = cN^{3/2}$, $c > 0$), but without positivity constraint, is treated in Caravenna and Deuschel (2008a) and Caravenna and Deuschel (2008b).

2. A general convergence result

In this section we recall the results of Ambrosio et al. (2008), already mentioned in the introduction. It turns out that the processes (ϕ_t) and $(u(t,\cdot)$, solutions of (1.1) and (1.5) respectively, are both *monotone gradient systems*, i.e. the equation they satisfy can be interpreted as follows

$$dX = -\nabla U(X) \, dt + \sqrt{2} \, dW$$

where W is a Wiener process in a Hilbert space H and $U : H \mapsto \mathbb{R} \cup \{+\infty\}$ is a convex potential. These processes are reversible and associated with a gradient-type Dirichlet form. The general results of existence and convergence of such processes given in Ambrosio et al. (2008), have a nice application in the present setting. Hence we devote this section to recall them.

Let H be a separable Hilbert space with scalar product $\langle \cdot, \cdot \rangle_H$ and let γ be a probability measure on H. We suppose that γ is *log-concave*, i.e. for all pairs of open sets $B, C \subset H$

$$\log \gamma\left((1-t)B + tC\right) \geq (1-t)\log \gamma(B) + t\log \gamma(C) \qquad \forall t \in (0,1). \tag{2.1}$$

If $H = \mathbb{R}^k$, then the class of log-concave probability measures contains all measures of the form (here $\mathcal{L}_k$ stands for Lebesgue measure)

$$\gamma := \frac{1}{Z}\, e^{-U} \mathcal{L}_k, \tag{2.2}$$

where $U : H = \mathbb{R}^k \to \mathbb{R} \cup \{+\infty\}$ is convex and $Z := \int_{\mathbb{R}^k} e^{-U}\, dx < +\infty$, see Theorem 9.4.11 in Ambrosio et al. (2005), in particular all Gaussian measures. Notice that the class of log-concave measures is closed under weak convergence. Moreover, if γ is log-concave and K is a convex set with $\gamma(K) > 0$, then the conditional measure $\gamma(\cdot|K) := \gamma(\cdot \cap K)/\gamma(K)$ is also log-concave.

We denote the support of γ by $K = K(\gamma)$ and the smallest closed affine subspace of H containing K by $A = A(\gamma)$. We write canonically

$$A \,=\, H^0 \,+\, h^0, \tag{2.3}$$

where $H^0 = H^0(\gamma)$ is a closed linear subspace of H and $h^0 = h^0(\gamma)$ is the element of minimal norm in A. We endow H^0 with the scalar product $\langle \cdot, \cdot \rangle_{H^0}$ induced by H.

We want to consider a stochastic processes with values in A and reversible with respect to γ. We denote by $C_b(H)$ the space of bounded continuous functions in H and by $C_b^1(A)$ the space of all $\Phi : A \mapsto \mathbb{R}$ which are bounded, continuous and Fréchet differentiable. To $\varphi \in C_b^1(A)$ we associate a gradient $\nabla_{H^0}\varphi : A \mapsto H^0$, defined by

$$\frac{d}{d\varepsilon}\, \varphi(k + \varepsilon\, h)\bigg|_{\varepsilon=0} \,=\, \langle \nabla_{H^0}\varphi(k), h \rangle_{H^0}, \qquad \forall\; k \in A,\;\; h \in H^0. \tag{2.4}$$

We denote by $X_t : K^{[0,+\infty[} \to K$ the coordinate process $X_t(\omega) := \omega_t$, $t \geq 0$. Finally, we denote the set of probability measures on H by $\mathscr{P}(H)$ and we set

$$\mathscr{P}_2(H) := \left\{ \mu \in \mathscr{P}(H) : \int_H \|x\|_H^2 \, d\mu(x) < \infty \right\},$$

Then we recall one of the main results of Ambrosio et al. (2008).

Theorem 2.1 (Markov process and Dirichlet form associated with γ and $\|\cdot\|_{H^0}$)**.**

(a) *The bilinear form $\mathcal{E} = \mathcal{E}_{\gamma, \|\cdot\|_{H^0}}$ given by*

$$\mathcal{E}(u,v) \,:=\, \int_K \langle \nabla_{H^0} u, \nabla_{H^0} v \rangle_{H^0}\, d\gamma, \qquad u,\, v \in C_b^1(A), \tag{2.5}$$

is closable in $L^2(\gamma)$ and its closure $(\mathcal{E}, D(\mathcal{E}))$ is a symmetric Dirichlet Form. Furthermore, the associated semigroup $(P_t)_{t\geq 0}$ in $L^2(\gamma)$ maps $L^\infty(\gamma)$ in $C_b(K)$.

(b) *There exists a unique Markov family $(\mathbb{P}_x : x \in K)$ of probability measures on $K^{[0,+\infty[}$ associated with $\mathcal{E}$. More precisely, $\mathbb{E}_x[f(X_t)] = P_t f(x)$ for all bounded Borel functions and all $x \in K$.*

(c) *For all $x \in K$, $\mathbb{P}_x^*\left(C(]0,+\infty[; H)\right) = 1$ and $\mathbb{E}_x[\|X_t - x\|^2] \to 0$ as $t \downarrow 0$. Moreover, $\mathbb{P}_x^*\left(C([0,+\infty[; H)\right) = 1$ for γ-a.e. $x \in K$.*

(d) $(\mathbb{P}_x : x \in K)$ *is reversible with respect to* γ, *i.e. the transition semigroup* $(P_t)_{t\geq 0}$ *is symmetric in* $L^2(\gamma)$*; moreover* γ *is invariant for* (P_t), *i.e.* $\gamma(P_t f) = \gamma(f)$ *for all* $f \in C_b(K)$ *and* $t \geq 0$.

(e) *If* $\gamma \in \mathscr{P}_2(H)$, *then* γ *is the only invariant probability measure for* (P_t) *in* $\mathscr{P}_2(H)$.

We shall see below that the solutions of (1.1), (1.5), (1.7) and (1.8) are all particular cases of the class of Markov processes described in Theorem 2.1. This fact will be crucial in the proof of Theorem 1.1.

We consider now a sequence (γ_N) of log-concave probability measures on H such that γ_N converge weakly in H to γ. We denote by K_N the support of γ_N, and by A_N the smallest closed affine subspace of H containing K_N. We suppose that $A_N \subseteq A$ for all N.

We write $A_N = h_N^0 + H_N^0$, where $h_N^0 \in A_N$ and $H_N^0 \subseteq H^0$ is a closed linear subspace of H. We want to consider situations where each H_N^0 is a Hilbert space endowed with a scalar product $\langle \cdot, \cdot \rangle_{H_N^0}$, possibly different from the scalar product induced by H^0. In order to ensure that this family of scalar products converges (in a suitable sense) to the scalar product of H^0 as $N \to \infty$, we will make the following assumptions.

(1) There exists a finite constant $\kappa \geq 1$ such that

$$\frac{1}{\kappa}\|h\|_{H^0} \leq \|h\|_{H_N^0} \leq \kappa \|h\|_{H^0} \qquad \forall\, h \in H_N^0,\ N \in \mathbb{N}. \tag{2.6}$$

(2) Denoting by $\Pi_N : H^0 \to H_N^0$ the orthogonal projections induced by the scalar product of H^0, we have

$$\lim_{N\to\infty} \|\Pi_N h\|_{H_N^0} = \|h\|_{H^0} \qquad \forall\, h \in H^0. \tag{2.7}$$

These assumptions guarantee in some weak sense that the geometry of H_N^0 converges to the geometry of H^0; the case when all scalar products coincide with $\langle \cdot, \cdot \rangle_H$, $H_N^0 \subset H_{N+1}^0$ and $\cup_N H_N^0$ is dense in H^0 is obviously included.

Let $(\mathbb{P}_x^N : x \in K_N)$ (respectively $(\mathbb{P}_x : x \in K)$) be the Markov process in $[0, +\infty[^{K_N}$ associated to γ_N (resp. in $[0, +\infty[^K$ associated to γ) given by Theorem 2.1. We denote by $\mathbb{P}_{\gamma_N}^N := \int \mathbb{P}_x^N \, d\gamma_N(x)$ (resp. $\mathbb{P}_\gamma := \int \mathbb{P}_x \, d\gamma(x)$) the associated stationary measures.

With an abuse of notation, we say that a sequence of measures $(\mathbf{P}_n)$ on the space $C([a,b];H)$ converges weakly in $C([a,b];H_w)$ if, for all $m \in \mathbb{N}$ and $h_1, \ldots, h_m \in H$, the process $(\langle X_\cdot, h_i \rangle_H,\, i = 1, \ldots, m)$ under $(\mathbf{P}_n)$ converges weakly in $C([a,b];\mathbb{R}^m)$ as $n \to \infty$.

In this setting we have the following stability and tightness result, also proven in Ambrosio et al. (2008).

Theorem 2.2 (Stability and tightness)**.** *Suppose that* $\gamma_N \to \gamma$ *weakly in* H *and that the norms of* H_N^0 *satisfy* (2.6) *and* (2.7). *Then, for any* $x_N \in K_N$ *such that* $x_N \to x \in K$ *in* H, *for any* $0 < \varepsilon \leq T < +\infty$, $\mathbb{P}_{x_N}^N \to \mathbb{P}_x$ *weakly in* $C([\varepsilon, T]; H_w)$*;*

This stability property means that the weak convergence of the invariant measures γ_N and a suitable convergence of the norms $\|\cdot\|_{H_N^0}$ to $\|\cdot\|_{H^0}$ imply the convergence in law of the associated processes, starting from any initial condition.

We recall that the above results, proven in Ambrosio et al. (2008), are based on the interpretation of the Markov semigroup (P_t) as the solution of a gradient flow in $\mathscr{P}_2(H)$ with respect to the relative entropy functional $\mathcal{H}(\cdot|\gamma)$ in the Wasserstein metric: see Ambrosio et al. (2008) for details.

In the rest of the paper we show how the results of this section apply to Theorem 1.1.

3. The microscopic dynamics

On $\mathbb{R}^N$ we consider the canonical scalar product and we denote it by $\langle \cdot, \cdot \rangle_{\mathbb{R}^N}$, with associated norm $\| \cdot \|_{\mathbb{R}^N}$.

We define $\mathbf{1} := (1, \ldots, 1) \in \mathbb{R}^N$ and the vector space $\mathbb{V}_N := \{v \in \mathbb{R}^N : v_1 + \cdots + v_N = 0\} = \mathbf{1}^\perp$. It is easy to see that the kernels of σ and σ^T are respectively $\mathrm{Ker}(\sigma) = \{(0, \ldots, 0, t) : t \in \mathbb{R}\}$ and $\mathrm{Ker}(\sigma^T) = \{t \cdot \mathbf{1} \in \mathbb{R}^N : t \in \mathbb{R}\}$; it follows that the image of σ is $\mathrm{Im}(\sigma) = (\mathrm{Ker}(\sigma^T))^\perp = \mathbb{V}_N$ and that $\mathrm{Ker}(\sigma) \cap \mathbb{V}_N = \{0\}$; therefore $\sigma : \mathbb{V}_N \mapsto \mathbb{V}_N$ is bijective, $\sigma^{-1} : \mathbb{V}_N \mapsto \mathbb{V}_N$ is well defined and we can define the scalar product in $\mathbb{V}_N$

$$\langle v_1, v_2 \rangle_{\mathbb{V}_N} := \langle \sigma^{-1} v_1, \sigma^{-1} v_2 \rangle_{\mathbb{R}^N}, \qquad \forall\, v_1, v_2 \in \mathbb{V}_N.$$

We want now to give a useful representation of $\langle \cdot, \cdot \rangle_{\mathbb{V}_N}$. Let $(B_t, t \geq 0)$ be a standard Brownian motion and set

$$D_i := B_i - \frac{B_1 + B_2 + \cdots + B_N}{N}, \quad i = 1, \ldots, N, \qquad D := (D_1, \ldots, D_N) \in \mathbb{V}_N. \tag{3.1}$$

Lemma 3.1. *For all $v \in \mathbb{V}_N$*

$$\|v\|^2_{\mathbb{V}_N} = \mathbb{E}\left[\langle v, D \rangle^2_{\mathbb{R}^N}\right] = \sum_{i=1}^{N-1} \left(\sum_{j=1}^{i} v_j \right)^2.$$

Proof. Let $V \in \mathbb{V}_N$ such that $\sigma V = v$. Then $\|v\|^2_{\mathbb{V}_N} = \|V\|^2_{\mathbb{R}^N}$. Moreover $V_i = \sum_{j=1}^{i} v_j$, $i = 1, \ldots, N$, and in particular $V_N = 0$ since $v \in \mathbb{V}_N$. Since $\sigma^T D = (B_2 - B_1, \ldots, B_N - B_{N-1}, 0)$ and $V_N = 0$

$$\mathbb{E}\left[\langle v, D \rangle^2_{\mathbb{R}^N}\right] = \mathbb{E}\left[\langle V, \sigma^T D \rangle^2_{\mathbb{R}^N}\right] = \|V\|^2_{\mathbb{R}^N} = \|v\|^2_{\mathbb{V}_N}. \quad \square$$

Recall that $\{(w_t(x))_{t \geq 0} : x = 1, \ldots, N\}$ is an independent family of standard Brownian motions; then $w = (w(1), \ldots, w(N))$ is a Wiener process in $\mathbb{R}^N$ and σw is a Wiener process in $\mathbb{V}_N$, i.e. for all $t \geq 0$

$$\mathbb{E}\left[\langle h, w_t \rangle^2_{\mathbb{R}^N}\right] = t \|h\|^2_{\mathbb{R}^N}, \quad \forall\, h \in \mathbb{R}^N, \qquad \mathbb{E}\left[\langle v, \sigma w_t \rangle^2_{\mathbb{V}_N}\right] = t \|v\|^2_{\mathbb{V}_N}, \quad \forall\, v \in \mathbb{V}_N.$$

Lemma 3.2. *For all $\phi_0 \in K_N$ there exists a unique pair $(\phi_t, l_t)_{t \geq 0}$, solution of* (1.1). *We use the notation $\phi(t, \phi_0) = \phi_t$, $t \geq 0$.*

Proof. We start by (pathwise) uniqueness. Let (ϕ, l) and $(\overline{\phi}, \overline{l})$ be solutions of (1.1) with initial condition ϕ_0, resp. $\overline{\phi}_0$. Setting $\psi_t := \phi_t - \overline{\phi}_t$, by Itô's formula we obtain

$$d\langle \psi_t, \mathbf{1} \rangle_{\mathbb{R}^N} = \langle \sigma^T \mathbf{1}, -\sigma^T \{\sigma(V'(\sigma^T \phi_t) - V'(\sigma^T \overline{\phi}_t))\, dt + dl_t - d\overline{l}_t\} \rangle_{\mathbb{R}^N} = 0$$

so that $\langle \psi_t, \mathbf{1} \rangle = 0$ for all $t \geq 0$ and therefore $\psi_t \in \mathbb{V}_N$. Then, again by Itô's formula

$$d\langle \psi_t, \psi_t \rangle_{\mathbb{V}_N} = -\langle \sigma^T \psi_t, V'(\sigma^T \phi_t) - V'(\sigma^T \overline{\phi}_t) \rangle\, dt + \langle \psi, dl_t - d\overline{l}_t \rangle_{\mathbb{R}^N} \leq 0$$

since V' is monotone non-decreasing and by (1.2).

For or existence of (strong) solutions, we can refer to Cépa (1998). Indeed, setting $\mathbf{1}_\phi := \langle \phi_0, \mathbf{1}\rangle_{\mathbb{R}^N} \mathbf{1}$ and $\zeta_t := \phi_t - \mathbf{1}_\phi$, (1.1) is equivalent to

$$d\zeta_t \;=\; -\,\sigma\sigma^T\{\sigma V'(\sigma^T(\zeta_t + \mathbf{1}_\phi))\,dt + \,dl_t\} + \sqrt{2}\,\sigma\,dw_t \tag{3.2}$$

for all $x \in \Gamma_N$, subject to the conditions

$$t \mapsto l_t(x) \text{ continuous and non}-\text{decreasing}, \qquad l_0(x) = 0,$$

$$\zeta_t(x) + \langle \phi_0, \mathbf{1}\rangle_{\mathbb{R}^N} \;\geq\; 0, \qquad \int_0^\infty (\zeta_t(x) + \langle \phi_0, \mathbf{1}\rangle_{\mathbb{R}^N})\; dl_t(x) \;=\; 0, \qquad x \in \Gamma_N.$$

Equation (3.2) is a Skorohod problem in the convex set $[0,\infty[^{\Gamma_N} \cap \mathbb{V}_N$; in other words, ζ solves the stochastic differential *inclusion*

$$d\zeta \in -\partial U(\zeta_t)\,dt + \sqrt{2}\,\sigma\,dw_t$$

where $U : \mathbb{V}_N \mapsto \mathbb{R}$ is the convex potential

$$U(\zeta) := \begin{cases} \displaystyle\sum_{x=2}^N V(\zeta(x) - \zeta(x-1)), & \text{if } \; \zeta + \mathbf{1}_\phi \in [0,\infty[^{\Gamma_N} \cap \mathbb{V}_N \\ \\ +\infty, & \text{otherwise,} \end{cases}$$

see in particular Proposition 3.1 in Cépa (1998). Therefore existence of a strong solution of 3.2 follows from Theorem 5.1 of Cépa (1998). □

4. The microscopic invariant measure

In this section we study invariant measures of (1.1) and the associated Dirichlet forms. Since (1.1) conserves the sum $\sum_{x=1}^N \phi_t(x) = \sum_{x=1}^N \phi_0(x)$ for all $t \geq 0$, each subspace $\mathbb{V}_N^c = \mathbb{V}_N + c\mathbf{1}$, with $c > 0$, supports an invariant measure. Therefore it is natural to fix $c > 0$ and consider only initial conditions ϕ_0 in $\mathbb{V}_N^c$.

We consider a sequence of i.i.d. real random variables $(X_i)_{i\in\mathbb{N}}$, such that X_i has probability density $\exp(-V)dr$ on $\mathbb{R}$. Then $q = \mathbb{E}\left[X_1^2\right]$, see (1.3). For $n \in \mathbb{N}$ we set $S_n := X_1 + \cdots + X_n$, $S_0 := 0$. Moreover, for any $c \in \mathbb{R}$ and $N \in \mathbb{N}$ we set

$$T_i^{N,c} := S_{i-1} - \frac{1}{N}\sum_{j=1}^{N-1} S_j + c\,N^{1/2}, \qquad i = 1, \ldots, N,$$

and

$$\mathbb{V}_N^c \;:=\; \left\{ \phi \in \mathbb{R}^N : \sum_{i=1}^N \phi_i \;=\; c\,N^{3/2} \right\} \;=\; \mathbb{V}_N + c\,N^{1/2}\,\mathbf{1}.$$

Notice that a.s. $T^{N,c} = (T_1^{N,c}, \ldots, T_N^{N,c}) \in \mathbb{V}_N^c$. Clearly $\mathbb{V}_N^c$ is a $(N-1)$-dimensional affine subspace of $\mathbb{R}^N$; we denote by $\mathcal{L}^{N-1}(d\phi)$ the induced $(N-1)$-dimensional Lebesgue measure.

Lemma 4.1. *The law of* $(T_1^{N,c}, \ldots, T_N^{N,c})$ *on* $\mathbb{V}_N^c$ *is*

$$\mathbf{P}_N^c(d\phi) \;:=\; \frac{1}{Z_N^c}\, 1_{(\phi \in \mathbb{V}_N^c)}\, \exp\{-\mathcal{H}_N(\phi)\}\; \mathcal{L}^{N-1}(d\phi), \tag{4.1}$$

where Z_N^c *is a normalization constant and* $\mathcal{H}_N$ *is the Hamiltonian*

$$\mathcal{H}_N(\phi) \;:=\; \sum_{x=2}^N V(\phi(x) - \phi(x-1)), \qquad \phi \in \mathbb{R}^N.$$

Proof. It is enough to prove the case $c = 0$. We set $\tau : \mathbb{R}^{n-1} \mapsto \mathbb{R}^N$,

$$\tau(y) := -\frac{1}{N} \sum_{k=1}^{N-1} y_k \cdot \mathbf{1} + (0, y_1, \ldots, y_{N-1}), \quad y \in \mathbb{R}^{N-1}.$$

For all $f \in C_b(\mathbb{R}^N)$, we have

$$\mathbb{E}[f(T^{N,0})] = \int_{\mathbb{R}^{N-1}} f(\tau(y))\, e^{-V(y_1)-V(y_2-y_1)-\cdots-V(y_{N-1}-y_{N-2})}\, dy_1 \cdots dy_{N-1}.$$

Now we define the $(N-1) \times (N-1)$ matrix

$$L := (L_{ij}), \qquad L_{ij} = 1_{(i=j)} - \frac{1}{N},$$

so that $\tau_i(y) = (Ly)_{i-1}$ for all $i = 2, \ldots, N$. Let us now use the following change of variable

$$\mathbb{R}^{N-1} \ni y \mapsto (\phi_2, \ldots, \phi_N) \in \mathbb{R}^{N-1}, \qquad \phi_i := (Ly)_{i-1}, \quad i = 2, \ldots, N.$$

Moreover we set

$$\phi_1 := -\frac{1}{N} \sum_{k=1}^{N-1} y_k = -(\phi_2 + \cdots + \phi_N).$$

Then $(\phi_1, \ldots, \phi_N) \in \mathbb{V}_N$ and $y_1 = \phi_2 - \phi_1$, $y_i - y_{i-1} = \phi_{i+1} - \phi_i$, for all $i = 1, \ldots, N-1$. Finally

$$\mathbb{E}[f(T^{N,0})] = \int_{\mathbb{R}^{N-1}} f(\phi_1, \ldots, \phi_N)\, \frac{e^{-V(\phi_2-\phi_1)-\cdots-V(\phi_N-\phi_{N-1})}}{|\det L|}\, d\phi_2 \cdots d\phi_N. \quad \square$$

We also set $\mathbf{P}_N^{c,+} = \mathbf{P}_N^c(\,\cdot\,|\,\Omega_N^+)$. Then

$$\mathbf{P}_N^{c,+}(d\phi) \;=\; \frac{1}{Z_N^{c,+}}\, 1_{(\phi \in \mathbb{V}_N^c \cap \Omega_N^+)}\, \exp\{-\mathcal{H}_N(\phi)\}\; \mathcal{L}^{N-1}(d\phi), \tag{4.2}$$

where $Z_N^{c,+} = \mathbf{P}_N^c(\Omega_N^+)$ is a normalization constant.

Since $\mathbb{V}_N^c = c\mathbf{1} + \mathbb{V}_N$ is an affine space obtained by a translation of $\mathbb{V}_N$, it is natural to consider $\mathbb{V}_N$ as its tangent space. More precisely, for any $F : \mathbb{V}_N^c \mapsto \mathbb{R}$ in C^1, one can define a gradient $\nabla_{\mathbb{V}_N} F : \mathbb{V}_N^c \mapsto \mathbb{V}_N$ as follows

$$\frac{d}{d\varepsilon}\, F(\phi + \varepsilon\, v)\bigg|_{\varepsilon=0} = \langle \nabla_{\mathbb{V}_N} F(\phi), v \rangle_{\mathbb{V}_N}, \qquad \forall\; \phi \in \mathbb{V}_N^c,\; v \in \mathbb{V}_N,$$

recall (2.4). Notice that $\nabla_{\mathbb{V}_N}$ is the gradient operator in $\mathbb{V}_N$ with respect to the scalar product $\langle \cdot, \cdot \rangle_{\mathbb{V}_N}$. If $F \in C^1(\mathbb{R}^N)$ and $\phi \in \mathbb{V}_N^c$, then it is possible to compare the gradient in $\mathbb{V}_N$ and the standard gradient $\nabla F = (\frac{\partial F}{\partial \phi_i}, i = 1, \ldots, N)$

$$\nabla_{\mathbb{V}_N} F = \sigma\sigma^T \nabla F, \qquad \|\nabla_{\mathbb{V}_N} F\|^2_{\mathbb{V}_N} = \|\sigma^T \nabla F\|^2_{\mathbb{R}^N} = \langle \nabla F, \sigma\sigma^T \nabla F \rangle_{\mathbb{R}^N}.$$

Proposition 4.2. *Let $c > 0$.*

(1) The Markov process $(\phi(t, \phi_0))_{t \geq 0, \phi_0 \in \mathbb{V}_N^c \cap \Omega_N^+}$ is the diffusion generated by the symmetric Dirichlet Form in $L^2(\Omega_N^+, \mathbf{P}_N^{c,+})$, closure of

$$C_b^1(\Omega_N^+) \;\ni\; F \;\mapsto\; e^{c,N}(F,F) \;:=\; \int \sum_{x,y \in \Gamma_N} \frac{\partial F}{\partial \phi(x)}\, [\sigma\sigma^T]_{xy}\, \frac{\partial F}{\partial \phi(y)}\, d\mathbf{P}_N^{c,+}$$

$$= \int \|\nabla_{\mathbb{V}_N} F\|^2_{\mathbb{V}_N}\, d\mathbf{P}_N^{c,+}.$$

(2) $\mathbf{P}_N^{c,+}$ *is the only tempered invariant probability measure of* ϕ *on* $\mathbb{V}_N^c \cap \Omega_N^+$*, where temperedness means having finite second moment.*

Proof. Closability of $e^{c,N}$ on $C_b^1(\Omega_N^+)$ follows from Theorem 2.1, since the Hamiltonian $\mathcal{H}_N$ and the set $\mathbb{V}_N^c \cap \Omega_N^+$ are convex and $\mathbf{P}_N^{c,+}$ is therefore log-concave (see Theorem 9.4.11 of Ambrosio et al. (2005)). Since $\mathbb{V}_N^c \cap \Omega_N^+$ is locally compact, by Fukushima's theory of Dirichlet forms there exists a continuous Markov process $(\psi_t, t \geq 0)$ in $\mathbb{V}_N^c \cap \Omega_N^+$, starting from quasi-every $\psi_0 \in \mathbb{V}_N^c \cap \Omega_N^+$, weak solution of (1.1). By the pathwise uniqueness result of Lemma 3.2, $(\psi_t, t \geq 0)$ and $(\phi_t, t \geq 0)$ are identical in law if $\psi_0 = \phi_0$ and therefore $(\phi_t, t \geq 0, \phi_0 \in \mathbb{V}_N^c \cap \Omega_N^+)$ is the Markov process associated with $e^{c,N}$.

The second assertion follows from point (e) of Theorem 2.1, since $\mathbf{P}_N^{c,+} \in \mathscr{P}_2(\mathbb{R}^N)$ by the convexity of V and in particular (1.3). □

5. The rescaling

Recall now the rescaling map $\Lambda_N : \mathbb{R}^N \mapsto L^2(0,1)$, defined in (1.4). In this section we show how the scalar product of $\mathbb{V}_N$ is transformed under this map. This issue is crucial for the proof of (2.6) and (2.7) in our setting, see Proposition 6.3 below.

We define the linear subspace H_N of $L^2(0,1)$ as the image of Λ_N. We denote by $1_{I(x)}$ the indicator function of the interval $I(x)$, where

$$I(0) := \emptyset, \qquad I(x) := [(x-1)/N, x/N), \quad x \in \Gamma_N.$$

Then, by the definition of Λ_N

$$H_N = \left\{ \sum_{i=1}^N a_i \, 1_{I_i}, \quad (a_1, \ldots, a_N) \in \mathbb{R}^N \right\},$$

i.e. H_N can be identified with the space of functions on $[0,1)$ being constant on $I(x)$ for all $x \in \Gamma_N$.

Let B denote a standard Brownian motion in $\mathbb{R}$ with $B_0 = 0$. We set

$$\overline{B}_N := \frac{B_{\frac{1}{N}} + B_{\frac{2}{N}} + \cdots + B_1}{N}, \qquad \overline{B} := \int_0^1 B_r \, dr.$$

Then we define the process

$$Y_r^N := B_{\lfloor Nr+1 \rfloor / N} - \overline{B}_N, \qquad r \in [0,1),$$
$$Y_r := B_r - \overline{B}, \qquad r \in [0,1],$$

where $\lfloor \cdot \rfloor$ denotes the integer part. Notice that almost surely

$$\langle Y^N, 1 \rangle = \langle Y, 1 \rangle = 0, \qquad Y_r^N \to Y_r, \quad \forall \, r \in [0,1)$$

as $N \to \infty$. Both processes are centered Gaussian. Recall that $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{L^2(0,1)}$ denotes the scalar product in $L^2(0,1)$. Now we define

$$\langle h, k \rangle_{H_N} := \mathbb{E}\left[\langle h, Y^N \rangle \langle k, Y^N \rangle \right] + \langle h, 1 \rangle \langle k, 1 \rangle, \qquad \forall \, h, k \in H_N,$$
$$\langle h, k \rangle_H := \mathbb{E}\left[\langle h, Y \rangle \langle k, Y \rangle \right] + \langle h, 1 \rangle \langle k, 1 \rangle, \qquad \forall \, h, k \in L^2(0,1).$$

Lemma 5.1.

- *For any $N \in \mathbb{N}$ and $h \in H_N$*

$$\langle h, h\rangle_{H_N} = \langle h, 1\rangle^2 + \frac{1}{N}\sum_{i=1}^{N-1}\left(\sum_{j=1}^{i}\langle h - \langle h, 1\rangle, 1_{I(j)}\rangle\right)^2 \tag{5.1}$$
$$= \langle h, 1\rangle^2 + \mathbb{E}\left[\langle h, \Lambda_N D\rangle^2\right],$$

where D is defined in (3.1). *In particular, if $h \neq 0$ then $\langle h, h\rangle_{H_N} > 0$.*

- *For any $h \in L^2(0,1)$*

$$\langle h, h\rangle_H = \langle h, 1\rangle^2 + \int_0^1 \left(-\langle h, 1\rangle + \int_0^t h(s)\, ds\right)^2 dt.$$

In particular, if $h \neq 0$, then $\langle h, h\rangle_H > 0$.

Proof. Let $h \in H_N$ and set

$$k := \sum_i \langle h - \langle h, 1\rangle, 1_{I(1)} + \cdots + 1_{I(i)}\rangle\, 1_{I(i)},$$

and notice that $\langle k, 1_{I(N)}\rangle = 0$. Then

$$\begin{aligned}
\langle h, h\rangle_{H_N} - \langle h, 1\rangle^2 &= \mathbb{E}\left[\langle h - \langle h, 1\rangle, B_{\lfloor N\cdot +1\rfloor/N}\rangle^2\right] \\
&= \mathbb{E}\left[\left(\sum_{i=1}^{N}\langle h - \langle h, 1\rangle, 1_{(i)}\rangle B_{\frac{i}{N}}\right)^2\right] \\
&= \mathbb{E}\left[\left(\langle k, 1_{I(N)}\rangle\, B_1 - \sum_{i=1}^{N-1}\langle k, 1_{I(i)}\rangle\left(B_{\frac{i+1}{N}} - B_{\frac{i}{N}}\right)\right)^2\right] \\
&= \frac{1}{N}\sum_{i=1}^{N-1}\langle k, 1_{I(i)}\rangle^2,
\end{aligned}$$

and (5.1) is proven, also recalling Lemma 3.1.

Analogously, for any $h \in L^2(0,1)$ we set $k_r := \int_0^r (h - \langle h, 1\rangle)$. Then we find $k_1 = 0$ and

$$\langle h, h\rangle_H - \langle h, 1\rangle^2 = \mathbb{E}\left[\langle h - \langle h, 1\rangle, B\rangle^2\right] = \mathbb{E}\left[\left(k_1\, B_1 - \int_0^1 k\; dB\right)^2\right] = \int_0^1 k^2. \quad \square$$

Therefore $\langle\cdot,\cdot\rangle_{H_N}$, respectively $\langle\cdot,\cdot\rangle_H$, defines a scalar product on H_N, resp. on $L^2(0,1)$. We define the Hilbert space H, completion of $L^2(0,1)$ with respect to the scalar product $\langle\cdot,\cdot\rangle_H$. Notice that the associated norms are controlled by the $L^2(0,1)$ norm.

Lemma 5.2. *For all $N \in \mathbb{N}$ and $h \in H_N$*

$$\|h\|_{H_N}^2 \leq \|h\|_{L^2(0,1)}^2.$$

For all $h \in L^2(0,1)$

$$\|h\|_H^2 \leq \|h\|_{L^2(0,1)}^2.$$

Proof. For any $N \in \mathbb{N}$ and $h \in H_N$

$$\langle h, h\rangle_{H_N} - \langle h, 1\rangle^2 = \mathbb{E}\left[\langle h - \langle h, 1\rangle, B_{\lfloor N\cdot+1\rfloor/N}\rangle^2\right]$$
$$\leq \|h - \langle h, 1\rangle\|^2_{L^2(0,1)} \; \mathbb{E}\left[\|B_{\lfloor N\cdot+1\rfloor/N}\|^2_{L^2(0,1)}\right] = \|h - \langle h, 1\rangle\|^2_{L^2(0,1)} \; \frac{1}{N}\sum_{i=1}^{N}\frac{i}{N}$$
$$\leq \|h - \langle h, 1\rangle\|^2_{L^2(0,1)}.$$

Therefore

$$\langle h, h\rangle_{H_N} \leq \langle h, 1\rangle^2 + \|h - \langle h, 1\rangle\|^2_{L^2(0,1)} = \|h\|^2_{L^2(0,1)}.$$

Analogously, for any $h \in L^2(0,1)$

$$\langle h, h\rangle_{H} - \langle h, 1\rangle^2 = \mathbb{E}\left[\langle h - \langle h, 1\rangle, B\rangle^2\right] \leq \|h - \langle h, 1\rangle\|^2_{L^2(0,1)} \; \mathbb{E}\left[\|B\|^2_{L^2(0,1)}\right]$$
$$= \|h - \langle h, 1\rangle\|^2_{L^2(0,1)} \int_0^1 t\, dt \leq \|h - \langle h, 1\rangle\|^2_{L^2(0,1)}. \quad \square$$

We define now the image measures of $\mathbf{P}^c_N$ and $\mathbf{P}^{c,+}_N$ under Λ_N,

$$\nu^c_N := \Lambda^*_N(\mathbf{P}^c_N), \qquad \nu^{c,+}_N := \Lambda^*_N(\mathbf{P}^{c,+}_N), \qquad c > 0,$$

where Λ_N, $\mathbf{P}^c_N$ and $\mathbf{P}^{c,+}_N$ are defined, respectively, in (1.4), (4.1) and (4.2), and e.g. $\Lambda^*_N(\mathbf{P}^c_N)$ denotes the image measure of $\mathbf{P}^c_N$ under the map Λ_N. Finally, we set for all $c \in \mathbb{R}$

$$H^c_N := \{h \in H_N,\ \langle h, 1\rangle = c\}, \qquad H^c := \{h \in H,\ \langle h, 1\rangle = c\};$$

in particular, H^0_N and H^0 are Hilbert space w.r.t. to the restrictions of $\langle\cdot,\cdot\rangle_{H_N}$, respectively $\langle\cdot,\cdot\rangle_H$, that we denote

$$\langle h, k\rangle_{H^0_N} := \mathbb{E}\left[\langle h, Y^N\rangle\,\langle k, Y^N\rangle\right], \qquad \forall\ h, k \in H^0_N,$$
$$\langle h, k\rangle_{H^0} := \mathbb{E}\left[\langle h, Y\rangle\,\langle k, Y\rangle\right], \qquad \forall\ h, k \in H^0.$$

By (5.1) and Lemma 3.1, we see that the scalar product in H^0_N is the push-forward of the scalar product in $\mathbb{V}_N$ under Λ_N, i.e. for all $h \in H^0_N$

$$\|h\|^2_{H^0_N} = \|\Lambda^{-1}_N h\|^2_{\mathbb{V}_N}. \tag{5.2}$$

As in the case of $\mathbb{V}^c_N$, for a differentiable $F : H^c_N \mapsto \mathbb{R}$ we can define a gradient $\nabla_{H^0_N} F : H^c_N \mapsto H^0_N$

$$\frac{d}{d\varepsilon}\, F(k + \varepsilon\, h)\bigg|_{\varepsilon=0} = \langle \nabla_{H^0_N} F(k), h\rangle_{H^0_N}, \qquad \forall\ k \in H^c_N,\ h \in H^0_N.$$

Analogously for a differentiable $F : H^c \mapsto \mathbb{R}$ we can define a gradient $\nabla_{H^0} F : H^c \mapsto H^0$

$$\frac{d}{d\varepsilon}\, F(k + \varepsilon\, h)\bigg|_{\varepsilon=0} = \langle \nabla_{H^0} F(k), h\rangle_{H^0}, \qquad \forall\ k \in H^c,\ h \in H^0.$$

Moreover, $\Lambda_N : \mathbb{V}^c_N \mapsto H^c_N$ is bijective. Then, for any $f \in C^1_b(H^c_N)$ we have $f \circ \Lambda_N \in C^1_b(\mathbb{V}^c_N)$ and

$$\sum_{x,y\in\Gamma_N} \frac{\partial(f\circ\Lambda_N)}{\partial\phi(x)}\ [\sigma\sigma^T]_{xy}\ \frac{\partial(f\circ\Lambda_N)}{\partial\phi(y)} = \frac{1}{N^4}\,\|\nabla_{H^0_N} f\|^2_{H^0_N} \circ \Lambda_N. \tag{5.3}$$

Then we have for any $\varphi, \psi \in C_b^1(H_N^c)$

$$\mathcal{E}^{c,N}(f,g) := \int_{K_N} \langle \nabla_{H_N^0} \varphi, \nabla_{H_N^0} \psi \rangle_{H_N^0} \, d\nu_N^{c,+} = N^4 \, e^{c,N}(\varphi \circ \Lambda_N, \psi \circ \Lambda_N).$$

We obtain readily from Proposition 4.2

Proposition 5.3. *The bilinear form $(\mathcal{E}^{c,N}, C_b^1(H_N^c))$ is closable in $L^2(\nu_N^{c,+})$ and the closure $(\mathcal{E}^{c,N}, D(\mathcal{E}^{c,N}))$ is a symmetric Dirichlet form with associated Markov process Φ^N.*

6. Proof of Theorem 1.1

6.1. *The limit equation.* We recall that B denotes a standard real Brownian motion and

$$\overline{B} := \int_0^1 B_r \, dr,$$

We define the process

$$Y_\theta^c := q^{1/2} \left(B_\theta - \overline{B} \right) + c, \qquad \theta \in [0,1],$$

where q is defined in (1.3).

Lemma 6.1. *For all $c > 0$, the probability of the event $\{\inf_{\theta\in[0,1]} Y_\theta^c > 0\}$ is positive.*

Proof. Notice that $\mathbb{P}(\inf_{\theta\in[0,1]} |B_\theta| \le \varepsilon) > 0$ for all $\varepsilon > 0$, and $\{\inf_{\theta\in[0,1]} |B_\theta| \le q^{-1/2}c/4\} \subset \{\inf_{\theta\in[0,1]} Y_\theta^c > 0\}$. □

In particular, $\nu^c(K) > 0$, where $K := \{h \in L^2(0,1), h \ge 0\}$ and ν^c is the law of Y^c. Moreover, if $\nu^{c,+}$ is the law of Y^c conditioned to be non-negative on $[0,1]$, then $\nu^{c,+} = \nu^c(\,\cdot\,|K)$. The following result has been proven in Debussche and Zambotti (2007).

Proposition 6.2.

(1) For all $u_0 \in H^c \cap K$ there exists a unique strong solution of (1.5). We denote $X_t(u_0) := u(t,\cdot) \in H^c \cap K$

(2) The process $(X_t(u_0))_{t\ge0, u_0\in H^c\cap K}$ is the diffusion associated with the Dirichlet form $(\mathcal{E}^c, D(\mathcal{E}^c))$, closure of the symmetric form

$$\mathcal{E}^c(\varphi,\psi) := \int \langle \nabla_{H^0}\varphi, \nabla_{H^0}\psi \rangle_{H^0} \, d\nu^{c,+}, \qquad \forall\, \varphi,\psi \in C_b^1(H^c).$$

(3) $\nu^{c,+}$ is the only invariant measure of $(X_t(u_0))_{t\ge0,u_0\in H^c\cap K}$.

6.2. *Proof of* (2.6) *and* (2.7). We are going to show now that, as $N \to \infty$, $\nu_N^{c,+}$ converges weakly to $\nu^{c,+}$ and the norm $\|\cdot\|_{H_N^0}$ converges to $\|\cdot\|_{H^0}$, in the sense of (2.6) and (2.7).

Proposition 6.3. *In the notation of section 5*

(1) If $c > 0$ then $\nu_N^{c,+}$ converges weakly in H to $\nu^{c,+}$ as $N \to +\infty$.

(2) We have

$$\frac{1}{6}\|h\|_{H^0} \le \|h\|_{H_N^0} \le \|h\|_{H^0} \qquad \forall\, h \in H_N^0,\ N \in \mathbb{N}. \tag{6.1}$$

(3) *Denoting by* $\Pi_N : H^0 \to H^0_N$ *the orthogonal projections induced by the scalar product of* H^0*, we have*

$$\lim_{N\to\infty} \|\Pi_N h\|_{H^0_N} = \|h\|_{H^0} \qquad \forall\, h \in H^0. \tag{6.2}$$

Proof. We start with weak convergence of $\nu_N^{c,+}$ to $\nu^{c,+}$. We set $\nu_N^c := \Lambda_N^*(\mathbf{P}_N^c)$, i.e. ν_N^c is the law of the process $Y^{c,N}$

$$Y^{c,N}_\theta := \frac{S_{\lfloor N\theta\rfloor} - \overline{S}_N}{\sqrt{N}} + c, \qquad \theta \in [0,1).$$

By the invariance principle, ν_N^c converges weakly to the law ν^c of Y^c. We have to prove now that for $c > 0$

$$\nu^c(\partial K) = \mathbb{P}\left(\inf_{\theta\in[0,1]} Y^c_\theta = 0\right) = 0.$$

Notice that, by the symmetry of Y^c with respect to time inversion $\theta \mapsto 1-\theta$, we have

$$\mathbb{P}\left(\inf_{\theta\in[0,1]} Y^c_\theta = 0\right) \le 2\,\mathbb{P}\left(\inf_{\theta\in[0,1/2]} Y^c_\theta = 0\right).$$

Notice that $\overline{B} \sim \mathcal{N}(0,1/3)$. By a standard Gaussian computation, it is easy to see that the law of $(Y^c_\theta, \theta \in [0,1/2])$ is equivalent to the law of

$$V_\theta := q^{1/2}(B_\theta - Z) + c, \qquad \theta \in [0,1/2],$$

where $Z \sim \mathcal{N}(0,1/3)$ is independent of B. Since the minimum value of B over $[0,1/2]$ has the law of $|B_{1/2}|$, we obtain that

$$\mathbb{P}\left(\inf_{\theta\in[0,1/2]} V_\theta = 0\right) = \mathbb{P}\left(|B_{1/2}| = Z - q^{-1/2}c\right) = 0$$

and therefore $\mathbb{P}\left(\inf_{\theta\in[0,1/2]} Y^c_\theta = 0\right) = 0$. Then $\nu^c(\partial K) = 0$ and $\nu_N^c(\,\cdot\,|K) = \nu_N^{c,+}$ converges weakly to $\nu^c(\,\cdot\,|K) = \nu^{c,+}$.

We prove now (6.1) and (6.2). The key result is the following lemma.

Lemma 6.4. *For all* $N \in \mathbb{N}$ *and* $h \in H_N$

$$\|h\|^2_{H_N} + \frac{1}{6N^2}\langle h, 1\rangle^2 \;=\; \|h\|^2_H + \frac{1}{6N^2}\|h\|^2_{L^2(0,1)}. \tag{6.3}$$

Proof. Since $\langle h,1\rangle_H = \langle h,1\rangle_{H_N} = \langle h,1\rangle$, then (6.3) is equivalent to

$$\|h - \langle h,1\rangle\|^2_{H_N} \;=\; \|h-\langle h,1\rangle\|^2_H + \frac{1}{6N^2}\|h-\langle h,1\rangle\|^2_{L^2(0,1)}, \qquad \forall\, h \in H_N.$$

This, in turn, is equivalent to

$$\mathbb{E}\left[\langle h, B_{\lfloor N\cdot+1\rfloor/N}\rangle^2\right] \;=\; \mathbb{E}\left[\langle h, B\rangle^2\right] + \frac{1}{6N^2}\|h\|^2_{L^2(0,1)}, \qquad \forall\, h \in H^0_N.$$

This formula can be proven by noting that for all $i = 1,\dots,N$

$$B_{\frac{i}{N}} \;=\; N\int_{\frac{i-1}{N}}^{\frac{i}{N}} B_s\,ds + N\int_{\frac{i-1}{N}}^{\frac{i}{N}} \left(B_{\frac{i}{N}} - B_s\right) ds.$$

Indeed, it follows that for all $h \in H_N$

$$\mathbb{E}\left[\langle h, B_{\lfloor N\cdot+1\rfloor/N}\rangle^2\right] \;=\; \mathbb{E}\left[\left(\sum_{i=1}^{N}\langle h, 1_{(i)}\rangle B_{\frac{i}{N}}\right)^2\right]$$

$$= \mathbb{E}\left[\langle h, B\rangle^2\right] + \mathbb{E}\left[\left(\sum_{i=1}^{N}\langle h, 1_{(i)}\rangle N\int_{\frac{i-1}{N}}^{\frac{i}{N}}\left(B_{\frac{i}{N}} - B_s\right)\,ds\right)^2\right]$$

$$+\;2\,N^2\,\mathbb{E}\left[\langle h, B\rangle\sum_{i,j=1}^{N}\langle h, 1_{(i)}\rangle\,\langle h, 1_{(j)}\rangle\int_{\frac{j-1}{N}}^{\frac{j}{N}}B_r\,dr\int_{\frac{i-1}{N}}^{\frac{i}{N}}\left(B_{\frac{i}{N}} - B_s\right)ds\right]$$

By independence of increments of the Brownian motion, the second term in the right hand side is

$$\mathbb{E}\left[\left(\sum_{i=1}^{N}\langle h, 1_{(i)}\rangle N\int_{\frac{i-1}{N}}^{\frac{i}{N}}\left(B_{\frac{i}{N}} - B_s\right)\,ds\right)^2\right] = \frac{1}{3N}\sum_{i=1}^{N}\langle h, 1_{(i)}\rangle^2 \;=\; \frac{1}{3N^2}\|h\|^2_{L^2(0,1)}.$$

Now, for the third term, we need to calculate

$$I_{ij} \;:=\; \mathbb{E}\left[\int_{\frac{j-1}{N}}^{\frac{j}{N}}B_r\,dr\int_{\frac{i-1}{N}}^{\frac{i}{N}}\left(B_{\frac{i}{N}} - B_s\right)ds\right].$$

Again by independence we have $I_{ij} = 0$ if $j < i$. On the other hand

$$i < j \implies I_{ij} \;=\; \int_{\frac{j-1}{N}}^{\frac{j}{N}}dr\int_{\frac{i-1}{N}}^{\frac{i}{N}}\left(\frac{i}{N} - s\right)ds \;=\; \frac{1}{2N^3},$$

$$i = j \implies I_{ii} \;=\; \int_{\frac{i-1}{N}}^{\frac{i}{N}}dr\int_{\frac{i-1}{N}}^{\frac{i}{N}}(s - r)\,ds \;=\; \frac{1}{6N^3}.$$

Then we must compute for all $h \in H_N$

$$\frac{1}{N}\sum_{i<j}\langle h, 1_{(i)}\rangle\,\langle h, 1_{(j)}\rangle + \frac{1}{3N}\sum_{i}\langle h, 1_{(i)}\rangle^2$$

$$= \frac{1}{2N}\sum_{i\neq j}\langle h, 1_{(i)}\rangle\,\langle h, 1_{(j)}\rangle + \frac{1}{3N}\sum_{i}\langle h, 1_{(i)}\rangle^2$$

$$= \frac{1}{2N}\sum_{i,j}\langle h, 1_{(i)}\rangle\,\langle h, 1_{(j)}\rangle - \frac{1}{6N}\sum_{i}\langle h, 1_{(i)}\rangle^2 \;=\; \frac{1}{2N}\langle h, 1\rangle^2 - \frac{1}{6N^2}\|h\|^2_{L^2(0,1)}.$$

Finally, we have proven that for all $h \in H_N$

$$\mathbb{E}\left[\langle h, B_{\lfloor N\cdot+1\rfloor/N}\rangle^2\right] \;=\; \mathbb{E}\left[\langle h, B\rangle^2\right] + \frac{1}{6N^2}\|h\|^2_{L^2(0,1)} \;+\; \frac{1}{2N}\langle h, 1\rangle^2$$

and choosing h such that $\langle h, 1\rangle = 0$ we have the desired result. □

End of the proof of Proposition 6.3. We prove now (6.1), namely the estimate

$$\frac{1}{6}\,\|h\|^2_{H^0_N} \;\leq\; \|h\|^2_{H^0} \;\leq\; \|h\|^2_{H^0_N}, \qquad \forall\; N \in \mathbb{N},\; h \in H^0_N. \tag{6.4}$$

The second inequality of (6.4) follows from (6.3). For the first inequality, recall now (5.1), where we proved that for all $h \in H_N^0$

$$\|h\|^2_{H_N^0} = \frac{1}{N} \sum_{i=1}^{N-1} \left(\sum_{j=1}^{i} \langle 1_{(j)}, h \rangle \right)^2 .$$

Then we obtain for all $h \in H_N^0$

$$\begin{aligned} \|h\|^2_{L^2(0,1)} &= N \sum_{i=1}^{N} \langle 1_{(i)}, h \rangle^2 \\ &= N \sum_{i=1}^{N-1} \left(\sum_{j=1}^{i} \langle 1_{(j)}, h \rangle - \sum_{j=1}^{i-1} \langle 1_{(j)}, h \rangle \right)^2 + N \left(\sum_{j=1}^{N-1} \langle 1_{(j)}, h \rangle \right)^2 \\ &\le 4\, N \sum_{i=1}^{N-1} \left(\sum_{j=1}^{i} \langle 1_{(j)}, h \rangle \right)^2 + N \left(\sum_{j=1}^{N-1} \langle 1_{(j)}, h \rangle \right)^2 \le 5\, N^2 \, \|h\|^2_{H_N^0}. \end{aligned}$$

Using (6.3) we obtain the first inequality and (6.4) is proven.

We prove now (6.2), namely we prove that, denoting by $\Pi_N : H^0 \to H_N$ the orthogonal projections induced by the scalar product of H^0, we have

$$\lim_{N\to\infty} \|\Pi_N h\|_{H_N^0} = \|h\|_{H^0} \qquad \forall\, h \in H^0.$$

We denote by $P_N : L^2(0,1) \mapsto L^2(0,1)$ the following projection

$$P_N h := \sum_{i=1}^{N} N \, \langle h, 1_{I(i)} \rangle \, 1_{I(i)}, \qquad h \in L^2(0,1). \tag{6.5}$$

Then P_N is an orthogonal projector with respect to the scalar product of $L^2(0,1)$ and for all $h \in L^2(0,1)$, $\|h - P_N h\|_{L^2(0,1)} \to 0$ as $N \to \infty$. Now, let us fix $h \in L^2(0,1) \cap H^0$; then we have

$$\|P_N h\|^2_{H_N^0} = \mathbb{E}\left[\langle Y^N, h\rangle^2\right] \to \mathbb{E}\left[\langle Y, h\rangle^2\right] = \|h\|^2_{H^0}, \qquad N \to \infty. \tag{6.6}$$

Now we claim that $\|\Pi_N h\|^2_{H^0} \to \|h\|^2_{H^0}$, as $N \to \infty$. Indeed, Π_N is the element of minimal H^0-distance from h in H_N^0. Then, since $P_N h$ belongs to H_N^0, by Lemma 5.2

$$\|\Pi_N h - h\|_{H^0} \le \|P_N h - h\|_{H^0} \le \|P_N h - h\|_{L^2(0,1)} \to 0, \qquad N \to \infty. \tag{6.7}$$

Now, by (6.3)

$$\|\Pi_N h\|^2_{H_N^0} = \|\Pi_N h\|^2_{H^0} + \frac{1}{6N^2} \|\Pi_N h\|^2_{L^2(0,1)} \ge \|\Pi_N h\|^2_{H^0} \to \|h\|^2_{H^0}, \qquad N \to \infty.$$

In particular

$$\liminf_{N\to\infty} \|\Pi_N h\|_{H_N^0} \ge \|h\|_{H^0}.$$

On the other hand, by (6.4)

$$\|\Pi_N h\|_{H_N^0} \le \|P_N h\|_{H_N^0} + \|P_N h - \Pi_N h\|_{H_N^0} \le \|P_N h\|_{H_N^0} + \|P_N h - \Pi_N h\|_{H^0}.$$

Since $\lim_N (P_N h - \Pi_N h) = 0$ in H^0 by (6.7), then by (6.6) we find

$$\limsup_{N\to\infty} \|\Pi_N h\|_{H_N^0} \le \|h\|_{H^0}.$$

If we set now

$$\psi_N : H^0 \mapsto \mathbb{R}, \qquad \psi_N(h) = \|\Pi_N h\|_{H^0_N},$$

then ψ_N is Lipschitz-continuous in the H^0-norm uniformly in N, since

$$\|\Pi_N h\|_{H^0_N} \le \|\Pi_N h\|_{H^0} \le \|h\|_{H^0}$$

by (6.1) and by the definition of Π_N. Moreover and $\psi_N(h) \to \|h\|_{H^0}$ as $N \to \infty$ for all h in $L^2(0,1) \cap H^0$. Since $L^2(0,1) \cap H^0$ is dense in H^0, this concludes the proof of Proposition 6.3. □

6.3. *Proof of Theorem 1.1.* In order to prove Theorem 1.1, it is now enough to notice that by Propositions 5.3, 6.2 and 6.3, Theorems 2.1 and 2.2 apply and yield the desidered convergence result.

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