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FLUCTUATIONS IN THE X-RAY BACKGROUND

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SUMMARY

The fluctuations in the X-ray background at high galactic latitudes give an upper limit to the counts of extragalactic X-ray sources near the intensity level corresponding to one source per beam width. The general mathematical relation between the source counts and the probability distribution of the resulting contribution to the fluctuations is known, and is here computed for various extrapolations of the observed source counts to fainter sources. The observed limits to background fluctuations are consistent with a uniform distribution of sources in space.

SCIENTIFIC ENVIRONMENT

Various attempts have been made to decide whether the almost isotropic X-ray background arises from (i) diffuse emission from the intergalactic medium, (ii) discrete extragalactic sources, or (iii) diffuse emission, or (iv) discrete sources in a more local region around our Galaxy. The lack of noticeable absorption in the direction of the Large Magellanic Cloud (McCammon *et al.* 1971) indicates that much of the background radiation below 1 keV is local in origin; on the other hand, high latitude sources have been identified with extragalactic objects of small redshift, and sources of these types could well contribute an important fraction of the background above 1 keV (Schwartz & Gursky 1973).

One question which has been raised is whether the angular fluctuations in the background are consistent with an origin in discrete sources. In its simplest form the question is:

' If the background is due to N sources per steradian, the fluctuations observed with a beam of Ω steradians are $(N\Omega)^{-1/2}$ of the whole. How big do we have to make N to reduce fluctuations to the observed upper limit? '

However, it was soon realized that this form of the question is relevant only if all the sources have similar intensities (e.g. sources of comparable luminosities which flourished only at some well-defined fairly early epoch of the Universe). If we have a volume distribution of sources rather than a surface distribution, the number of sources in the intensity range S to $S+dS$ is

$$N(S)\,dS = \tfrac{1}{2}\rho P^{3/2} S^{-5/2}\,dS \tag{1}$$

for a uniform distribution of ρ sources per unit volume, each of luminosity $4\pi P$. Thus the source counts tell us about $\rho P^{3/2}$ while the background intensity sets an upper limit to ρPR, where R is the radius of the volume filled with sources (in the cosmological context, R = (Hubble radius) × (model-dependent factor of order unity)). If sources of many different X-ray luminosities are present (which is certainly true) we have to split them up into luminosity classes with luminosities $4\pi P_i$ and densities ρ_i, and replace $\rho P^{3/2}$, ρP by $\Sigma\rho_i P_i^{3/2}$, $\Sigma\rho_i P_i$ respectively. These arguments are identical with the arguments which were used to infer a lower limit to the distances of 'typical' radio sources in the 1950s, and which were set out very clearly by Ryle (1958).

Finally, the effective P_i and ρ_i for particular kinds of source may be correlated with distance (or, in the cosmological context, epoch). These evolutionary changes and geometrical effects of the metric will modify the relation (1) somewhat, and the evolutionary changes required in the case of radio sources have been discussed in a voluminous literature over the past decade.

Whenever there is a distribution of intensities which does not rise too much more rapidly than the relation (1) with decreasing S, the fluctuations are dominated by sources at a level S such that one or two sources brighter than S occur per beamwidth Ω; the very numerous fainter sources may contribute much to the background but affect the fluctuations very little. Thus, given sufficiently good sensitivity, observations of the background fluctuations tell us something about the source counts at the level of 1 source/beamwidth. The fluctuations may be due in part to, say, fluctuations in diffuse emission or absorption, so that really we have only an upper limit to the fluctuations caused by the discreteness of sources.

To use the observations of fluctuations to best advantage, we first have to solve a mathematical problem: Given some guessed extrapolation $N(S)$ of the source counts, what probability distribution of fluctuations do we expect? Once that problem is solved, we can test whether our guess $N(S)$ is consistent with the observed fluctuations.* The problem can be solved using the well-known method of characteristic functions (e.g. Lindley 1969), and has been treated in just this way for the case of radio sources (Scheuer 1957). The details are a little different, as the radio interferometer observations required a two-dimensional treatment leading to Hankel transforms, whereas the pencil-beam X-ray observations require a one-dimensional addition of contributing intensities, which leads to Fourier transforms and is thus a little simpler.

THE PROBABILITY DISTRIBUTION OF FLUCTUATIONS

We are interested in the probability distribution of the sum of the intensities of the sources in solid angle Ω. The probability of finding one source in the beam with intensity S to $S+dS$ is $\exp(-N(S)\,dS).(N(S)\,dS)$ (Poisson distribution). If dS is small enough the probabilities of finding 2, 3, ... such sources, which are of order dS^2, dS^3, ..., are negligible, and the probability of finding one such source is just $N(S)\,dS$. This statement requires some refinement in practice, as the sensitivity of the X-ray telescope is not uniform over the beam; we really want the probability distribution of the sum of the count rates due to the sources in the beam. The probability that there is a source in the beam giving count rate x to $x+dx$ is

$$N'(x)\,dx = \int_0^1 N(x/\eta)(dx/\eta)(d\Omega/d\eta)\,d\eta \tag{2}$$

where $d\Omega$ of solid angle in the beam gives count rate η to $\eta+d\eta$ for a source which would give unit count rate at the beam centre (Fig. 1). For the pyramidal beam defined by a honeycomb structure of the kind used in simple X-ray telescopes, and a power law $N(S) = KS^{-\beta}$ for the source counts, $d\Omega \propto (1-\eta)\,d\eta$, and

* The converse problem of deducing $N(S)$ from the distribution of fluctuations runs into real, not merely mathematical, difficulties; this point is discussed in Scheuer (1957).

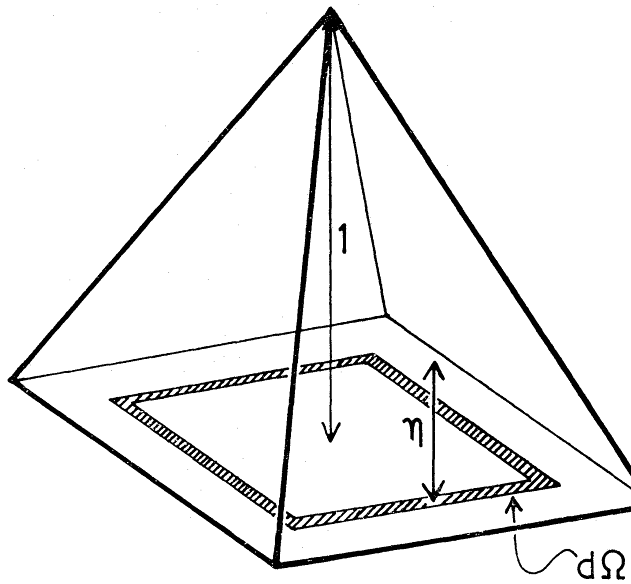

FIG. 1. *Sketch of the beam shape, illustrating the derivation of equations* (2)–(4).

$$N'(x) = K'x^{-\beta} = K\Omega_{\text{eff}} x^{-\beta} \tag{3}$$

where the effective solid angle of the beam is

$$\Omega_{\text{eff}} = \frac{2}{\beta(\beta+1)} \text{ (total solid angle).} \tag{4}$$

We shall also want to consider truncated power laws:

$$N(S) = \begin{cases} 0 & S < S_0 \\ KS^{-\beta} & S > S_0 \end{cases} \tag{5}$$

which serve to illustrate the cosmological effects at large redshifts. For these, equation (2) leads to

$$N'(x) = \begin{cases} K\Omega_{\text{eff}} S_0^{-\beta}[1+\beta(1-x/S_0)] & x < S_0 \\ K\Omega_{\text{eff}} x^{-\beta} & x > S_0 \end{cases}. \tag{6}$$

The characteristic function of the sum X of the x's is then

$$\exp\,(n(\omega)-n(0)) \tag{7}$$

where $n(\omega)$ is the characteristic function of x,

$$n(\omega) = \int_0^\infty N'(x)\, e^{i\omega x}\, dx \tag{8}$$

(e.g. Scheuer 1957).

For a power law with $2 < \beta$ the total background diverges (Olbers' paradox), but so long as $\beta < 3$ the fluctuations $X - \langle X \rangle$ are not affected by the numerous very faint sources, and converge. To find the characteristic function for $X - \langle X \rangle$ we multiply (7) by the characteristic function of the δ-function probability distribution $\delta(X + \langle X \rangle)$ which gives with unit probability a contribution $-\langle X \rangle$; the result is

$$\text{CF} = \exp\,(n(\omega) - n(0) - i\omega\langle X \rangle) = \exp \int_0^\infty N'(x)\,(e^{i\omega x} - 1 - i\omega x)\, dx \tag{9}$$

and the probability distribution of $D \equiv X - \langle X \rangle$ is its Fourier transform

$$P(D) = \frac{1}{2\pi} \int_{-\infty}^{\infty} (\mathrm{CF})\, \mathrm{e}^{-i\omega X}\, d\omega. \tag{10}$$

For $N(S) = KS^{-\beta}$, $(2 < \beta < 3)$, expression (9) can be evaluated after integrating by parts, and turns out to be

$$\exp\,[K\Omega_{\mathrm{eff}}(-\beta)!\,(-i\omega)^{\beta-1}] \tag{11};$$

$P(D)$ can then be expanded in ascending power of D:

$$P(D) = \frac{1}{\pi(\beta-1)} \sum_{r=0}^{\infty} \frac{(t-1)!\, \sin t\pi}{\{K\Omega_{\mathrm{eff}}(-\beta)!\}^t} \frac{D^r}{r!} \tag{12}$$

where $t \equiv (r+1)/(\beta-1)$.

Expression (9) can also be expressed analytically for power laws with a low flux density cut-off of the form

$$N'(x) = K'x^{-\beta}(1-\mathrm{e}^{-\alpha x})^M, \quad M = \text{integer}$$

but as the Fourier transform (10) still has to be performed numerically, this is hardly worth while.

The distribution $P(D)$ of fluctuations has been computed from (12) for power laws with various β, and the results are shown in Fig. 2. The distribution $P(X)$ of source background + fluctuations has also been computed for truncated power laws with $\beta = 2{\cdot}5$ (uniform distribution in Euclidean space up to some finite

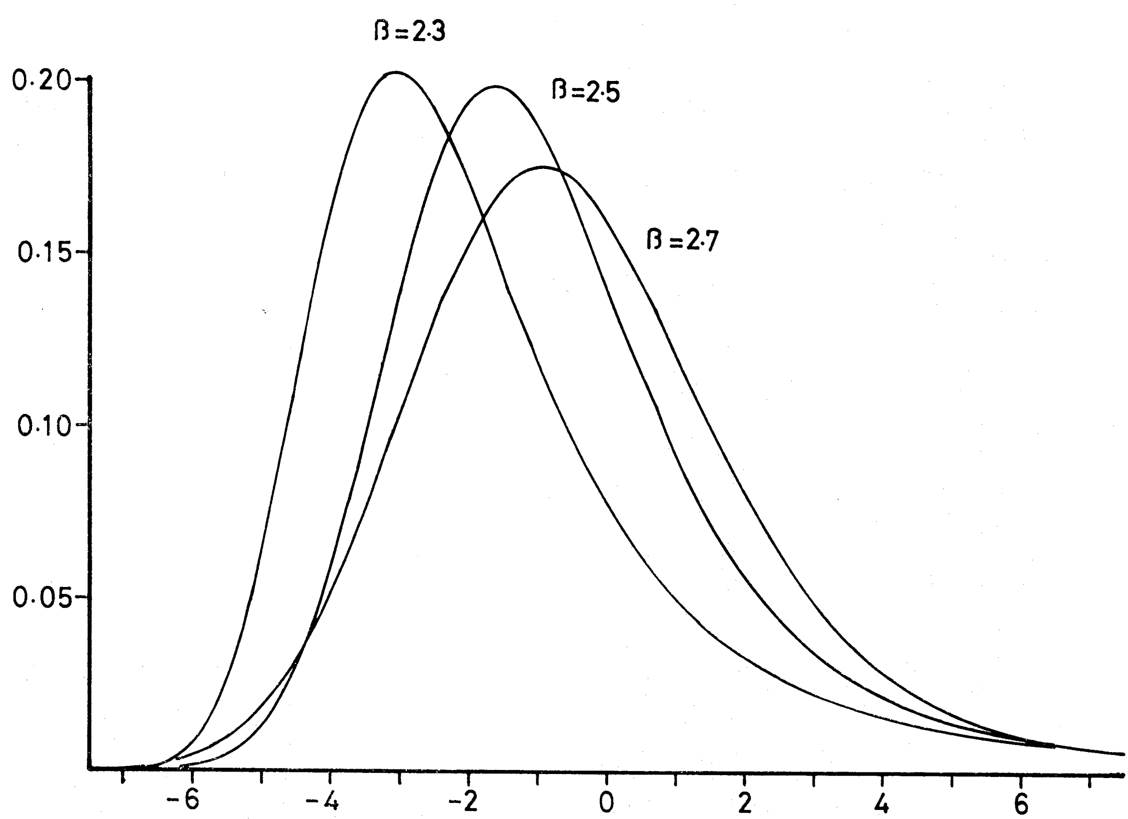

FIG. 2. *The probability distribution of count rates when the source counts follow power laws $N(S)\,dS \propto S^{-\beta}\,dS$ with $\beta = 2{\cdot}3$, $2{\cdot}5$ and $2{\cdot}7$. Each curve encloses unit area and is asymptotic to $S^{-\beta}$ for large S.*

distance), using a computer program which performs the operations (8), (7) and (10) on a distribution of the form (6) by means of a fast Fourier routine, and the results are shown in Fig. 3. In each case the scales are normalized such that the

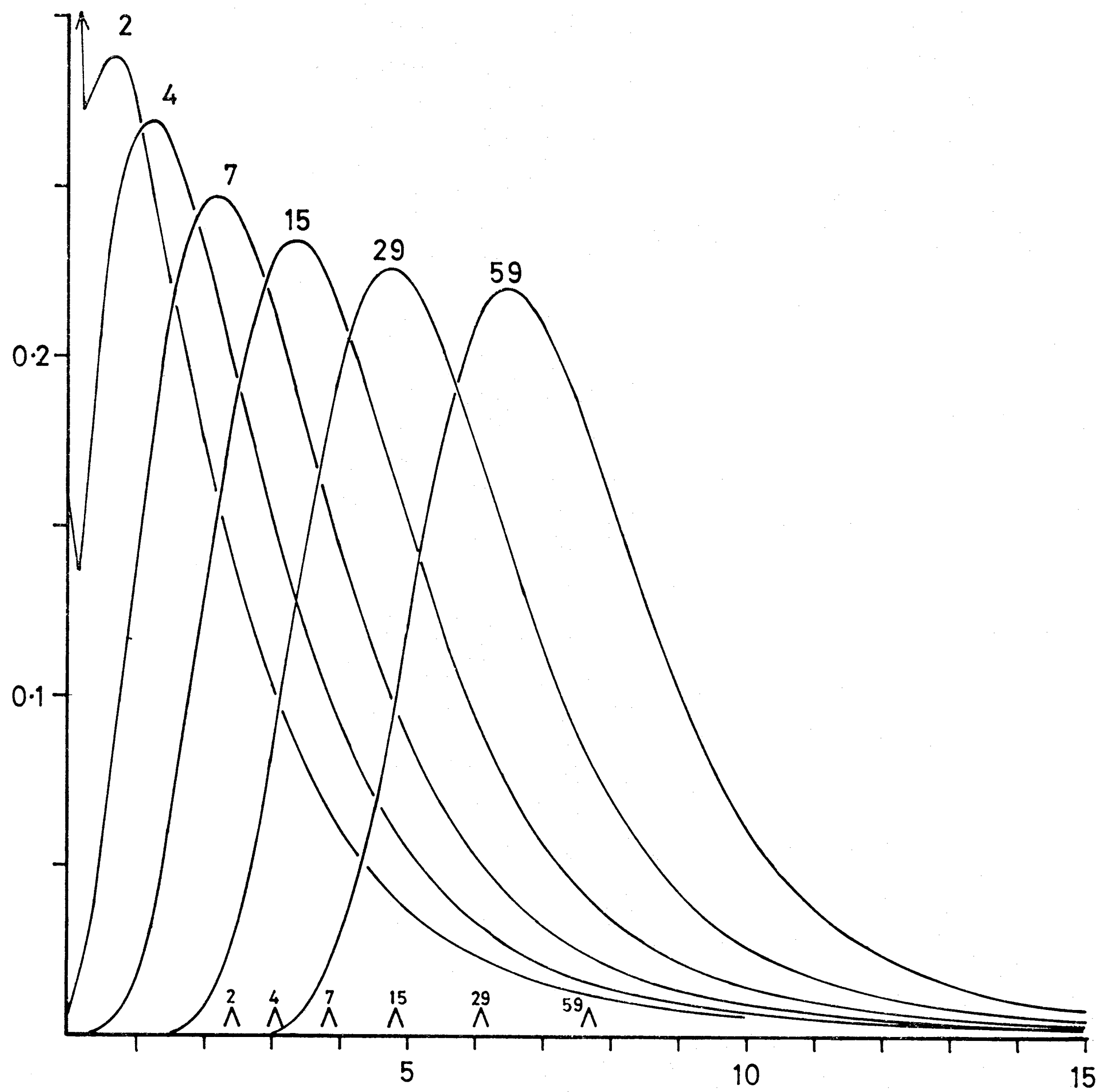

FIG. 3. *The probability distribution of count rates when the source counts follow truncated power laws:* $N(S) \propto S^{-2 \cdot 5}$ *if* $S > S_0$, $N(S) = 0$ *if* $S < S_0$. *Each curve encloses unit area and is asymptotic to* $S^{-2 \cdot 5}$ *for large* S. *It is assumed that the beam of the X-ray telescope is pyramidal (or conical) and each curve is marked with the corresponding total number of sources per beam area. Whereas in Fig. 2* $D = X - \langle X \rangle$ = *(count rate) − (mean count rate) is plotted along the abscissa, in Fig. 3 the whole count rate due to sources is plotted along the abscissa and the mean* X *for each curve is marked.*

total area under the curve is unity and in the units shown in Fig. 3 the ordinate approaches (abscissa)$^{-\beta}$ asymptotically. Each curve in Fig. 3 can be characterized by the mean number of sources per beam area of the X-ray telescope; as that number becomes large the curves in Fig. 3 should approximate the curve in Fig. 2 for $\beta = 2 \cdot 5$, except for a displacement in the abscissa, thus providing a check on the calculations.

COMPARISON WITH OBSERVED FLUCTUATIONS AND SOURCE COUNTS

The difficult part of the exercise begins here.

The first task is to display on the same scale the computed probability distribution of fluctuations (fitted at high intensities to the observed source counts) and the observed fluctuations.

Source counts from the UHURU catalogues have been published by Matilsky *et al.* (1973); the only published data on background fluctuations in a similar energy range are those of Fabian & Sanford (1971) of the MSSL group. In view of the uncertainties in calibration and the slightly different spectral responses of the UHURU and MSSL instruments, it seems best to express UHURU source intensities and MSSL fluctuations each as a fraction of the mean X-ray background observed with the same instrument. This procedure leads to some error in so far as the mean source spectrum differs from the background spectrum, and the spectral responses of the instruments differ somewhat, but that error is probably smaller than other uncertainties at present. The source counts of Matilsky *et al.* may be represented by

$$N(>S) = 30\, S^{-1\cdot 5} \text{ steradians}^{-1} \tag{13}$$

if S is in UHURU counts, and we take the UHURU X-ray background to be 2400 counts per steradian. We take the angular response of Fabian & Sanford's instrument to be a square pyramid of side 10°, so that the background received by it is

$$2400 \times (1/3) \times (10\pi/180)^2 = 24\cdot 37 \text{ UHURU counts}$$

and the effective solid angle for sources (equation (4)) is

$$\Omega_{\text{eff}} = 6\cdot 96 \times 10^{-3} \text{ steradians} \tag{14}$$

for $\beta = 2\cdot 5$. For that solid angle, the source counts are

$$\begin{aligned} dN &= 30 \times 1\cdot 5\, S^{-2\cdot 5}\, dS\Omega_{\text{eff}} = 0\cdot 3133\, S^{-2\cdot 5}\, dS \ (S \text{ in UHURU counts}) \qquad (15)\\ &= 0\cdot 002605\, S^{-2\cdot 5}\, dS \ (S \text{ in units in which background} = 1\cdot 0)\\ &= S^{-2\cdot 5}\, dS \ (S \text{ in units of } 0\cdot 01893 \times \text{background}). \end{aligned}$$

The third expression for dN shows that the appropriate unit of S in Fig. 3 is $0\cdot 01893 \times$ background $= 0\cdot 4613$ UHURU counts. Fabian & Sanford observed counts of 2495, 2510, 2468, 2416, 2407, 2379, 2431, 2352, 2370 at nine points in the sky; the mean is 2425. Of these total counts, 40 per cent is believed to be due to X-rays and 60 per cent to charged particles (Fabian, private communication), so that the fluctuations expressed as fractions of the mean X-ray background are $+0\cdot 072$, $+0\cdot 088$, $+0\cdot 044$, $-0\cdot 009$, $-0\cdot 019$, $-0\cdot 047$, $+0\cdot 006$, $-0\cdot 075$, $-0\cdot 057$. From counting statistics alone, we must expect a Gaussian scatter with variance 2425; as Fabian & Sanford point out the observed fluctuations are consistent with the scatter due to counting statistics alone, and only an upper limit to the true background fluctuations can be given with confidence. The most direct comparison with models is obtained by convolving the model distributions of Figs 2 or 3 with the Gaussian distribution due to counting statistics, and that is done in Fig. 4. To allow for the fact that Fabian & Sanford deliberately chose their observing points well away from known sources, a rough estimate suggests that we should add ~ 1 fairly large positive fluctuation to their data.

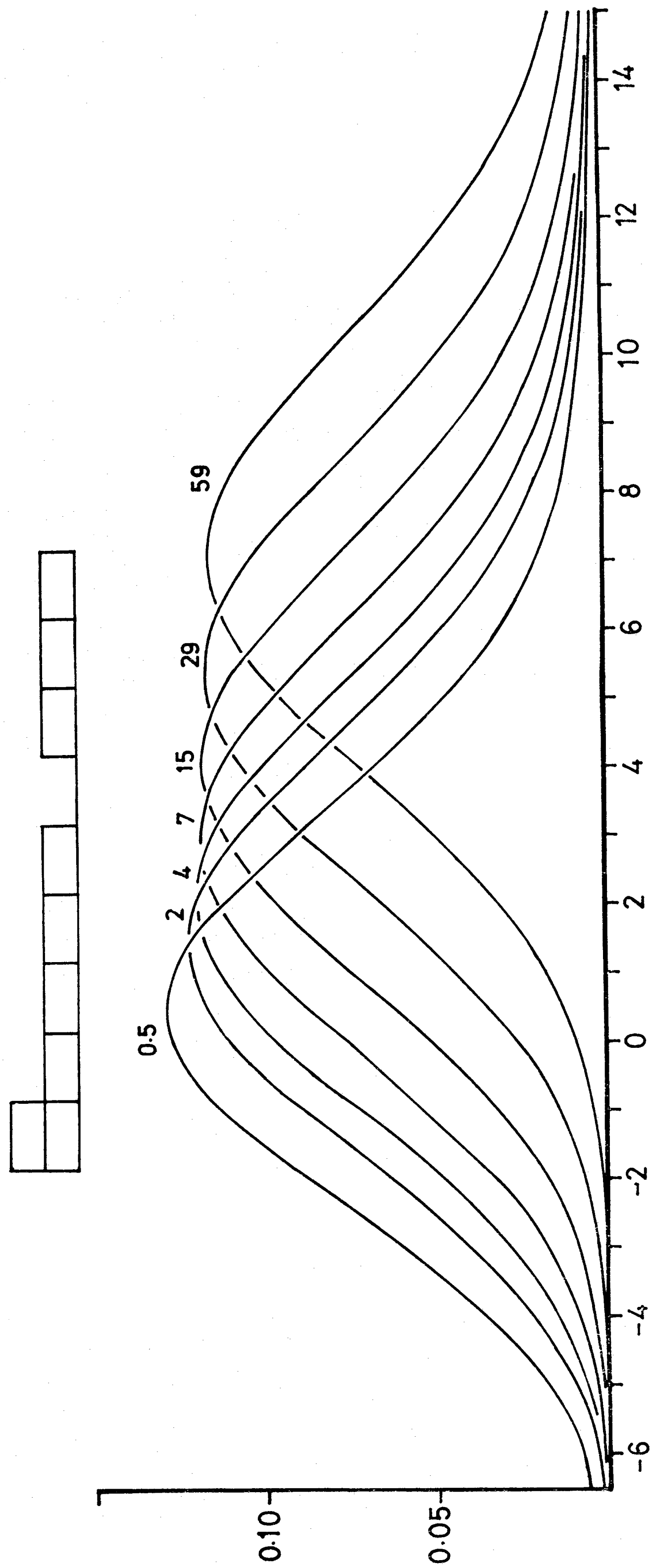

Fig. 4. *The curves of Fig. 3 convolved with the Gaussian distribution due to counting statistics in Fabian & Sanford's observations. A histogram showing the distribution of their observations is also plotted; its position is arbitrary in the sense that it may be moved as a whole along the X axis.*

The curves in Fig. 4 are almost indistinguishable except for a displacement to the right, i.e. to a larger background; with these counting statistics the differences still perceptible in Fig. 3 have gone, and we cannot distinguish between one source per beamwidth and an indefinitely large number.

The observations of fluctuations are quite consistent with a simple extrapolation of the source counts to indefinitely small intensities, with the $\beta = 2{\cdot}5$ power law appropriate to a uniform Euclidean universe.

PEDAGOGICAL DIGRESSION

If one knows the fluctuations in the background from observation, one can gain some advance knowledge of the source counts that will be determined with X-ray telescopes of better angular resolution in a few years' time. The information extends down to the level of intensity where there is about one source per beam width, and no further; this latter point does not seem to have been generally appreciated among X-ray astronomers and it may therefore be useful to show crudely but simply why it is true and in what circumstances it might conceivably be false.

Suppose for the moment that $N(>S) \propto S^{-1{\cdot}5}$; then the sources in one typical beam have intensities (in decreasing order) of 1 (brightest source in beam): $2^{-2/3}$ (second brightest): $3^{-2/3}$: $4^{-2/3}$: $5^{-2/3}$: .... The mean background (obtained by adding these intensities) would diverge, were there not some cut-off at very low intensities, but to obtain the variance one must perform some sort of random walk with step lengths proportional to $1^{-2/3}$, $2^{-2/3}$, $3^{-2/3}$, ..., and the variance of the resultant is the sum of the squares of the steps:

$$1^{-4/3}+2^{-4/3}+3^{-4/3}\ldots \simeq \int_1^\infty x^{-4/3}\,dx = 3. \tag{16}$$

Therefore we expect a typical fluctuation to be $\sim\sqrt{3} = 1{\cdot}7$ times the intensity of the brightest source in a typical beam. Evidently the numerical constant ($1{\cdot}7$) is of the order of 1 and not sensitive to the form of the log N–log S relation so long as the series corresponding to (16) converges, which happens so long as the extrapolation of the integral log N–log S relation falls below a line of slope $-2{\cdot}0$ (i.e. $\beta < 3$, a result we found earlier using more precise mathematics). Thus the fluctuations give no information on sources much below the level of 1 source per beamwidth unless there is extremely strong cosmological evolution, evolution even stronger than that required for radio sources.

CONCLUSION

The fluctuations observed by Fabian & Sanford (1971) are consistent with a 5/2 power law extrapolation of the source counts of Matilsky *et al.* (1973); they provide no evidence of a cosmological cut-off at the level of 1 source per beam width ($\sim 0{\cdot}5$ UHURU counts). Thus the only evidence of the distances of sources comes from optical identifications, as discussed by Schwartz & Gursky (1973), and the extragalactic sources may indeed be responsible for much of the X-ray background. A larger amount of observational material on fluctuations exists in the UHURU data, and may well yield a more positive conclusion.

It remains only to advertise that it is a simple matter to use the existing program to compute probability distributions of fluctuations for other assumed log N–log S

extrapolations and/or beam shapes (including e.g. data differenced to eliminate background gradients), and the author hereby offers this service to owners of relevant X-ray observations.

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I am grateful to the referee for informing me that work similar to this paper has been done by P. Craven in Oxford, and has been referred to by M. J. Rees (Proc. I.A.U. Symposium on X-ray astronomy, pp. 250–7).

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