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FLUID DYNAMIC LIMIT FOR A MODIFIED BROADWELL SYSTEM

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This report discusses a new approach to the resolution of the fluid dynamic limit for the Broadwell system modeling gas dynamics. The main idea is to replace the Knudsen number ε in the Broadwell model by εt, t the time variable to obtain self similar solutions in $\xi = x/t$ and then let $\varepsilon \to 0+$. The limit thus obtained is a solution of the Riemann problem for the fluid dynamic limit equations.

The Broadwell system of discrete kinetic theory is given by the system of partial differential equations

$$
\begin{aligned}
&\frac{\partial f_1}{\partial t} + c\frac{\partial f_1}{\partial x} = \sigma(f_3 f_4 + f_5 f_6 - 2 f_1 f_2)\,,\\
&\frac{\partial f_2}{\partial t} - c\frac{\partial f_2}{\partial x} = \sigma(f_3 f_4 + f_5 f_6 - 2 f_1 f_2)\,,\\
&\frac{\partial f_3}{\partial t} + c\frac{\partial f_3}{\partial y} = \sigma(f_1 f_2 + f_5 f_6 - 2 f_3 f_4)\,,\\
&\frac{\partial f_4}{\partial t} - c\frac{\partial f_4}{\partial y} = \sigma(f_1 f_2 + f_5 f_6 - 2 f_3 f_4)\,,\\
&\frac{\partial f_5}{\partial t} + c\frac{\partial f_5}{\partial z} = \sigma(f_1 f_2 + f_3 f_4 - 2 f_5 f_6)\,,\\
&\frac{\partial f_6}{\partial t} - c\frac{\partial f_6}{\partial z} = \sigma(f_1 f_2 + f_3 f_4 - 2 f_5 f_6)\,.
\end{aligned}
\tag{1}
$$

The model describes a gas of particles with identical masses moving along three perpendicular coordinate axes with the same speed c. Results of a particular collision have the same probability and only binary collisions are considered. The functions $f_i = f_i(x, y, t, t)$, $i = 1, \ldots, 6$ denote the densities of particles moving in the six allowed directions; $\sigma/2c$ is the cross section for binary collisions.

For flows which are independent of y, z and for which $f_3 = f_4 = f_5 = f_6$ the above six velocity Broadwell model reduces to the simpler form

$$
\begin{aligned}
&\frac{\partial f_1}{\partial t} + \frac{\partial f_1}{\partial x} = \frac{1}{\varepsilon}\left(f_3^2 - f_1 f_1\right) \\
(2) \qquad &\frac{\partial f_2}{\partial t} - \frac{\partial f_2}{\partial x} = \frac{1}{\varepsilon}\left(f_3^2 - f_1 f_1\right) \\
&\frac{\partial f_3}{\partial t} = \frac{1}{2\varepsilon}\left(f_1 f_2 - f_3^2\right)
\end{aligned}
$$

where for simplicity we have set $c = 1$ and $\sigma = \frac{1}{2\varepsilon}$; ε the Knudsen number or "mean free path" of the gas. As the "mean free path" is the distance between successive collisions, a small mean free path means the gas becomes less rarefied and a "macroscropic" description of the gas based on fluid dynamic Euler and/or Navier-Stokes equations should become meaningful.

The problem of rigorously passing to the fluid dynamic limit has a long history. Here I give a quick summary of relevant results. Additional references may be found in the book of Cercignani [4] for work on the Boltzmann equation and the review paper of Platkowski and Illner [11] with regard to research on discrete velocity model in the kinetic theory of gases.

First within the realm of discrete velocity models the Carleman model does allow for rigorous passage to the hydrodynamic limit. This was done in the work of Kurtz [10]. But as the Carleman model does not conserve momentum it is perhaps a poor test case.

For the Broadwell model the basic result is due to Broadwell himself [1]. One begins by rewriting the system (2) as

$$\frac{\partial}{\partial t}(f_1 + f_2 + 4f_3) + \frac{\partial}{\partial x}(f_1 - f_2) = 0\,.$$

(3)
$$\frac{\partial}{\partial t}(f_1 - f_2) + \frac{\partial}{\partial x}(f_1 + f_2) = 0\,,$$

$$\frac{\partial f_3}{\partial t} = \frac{1}{2\varepsilon}(f_1 f_2 - f_3^2)\,.$$

Next one makes the ansatz of travelling wave solutions $f_1 = f_1(\theta)$, $f_2 = f_2(\theta)$, $f_3 = f_3(\theta)$, $\theta = \dfrac{x - st}{\varepsilon}$ where s will be the speed of the wave. Substitution of this ansatz into (3) yields the system of ordinary differential equations

$$-s(f_1 + f_2 + 4f_3)' + (f_1 - f_2)' = 0$$

(4)
$$-s(f_1 - f_2)' + (f_1 + f_2)' = 0$$

$$-sf_3' = \frac{1}{2}(f_1 f_2 - f_3^2)\,.$$

We pose downstream and upstream positive, constant data $(f_1, f_2, f_3) \to \left(f_1^{\pm}, f_2^{\pm}, f_3^{\pm}\right)$ as $\theta \to \pm\infty$ which of course is consistent with (4) if and only if the data are Maxwellians, i.e. $f_1^{\pm} f_2^{\pm} = f_3^{\pm 2}$. Next since s is a constant (4a), (4b) may be integrated from $-\infty$ to θ to yield:

$$-s(f_1 + f_2 + 4f_3) + (f_1 - f_2) = -s(f_1^- + f_2^- + 4f_3^-) + (f_1^- - f_2^-)\,,$$

$$-s(f_1 - f_2) + (f_1 + f_2) = -s(f_1^- - f_2^-) + (f_1^- + f_2^-)\,.$$

These two equations determine f_1, f_2 as functions of $f_3(\theta)$ and s. Substitution of these functions into (4c) will yield an autonomous scalar ordinary differential equation for f_3 with precisely two equilibrium points at $f_3 = f_3^{\pm} = \sqrt{f_1^{\pm} f_2^{\pm}}$. Since such boundary value problems must possess solutions it follows that such a travelling wave solution must exist. The value of s is found by integrating (4) from $-\infty$ to $+\infty$:

$$-s\left(f_1^+ + f_2^+ + 4(f_1^+ f_2^+)^{1/2}\right) + (f_1^+ - f_2^+)$$

(5)
$$= -s\left(f_1^- + f_2^- + 4(f_1^- f_2^-)^{1/2}\right) + (f_1^- - f_2^-)\,,$$

$$-s(f_1^+ - f_2^+) + (f_1^+ + f_2^+) = -s(f_1^- - f_2^-) + (f_1^- + f_2^-)\,,$$

and solving the system (5) for s. This is just the Rankine-Hugoniot jump condition.

Once the existence of a travelling wave solution to (3) is established we can immediately let $\varepsilon \to 0+$ to see that

$$(f_1, f_2, f_3) \to \left(f_1^-, f_2^-, \sqrt{f_1^- f_2^-}\right) \quad \text{if} \quad x < st ;$$

$$\left(f_1^+, f_2^+, \sqrt{f_1^+ f_2^+}\right) \quad \text{if} \quad x > st \quad \text{as} \quad \varepsilon \to 0+ .$$

The limit function is a distributional solution of the limiting fluid dynamic conservation laws

$$\begin{aligned} &\frac{\partial}{\partial t}\left(f_1 + f_2 + 4(f_1 f_2)^{1/2}\right) + \frac{\partial}{\partial x}(f_1 - f_2) = 0 , \\ &\frac{\partial}{\partial t}(f_1 - f_2) + \frac{\partial}{\partial x}(f_1 + f_2) = 0 . \end{aligned} \tag{6}$$

This is because the limit function is piecewise constant possessing a jump discontinuity across the shock $x = st$ and across $x = st$ the limit function satisfies the jump condition (5). In fact the limit function is a solution to the *Riemann problem* (6) with piecewise constant initial data

$$\begin{aligned} f_1 &= f_1^- \quad (x < 0) , \; f_1^+ \quad (x > 0) ; \\ f_2 &= f_2^- \quad (x < 0) , \; f_2^+ \quad (x > 0) . \end{aligned} \tag{7}$$

It should be noted that the Boltzmann equation also possesses a travelling wave solution for Maxwellian data which are close (see Caflisch and Nicolaenko [2]). Of course their data must also be consistent with relevant fluid dynamic jump condition which is the Rankine-Hugoniot jump condition for a shock wave in an ideal fluid.

In summary we see for *Riemann data satisfying the Rankine-Hugoniot jump condition associated with the fluid dynamic limit equations* passage to the fluid dynamic limit for the Broadwell model can be achieved (and with a smallness assumption on the variation of the data for the Boltzmann equation also).

What can be said regarding the fluid dynamic limit for arbitrary data or for that manner even the more restricted case of *arbitrary Riemann data*?

For the case of smooth data Inoue and Nishida [9] have shown that one can pass to the fluid dynamic limit for the Broadwell system on a sufficiently small time interval to yield a smooth solution of the fluid dynamic limit equations. Thus they were able to show *compactness* of the parametrized sequence $\{f_1^\varepsilon, f_2^\varepsilon, f_3^\varepsilon\}$ satsfying the Broadwell system in a space which allowed passage to the fluid dynamic limit. Of course this result misses the fundamental issue of recovery of the fluid dynamic limit when the limit equations possess solutions with shocks.

In a similar spirit to Inoue and Nishida's result, Caflisch and Papanicalaou [3] showed that a given *smooth* solution (shockless) to the limit equations can be *approximated* by a solution to the Broadwell system when ε is small. This *approximation* approach has been continued by Xin [14] where he showed that if the Broadwell fluid dynamic limit equation has a piecewise smooth solution with finitely many noninteracting shocks of suitably small oscillation, then he can construct solutions to the Broadwell system which converge asymptotically to the fluid dynamical solution as $\varepsilon \to 0+$. *However* this convergence does not hold in thin shock layers. Of course the whole notion of the *approximation* program of Caflisch, Papanicalaou, and Xin presupposed knowledge of an "admissible" solution to the underlying limit conservation laws (6) and is intended as a method to solve the Broadwell system (2) based on solutions to (6). The *compactness* method is the reverse, one attempts to solve (6) as a limit of solutions of (2).

In the research discussed here I continue in the spirit of the *compactness* issue. The goal is to extend the success of Broadwell's original travelling wave idea to more general Riemann data, i.e. to data not necessarily consistent with the Rankine-Hugoniot jump conditions. The idea is based on the following observation: For any system of conservation laws

$$\frac{\partial F(U)}{\partial t} + \frac{\partial G(U)}{\partial x} = 0, \tag{8}$$

$U : (-\infty,\infty) \times (0,\infty) \to \mathbb{R}^N$, $F, G : \mathbb{R}^N \to \mathbb{R}^N$ with Riemann data

$$\begin{gathered} U(x,0) = U_0(x), \\ U_0(x) = U^- \quad x < 0, \; U = U^+ \quad x > 0, \end{gathered} \tag{9}$$

must possess space-time dilation invariance. This means that for any positive constant $\alpha > 0$, the change of variable $(x,t) \to (\alpha x, \alpha t)$ preserves both the equations and the initial data. Hence solutions of Riemann problems should depend only on the similarity variable $\xi = \frac{x}{t}$ i.e. $U(x,t) = U(\xi)$.

For example if one was attempting to solve the Riemann problem for a system of conservation laws as a "viscous" limit of the system

$$\frac{\partial F(U)}{\partial t} + \frac{\partial G(U)}{\partial x} = \varepsilon \frac{\partial^2 U}{\partial x^2} \tag{10}$$

one might first consider substitution of the ansatz $U(x,t) = U(\xi)$ into (10). But unfortunately (10) does not possess space-time dilational invariance. It was for this reason that Dafermos [5] suggested a new type of "viscous" limit problem

$$\frac{\partial F(U)}{\partial t} + \frac{\partial G(U)}{\partial x} = \varepsilon t \frac{\partial^2 U}{\partial x^2} \tag{11}$$

$t > 0$, which does possess space-time dilation invariance. Substitution of $U(x,t) = U(\xi)$ into (11) yields the system of ordinary differential equations

$$-\xi F(U(\xi))' + G(U(\xi))' = \varepsilon U'' \tag{12}$$

and (9) implies boundary conditions

$$U(-\infty) = U^- \,,\ U(+\infty) = U^+ \,. \tag{13}$$

In papers [5], [6] Dafermos and DiPerna showed that for $N = 2$ a large class of Riemann problems for hyperbolic conservation laws may be solved as limits of solutions of (12), (13) as $\varepsilon \to 0+$. The program has been continued in the work of Slemrod [12], Fan [7], and Slemrod and Tzavaras [13].

In the same spirit we easily recognize that the Broadwell system does not possess space-time dilational invariance. Hence we are motivated to consider an artificial Broadwell system

$$\begin{aligned} \frac{\partial f_1}{\partial t} + \frac{\partial f_1}{\partial x} &= \frac{1}{\varepsilon t}(f_3^2 - f_1 f_2), \\ \frac{\partial f_2}{\partial t} - \frac{\partial f_2}{\partial x} &= \frac{1}{\varepsilon t}(f_3^2 - f_1 f_2), \\ \frac{\partial f_3}{\partial t} &= \frac{1}{2\varepsilon t}(f_1 f_2 - f_3^2), \end{aligned} \tag{14}$$

which does possess space-time dilational invariance. (Of course the same observation is true for the Boltzmann equation and any of the standard discrete velocity models in the kinetic theory of gases.)

We now make the ansatz $f_1(x,t) = f_1(\xi)$, $f_2(x,t) = f_2(\xi)$, $f_3(x,t) = f_3(\xi)$, and substitute into (14) to obtain the system of non-autonomous ordinary differential equations

$$\begin{aligned} -(\xi - 1)f_1'(\xi) &= (f_3^2 - f_1 f_2)/\varepsilon, \\ -(\xi + 1)f_2'(\xi) &= (f_3^2 - f_1 f_2)/\varepsilon, \\ -\xi f_3' &= (f_1 f_2 - f_3^2)/2\varepsilon. \end{aligned} \tag{15}$$

Since we wish $f_j(x,t) \to f_j^{\pm}$ for $x \lessgtr 0$ as $t \to 0+$ for $j = 1,2,3$ we impose boundary data

$$f_j(-\infty) = f_j^-, \; f_j(+\infty) = f_j^+, \; j = 1,2,3 \tag{16}$$

where $f_1^- f_2^- = f_3^{-2}$, $f_1^+ f_2^+ = f_3^{+2}$.

System (15), (16) is considerably harder to analyze than system (4) obtained from the travelling wave ansatz. The reasons are obvious: (i) (15) is non-autonomous in the similarity variable ξ and (ii) (15) does not possess any first integrals that will allow us to reduce the number of dependent variables.

System (15), (16) does possess one small simplification. Since $(f_1, f_2, f_3) = (f_1^{\pm}, f_2^{\pm}, f_3^{\pm})$ are equilibria on $-\infty < \xi < -1$ and $1 < \xi < \infty$ we must have $(f_1, f_2, f_3)(\xi) = (f_1, f_2, f_3)$ and $(f_1, f_2, f_3)(\xi) = (f_1^+, f_2^+, f_3^+)$ respectively. Hence the boundary conditions at $\xi = \pm\infty$ are replaced by

$$f_j(-1) = f_j^-, \; f_j(+1) = f_j^+, \; j = 1,2,3 \tag{17}$$

and (15) need only be considered on $-1 < \xi < 1$.

The goal now is twofold. First we must solve the boundary value problem (15), (17) for $\varepsilon > 0$ fixed. Second we must show that as $\varepsilon \to 0+$ solutions $(f_1^\varepsilon, f_2^\varepsilon, f_3^\varepsilon)$ of (15), (17) possess a limit which is weak solution to the Riemann problem (6), (7) for the fluid limit system. Remarkably the second part of this program is easier than the first. At the moment (in joint work with A. Tzavaras) only a class of Maxwellian data f_j^-, f_j^+ $j = 1, 2, 3$ has been found for which solutions exist to (15), (17) for all $\varepsilon > 0$. However for any data for which solutions $(f_1^\varepsilon, f_2^\varepsilon, f_3^\varepsilon)$ of (15), (17) are known to exist for all $\varepsilon > 0$ we are guaranteed that we can extract a convergence subsequence so that $(f_1^\varepsilon, f_2^\varepsilon, f_3^\varepsilon)(\xi) \to (f_1, f_2, f_3)$ boundedly a.e. in $-\infty < \xi < \infty$ where $f_3 = \sqrt{f_1 f_2}$ a.e. and f_1, f_2, f_3 is a weak solution of the Riemann problem (6), (7). The proof is based on obtaining estimates on the total variation of solution of (16), (17) and applying Helly's theorem.

Finally it is interesting to compare the approach given here to a recent paper of F. Golse [8]. In his paper Golse makes the self-similar ansatz for the Broadwell system

$$f_j(x,t) = F_j(\xi)/t \quad , \quad j = 1, 2, 3 \tag{18}$$

where again $\xi = \dfrac{x}{t}$.

Substitution of (18) into (2) yields the system of ordinary differential equations

$$\begin{aligned} -[(\xi - 1)F_1(\xi)]' &= (F_3^2 - F_1F_2)/\varepsilon \ , \\ -[(\xi + 1)F_2(\xi)]' &= (F_3^2 - F_1F_2)/\varepsilon \ , \\ -[\xi F_3(\xi)]' &= (F_1F_2 - F_3^2)/2\varepsilon \ , \end{aligned} \tag{19}$$

which differs from (15) in the fact that the left hand side of (15) has differentiation followed by multiplication while (19) has the reverse. System (19) then has the same properties as (4) of possessing two first integrals and hence the ability to sufficiently simplify the analysis. Golse exploits this property to show the existence of an analytic on $-1 < \xi < 1$ solutions F_1, F_2, F_3 of (19). The importance of the result is that it displays explicitly the large time $O\left(\frac{1}{t}\right)$ behavior of a class of solutions to the Broadwell system (2). The solutions f_1, f_2, f_3 of course do not possess space-time dilational invariance and will not be useful in resolution of the Riemann problem for the limit fluid dynamic system (6), (7).

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