

Fluid Motion Induced by Surface-Tension Variation

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Fluid motion in a long straight channel induced by longitudinally varying surface tension has been discussed by Levich. This problem is re-examined and a different solution is given. In addition, the stability of laminar flows involving surface-tension variation is briefly discussed, and a correction of a previous result [C.-S. Yih, *J. Fluid Mech.* **28**, 493 (1967)] is made.

I. INTRODUCTION

It is now common knowledge that at the free surface of a liquid, the pressure at the liquid side of the surface depends on the local surface tension and the local curvature of the surface, and the shear stress depends on the local gradient of surface tension. If the surface tension is nonuniform, motion will be induced in the liquid.

Levich has written an impressive book with the title *Physicochemical Hydrodynamics*, in which fluid motion induced by surface tension has been discussed in some detail. A solution was given by Levich¹ for the motion of a thin and wide fluid film induced by surface-tension variation. Upon close examination of Levich's solution many inconsistencies are found. The correct solution of Levich's problem is presented here as a development resulting from the stimulation of his admirable work.

II. SOLUTION OF LEVICH'S PROBLEM

Levich's problem concerns the flow of a liquid layer induced by longitudinal variation of surface tension. The channel joining two reservoirs (Fig. 1) is supposed to be very much wider than the depth of the liquid supported by a horizontal bottom at $z = 0$. The reservoirs are supposed to be even wider, and the channel is supposed to be much longer than it is wide. Thus nonuniformities at the ends ($x = 0$ and $x = L$) of the channel can be neglected, and the flow can be assumed independent of the coordinate y measured in a direction across the channel. The flow is then two-dimensional. Since the depth is small compared with L , the flow is nearly, though not strictly, unidirectional. That is to say, it is nearly parallel to the x axis, but not quite, since the depth of the fluid, as will be seen, changes from one value of x to another.

The variation of surface tension is accomplished by the presence of surface-active material in the

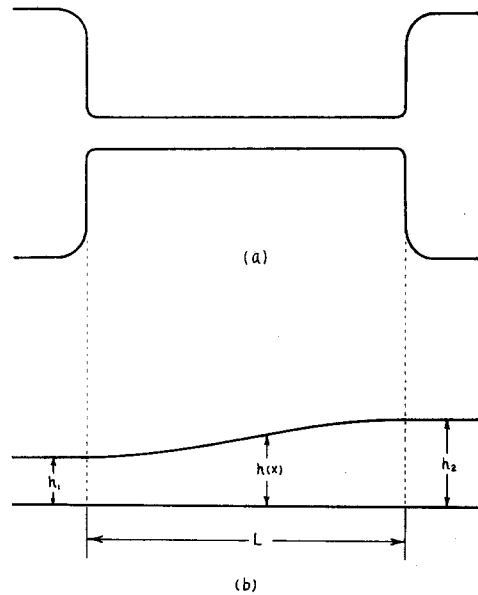


FIG. 1. Definition sketch. (a) Plan. (b) Vertical cross section along the longitudinal centerline. The vertical scale is exaggerated.

reservoir to the left of the position $x = 0$. If the surface concentration of that material is γ , the surface tension σ is a function of γ . For simplicity we shall assume $d\sigma/d\gamma$ to be constant. For steady flows the diffusion equation for the surface material can then be written as

$$\frac{\partial}{\partial x}(u\sigma) = \frac{d}{dx}\left(D\frac{d\sigma}{dx}\right), \quad (1)$$

in which u is the velocity component in the x direction at the surface, and D is the diffusivity. Strictly speaking x should be replaced by a curvilinear distance in a direction along the curved free surface, and u should be the velocity in the same direction. We state without delay that the theory to be presented here is a shallow-water theory, that even though the depth changes, the vertical component of the velocity is much smaller than the horizontal component, and that all effects of the curvature of the free surface are neglected.

¹ V. G. Levich, *Physicochemical Hydrodynamics* (Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1962), 2nd ed. pp. 384-388.

Following Levich, we shall only treat cases in which the inertial effects are negligible. That is to say, we assume the Reynolds number VH/ν to be very small, in which H is the maximum depth of the liquid film, V is a representative velocity, and ν is the kinematic viscosity. [Note that the Péclet number VL/D , in which L is a representative horizontal scale, may not be small. In fact it may be so large that the diffusive term in Eq. (1) may be neglected.] The equations of steady motion are then

$$\frac{\partial p}{\partial x} = \mu \frac{\partial^2 u}{\partial z^2}, \quad \frac{\partial p}{\partial z} = -\rho g, \quad (2a, b)$$

in which p is the pressure, μ the viscosity, ρ the density, and g the gravitational acceleration. Both μ and ρ are assumed constant, and of the term $\nabla^2 u$ only the dominant term $\partial^2 u/\partial z^2$ is retained in (2a). Furthermore w (velocity component in the z direction) is small, so that the term $\mu \nabla^2 w$ is entirely neglected in (2b). The equation of continuity is

$$\frac{d}{dx} \int_0^h u \, dz = 0. \quad (3)$$

The boundary condition at the bottom is simply

$$u = 0 \quad \text{at} \quad z = 0. \quad (4)$$

The boundary conditions at the free surface are

$$\mu \frac{\partial u}{\partial z} = \frac{d\sigma}{dx} \quad \text{at} \quad z = h(x) \quad (5a)$$

and

$$p = 0 \quad \text{at} \quad z = h(x), \quad (5b)$$

$h(x)$ being the depth of the liquid film. Note that in (5) again the effect of curvature has been neglected. Suppose that we also specify the values of σ and of h at the two ends as follows:

$$\sigma = \sigma_1 \quad \text{and} \quad h = h_1 \quad \text{at} \quad x = 0, \quad (6)$$

$$\sigma = \sigma_2 \quad \text{and} \quad h = h_2 \quad \text{at} \quad x = L. \quad (7)$$

Equations (1), (2), and (3), and the boundary conditions (4)–(7) constitute the differential system to be solved.

At this time it is appropriate to mention the chief features of Levich's solution. He assumed

- (a) that σ is linear with x ,
- (b) that h is constant,
- (c) that there is a pressure gradient $\partial p/\partial x$, and
- (d) that the net discharge across any section is zero.

Assumption (a) is made in Eq. (68.2) on p. 385 of his book¹, (b) on the first two lines on the same

page (note the two uses of the word plane), (c) on pp. 385–387, and (d) in (68.7) on p. 386. We note that the diffusion equation (1) may not allow (a), that (b) and (c) are inconsistent, and (d) is a special case only. Levich never mentioned how his $\partial p/\partial x$ is to be evaluated. This may well be connected with his equation

$$\frac{\partial p}{\partial z} = 0,$$

which seems to indicate that the role of gravity was not recognized by him.

We shall satisfy (2a), (4), and (5a) by assuming

$$u = \frac{1}{\mu} \frac{d\sigma}{dx} z - \frac{1}{2\mu} \frac{\partial p}{\partial x} z(2h - z). \quad (8)$$

This was obtained by Levich. On the surface

$$u = \frac{h}{\mu} \left(\frac{d\sigma}{dx} - \frac{h}{2} \frac{\partial p}{\partial x} \right). \quad (9)$$

Equations (2b) and (5b) allow us to write

$$p = \rho g(h - y). \quad (10)$$

Thus

$$\frac{\partial p}{\partial x} = \rho g \frac{dh}{dx}. \quad (11)$$

Now we can use Eqs. (9) and (11), together with Eqs. (1) and (3), and forget about the boundary conditions (4) and (5) which have been satisfied. First, Eqs. (3), (8), and (11) yield

$$\frac{h^2}{6\mu} \left(3 \frac{d\sigma}{dx} - 2g\rho h \frac{dh}{dx} \right) = Q, \quad (12)$$

in which Q , a constant, is the discharge per unit width. With Eqs. (11) and (12), we can write Eq. (9) as

$$u = \frac{h}{4\mu} \left(\frac{d\sigma}{dx} + \frac{6\mu Q}{h^2} \right) \quad \text{at} \quad z = h. \quad (13)$$

Equation (1) can be integrated to give

$$u\sigma = D \frac{d\sigma}{dx} + q, \quad (14)$$

in which q is the (constant) discharge of σ per unit width. (Perhaps it is better to say that $q \, d\gamma/d\sigma$ is the discharge of the surface material per unit width.) We can, if we so choose, eliminate u and σ between Eqs. (12), (13), and (14), and obtain a differential equation in h alone. But this will only produce a highly nonlinear equation that in general cannot be solved analytically. For the general case, we prefer to resort to numerical integration in the

following way. Substituting (13) into (14), we have

$$\left(\frac{h\sigma}{4\mu} - D\right) \frac{d\sigma}{dx} + \frac{3Q\sigma}{2h} = q. \tag{15}$$

Given h_1 and σ_1 , we can assume Q and q and compute $d\sigma/dx$ at $x = 0$. Then compute dh/dx at $x = 0$ by (12). We then know σ and h at $x = \Delta x$, and can proceed further, until we come to $x = L$, where σ should be equal to σ_2 and h equal to h_2 . These conditions are to be satisfied by the proper choice of Q and q . This is obviously a tedious calculation, and it may happen that the end values σ_1, σ_2, h_1 , and h_2 cannot be satisfied by a steady flow. To proceed further and to bring out the possible lack of steady-state solutions, we shall consider the following special cases.

A. The Case of Zero Discharge

In case $Q = 0$, Eq. (12) can be immediately integrated to give

$$\sigma = \frac{\rho g}{3} h^2 + C_1, \tag{16}$$

and (15) can be integrated to produce

$$\frac{\rho g}{6\mu} \left(\frac{\rho g h^5}{15} + \frac{C_1 h^3}{3} \right) - \frac{1}{3} D \rho g h^2 = qx + C_2. \tag{17}$$

The constants are determined by Eqs. (6) and (7). Thus,

$$C_1 = \sigma_1 - \frac{\rho g}{3} h_1^2. \tag{18}$$

Due to the requirement that Q be zero, there is no freedom in choosing h_2 if σ_2 is given, since

$$\sigma_2 = \frac{\rho g}{3} h_2^2 + C_1. \tag{19}$$

We can now determine q and C_2 such that

$$h = h_1 \text{ at } x = 0 \text{ and } h = h_2 \text{ at } x = L,$$

and the problem is solved.

But it is important to note the singular nature of the differential equation (15). Let us consider cases in which $\sigma_2 > \sigma_1, h_2 > h_1$. If

$$D > \frac{h_2 \sigma_2}{4\mu},$$

q is negative according to Eq. (15), and by choosing a proper q we can satisfy the condition $\sigma = \sigma_2$ at $x = L$ on integrating Eq. (15) with the aid of Eqs. (16) and (18). Similarly, if

$$D < \frac{h_1 \sigma_1}{4\mu},$$

a positive q can satisfy the condition for σ at $x = L$. But if

$$\frac{h_1 \sigma_1}{4\mu} < D < \frac{h_2 \sigma_2}{4\mu}, \tag{20}$$

for some intermediate value of x (15) is singular, and there will be a cusp there if the end conditions (6) and (7) are to be satisfied. Since cusps must be ruled out as physically inadmissible, for end values satisfying (20) steady-state solutions are impossible, and the flow will be transient until the liquid levels or the σ values at the ends have reached values for which steady flow is possible. When steady flow is possible, the profile is given by (17), and the surface-material discharge by $q \, d\gamma/d\sigma$.

B. The Case of Zero Surface Velocity

In this case the u in (9) is zero, so that

$$\frac{d\sigma}{dx} - \frac{h}{2} \frac{\partial p}{\partial x} = \frac{d\sigma}{dx} - \frac{\rho g h}{2} \frac{dh}{dx} = 0,$$

integration of which gives

$$\sigma = \frac{\rho g}{4} h^2 + C_3. \tag{21}$$

Since the u in (1) is zero, integration of (1) produces

$$\sigma = \sigma_1 + \frac{\sigma_2 - \sigma_1}{L} x. \tag{22}$$

Thus

$$\frac{\rho g}{4} h^2 = \sigma_1 + \frac{\sigma_2 - \sigma_1}{L} x - C_3. \tag{23}$$

The condition $h = h_1$ at $x = 0$ gives

$$C_3 = \sigma_1 - \frac{\rho g}{4} h_1^2. \tag{24}$$

But h_2 is no longer arbitrary. It is given by

$$\frac{\rho g}{4} h_2^2 = \sigma_2 - C_3. \tag{25}$$

The discharge of surface material per unit width is

$$q = D \frac{\sigma_2 - \sigma_1}{L} \frac{d\gamma}{d\sigma}. \tag{26}$$

One especially simple case of zero surface velocity and constant depth requires the bottom to be inclined. If the angle of inclination is β , and if we measure x along the inclined bottom and z in a direction normal to it, Eq. (1) remains valid but Eq. (2a) is replaced by

$$g\rho \sin \beta = \mu \frac{\partial^2 u}{\partial z^2} \tag{27}$$

and Eq. (2b) by

$$\frac{\partial p}{\partial z} = -\rho g \cos \beta. \quad (28)$$

Equation (28) gives the pressure p , and is not needed if the depth is constant. The solution of (27) satisfying (4) and $u = 0$ at $z = \text{const } h$ is

$$u = \frac{g}{2\nu} (\sin \beta) z(z - h). \quad (29)$$

Since $u = 0$ at $z = h$, the solution of (1) is simply (22), and (5a) is satisfied if

$$\frac{\rho g h}{2} \sin \beta = \frac{d\sigma}{dx}, \quad (30)$$

which determines h for a given β , or β for a given h . Of course, if Eq. (30) is not satisfied the constant-depth solution does not exist, and a steady-state solution may not even exist.

III. STABILITY OF LAMINAR FLOWS DRIVEN BY SURFACE TENSION

As shown in Yih,² laminar flows driven by or affected by surface-tension variation can be unstable, especially with respect to long waves. It must be kept in mind that if instability with respect to long waves is considered, the longitudinal variation of the velocity must not be ignored. In the paper of Yih, the flow is strictly unidirectional.

I should like to avail myself of the opportunity to correct an omission kindly pointed out to me by A. Craik. Since the surface-diffusion equation must be applied on the free surface, the quantity u' should be replaced by $u' + \eta$ in Eq. (28) of Ref. 2. When this correction is made, and the anal-

ysis is followed through, the correct criterion replacing the final formula in that paper is

$$\left(\frac{\alpha_0}{\omega} + 2 \frac{\omega}{\alpha_0} \right) \Delta\alpha = \frac{i\omega^2 R}{60} \left\{ -90 - 12 \frac{\alpha_0}{\omega} - 3 \left(\frac{\alpha_0}{\omega} \right)^2 + \left(2 - \frac{9\alpha_0}{\omega} \right) F^{-2} \right\} - i2\omega^2 \left(\frac{\bar{\gamma}}{\gamma_1 d} \frac{\alpha_0}{\omega} - \frac{1}{P\epsilon} \frac{\alpha_0^2}{\omega^2} \right), \quad (31)$$

in which

$$\frac{\alpha_0}{\omega} = 1 \pm \sqrt{3}, \quad (32)$$

the symbols being defined in Yih². The interesting feature is that there are two modes, one traveling upstream and one traveling downstream. The former corresponds to the negative sign in (39), and for that mode instability corresponds to positive values of α_i ($\Delta\alpha = i\alpha_i$). The latter corresponds to the positive sign in (39), and for it instability corresponds to negative values of α_i , because of the form of the assumed exponential factor (for all perturbation quantities)

$$\exp i \left(\int \alpha dx - \omega\tau \right).$$

It turns out that both modes can be unstable! Note that the coefficient of $\Delta\alpha$ in (31) is simply $\pm 2\sqrt{3}$. A numerical verification can be given to show the actual possibility of instability, as was done in Yih². The conclusion that there are realistic cases of instability remain valid.

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² C.-S. Yih, *J. Fluid Mech.* **28**, 493 (1967).