FLUID QUEUES DRIVEN BY A BIRTH AND DEATH PROCESS WITH ALTERNATING FLOW RATES

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Fluid queue driven by a birth and death process (BDP) with only one negative effective input rate has been considered in the literature. As an alternative, here we consider a fluid queue in which the input is characterized by a BDP with alternating positive and negative flow rates on a finite state space. Also, the BDP has two alternating arrival rates and two alternating service rates. Explicit expression for the distribution function of the buffer occupancy is obtained. The case where the state space is infinite is also discussed. Graphs are presented to visualize the buffer content distribution.

1. Introduction

Recent measurements have revealed that in high-speed telecommunication networks, like the ATM-based broadband ISDN, traffic conditions exhibit long-range dependence and burstiness over a wide range of time scales. Fluid models characterize such a traffic as a continuous stream with a parameterized flow rate. Whitt [6] establishes heavy-traffic stochastic process limits for fluid queue models with multiple on-off sources.

A fluid model that is typically used to model such a traffic is a *Markov Modulated Fluid Model* wherein the current state of the underlying Markov process determines the flow rate. Fluid models driven by finite state Markov processes that modulate the input rate in the fluid buffer have been analyzed by many authors. Lenin and Parthasarathy [2] provide closed-form expressions for the eigenvalues and eigenvectors for fluid queues driven by an $M/M/1/N$ queue. The case where the state space is infinite has been analyzed by van Doorn and Scheinhardt [5] for a birth and death process (BDP).

In most studies dealing with Markov modulated fluid queues, a single negative effective flow rate is assumed. Here, we consider a more general setting of a fluid queue driven by a BDP on a finite state space in which the flow rates are alternatively positive and negative. Our aim is to obtain the stationary distribution function of the buffer occupancy for this fluid model which is modulated by a BDP with two alternating arrival rates and two alternating service rates. This modulating Markov process can be visualized as a simple case of a two-state *Markov Modulated Poisson Process* which is characterized

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by a Markov process with an infinitesimal generator $\mathbf{Q} = \begin{bmatrix} -\lambda_1 & \lambda_1 \\ \lambda_2 & -\lambda_2 \end{bmatrix}$ and a diagonal matrix $\mathbf{\Lambda} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ of arrival probabilities, where λ_1 and λ_2 denote the rates of arrival when the traffic is bursty and slow, respectively. The case where the state space of the BDP is infinite is also discussed. Some interesting identities of tridiagonal determinants are used for the finite state space, and continued fraction methodology is employed for the infinite state space. Graphs are presented to visualize the buffer content distribution.

The model under consideration finds a wide range of application in modelling a communication switch as a fluid stochastic petri net in which two streams of traffic arrive (Horton et al. [1]). One stream is bursty with a high flow rate when the server is busy and the other stream is slow with a low flow rate when the workload of the server is less. We will designate the fluid commodity accumulating in the infinite capacity buffer as *credit*. It may be helpful to think of credit as the energy which the server gathers during lean traffic period and consumes when the traffic is bursty.

2. Model description

We consider an infinite capacity buffer which receives and releases fluid flows modulated by a BDP evolving in the background. We denote the background birth-death process by $\mathscr{X} := \{X(t),\ t \geq 0\}$ taking values in the state space $\mathscr{S}$, where $X(t)$ denotes the state of the process at time t. Let λ_n and μ_n denote the mean arrival and service rates, respectively, when there are n units in the system.

The flow rates of the fluid into and out of an infinite capacity buffer are determined by the actual state of the background process. Let r_j denote the flow rate of the fluid when the background process is in state j. The rate of change of content of the buffer $C(t)$ when $X(t) = j$ is given by

$$\frac{dC(t)}{dt} = \begin{cases} r_j & \text{if } C(t) > 0, \\ 0 & \text{if } C(t) = 0,\ r_j < 0. \end{cases} \tag{2.1}$$

Clearly, the two-dimensional process $\{(X(t), C(t)),\ t \geq 0\}$ constitutes a Markov process which possesses a unique stationary distribution under a suitable stability condition.

The stationary state probabilities p_i, $i \in \mathscr{S}$, of the BDP can be represented as

$$p_i = \frac{\pi_i}{\sum_{j \in \mathscr{S}} \pi_j}, \quad i \in \mathscr{S}, \tag{2.2}$$

where $\pi_i = \lambda_0 \lambda_1 \cdots \lambda_{i-1} / \mu_1 \mu_2 \cdots \mu_i$, $i = 1, 2, 3, \ldots$, and $\pi_0 = 1$ are called the potential coefficients. In order that a limit distribution for $C(t)$ exists as $t \to \infty$, the stationary net input rate should be negative, that is,

$$\sum_{i=0}^{\infty} \pi_i r_i < 0. \tag{2.3}$$

Letting

$$F_j(t,u) \equiv P\big(X(t) = j,\ C(t) \le u\big), \quad j \in \mathcal{S},\ t,u \ge 0, \tag{2.4}$$

the Kolmogorov forward equations for the Markov process $\{X(t), C(t)\}$ are given by

$$\begin{aligned}
\frac{\partial F_0(t,u)}{\partial t} &= -r_0 \frac{\partial F_0(t,u)}{\partial u} - \lambda_0 F_0(t,u) + \mu_1 F_1(t,u), \\
\frac{\partial F_j(t,u)}{\partial t} &= -r_j \frac{\partial F_j(t,u)}{\partial u} - (\lambda_j + \mu_j) F_j(t,u) \\
&\quad + \lambda_{j-1} F_{j-1}(t,u) + \mu_{j+1} F_{j+1}(t,u), \quad j \in \mathcal{S} \setminus \{0\},\ t,u \ge 0.
\end{aligned} \tag{2.5}$$

(See van Doorn and Scheinhardt [5]). When the process is in equilibrium, $\partial F_j(t,u)/\partial t \equiv 0$, and let $\lim_{t\to\infty} F_j(t,u) \equiv F_j(u)$.

3. Finite state space

This section deals with a fluid queue modulated by a finite BDP with state space $\mathcal{S} = \{0,1,2,\dots,N\}$. The system of equations governing the two-dimensional process $\{(X(t), C(t)),\ t \ge 0\}$ in equilibrium is

$$\begin{aligned}
r_0 \frac{dF_0(u)}{du} &= -\lambda_0 F_0(u) + \mu_1 F_1(u), \\
r_j \frac{dF_j(u)}{du} &= \lambda_{j-1} F_{j-1}(u) - (\lambda_j + \mu_j) F_j(u) + \mu_{j+1} F_{j+1}(u), \quad \text{for } j \in \mathcal{S} \setminus \{0\},\ u \ge 0.
\end{aligned} \tag{3.1}$$

In matrix notation (3.1) can be written as

$$\frac{d\mathbf{F}(u)}{du} = \mathbf{R}^{-1}\mathbf{Q}^T \mathbf{F}(u), \quad u \ge 0, \tag{3.2}$$

where $\mathbf{F}(u) = [F_0(u), F_1(u), \dots, F_N(u)]^T$, $\mathbf{R} = \operatorname{diag}(r_0, r_1, \dots, r_N)$, and

$$\mathbf{Q} = \begin{bmatrix}
-\lambda_0 & \lambda_0 & & & \\
\mu_1 & -(\lambda_1 + \mu_1) & \lambda_1 & & \\
& & \ddots & & \\
& & \mu_{N-1} & -(\lambda_{N-1} + \mu_{N-1}) & \lambda_{N-1} \\
& & & \mu_N & -\mu_N
\end{bmatrix}_{(N+1)\times(N+1)}. \tag{3.3}$$

Hence

$$\mathbf{R}^{-1}\mathbf{Q}^{T}=\begin{bmatrix} -\frac{\lambda_0}{r_0} & \frac{\mu_1}{r_0} & & & & \\ \frac{\lambda_0}{r_1} & -\frac{(\lambda_1+\mu_1)}{r_1} & \frac{\mu_2}{r_1} & & & \\ & & & \ddots & & \\ & & & \frac{\lambda_{N-2}}{r_{N-1}} & -\frac{(\lambda_{N-1}+\mu_{N-1})}{r_{N-1}} & \frac{\mu_N}{r_{N-1}} \\ & & & & \frac{\lambda_{N-1}}{r_N} & -\frac{\mu_N}{r_N} \end{bmatrix}_{(N+1)\times(N+1)}. \tag{3.4}$$

Mitra [3] has shown that $\mathbf{R}^{-1}\mathbf{Q}^{T}$ has exactly N_+ negative eigenvalues, $N_- - 1$ positive eigenvalues, and one zero-eigenvalue, where N_+ is the cardinality of the set

$$\mathcal{S}^{+} := \{j \in \mathcal{S} : r_j > 0\} \tag{3.5}$$

and N^- is that of

$$\mathcal{S}^{-} := \{j \in \mathcal{S} : r_j < 0\}. \tag{3.6}$$

Let ξ_j, $j = 0,1,2,\dots,N$, be the eigenvalues of the matrix $\mathbf{R}^{-1}\mathbf{Q}^{T}$ such that

$$\xi_j < 0, \quad j = 0,1,2,\dots,N_+ - 1, \qquad \xi_{N^+} = 0, \qquad \xi_j > 0, \quad j = N_+ + 1,\dots,N. \tag{3.7}$$

Since the content of the buffer increases when the net input rate of fluid flow into the buffer is positive, it follows that $F_j(u)$ must satisfy the boundary condition

$$F_j(0) = 0 \quad \text{for } j \in \mathcal{S}^{+}. \tag{3.8}$$

Also, we have

$$\lim_{u\to\infty} F_j(u) = p_j \quad \text{for } j \in \mathcal{S}, \tag{3.9}$$

where the p_j's are the stationary state probabilities of the background BDP. The solution to the matrix equation (3.2) is given by

$$F_j(u) = p_j + \sum_{l=0}^{N_+-1} \eta_{l,j} e^{\xi_l u}, \quad j \in \mathcal{S}, \tag{3.10}$$

where

$$\eta_{l,j} = k_l \frac{B_j(\xi_l)}{c_{j0}}, \qquad c_{j0} = \frac{\mu_1\mu_2\cdots\mu_j}{r_0 r_1\cdots r_{j-1}}. \tag{3.11}$$

The constants k_l are obtained by solving

$$p_j + \sum_{l=0}^{N+-1} k_l \frac{B_j(\xi_l)}{c_{j0}} = 0, \quad \text{for } j \in \mathcal{S}^+. \tag{3.12}$$

The polynomials $B_j(s)$ are defined recursively as

$$\begin{gathered} B_0(s) = 1, \qquad B_1(s) = s + \frac{\lambda_0}{r_0}, \\ B_j(s) = \left(s + \frac{\lambda_{j-1} + \mu_{j-1}}{r_{j-1}}\right) B_{j-1}(s) - \frac{\lambda_{j-2}\mu_{j-1}}{r_{j-2}r_{j-1}} B_{j-2}(s), \quad j = 2,3,4,\ldots,N, \\ B_{N+1}(s) = \left(s + \frac{\mu_N}{r_N}\right) B_N(s) - \frac{\lambda_{N-1}\mu_N}{r_{N-1}r_N} B_{N-1}(s). \end{gathered} \tag{3.13}$$

Also $B_{N+1}(s) = \det(s\mathbf{I} - \mathbf{R}^{-1}\mathbf{Q}^T)$ and $B_j(s)$ is the determinant obtained by considering the first j rows and columns of $B_{N+1}(s)$. More specifically, we consider a fluid queue model with effective input rates $r_{2j} < 0$ and $r_{2j+1} > 0$ for $j = 0,1,2,\ldots,(N-1)/2$. Under this assumption, the system of equations involved in the determination of the constant k_l is given in matrix form as

$$\begin{bmatrix} \frac{B_1(\xi_0)}{c_{10}} & \frac{B_1(\xi_1)}{c_{10}} & \cdots & \frac{B_1(\xi_{[N/2]})}{c_{10}} \\ \frac{B_3(\xi_0)}{c_{30}} & \frac{B_3(\xi_1)}{c_{30}} & \cdots & \frac{B_3(\xi_{[N/2]})}{c_{30}} \\ \frac{B_5(\xi_0)}{c_{50}} & \frac{B_5(\xi_1)}{c_{50}} & \cdots & \frac{B_5(\xi_{[N/2]})}{c_{50}} \\ \vdots & \vdots & & \vdots \end{bmatrix}_{[N/2]} \begin{bmatrix} k_0 \\ k_1 \\ k_2 \\ \vdots \end{bmatrix}_{[N/2]} = \begin{bmatrix} p_1 \\ p_3 \\ p_5 \\ \vdots \end{bmatrix}_{[N/2]}. \tag{3.14}$$

Hence the problem of determining the stationary distribution of the content in the fluid buffer is reduced to that of solving the above matrix equation via Cramer's rule.

We now give three examples to illustrate the above discussion.

Example 3.1 ($N = 1$). The steady state probabilities for this two-state Markov process are given by $p_0 = \mu_1/(\lambda_0 + \mu_1)$ and $p_1 = \lambda_0/(\lambda_0 + \mu_1)$. It follows from (2.3) that the condition $\mu_1 r_0 + \lambda_0 r_1 < 0$ ensures the stability of the Markov process $\{(X(t), C(t)),\ t \geq 0\}$.

The matrix $\mathbf{R}^{-1}\mathbf{Q}^T$ takes the form

$$\mathbf{R}^{-1}\mathbf{Q}^T = \begin{bmatrix} -\dfrac{\lambda_0}{r_0} & \dfrac{\mu_1}{r_0} \\ \dfrac{\lambda_0}{r_1} & -\dfrac{\mu_1}{r_1} \end{bmatrix} \tag{3.15}$$

with eigenvalues $\xi_0 = -(\lambda_0/r_0 + \mu_1/r_1)$ and $\xi_1 = 0$. The system has one negative root provided $\lambda_0/r_0 + \mu_1/r_1 > 0$ (which follows from the stability condition). Further, $\eta_{00} = k_0$, $\eta_{01} = -k_0 r_0/r_1$, and from $p_1 + k_0(B_1(\xi_0)/c_{10}) = 0$, we obtain $k_0 = (r_1/r_0)p_1$. Therefore the final solution is given by

$$\begin{aligned} F_0(u) &= p_0 + \frac{r_1}{r_0} p_1 e^{-(\lambda_0/r_0 + \mu_1/r_1)u}, \\ F_1(u) &= p_1 - p_1 e^{-(\lambda_0/r_0 + \mu_1/r_1)u}. \end{aligned} \tag{3.16}$$

Observe that

$$P(C(t) < u) = F_0(u) + F_1(u) = 1 - \left(1 - \frac{r_1}{r_0}\right)\left(\frac{\lambda_0}{\lambda_0 + \mu_1}\right) e^{-(\lambda_0/r_0 + \mu_1/r_1)u}. \tag{3.17}$$

Example 3.2 ($N = 2$). The steady state probabilities of the modulating Markov process are given by

$$\begin{aligned} p_0 &= \frac{\mu_1\mu_2}{\lambda_0\lambda_1 + \lambda_0\mu_2 + \mu_1\mu_2}, \\ p_1 &= \frac{\lambda_0\mu_2}{\lambda_0\lambda_1 + \lambda_0\mu_2 + \mu_1\mu_2}, \\ p_2 &= \frac{\lambda_0\lambda_1}{\lambda_0\lambda_1 + \lambda_0\mu_2 + \mu_1\mu_2}. \end{aligned} \tag{3.18}$$

It follows from (2.3) that the stability condition for the Markov process $\{(X(t), C(t)),\ t \geq 0\}$ is $\mu_1\mu_2 r_0 + \lambda_0\mu_2 r_1 + \lambda_0\lambda_1 r_2 < 0$.

The matrix $\mathbf{R}^{-1}\mathbf{Q}^T$ takes the form

$$\mathbf{R}^{-1}\mathbf{Q}^T = \begin{bmatrix} -\dfrac{\lambda_0}{r_0} & \dfrac{\mu_1}{r_0} & \\ \dfrac{\lambda_0}{r_1} & -\dfrac{\lambda_1 + \mu_1}{r_1} & \dfrac{\mu_2}{r_1} \\ & \dfrac{\lambda_1}{r_2} & -\dfrac{\mu_2}{r_2} \end{bmatrix} \tag{3.19}$$

with eigenvalues given by

$$\xi_0 = -\frac{1}{2}\left(\frac{\lambda_0}{r_0}+\frac{\lambda_1}{r_1}+\frac{\mu_1}{r_1}+\frac{\mu_2}{r_2}\right) - \frac{1}{2}\sqrt{\left(\frac{\lambda_0}{r_0}+\frac{\lambda_1}{r_1}+\frac{\mu_1}{r_1}+\frac{\mu_2}{r_2}\right)^2 - 4\left(\frac{\lambda_0\lambda_1}{r_0r_1}+\frac{\mu_1\mu_2}{r_1r_2}+\frac{\lambda_0\mu_2}{r_0r_2}\right)},$$
$$\xi_1 = 0, \tag{3.20}$$
$$\xi_2 = -\frac{1}{2}\left(\frac{\lambda_0}{r_0}+\frac{\lambda_1}{r_1}+\frac{\mu_1}{r_1}+\frac{\mu_2}{r_2}\right) + \frac{1}{2}\sqrt{\left(\frac{\lambda_0}{r_0}+\frac{\lambda_1}{r_1}+\frac{\mu_1}{r_1}+\frac{\mu_2}{r_2}\right)^2 - 4\left(\frac{\lambda_0\lambda_1}{r_0r_1}+\frac{\mu_1\mu_2}{r_1r_2}+\frac{\lambda_0\mu_2}{r_0r_2}\right)}.$$

The constant k_0 is determined as $k_0 = -p_1\mu_1/r_0B_1(\xi_0)$, where

$$\begin{aligned} B_0(s) &= 1, \\ B_1(s) &= \left(s+\frac{\lambda_0}{r_0}\right), \\ B_2(s) &= \left(s+\frac{\lambda_0}{r_0}\right)\left(s+\frac{\lambda_1+\mu_1}{r_1}\right) - \frac{\lambda_0\mu_1}{r_0r_1}. \end{aligned} \tag{3.21}$$

Therefore, the stationary distribution of the buffer content is given by

$$\begin{aligned} F_0(u) &= p_0 - p_1\frac{\mu_1}{r_0}\frac{1}{B_1(\xi_0)}e^{\xi_0 u}, \\ F_1(u) &= p_1 - p_1 e^{\xi_0 u}, \\ F_2(u) &= p_2 - p_1\frac{r_1}{\mu_2}\frac{B_2(\xi_0)}{B_1(\xi_0)}e^{\xi_0 u}. \end{aligned} \tag{3.22}$$

Observe that

$$\begin{aligned} P(C(t) < u) &= F_0(u)+F_1(u)+F_2(u) \\ &= 1 - p_1\left(1+\frac{\mu_1}{r_0B_1(\xi_0)}+\frac{r_1}{\mu_2}\frac{B_2(\xi_0)}{B_1(\xi_0)}\right)e^{\xi_0 u}. \end{aligned} \tag{3.23}$$

In the above discussion, we considered two examples in the general case. The forthcoming example deals with the model in which the birth and death rates alternate between two constant values, with even number of states. The case with odd number of states does not lead to explicit expression for eigenvalues.

Example 3.3 (alternating rates). Consider a fluid queue driven by a single server queuing model with state space $\mathcal{S} = \{0,1,2,\dots,2n-1\}$ whose birth and death rates are given by

$$\begin{aligned} \lambda_{2i} &= \lambda_1, \quad \mu_{2i} = \mu_2 \quad \text{for } i = 1,2,\dots,n-1, \\ \lambda_{2i+1} &= \lambda_2, \quad \mu_{2i+1} = \mu_1 \quad \text{for } i = 0,1,\dots,n-2, \end{aligned} \tag{3.24}$$

with $\lambda_0 = \lambda_1$, $\mu_{2n-1} = \mu_1$, and the effective input rates are $r_{2i} = r_1 < 0$ and $r_{2i+1} = r_2 > 0$ for $i = 0,1,2,\dots,n-1$. If $\rho = \lambda_1\lambda_2/\mu_1\mu_2 < 1$, the steady state probabilities are given by

$$\begin{aligned} p_{2k} &= \left(\frac{\lambda_1\lambda_2}{\mu_1\mu_2}\right)^k p_0, \quad k = 1,2,3,\dots,n-1, \\ p_{2k+1} &= \frac{\lambda_1}{\mu_1}\left(\frac{\lambda_1\lambda_2}{\mu_1\mu_2}\right)^k p_0, \quad k = 0,1,2,\dots,n-1, \end{aligned} \tag{3.25}$$

with $p_0 = (\mu_1/(\lambda_1+\mu_1))((1-\rho)/(1-\rho^n))$. From (2.3), it is observed that the condition $r_1\mu_1 + r_2\lambda_1 < 0$ ensures the stability of the Markov process $\{(X(t),C(t)),\ t \geq 0\}$.

For this specific model, the matrix $\mathbf{R}^{-1}\mathbf{Q}^T$ takes the form

$$\begin{aligned} \mathbf{R}^{-1}\mathbf{Q}^T &= \begin{vmatrix} -\frac{\lambda_1}{r_1} & \frac{\mu_1}{r_1} & & & & \\ \frac{\lambda_1}{r_2} & -\frac{\lambda_2+\mu_1}{r_2} & \frac{\mu_2}{r_2} & & & \\ & & & \ddots & & \\ & & & \frac{\lambda_2}{r_1} & -\frac{\lambda_1+\mu_2}{r_1} & \frac{\mu_1}{r_1} \\ & & & & \frac{\lambda_1}{r_2} & -\frac{\mu_1}{r_2} \end{vmatrix}_{2n} \\ &= \begin{vmatrix} -\frac{\lambda_1}{r_1} & \frac{\mu_1}{r_2} & & & & \\ \frac{\lambda_1}{r_1} & -\frac{\lambda_2+\mu_1}{r_2} & \frac{\mu_2}{r_1} & & & \\ & & & \ddots & & \\ & & & \frac{\lambda_2}{r_2} & -\frac{\lambda_1+\mu_2}{r_1} & \frac{\mu_1}{r_2} \\ & & & & \frac{\lambda_1}{r_1} & -\frac{\mu_1}{r_2} \end{vmatrix}_{2n}. \end{aligned} \tag{3.26}$$

Therefore,

$$
\begin{aligned}
&|s\mathbf{I}-\mathbf{R}^{-1}\mathbf{Q}^T| \\
&\quad = \begin{vmatrix}
s+\frac{\lambda_1}{r_1} & -\frac{\mu_1}{r_2} & & & & \\
-\frac{\lambda_1}{r_1} & s+\frac{\lambda_2+\mu_1}{r_2} & -\frac{\mu_2}{r_1} & & & \\
& & \ddots & & & \\
& & & -\frac{\lambda_2}{r_2} & s+\frac{\lambda_1+\mu_2}{r_1} & \frac{\mu_1}{r_2} \\
& & & & \frac{\lambda_1}{r_1} & s+\frac{\mu_1}{r_2}
\end{vmatrix}_{2n} \\
&\quad = s\times\begin{vmatrix}
s+\frac{\lambda_1}{r_1}+\frac{\mu_1}{r_2} & -\frac{\lambda_1\mu_2}{r_1^2} & & & \\
-1 & s+\frac{\lambda_2}{r_2}+\frac{\mu_2}{r_1} & -\frac{\lambda_2\mu_1}{r_2^2} & & \\
& & \ddots & & \\
& & -1 & s+\frac{\lambda_2}{r_2}+\frac{\mu_2}{r_1} & -\frac{\lambda_2\mu_1}{r_2^2} \\
& & & -1 & s+\frac{\lambda_1}{r_1}+\frac{\mu_1}{r_2}
\end{vmatrix}_{2n-1}.
\end{aligned}
\tag{3.27}
$$

Using the Identities A.2, A.1, and A.3 given in the appendix in succession, with $\theta = s+\lambda_1/r_1+\mu_1/r_2$, $\phi = s+\lambda_2/r_2+\mu_2/r_1$, and $\omega = \theta\phi-\lambda_2\mu_1/r_2^2-\lambda_1\mu_2/r_1^2$, we obtain

$$
\begin{aligned}
B_{2n}(s) &= \frac{s}{\phi}\times\begin{vmatrix}
\theta\phi-\frac{\lambda_1\mu_2}{r_1^2} & -\frac{\lambda_1\mu_2}{r_1^2} & & & & \\
-\frac{\lambda_2\mu_1}{r_2^2} & \omega & -\frac{\lambda_1\mu_2}{r_1^2} & & & \\
& -\frac{\lambda_2\mu_1}{r_2^2} & \omega & -\frac{\lambda_1\mu_2}{r_1^2} & & \\
& & & \ddots & & \\
& & & & -\frac{\lambda_2\mu_1}{r_2^2} & \theta\phi-\frac{\lambda_2\mu_1}{r_2^2}
\end{vmatrix}_{n} \\
&= \frac{s}{\phi}\times\theta\phi\times\begin{vmatrix}
\omega & -\frac{\lambda_2\mu_1}{r_2^2} & & \\
-\frac{\lambda_1\mu_2}{r_1^2} & \omega & -\frac{\lambda_2\mu_1}{r_2^2} & \\
& & \ddots & \\
& & -\frac{\lambda_1\mu_2}{r_1^2} & \omega
\end{vmatrix}_{n-1}
\end{aligned}
$$

$$= s\theta \times \prod_{r=1}^{n-1}\left[\theta\phi - \frac{\lambda_1\mu_2}{r_1^2} - \frac{\lambda_2\mu_1}{r_2^2} - 2\sqrt{\frac{\lambda_1\lambda_2\mu_1\mu_2}{r_1^2r_2^2}}\cos\frac{r\pi}{n}\right]. \tag{3.28}$$

Substituting for θ and ϕ, we obtain

$$\begin{aligned} B_{2n}(s) = s&\left(s+\frac{\lambda_1}{r_1}+\frac{\mu_1}{r_2}\right) \\ &\times\prod_{r=1}^{n-1}\left[\left(s+\frac{\lambda_1}{r_1}+\frac{\mu_1}{r_2}\right)\left(s+\frac{\lambda_2}{r_2}+\frac{\mu_2}{r_1}\right)-\frac{\lambda_1\mu_2}{r_1^2}\right. \\ &\left.\qquad -\frac{\lambda_2\mu_1}{r_2^2}-2\frac{\sqrt{\lambda_1\lambda_2\mu_1\mu_2}}{r_1r_2}\cos\frac{r\pi}{n}\right]. \end{aligned} \tag{3.29}$$

We observe that $B_{2n}(s)$ is zero when $-s$ is the eigenvalue of the tridiagonal matrix $\mathbf{R}^{-1}\mathbf{Q}^T$. Therefore, the eigenvalues of $\mathbf{R}^{-1}\mathbf{Q}^T$ are given by

$$\begin{aligned} \xi_0 &= -\left(\frac{\lambda_1}{r_1}+\frac{\mu_1}{r_2}\right), \\ \xi_j &= \frac{1}{2}\left[-\left(\frac{\lambda_1+\mu_2}{r_1}+\frac{\lambda_2+\mu_1}{r_2}\right)-\left(\left(\frac{\lambda_1+\mu_2}{r_1}+\frac{\lambda_2+\mu_1}{r_2}\right)^2-4\frac{\lambda_1\lambda_2}{r_1r_2}-4\frac{\mu_1\mu_2}{r_1r_2}\right.\right. \\ &\qquad\left.\left.+8\frac{\sqrt{\lambda_1\lambda_2\mu_1\mu_2}}{r_1r_2}\cos\frac{j\pi}{n}\right)^{1/2}\right], \quad j=1,2,\dots,n-1, \\ \xi_n &= 0, \\ \xi_j &= \frac{1}{2}\left[-\left(\frac{\lambda_1+\mu_2}{r_1}+\frac{\lambda_2+\mu_1}{r_2}\right)+\left(\left(\frac{\lambda_1+\mu_2}{r_1}+\frac{\lambda_2+\mu_1}{r_2}\right)^2-4\frac{\lambda_1\lambda_2}{r_1r_2}-4\frac{\mu_1\mu_2}{r_1r_2}\right.\right. \\ &\qquad\left.\left.+8\frac{\sqrt{\lambda_1\lambda_2\mu_1\mu_2}}{r_1r_2}\cos\frac{j\pi}{n}\right)^{1/2}\right], \quad j=n+1,n+2,\dots,2n-1. \end{aligned} \tag{3.30}$$

We give below closed-form expressions for the terms $B_{2k}(s)$ and $B_{2k+1}(s)$ for $k = 0,\ldots,n-1$, using the well-known identities of continuants given in the appendix. Consider

$$
\begin{aligned}
B_{2k}(s) &= \begin{vmatrix}
s+\frac{\lambda_1}{r_1} & \frac{\mu_1}{r_2} & & & \\
\frac{\lambda_1}{r_1} & s+\frac{\lambda_2+\mu_1}{r_2} & \frac{\mu_2}{r_1} & & \\
 & & \ddots & & \\
 & & \frac{\lambda_2}{r_2} & s+\frac{\lambda_1+\mu_2}{r_1} & \frac{\mu_1}{r_2} \\
 & & & \frac{\lambda_1}{r_1} & s+\frac{\lambda_2+\mu_1}{r_2}
\end{vmatrix}_{2k} \\
&= \begin{vmatrix}
s+\frac{\lambda_1+\mu_2}{r_1} & \frac{\mu_1}{r_2} & & & \\
\frac{\lambda_1}{r_1} & s+\frac{\lambda_2+\mu_1}{r_2} & \frac{\mu_2}{r_1} & & \\
 & & \ddots & & \\
 & & \frac{\lambda_2}{r_2} & s+\frac{\lambda_1+\mu_2}{r_1} & \frac{\mu_1}{r_2} \\
 & & & \frac{\lambda_1}{r_1} & s+\frac{\lambda_2+\mu_1}{r_2}
\end{vmatrix}_{2k} \\
&\quad -\frac{\mu_2}{r_1}\begin{vmatrix}
s+\frac{\lambda_2+\mu_1}{r_2} & \frac{\mu_2}{r_1} & & & \\
\frac{\lambda_2}{r_2} & s+\frac{\lambda_1+\mu_2}{r_1} & \frac{\mu_1}{r_2} & & \\
 & & \ddots & & \\
 & & \frac{\lambda_2}{r_2} & s+\frac{\lambda_1+\mu_2}{r_1} & \frac{\mu_1}{r_2} \\
 & & & \frac{\lambda_1}{r_1} & s+\frac{\lambda_2+\mu_1}{r_2}
\end{vmatrix}_{2k-1}.
\end{aligned}
\tag{3.31}
$$

Using Identities A.2 and A.4,

$$B_{2k}(s)=\begin{vmatrix} \omega_1+\frac{\lambda_2\mu_2}{r_1r_2} & -\frac{\lambda_1\mu_1}{r_1r_2} & & & \\ -\frac{\lambda_2\mu_2}{r_1r_2} & \omega_1 & -\frac{\lambda_1\mu_1}{r_1r_2} & & \\ & & \ddots & & \\ & & & \omega_1 & -\frac{\lambda_1\mu_1}{r_1r_2} \\ & & & -\frac{\lambda_2\mu_2}{r_1r_2} & \omega_1 \end{vmatrix}_k$$

$$-\frac{\mu_2}{r_1s+\lambda_1+\mu_2}\begin{vmatrix} \omega_1+\frac{\lambda_2\mu_2}{r_1r_2} & -\frac{\lambda_1\mu_1}{r_1r_2} & & & \\ -\frac{\lambda_2\mu_2}{r_1r_2} & \omega_1 & -\frac{\lambda_1\mu_1}{r_1r_2} & & \\ & & \ddots & & \\ & & & \omega_1 & -\frac{\lambda_1\mu_1}{r_1r_2} \\ & & & -\frac{\lambda_2\mu_2}{r_1r_2} & \omega_1+\frac{\lambda_2\mu_2}{r_1r_2} \end{vmatrix}_k, \tag{3.32}$$

where $\omega_1=(s+(\lambda_1+\mu_2)/r_1)(s+(\lambda_2+\mu_1)/r_2)-\lambda_1\mu_1/r_1r_2-\lambda_2\mu_2/r_1r_2$. Now, expanding the first determinant and using Identity A.1 for the second, we obtain

$$B_{2k}(s)=\begin{vmatrix} \omega_1 & \frac{\lambda_1\mu_1}{r_1r_2} & & & \\ \frac{\lambda_2\mu_2}{r_1r_2} & \omega_1 & \frac{\lambda_1\mu_1}{r_1r_2} & & \\ & & \ddots & & \\ & & & \omega_1 & \frac{\lambda_1\mu_1}{r_1r_2} \\ & & & \frac{\lambda_2\mu_2}{r_1r_2} & \omega_1 \end{vmatrix}_k$$

$$+\frac{\lambda_2\mu_2}{r_1r_2}\begin{vmatrix} \omega_1 & \frac{\lambda_1\mu_1}{r_1r_2} & & & \\ \frac{\lambda_2\mu_2}{r_1r_2} & \omega_1 & \frac{\lambda_1\mu_1}{r_1r_2} & & \\ & & \ddots & & \\ & & & \omega_1 & \frac{\lambda_1\mu_1}{r_1r_2} \\ & & & \frac{\lambda_2\mu_2}{r_1r_2} & \omega_1 \end{vmatrix}_{k-1}$$

$$-\frac{\mu_2}{r_1}\left(s+\frac{\lambda_2+\mu_1}{r_2}\right)\begin{vmatrix} \omega_1 & \frac{\lambda_2\mu_2}{r_1r_2} & & & \\ \frac{\lambda_1\mu_1}{r_1r_2} & \omega_1 & \frac{\lambda_2\mu_2}{r_1r_2} & & \\ & & \ddots & & \\ & & & \omega_1 & \frac{\lambda_2\mu_2}{r_1r_2} \\ & & & \frac{\lambda_1\mu_1}{r_1r_2} & \omega_1 \end{vmatrix}_{k-1}. \tag{3.33}$$

Therefore, from Identity A.3, the quantity $B_{2k}(s)$ can be expressed in closed form as follows:

$$\begin{aligned} B_{2k}(s) &= \left(\frac{\lambda_1\lambda_2\mu_1\mu_2}{r_1^2r_2^2}\right)^{k/2}\left[U_k\left(\frac{x}{2}\right)+\left(\frac{\lambda_2\mu_2}{\lambda_1\mu_1}\right)^{1/2}U_{k-1}\left(\frac{x}{2}\right)\right] \\ &\quad -\frac{\mu_2}{r_1}\left(s+\frac{\lambda_2+\mu_1}{r_2}\right)\left(\frac{\lambda_1\lambda_2\mu_1\mu_2}{r_1^2r_2^2}\right)^{(k-1)/2}U_{k-1}\left(\frac{x}{2}\right), \end{aligned} \tag{3.34}$$

where

$$x=2\left[\frac{(r_1s+\lambda_1+\mu_2)(r_2s+\lambda_2+\mu_1)-\lambda_1\mu_1-\lambda_2\mu_2}{\sqrt{\lambda_1\lambda_2\mu_1\mu_2}}\right] \tag{3.35}$$

and $U_k(x)$ is the Chebyshev polynomial of the second kind. Similarly, $B_{2k+1}(s)$ can also be expressed as

$$\begin{aligned} B_{2k+1}(s) &= \left(s+\frac{\lambda_1+\mu_2}{r_1}\right)\left(\frac{\lambda_1\lambda_2\mu_1\mu_2}{r_1^2r_2^2}\right)^{k/2}U_k\left(\frac{x}{2}\right) \\ &\quad -\frac{\mu_2}{r_1}\left(\frac{\lambda_1\lambda_2\mu_1\mu_2}{r_1^2r_2^2}\right)^{k/2}\left[U_k\left(\frac{x}{2}\right)+\left(\frac{\lambda_1\mu_1}{\lambda_2\mu_2}\right)^{1/2}U_{k-1}\left(\frac{x}{2}\right)\right]. \end{aligned} \tag{3.36}$$

Further,

$$c_{j0} = \begin{cases} \left(\dfrac{\mu_1\mu_2}{r_1 r_2}\right)^k & \text{if } j = 2k, \\ \dfrac{\mu_1}{r_1}\left(\dfrac{\mu_1\mu_2}{r_1 r_2}\right)^k & \text{if } j = 2k+1. \end{cases} \tag{3.37}$$

Having determined $B_j(s)$ and the roots of $\mathbf{R}^{-1}\mathbf{Q}^T$ explicitly, the constants k_l, and hence the buffer occupancy distributions, are obtained using (3.10) and (3.14).

Remark 3.4. From the above discussion, we observe that, without loss of generality, r_1 and r_2 can be taken as -1 and $+1$, respectively. This is because the solution remains unaltered when λ_1 (μ_2) and λ_2 (μ_1) are replaced by $-\lambda_1 r_1$ ($-\mu_2 r_1$) and $\lambda_2 r_2$ ($\mu_1 r_2$), respectively.

4. Infinite state space

In the previous section, we discussed the fluid queue model on a finite state space with alternating positive and negative flow rates. Similar analysis on an infinite state space does not lead to explicit expression for the buffer content distribution. Hence we analyze the fluid queue driven by an infinite state BDP with a *single* negative effective flow rate, say r_0 (< 0). We employ the continued fraction methodology to obtain the stationary distribution of the content of the buffer. Let $F_j(t,u)$ represent the probability that there are j units in the system and the content of the buffer does not exceed u at time t, and let $\widehat{F}_j(t,s)$ denote the corresponding Laplace transform. If $\lim_{t\to\infty} F_j(t,u) = F_j(u)$, then the governing system of differential-difference equations is

$$\begin{gathered} r_0 F_0'(u) = -\lambda_0 F_0(u) + \mu_1 F_1(u), \\ r_j F_j'(u) = \lambda_{j-1} F_{j-1}(u) - (\lambda_j + \mu_j) F_j(u) + \mu_{j+1} F_{j+1}(u), \quad j = 1,2,\ldots. \end{gathered} \tag{4.1}$$

Using the initial condition $F_0(0) = a$, the Laplace transform of (4.1) yields

$$\begin{gathered} \widehat{F}_0(s) = \cfrac{a}{s + \cfrac{\lambda_0}{r_0} - \cfrac{\mu_1}{r_0}\cfrac{\widehat{F}_1(s)}{\widehat{F}_0(s)}}, \\ \frac{\widehat{F}_j(s)}{\widehat{F}_{j-1}(s)} = \cfrac{\cfrac{\lambda_{j-1}}{r_j}}{s + \cfrac{\lambda_j + \mu_j}{r_j} - \cfrac{\mu_{j+1}}{r_j}\cfrac{\widehat{F}_{j+1}(s)}{\widehat{F}_j(s)}}, \quad j = 1,2,3,\ldots. \end{gathered} \tag{4.2}$$

This leads to the continued fractions

$$
\begin{aligned}
\widehat{F}_0(s) &= \frac{a}{s+\dfrac{\lambda_0}{r_0}-}\;\frac{\dfrac{\lambda_0\mu_1}{r_0r_1}}{s+\dfrac{\lambda_1+\mu_1}{r_1}-}\;\frac{\dfrac{\lambda_1\mu_2}{r_1r_2}}{s+\dfrac{\lambda_2+\mu_2}{r_2}-}\cdots,\\
\frac{\widehat{F}_j(s)}{\widehat{F}_{j-1}(s)} &= \frac{\dfrac{\lambda_{j-1}}{r_j}}{s+\dfrac{\lambda_j+\mu_j}{r_j}-}\;\frac{\dfrac{\lambda_j\mu_{j+1}}{r_jr_{j+1}}}{s+\dfrac{\lambda_{j+1}+\mu_{j+1}}{r_{j+1}}-}\;\frac{\dfrac{\lambda_{j+1}\mu_{j+2}}{r_{j+1}r_{j+2}}}{s+\dfrac{\lambda_{j+2}+\mu_{j+2}}{r_{j+2}}-}\cdots.
\end{aligned}
\tag{4.3}
$$

Define

$$
\phi_j(s) := \frac{\dfrac{\lambda_{j-1}}{r_j}}{s+\dfrac{\lambda_j+\mu_j}{r_j}-}\;\frac{\dfrac{\lambda_j\mu_{j+1}}{r_jr_{j+1}}}{s+\dfrac{\lambda_{j+1}+\mu_{j+1}}{r_{j+1}}-}\;\frac{\dfrac{\lambda_{j+1}\mu_{j+2}}{r_{j+1}r_{j+2}}}{s+\dfrac{\lambda_{j+2}+\mu_{j+2}}{r_{j+2}}-}\cdots. \tag{4.4}
$$

Then,

$$
\widehat{F}_0(s) = \frac{a}{s+\dfrac{\lambda_0}{r_0}-\dfrac{\mu_1}{r_0}\phi_1(s)}, \qquad \frac{\widehat{F}_j(s)}{\widehat{F}_{j-1}(s)} = \phi_j(s). \tag{4.5}
$$

After certain algebra, we obtain

$$
\begin{aligned}
\widehat{F}_0(s) &= a\sum_{k=0}^{\infty}\left(\frac{\mu_1}{r_0}\right)^k \frac{\left(\phi_1(s)\right)^k}{\left(s+\dfrac{\lambda_0}{r_0}\right)^{k+1}},\\
\widehat{F}_j(s) &= \left(\prod_{k=1}^{j}\phi_k(s)\right)\widehat{F}_0(s), \quad j=1,2,3,\ldots.
\end{aligned}
\tag{4.6}
$$

On inversion, we get

$$
\begin{aligned}
F_0(u) &= a\sum_{k=0}^{\infty}\left(\frac{\mu_1}{r_0}\right)^k\left(\frac{u^k e^{-(\lambda_0/r_0)u}}{k!}\right) * \phi_1^{*(k)}(u),\\
F_j(u) &= \phi_1(u)*\phi_2(u)*\cdots*\phi_j(u)*F_0(u), \quad j=1,2,3,\ldots.
\end{aligned}
\tag{4.7}
$$

In the following argument we consider a specific nature of birth and death rates.

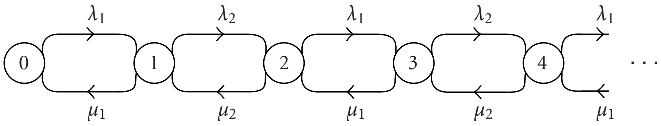

Figure 4.1. State-transition diagram of the background Markov process.

Example 4.1. Consider a fluid queue driven by an *M/M/*1 queue with two alternating arrival rates and two alternating service rates as shown in Figure 4.1.

We assume the net input rate of fluid to be $r_0 < 0$, $r_{2j} = r_1 > 0$ for $j = 1,2,3,\ldots$ and $r_{2j+1} = r_2 > 0$ for $j = 0,1,2,\ldots$. The steady state probabilities are given by (3.25) with $p_0 = (\mu_1\mu_2 - \lambda_1\lambda_2)/\mu_2(\lambda_1+\mu_1)$. From (2.3), the condition $\mu_1(r_0+r_1)+\lambda_1 r_2 < 0$ ensures the stability of the process $\{(X(t),C(t)),\ t \geq 0\}$. For the model under consideration,

$$\phi_j(s) = \begin{cases} f(s), & \text{if } j \text{ is odd,} \\ g(s), & \text{if } j \text{ is even,} \end{cases} \tag{4.8}$$

where

$$\begin{aligned} f(s) &= \frac{\frac{\lambda_1}{r_2}}{s + \frac{\lambda_2+\mu_1}{r_2} -} \; \frac{\frac{\lambda_2\mu_2}{r_1 r_2}}{s + \frac{\lambda_1+\mu_2}{r_1} -} \; \frac{\frac{\lambda_1\mu_1}{r_1 r_2}}{s + \frac{\lambda_2+\mu_1}{r_2} -} \; \cdots \\ &= \frac{\lambda_1}{r_2 s + \lambda_2 + \mu_1 - \dfrac{\lambda_2\mu_2}{r_1 s + \lambda_1 + \mu_2 - \mu_1 f(s)}}. \end{aligned} \tag{4.9}$$

Or

$$\begin{aligned} &\mu_1(r_2 s + \lambda_2 + \mu_1)(f(s))^2 \\ &\quad - \left[(r_1 s + \lambda_1 + \mu_2)(r_2 s + \lambda_2 + \mu_1) - \lambda_2\mu_2 + \lambda_1\mu_1\right] f(s) + \lambda_1(r_1 s + \lambda_1 + \mu_2) = 0. \end{aligned} \tag{4.10}$$

Using Rouches theorem, considering the root that lies inside the unit circle, we get

$$\begin{aligned} f(s) = \frac{1}{2\mu_1(r_2 s + \lambda_2 + \mu_1)} \Big[&(r_1 s + \lambda_1 + \mu_2)(r_2 s + \lambda_2 + \mu_1) + \lambda_1\mu_1 - \lambda_2\mu_2 \\ &+ \left(\left[(r_1 s + \lambda_1 + \mu_2)(r_2 s + \lambda_2 + \mu_1) + \lambda_2\mu_2 - \lambda_1\mu_1\right]^2 \right. \\ &\left. - 4\lambda_1\mu_1(r_1 s + \lambda_1 + \mu_2)(r_2 s + \lambda_2 + \mu_1)\right)^{1/2} \Big]. \end{aligned} \tag{4.11}$$

$g(s)$ is obtained from $f(s)$ by interchanging r_1 and r_2, λ_1 and λ_2, and μ_1 and μ_2. The expression for $f(s)$ can be written as

$$f(s)=\frac{r_1r_2\left[(s+a)^2-b^2\right]-\sqrt{r_1^2r_2^2\left((s+a)^2-b^2\right)^2-4\lambda_1\mu_1r_1r_2\left((s+a)^2-c^2\right)}}{2\mu_1r_2\left((s+a)-c\right)}, \tag{4.12}$$

where

$$\begin{aligned} a&=\frac{1}{2}\left(\frac{\lambda_1+\mu_2}{r_1}+\frac{\lambda_2+\mu_1}{r_2}\right),\\ b^2&=a^2-\left(\frac{\lambda_1\lambda_2+\mu_1\mu_2+2\lambda_1\mu_1}{r_1r_2}\right),\\ c&=\frac{1}{2}\left(\frac{\lambda_1+\mu_2}{r_1}-\frac{\lambda_2+\mu_1}{r_2}\right). \end{aligned} \tag{4.13}$$

On inversion, we get

$$f(u)=2\left(\frac{\lambda_1}{r_2}\right)e^{-au}\left\{f_2(u)*\sum_{n=0}^{\infty}(-1)^n\sum_{k=0}^{n}\frac{(n/2)(n/2-1)\cdots(n/2-k+1)}{k!}f_3^{*(k)}(u)\right\}, \tag{4.14}$$

where

$$\begin{aligned} f_3(u)&=\left(\frac{4\lambda_1\mu_1}{r_1r_2}\right)\left(f_1(u)*f_2(u)\right),\\ f_1(u)&=\cosh bu+\frac{c}{b}\sinh bu,\\ f_2(u)&=\cosh bu-\frac{c}{b}\sinh bu, \end{aligned} \tag{4.15}$$

and $h^{*(k)}(u)$ is the k-fold convolution of $h(u)$ with itself. Also, from (4.5),

$$\frac{\hat{F}_{2k+1}(s)}{\hat{F}_{2k}(s)}=f(s),\qquad \frac{\hat{F}_{2k}(s)}{\hat{F}_{2k-1}(s)}=g(s), \tag{4.16}$$

which on solving yield

$$\begin{aligned} \hat{F}_{2k+1}(s)&=\left(f(s)\right)^{k+1}\left(g(s)\right)^{k}\hat{F}_0(s),\\ \hat{F}_{2k}(s)&=\left(f(s)\right)^{k}\left(g(s)\right)^{k}\hat{F}_0(s). \end{aligned} \tag{4.17}$$

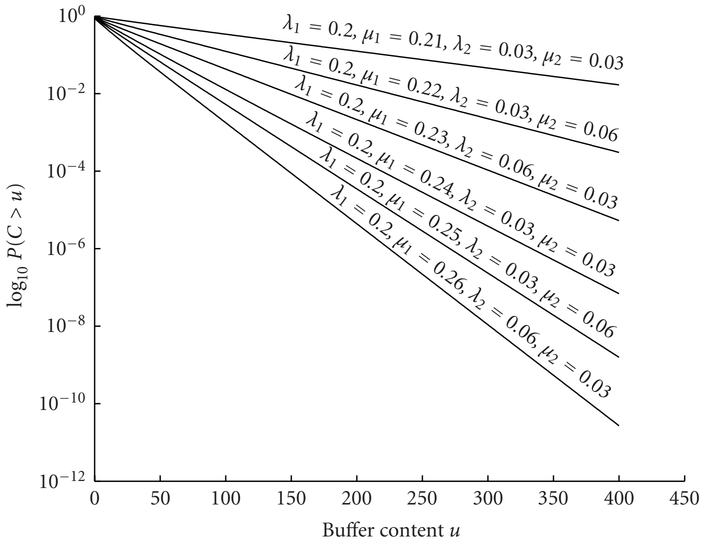

Figure 5.1. Buffer content distribution on a finite state space with $N = 21$, $r_1 = -1$, and $r_2 = 1$.

On inversion

$$\begin{aligned} F_{2k+1}(u) &= f^{*(k+1)}(u) * g^{*(k)}(u) * F_0(u), \quad k = 0,1,2,\ldots, \\ F_{2k}(u) &= f^{*(k)}(u) * g^{*(k)}(u) * F_0(u), \quad k = 1,2,3,\ldots, \end{aligned} \tag{4.18}$$

where $F_0(u)$ is obtained from (4.7) as

$$F_0(u) = a \sum_{k=0}^{\infty} \left(\frac{\mu_1}{r_0}\right)^k \left(\frac{u^k e^{-(\lambda_0/r_0)u}}{k!}\right) * f^{*(k)}(u). \tag{4.19}$$

5. Numerical illustrations

In this section, we provide numerical examples for the fluid queue driven by the finite and infinite queuing model discussed in the earlier sections.

In Figure 5.1 the buffer overflow probability corresponding to the varying values of the parameters λ_1, λ_2, μ_1, and μ_2 is plotted against buffer size for finite state space with $N = 21$, $r_1 = -1$, and $r_2 = 1$ (see Remark 3.4). The overflow probabilities are found to decrease rapidly with increase in μ_1. The graphs are plotted for two sets of parameters by considering the three different cases wherein the ratio λ_2/μ_2 is equal to one, less than one, and greater then one, respectively.

Figure 5.2 depicts an analogous setting on an infinite state space with N truncated at 25, $r_0 = -1$, and $r_i = 1$ for $i \geq 1$. For constant values of λ_1 and λ_2, the overflow probabilities decrease rapidly with increase in μ_1. The overflow probability is in the range of 10^{-10} to 10^{-15} and hence becomes a rare event.

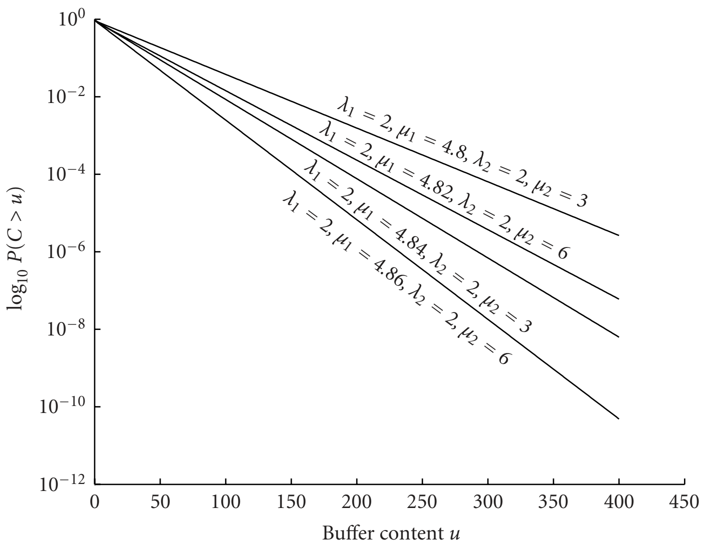

Figure 5.2. Buffer content distribution on an infinite state space with N truncated at 25, $r_0 = -1$, and $r_1 = 1 = r_2$.

Appendix

Some identities of tridiagonal determinants

Here we give some identities of tridiagonal determinants (Muir [4]), which are useful in determining the stationary buffer content distribution of the model under consideration.

Identity A.1.

$$\begin{vmatrix} A+a_0 & a_0 & & & \\ b_1 & A+a_1+b_1 & & & \\ & & \ddots & & \\ & & & A+a_{n-1}+b_{n-1} & a_{n-1} \\ & & & b_n & A+b_n \end{vmatrix}_{n+1} = A \times \begin{vmatrix} g_1 & b_1 & & & \\ a_1 & g_2 & b_2 & & \\ & & \ddots & & \\ & & a_{n-2} & g_{n-1} & b_{n-1} \\ & & & a_{n-1} & g_n \end{vmatrix}_{n}, \tag{A.1}$$

where $g_j = A + a_{j-1} + b_j$, $j = 1, 2, \ldots, n$.

Identity A.2.

$$\begin{vmatrix} \theta_1 & d_1 & & & & \\ -1 & \phi & d_2 & & & \\ & -1 & \theta_2 & d_3 & & \\ & & & \ddots & & \\ & & & -1 & \phi & d_{2n-2} \\ & & & & -1 & \theta_n \end{vmatrix}_{2n-1} = \frac{1}{\phi} \times \begin{vmatrix} \gamma_1 & d_1 & & & \\ d_2 & \gamma_2 & d_3 & & \\ & d_4 & \gamma_3 & d_5 & \\ & & & \ddots & \\ & & d_{2n-4} & \gamma_{n-1} & d_{2n-3} \\ & & & d_{2n-2} & \gamma_n \end{vmatrix}_n, \tag{A.2}$$

where $\gamma_j = d_{2j-2} + \theta_j \phi + d_{2j-1}$, $j = 1,2,\dots,n$, with $d_0 = 0$ and $d_{2n-1} = 0$.

Identity A.3.

$$\begin{vmatrix} A & a & & & \\ b & A & a & & \\ & & \ddots & & \\ & & b & A & a \\ & & & b & A \end{vmatrix}_{n\times n} = \prod_{j=1}^{n} \left(A - 2\sqrt{ab}\cos\left(\frac{j\pi}{n+1}\right)\right) = \left(\sqrt{ab}\right)^n U_n\left(\frac{A}{\sqrt{ab}}\right). \tag{A.3}$$

Identity A.4.

$$\begin{vmatrix} \theta_1 & d_1 & & & & \\ -1 & \phi & d_2 & & & \\ & -1 & \theta_2 & d_3 & & \\ & & & \ddots & & \\ & & & -1 & \theta_n & d_{2n-1} \\ & & & & -1 & \phi \end{vmatrix}_{2n} = \begin{vmatrix} \gamma_1 & d_1 & & & \\ d_2 & \gamma_2 & d_3 & & \\ & d_4 & \gamma_3 & d_5 & \\ & & & \ddots & \\ & & d_{2n-4} & \gamma_{n-1} & d_{2n-3} \\ & & & d_{2n-2} & \gamma_n \end{vmatrix}_n, \tag{A.4}$$

where $\gamma_j = d_{2j-2} + \theta_j \phi + d_{2j-1}$, $j = 1,2,\dots,n$, with $d_0 = 0$.

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