

## FOCAL SETS OF REGULAR MANIFOLDS $M_{n-1}$ IN $E_n$

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### 1. Introduction

The object of this paper is to prove the following theorem and give it a proper setting in differential topology.

**Theorem 1.1.** *There exists a regular connected  $(n-1)$ -dimensional manifold  $M_{n-1}$  of class  $C^\infty$  in a euclidean space  $E_n$  such that the focal points of  $M_{n-1}$  are everywhere dense in  $E_n$ .*

In particular there exists a simple regular curve  $M_1$  of class  $C^\infty$  in  $E_2$  whose centers of curvature are everywhere dense in  $E_2$ . See §2.

The manifold  $M_{n-1}$  of Theorem 1.1 is without any differentiable singularity in  $E_n$  and without self-intersection. However it cannot be compact by virtue of Theorem 1.2.

In Theorem 1.2 we refer to a subset of  $E_n$  of *J-content zero*. Given a positive constant  $e$ , such a set is characterized by the property that it is included in a finite number of  $n$ -rectangles whose total volume is less than  $e$ .

**Theorem 1.2.** *Let  $M_{n-1}$  be a regular manifold of class  $C^m$ ,  $m > 1$ , in  $E_n$ , and let  $\hat{M}_{n-1}$  and  $\hat{E}_n$  be, respectively, relatively compact open subsets of  $M_{n-1}$  and  $E_n$ .*

*Then the set of focal points of  $\hat{M}_{n-1}$  in  $\hat{E}_n$  has a J-content zero in  $E_n$ , implying that the set of focal points of  $M_{n-1}$  is nowhere dense in  $E_n$ .*

**Note.** It is not affirmed that the set of focal points of  $\hat{M}_{n-1}$  in  $E_n$  has *J-content zero*.

Theorem 1.2 admits an *extension* in which  $M_{n-1}$  is replaced by  $M_r$  where  $0 < r < n$ . Both Theorem 1.2 and its extension are provable by methods used by the author in his colloquium lectures in treating focal points of extremals "transverse" to a differentiable manifold. We shall establish Theorem 1.2 by non-variational methods later in this section. The extension of Theorem 1.2 can also be established by non-variational methods and this will be done in an introduction to critical point theory in global analysis and differential topology now being written.

*Theorem 1.2 implies, but is not implied by, the theorem that the set of focal points of the manifold  $M_{n-1}$  in Theorem 1.2 has a Lebesgue measure zero in  $E_n$ .*

We shall recall some essential definitions.

**Definition 1.1.** *Regular presentations.* With  $0 < r < n$  let  $V$  be an open subset of  $E_r$  and  $v^1, \dots, v^r$  rectangular coordinates of a point  $v \in V$ . Let  $x^1, \dots, x^n$  be rectangular coordinates of a point  $x \in E_n$ . A mapping

$$v \rightarrow F(v) = (F^1(v), \dots, F^n(v)) : V \rightarrow E_n$$

of class  $C^m$  is termed *regular* if the  $n$  by  $r$  functional matrix of the functions  $F^1, \dots, F^n$  has the rank  $r$  at each point  $v \in V$ . Set  $F(V) = X$ . If  $F$  is regular and a homeomorphism into  $E_n$ ,  $F$  is called a  $C^m$ -*embedding* of  $V$  in  $E_n$ , and a *regular  $C^m$ -presentation* ( $F : V, X$ ) of  $X$  in  $E_n$ .

**Definition 1.2.** *Regular  $C^m$ -manifolds in  $E_n$ .* For  $0 < r < n$  let  $\Gamma_r$  be a "topological  $r$ -manifold" which is a "subspace" of  $E_n$  in the sense of Bourbaki [1]. Suppose that there exists an ensemble of  $C^m$ -presentations ( $F : V, X$ ) of open subsets of  $\Gamma_r$  whose union is  $\Gamma_r$ . Then the set of *all* regular  $C^m$ -presentations of open subsets  $X$  of  $\Gamma_r$  defines a regular  $C^m$ -structure  $\mathcal{D}$  on  $\Gamma_r$ .  $\Gamma_r$  taken with such a  $C^m$ -structure is called a *regular  $C^m$ -manifold  $M_r$  in  $E_n$  with carrier  $|M_r| = \Gamma_r$  and set of presentations  $\mathcal{D}M_r$ .*

The inverse of a presentation  $F \in \mathcal{D}M_r$  is called a *regular chart of  $M_r$* . Given a presentation ( $F : V, X$ )  $\in \mathcal{D}M_r$  the coordinates  $v^1, \dots, v^r$  of a point  $v \in V$  are termed *local coordinates* of the point  $F(v)$  in the *coordinate domain  $X$  of  $M_r$* .

**Focal points of  $M_{n-1}$ .** Let  $M_{n-1}$  be a regular  $C^m$ -manifold in  $E_n$ , with  $m \geq 2$ . Let  $c = (c^1, \dots, c^n)$  be a point in  $E_n$ . let  $\zeta$  be a sensed straight line, meeting  $c$  and a point  $q \in M_r$ ,  $c \neq q$ , with  $\zeta$  normal to  $M_{n-1}$  at  $q$ . Then  $c$  can be defined as a *focal point of  $M_n$  on  $\zeta$  with base point  $q$*  in one of three equivalent ways, termed respectively definition by

- I. *critical point characteristics,*
- II. *a singular point of the field of normals to  $M_{n-1}$ ,*
- III. *a center of principal normal curvature of  $M_{n-1}$ .*

I. *Focal points as degenerate critical points of a distance function.* Let  $p$  be an arbitrary point on  $M_{n-1} - c$ , and  $p \rightarrow f(p)$  the function on  $M_{n-1}$  with values

$$(1.1) \quad f(p) = \|x - c\| \quad (p = x \in M_{n-1} - c).$$

Then  $f$  is of class  $C^m$  on  $M_{n-1} - c$ , and has a critical point when  $p$  is the above point  $q$ .

*The point  $c$  is called a focal point of  $M_{n-1}$  on  $\zeta$  with base point  $q$ , if  $q$  is a degenerate critical point of  $f$ .*

We are assuming that the reader is familiar with the invariant characterization of *degenerate and non-degenerate critical points* of a function  $f$  of class  $C^2$  on  $M_{n-1}$ .

II. *Focal points as singular points of the field of normals to  $M_{n-1}$ .* Let  $(F : V, X) \in \mathcal{D}M_{n-1}$  be a presentation of a neighborhood on  $M_{n-1}$  of the point  $q$  given on  $M_{n-1}$ . Set  $v_0 = F^{-1}(q)$ .

Let  $\lambda^0$  be the unit vector, normal to  $M_{n-1}$  at  $q$  with the direction of the given line  $\zeta$ . The family of unit vectors  $\lambda = (\lambda_1, \dots, \lambda_n)$  orthogonal to  $M_{n-1}$  at points  $F(v) \in X$ , including the vector  $\lambda^0$  when  $v = v_0$ , can be represented by  $C^{m-1}$  mappings

$$(1.2) \quad v \rightarrow \lambda_i(v) : \hat{V} \rightarrow R \quad (i = 1, \dots, n)$$

for  $\hat{V} \subset V$  a sufficiently small open neighborhood of  $v_0$ . Set  $F(\hat{V}) = \hat{X}$ . The normal to  $\hat{X}$  at the point  $F(v)$  with the direction  $\lambda(v)$  has a representation

$$(1.3) \quad x = F(v) + s\lambda(v) \quad (v \in \hat{V})$$

in terms of an algebraic distance coordinate  $s$ , measured from the point  $F(v)$  on the normal, and taken positive at points of the semi-normal with the direction of  $\lambda(v)$ .

We shall make use of the Jacobian

$$J(s, v) = \frac{D(x^1, \dots, x^n)}{D(s, v^1, \dots, v^{n-1})} \quad (s \in R, v \in \hat{V})$$

noting that

$$(1.4) \quad J(0, v) \neq 0 \quad (v \in \hat{V}).$$

Our second definition of focal points of  $M_{n-1}$  is as follows.

*The focal points of  $M_{n-1}$  with base points  $x = F(v)$ ,  $v \in \hat{V}$ , are the points  $x$  given by (1.3) when  $J(s, v) = 0$ .*

**Definition 1.3.** *Focal mappings.* A mapping

$$(1.5) \quad (s, v) \rightarrow F(v) + s\lambda(v) \quad (s \in R, v \in \hat{V})$$

conditioned as above, will be called a *focal mapping associated* with the base point  $q$  and directed line  $\zeta$  orthogonal to  $M_{n-1}$  at  $q$ .

III. *Focal points of  $M_{n-1}$  as centers of principal normal curvature of  $M_{n-1}$ .* We shall not need this type of definition and so refer to it only to complete the presentation. The interested reader may turn to treatises on differential geometry or to [2], page 403, and in particular to §21 of [2] on "Normals from a point to a manifold".

We state the following theorem.

**Theorem 1.3.** *The three definitions of focal points are equivalent.*

A proof of this theorem is implied by the analysis in [2].

A corollary of this theorem is that the definition of focal points by means of the vanishing of the Jacobian  $J(s, v)$  is independent of the choice of the presentation  $(F : V, X) \in \mathcal{DM}_{n-1}$  of a neighborhood  $X$  of the given point  $q \in M_{n-1}$ . One can also establish this independence by dealing directly with the focal mappings (1.3) which are involved.

**A proof of Theorem 1.2 recalled.** Theorem 1.2 is an almost trivial consequence of Lemma 1.1 below. In Lemma 1.1 we refer to a euclidean  $m$ -space  $U_m$  of coordinates  $u^1, \dots, u^m$  and to a euclidean  $m$ -space  $E_m$  of coordinates  $x^1, \dots, x^m$ . Given a  $C^1$ -mapping  $u \rightarrow x(u)$  of an open subset  $W$  of  $U_m$  into  $E_m$ , the points  $u \in W$  at which the Jacobian

$$\frac{D(x^1, \dots, x^m)}{D(u^1, \dots, u^m)} = 0$$

vanish are called the *singular points* of the mapping  $u \rightarrow x(u)$ , and the corresponding points in  $E_m$  *singular images*.

In proving Lemma 1.1 we shall refer to *generalized cylinders* in  $E_m$ . If  $r$  and  $t$  are positive integers such that  $r + t = m$  a generalized  $m$ -cylinder in  $E_m$  is the image in  $E_m$  under an orthogonal transformation of the product of a mutually orthogonal  $r$ -ball and euclidean  $t$ -rectangle, given in some auxiliary euclidean  $m$ -space. We shall refer to a generalized  $m$ -cylinder in  $E_m$  as an *elementary volume*  $V$ . It is clear that a subset  $Y$  of  $E_n$  has  $J$ -content zero if corresponding to a prescribed constant  $\eta$ ,  $Y$  can be included in a finite set of "elementary volumes"  $V$  whose total volume is less than  $\eta$ .

**Lemma 1.1.** *Let there be given a  $C^1$ -mapping*  
(1.6)

$$u \rightarrow x_i(u) = x_i \quad (u \in W; i = 1, \dots, m)$$

*of an open subset  $W$  of  $U_m$  into  $E_m$ .*

*If  $\hat{W}$  is a relatively compact open subset of  $W$  the image  $x(u)$  under the mapping (1.6) of the singular points  $u \in \hat{W}$  of the mapping (1.6), form a set of  $J$ -content zero in  $E_m$ .*

*Proof of Lemma 1.1.* The special case of Lemma 1.1 in which the mapping  $u \rightarrow x(u) = \text{grad } f(u)$ , where  $f$  is a function of class  $C''$  on  $W$ , was established and applied by the author frequently between 1926 and 1932. It was first applied in [3] in 1927 and applied several times in the author's colloquium lectures in 1932. It was noted by the author around 1932, too late to put the result into his colloquium lectures, that the proof of Lemma 1.1, in the case in which  $x(u) = \text{grad } f(u)$ , was applicable with at most trivial notational changes, to prove Lemma 1.1 in general.

To make clear the intimate relation between the general proof of Lemma 1.1 and the proof when  $x(u) = \text{grad } f(u)$ , we shall give the proof of Lemma 1.1 by quoting briefly a proof of a lemma on the density of conjugate points, as given on page 625 of [4] in 1930.

It should be noted that it is sufficient to prove Lemma 1.1 for the case in which  $\hat{W}$  is the open interior of a closed  $n$ -cube  $Q \subset W$ . In the 1930 quotation the "space ( $u$ )" means the space of the points  $u$ , and we shall replace the original phrase "conjugate points with  $s \leq d$ " by the phrase "singular images under (1.6) of points in  $Q$ ".

The following paragraph is quoted from pp. 625-6 of the 1930 paper, and, properly interpreted, gives a proof of Lemma 1.1.

“Let  $e$  now be an arbitrarily small positive constant. Let us break up the space ( $u$ ) into congruent  $m$ -cubes. If the diameter of each of these  $m$ -cubes be sufficiently small, then such of the corresponding sets  $[x(u)]$  as contain” (singular images under (1.6) of points in  $Q$ ) “can be enclosed in elementary volumes such as  $V$  whose ratios to that of the cubes will be less than  $e$ . The sum of these volumes  $V$  will be less than  $e$  times the total volume of the corresponding cubes. The sum of the elementary volumes will then be arbitrarily small”.

Lemma 1.1 follows.

See [5] for additional references.

Lemma 1.1 implies that the set of singular images in  $E_n$  of the singular points of the mapping (1.6) has a Lebesgue measure zero in  $E_n$ .

*This measure theorem is weaker than Lemma 1.1 because it does not conversely imply Lemma 1.1.*

*Lemma 1.1 applied to prove Theorem 1.2.* Let  $(c, q, \zeta)$  be a set of three elements of which  $c$  is a point in  $Cl\hat{E}_n$ ,  $q$  a point in  $Cl\hat{M}_{n-1} - c$ , and  $\zeta$  a sensed straight line normal to  $M_r$  at  $q$  and meeting  $c$ . With  $(c, q, \zeta)$  there can be associated a “focal mapping” of the form (1.5) such that  $q$  is a point  $F(v_0)$  with  $v_0 \in \hat{V}$  and  $\zeta$  has the direction  $\lambda(v_0)$ . We can suppose that  $\hat{V}$  is a relatively compact open subset of  $V$ , where  $V$  is given in a presentation  $(F:V, X) \in \mathcal{DM}_{n-1}$  of a neighborhood  $X$  of  $q$ .

Let  $d$  be a positive constant greater than the distance of an arbitrary point of  $Cl\hat{M}_{n-1}$  from an arbitrary point of  $Cl\hat{E}_n$ . We introduce the interval  $\hat{I} = (-d, d)$  and restrict the parameters  $(s, v)$  of the mapping (1.3) to the relatively compact subset

$$(1.7) \quad \hat{I} \times \hat{V} \text{ of } R \times V.$$

With  $n$  parameters  $s, v^1, \dots, v^{n-1}$  so restricted, it follows from Lemma 1.1 that the focal points of  $M_{n-1}$  with base points on  $\hat{X} = F(\hat{V})$ , on normals to  $\hat{X}$  with directions  $\lambda(v)$  for which  $v \in \hat{V}$ , and with algebraic distance coordinates  $s \in (-d, d)$ , form a set of focal points of  $J$ -content zero in  $E_n$ .

The focal points of  $M_{n-1}$  are bounded from their base points because of the condition (1.4). Because of this and because of the compactness of the sets  $Cl\hat{M}_{n-1}$  and  $Cl\hat{E}_n$  each triple  $(c, q, \zeta)$ , given and conditioned as above, is “associated” with one of a finite set of focal mappings such as (1.5).

Theorem 1.2 follows.

We turn to the proof of Theorem 1.1, beginning with the plane case  $n = 2$ . We shall make use of plane involutes and evolutes.

## 2. Theorem 1.1 in case $n = 2$

*Admissible spirals.* Let  $(\rho, \theta)$  be polar coordinates in a plane  $E_2$  of rectangular coordinates  $x^{(1)}, x^{(2)}$ . A curve  $\Lambda$  with a polar coordinate representation

$$(2.1) \quad \rho = \omega(\theta) > 0 \quad (\theta \geq 0)$$

in which the mapping  $\theta \rightarrow \omega(\theta)$  is unbounded, of class  $C^\infty$ , and such that

$$(2.2) \quad \omega(\theta + 2\pi) > \omega(\theta)$$

will be called an *admissible spiral*.

Such a curve is simple because of the condition (2.2). It is regular; if one sets  $\theta = t$  and

$$x^{(1)} = \omega(t) \cos t, \quad x^{(2)} = \omega(t) \sin t,$$

one obtains a "representation" of  $\Lambda$  such that

$$\|\dot{x}(t)\|^2 = \dot{\omega}^2(t) + \omega^2(t) > 0,$$

where differentiation as to  $t$  has been indicated by a superimposed dot.

In §6 we shall complete the proof of the following theorem.

**Theorem 2.1.** *There exists an admissible spiral whose focal points are everywhere dense in  $E_2$ .*

Use will be made of the theory of involutes. Graustein [6], p. 74, defines an involute of a regular plane curve  $h$  not a straight line, as a curve  $H$  which cuts each tangent to  $h$  at right angles.

This definition is not adequate for our purposes because such an involute of  $H$  of  $h$  is not necessarily regular or simple. We shall deal with arcs  $h$  and their involutes which are much more restricted. We begin with conditions on  $h$ .

Let  $R_+$  be the open positive axis of reals.

**Definition 2.1.** *Admissible arcs  $h$ .* Let  $h$  be a simple, sensed regular arc of the form

$$(x^{(1)}, x^{(2)}) = (\mu^{(1)}(\tau), \mu^{(2)}(\tau)) \quad (0 \leq \tau \leq \tau_0)$$

of class  $C^\infty$ . We suppose that  $\tau$  is the arc length on  $h$  measured from its initial point, and that the curvature

$$(2.3) \quad \kappa(\tau) = |\dot{\mu}^{(1)}(\tau)\ddot{\mu}^{(2)}(\tau) - \dot{\mu}^{(2)}(\tau)\ddot{\mu}^{(1)}(\tau)|$$

of  $h$  never vanishes. A final condition on  $h$  is that the mapping

$$(2.4) \quad (r, \tau) \rightarrow \mu(\tau) + r\dot{\mu}(\tau) = x : R_+ \times [0, \tau_0] \rightarrow E_2$$

of the subset  $R_+ \times [0, \tau_0]$  of the  $(r, \tau)$ -plane into  $E_2$  be a homeomorphism into  $E_2$ .

**Note.** To say that the mapping  $\tau \rightarrow \mu(\tau)$  is of class  $C^\infty$  means that it admits an extension of class  $C^\infty$  over some open interval containing  $[0, \tau_0]$ .

Extending  $\mu$  slightly, the mapping (2.4) has a jacobian in absolute value,

$$(2.5) \quad \left| \frac{D(x^{(1)}, x^{(2)})}{D(r, \tau)} \right| = r\kappa(\tau) > 0 \quad (r > 0, 0 \leq \tau \leq \tau_0).$$

Since the mapping (2.4) is by hypothesis a homeomorphism into  $E_2$  and has a non-vanishing jacobian, it is a diffeomorphism into  $E_2$ . We shall term the mapping (2.4), so conditioned, a *tangent diffeomorphism* ( $h : \tau_0$ ) into  $E_2$ .

**Example.** A closed subarc of an open plane semi-circle is an admissible arc  $h$ .

We state a lemma.

**Lemma 2.1.** *Given a "tangent diffeomorphism" ( $h : \tau_0$ ), for each constant  $\alpha > \tau_0$  there exists an involute  $H^\alpha$  of  $h$ , admitting a representation*

$$(2.6) \quad \tau \rightarrow H^\alpha(\tau) = \mu(\tau) + (\alpha - \tau)\dot{\mu}(\tau) \quad (0 \leq \tau \leq \tau_0)$$

and such that

- (a<sub>1</sub>)  $H^\alpha$  is a simple, regular arc of class  $C^\infty$ .
- (a<sub>2</sub>)  $H^\alpha$  is orthogonal at the point  $H^\alpha(\tau)$  of  $H^\alpha$  to the straight line tangent to  $h$  at the point  $\mu(\tau)$ .
- (a<sub>3</sub>) For  $0 < \tau < \tau_0$  there is one and only one "focal point" of  $H^\alpha$  with "base point"  $H^\alpha(\tau) \in H^\alpha$ , namely the point  $\mu(\tau) \in h$ .

Verification of (a<sub>1</sub>). The curve  $H^\alpha$  in  $E_2$  is the image in  $E_2$  under the diffeomorphism (2.4) of the arc in the  $(r, \tau)$ -plane of the form

$$(2.7) \quad r = \alpha - \tau \quad (0 \leq \tau \leq \tau_0, \alpha > \tau_0).$$

$H^\alpha$  accordingly has the properties (a<sub>1</sub>) since its antecedent (2.7) in the  $(r, \tau)$ -plane under the diffeomorphism (2.4) exists and has these properties in the  $(r, \tau)$ -plane.

Verification of (a<sub>2</sub>). A tangent to  $H^\alpha$  at the point represented by  $\tau$  has direction numbers  $\ddot{\mu}^{(1)}(\tau), \ddot{\mu}^{(2)}(\tau)$ . These numbers are not both zero since  $\kappa(\tau) \neq 0$ . Moreover  $\|\dot{\mu}(\tau)\| = 1$  identically by hypothesis, from which it follows that

$$(2.8) \quad \dot{\mu}^{(1)}(\tau)\ddot{\mu}^{(1)}(\tau) + \dot{\mu}^{(2)}(\tau)\ddot{\mu}^{(2)}(\tau) = 0,$$

establishing (a<sub>2</sub>).

Verification of (a<sub>3</sub>). The open subarc of  $H^\alpha$  on which  $0 < \tau < \tau_0$  may be considered as a 1-dimensional manifold  $M_1$ . A "focal mapping" based on  $M_1$  exists in the form (cf. (1.5))

$$(2.9) \quad (s, \tau) \rightarrow H^\alpha(\tau) + s\dot{\mu}(\tau) = x \quad (0 < \tau < \tau_0).$$

The normal to  $H^\alpha$  at the point  $H^\alpha(\tau)$  meets the point  $\mu(\tau)$  on  $h$  when  $s = \tau - \alpha$ , as one verifies using (2.6). A simple calculation shows that under (2.9)

$$(2.10) \quad \left| \frac{D(x^{(1)}, x^{(2)})}{D(s, \tau)} \right| = |(s - \tau + \alpha)\kappa(\tau)|,$$

where  $\kappa(\tau)$  is the curvature of  $h$ . Hence the jacobian (2.10) vanishes when  $s = \tau - \alpha$ , that is at the point  $s$  representing the point  $\mu(\tau) \in h$  on the normal to  $H^\alpha$  at  $H^\alpha(\tau)$ .

This establishes (a<sub>3</sub>) and completes the proof of Lemma 2.1.

**Definition 2.2.** *Mated evolute  $h$  and involute  $H^\alpha$ .* The arc  $h$  is admissible in the sense of Definition 2.1. We term such an arc  $h$  and the above involute  $H^\alpha$  of  $h$  a *mated evolute and involute*.

We shall restrict ourselves to mated evolutes and involutes.

### 3. Method of proof of Theorem 2.1

The only spirals of which we shall have need in proving Theorem 2.1 are of a limited type which we now characterize in a series of definitions.

*The annulus  $A(n', n'')$ .* Given two integers  $n'$  and  $n''$  such that  $n'' > n' > 0$ , by the *annulus  $A(n', n'')$*  is meant the set of points  $x \in E_2$  such that  $n' \leq \|x\| \leq n''$ .

*The interval  $I(i)$ .* Given a positive integer  $i$ ,  $I(i)$  shall denote the interval  $[(2i - 2)\pi, 2i\pi]$ .

By an arc  $\xi$  *spanning an annulus  $A(n', n'')$  over an interval  $I(i)$*  is meant a curve  $\xi$  in  $A(n', n'')$  with the properties (a<sub>1</sub>), (a<sub>2</sub>), (a<sub>3</sub>).

(a<sub>1</sub>)  $\xi$  shall have a polar coordinate representation  $\rho = \rho(\theta) > 0$  where the mapping  $\theta \rightarrow \rho(\theta)$  is defined and of class  $C^\infty$  over  $I(i)$ .

(a<sub>2</sub>)  $n' < \rho(\theta) < n''$  ( $\theta \in \overset{\circ}{I}(i)$ ),

where  $\overset{\circ}{I}(i)$  is the open interior of  $I(i)$ .

(a<sub>3</sub>) The mapping  $\theta \rightarrow \rho(\theta)$  shall admit an *extension* of class  $C^\infty$  over the  $\theta$ -axis such that

$$(3.0) \quad \begin{aligned} \rho(\theta) &= n' & (\theta \leq (2i - 2)\pi) \\ \rho(\theta) &= n'' & (\theta \geq 2i\pi). \end{aligned}$$

**Definition 3.1.** *Special spirals  $A$ .* To define such a spiral there is given an increasing sequence

$$(3.1) \quad n_0 < n_1 < n_2 < \dots \quad (n_0 = 1)$$

of integers and for each positive integer  $i$  an arc

$$(3.2) \quad \xi_i : \rho = \omega_i(\theta) > 0 \quad (\theta \in I(i))$$

in polar coordinates, "spanning" the annulus  $A(n_{i-1}, n_i)$  over the interval  $I(i)$ . A spiral  $A$ , admissible in the sense of §2, is defined by the



sequence  $\xi_1, \xi_2, \xi_3, \dots$  of the above arcs. Otherwise expressed  $\Lambda$  shall have a polar coordinate representation  $\rho = \omega(\theta) > 0$  for  $\theta \geq 0$  such that for each  $i$

$$(3.3) \quad \omega(\theta) = \omega_i(\theta) \quad (\theta \in I(i)).$$

To indicate how such a spiral  $\Lambda$ , if suitably chosen, will satisfy Theorem 2.1 a convention and definition are needed.

*A convention.* An arc  $\gamma$  with a representation

$$(3.4) \quad \rho = R(\theta) > 0 \quad (\theta' \leq \theta \leq \theta'')$$

in polar coordinates, will be regarded as real *analytic* if and only if the mapping  $\theta \rightarrow R(\theta)$  is analytic and admits a real analytic extension over an open interval which includes the interval  $[\theta', \theta'']$ .

**Definition 3.2.** A real analytic arc  $\gamma$  of form (3.4) will be said to *belong* to an annulus  $A(n', n'')$  and interval  $I(i)$  if  $\gamma$  is included in the open interior of  $A(n', n'')$  and if  $[\theta', \theta'']$  is included in the open interior  $\overset{\circ}{I}(i)$  of  $I(i)$ .

The following lemma is readily proved.

**Lemma 3.1.** *An analytic arc  $\gamma$  of form (3.4) which "belongs" to an annulus  $A(n', n'')$  and interval  $I(i)$ , admits an extension in polar coordinate form which "spans" the annulus  $A(n', n'')$  over  $I(i)$ .*

In §5 we shall study "mated" evolutes  $h$  and involutes  $H$  in which  $h$  is a subarc of a quarter circle so oriented and placed in  $E_2$  that  $H$  has the form (3.4) and "belongs" to an annulus  $A(n', n'')$  over an interval  $I(i)$ . The circular arcs  $h$  admitted are such that there are infinitely many choices of an annulus  $A(n', n'')$  and interval  $I(i)$  to which some mated involute  $H$  of  $h$  belongs. In §6 we shall take advantage of this freedom in the choice of the annuli  $A(n', n'')$  and intervals  $I(i)$  to prove the following lemma.

**Lemma 3.2.** *In the class  $K$  (cf. §4) of circular arcs  $h$  to be defined in §4, there exists a countable sequence  $h_1, h_2, h_3, \dots$  whose point set union*

$$(3.5) \quad X = h_1 \cup h_2 \cup h_3 \cup \dots$$

*is everywhere dense in  $E_2$  and can be associated with a "special" spiral  $\Lambda$  in the following way.*

*For each integer  $i > 0$  the circular arc  $h_i$  is the evolute of an involute  $H_i$  "mated to  $h_i$ " which is an analytic arc of the form (3.4), "belongs" to the annulus  $A(n_i, n_{i+1})$  and interval  $I(i)$  of  $\Lambda$ , and has an extension which is the subarc  $\xi_i$  of  $\Lambda$ .*

The spiral  $\Lambda$  of Lemma 3.2 has a focal set which includes the set  $X$  of (3.5) and so satisfies Theorem 2.1.

In §4 we characterize a class  $K$  of circular arcs  $h$  which, in retrospect (§6), will appear to be an adequate class of arcs to be used as evolutes in proving Theorem 2.1. In §4 we show that there is a countable subset of arcs  $h \in K$  which is everywhere dense in  $K$ .

#### 4. The class $K$ of circular arcs

The circular arc  $g$ . The class  $K$  shall contain that closed subarc  $g$  of the circle

$$(4.1) \quad (x^{(1)} + 1)^2 + (x^{(2)})^2 = 1$$

which meets the origin, is symmetric with respect to the  $x^{(1)}$ -axis and has a length  $b = 1/2$ . The arc  $g$  has a real analytic representation

$$(4.2) \quad x = g(t) = (g^{(1)}(t), g^{(2)}(t)) \quad (0 \leq t \leq b = 2a)$$

such that  $g^{(1)}(t) = \cos(t - a) - 1$ ,  $g^{(2)}(t) = \sin(t - a)$ . The parameter  $t$  is the arc length on  $g$  measured from  $g$ 's initial point  $(\cos a - 1, -\sin a)$ . The terminal point of  $g$  is  $(\cos a - 1, \sin a)$ . The arc  $g$  meets the origin when  $t = a$  and is there tangent to the  $x^{(2)}$ -axis.

We note that  $g$  is in the open disc

$$(4.3) \quad D_a = \{x \in E_2 \mid \|x\| < a\} \quad (2a = b).$$

The set  $K$  of circular arcs. We shall define  $K$  by means of linear transformations operating on  $g$ . With  $a$  as in (4.2) and  $c \in (-a, a)$  let  $T_c$  be the translation of  $E_2$  such that

$$(4.4) \quad T_c(x^{(1)}, x^{(2)}) = (x^{(1)} + c, x^{(2)}).$$

Let  $N$  be the ensemble of positive integers. For  $p \in N$  let  $T^p$  be the radial expansion of  $E_2$  such that

$$(4.5) \quad T^p(x^{(1)}, x^{(2)}) = (px^{(1)}, px^{(2)}).$$

Set  $T^p \circ T_c = T_c^p$ .

Let  $K_1$  be the ensemble of images  $h$  of  $g$  under  $T_c$  as  $c$  ranges on the interval  $(-a, a)$ . Set  $K_p = T^p K_1$  and  $K = \bigcup_{p \in N} K_p$ . Finally let  $|K_p|$  be the point set covered by the arcs in  $K_p$ . We see that  $|K_1|$  is a neighborhood of the origin and is included in the open disc  $D_{2a}$  with radius  $2a$  and center at the origin.

We shall verify the following lemma.

**Lemma 4.1.** *There is a countable subset  $\{h_i\}$  of the arcs  $h$  in  $K$  whose point set union is everywhere dense in  $E_2$ .*

To prove Lemma 4.1 let  $e_1, e_2, e_3, \dots$  be a decreasing sequence of positive numbers  $e_i$  such that  $e_i$  tends to zero as a limit as  $i \uparrow \infty$ . For each  $p \in N$  let  $(h)_p$  be a finite subset of the arcs  $h$  in  $K_p$  whose point set union contains points within a distance  $e_p$  of a prescribed point of  $|K_p|$ . The set of arcs

$$\{h_i\} = \text{Union}_{p \in N} (h)_p$$

is countable and satisfies the lemma.

In §5 we shall study a class of admissible involutes of arcs  $h \in K$ .

### 5. Involutes of the arcs $T_c^p g$

Recall that the length of the circular arc  $g$  introduced in (4.2) is  $b = 1/2$ . We have set  $b = 2a$ .

*Involutes  $G_k$  of  $g$ .* For each  $k > b$  an involute  $G_k$  of  $g$ , “mated” to  $g$ , is defined by the mapping (cf. (2.6))

$$(5.1) \quad t \rightarrow \mathbf{G}_k(t) = \mathbf{g}(t) + (k-t)\dot{\mathbf{g}}(t) \quad (0 \leq t \leq b).$$

This mapping is analytic, and according to Lemma 2.1, simple and regular. It is extendable as a simple, regular, analytic mapping over an open interval containing  $[0, b]$ .

We shall verify the following.

( $\lambda$ ) *For  $k > b$  the arc  $G_k$  is included in the half-plane of  $E_2$  on which  $x^{(2)} > 0$ .*

Statement ( $\lambda$ ) follows on noting that the minimum value of  $x^{(2)}$  on  $G_k$  is attained when  $t = b$  and is  $(k - b) \cos a + \sin a > 0$ .

The transformations  $T_c^p$ , introduced in §4, are linear, conformal homeomorphisms of  $E_2$  onto  $E_2$ . They carry circles into circles and simple, regular, analytic arcs  $\gamma$  into such arcs. If  $y$  is a focal point of  $\gamma$  with a base point  $x$  on  $\gamma$ ,  $T_c^p y$  is a focal point of  $T_c^p \gamma$  with the base point  $T_c^p x$  on  $T_c^p \gamma$ . Given  $g$  and a mated involute  $G_k$  of  $g$ , with  $k > b = 2a$ , we infer that the arcs

$$(5.2) \quad h = T_c^p g, \quad H = T_c^p G_k$$

are mated evolute and involute.

An arc  $h = T_c^p g$  has the representation

$$(5.3) \quad t \rightarrow \nu(t) = p(\mathbf{g}(t) + c) \quad (0 \leq t \leq b).$$

Its length parameter is  $pt$  and total length  $pb$ . An involute  $H = T_c^p G_k$ ,  $k > b$  of  $h$  has a representation

$$(5.4) \quad t \rightarrow p(\mathbf{G}_k(t) + c) = \nu(t) + (k-t)\dot{\nu}(t) \quad (0 \leq t \leq b).$$

We shall verify the following properties of the circular arc  $h = T_c^p g$  and its mated involute  $H$ , as given by (5.4).

*Subject to the condition that  $k > 3b$  the following is true:*

(a<sub>1</sub>) The arc  $H$  is simple, regular and analytic, with an open analytic extension. On  $H$ ,  $x^{(2)} > 0$ .

(a<sub>2</sub>) The arc  $H$  is included in the open annulus

$$(5.5) \quad p(k - 2b) < \|x\| < p(k + 2b).$$

(a<sub>3</sub>) No arc  $H$  is tangent to a ray from the origin.

(a<sub>4</sub>) An analytic polar coordinate representation

$$(5.6) \quad \rho = R(t) > 0, \quad \theta = \Theta(t), \quad (0 \leq t \leq b)$$

of  $H$  exists in which  $R(t), \Theta(t)$  are polar coordinates of the points  $T_c^p \mathbf{G}_k(t)$  of  $H$ . In such a representation  $\dot{\Theta}(t) > 0$ .

*Verification of (a<sub>1</sub>).* That  $H$  is simple, regular and analytic with an open, analytic extension follows from the fact that  $G_k$  has these properties. Since the point  $\mathbf{G}_k(t)$  is in the open upper half-plane by  $(\lambda)$ , the corresponding point  $p(\mathbf{G}_k(t) + c)$  of  $H$  is in the open upper half-plane.

*Verification of (a<sub>2</sub>).* We first examine the case in which  $p = 1$  and  $c = 0$ . In this case  $T_c^p G_k = G_k$ .

Let  $u$  and  $v$  be respectively points on  $g$  and  $G_k$  such that  $v$  is on the ray tangent to  $g$  at  $u$ . Let  $q$  be the origin. If  $x = v$  then  $\|x\| = d(q, v)$ . Consideration of the triangle with the vertices  $u, v, q$  shows that

$$(5.7) \quad d(u, v) - d(q, u) \leq \|x\| \leq d(u, v) + d(q, u).$$

Since  $d(q, u) < a$  and  $d(u, v) = k - t \geq k - 2a$  for points  $v \in G_k$  in accord with (5.1), it follows from (5.7) that

$$(5.8) \quad k - 3a < \|x\| < k + 3a.$$

For points  $x \in T_c G_k$  it follows from (5.8) that

$$(5.9) \quad k - 2b < \|x\| < k + 2b$$

since  $-a < c < a$  and  $T_c \mathbf{G}_k(t) = \mathbf{G}_k(t) + c$ . Since  $T_c^p = T^p \circ T_c$ , the inequalities (5.5) follow from (5.9) for  $x \in T_c^p G_k$ .

*Verification of (a<sub>3</sub>).* We begin by verifying (a<sub>3</sub>) for the case of an involute  $T_c G_k$  of  $T_c g$ . Since  $k > 3b$  by hypothesis it follows from (5.9) that  $T_c G_k$  does not meet the disc  $D_b$ . On the other hand  $T_c g$  is included in  $D_b$ .

Suppose (a<sub>3</sub>) false for the case of  $T_c G_k$ . There then exists a ray  $\zeta$  from the origin, tangent to  $T_c G_k$  at a point  $v$ . Such a ray  $\zeta$  would be orthogonal at  $v$  to a line  $\tau$  tangent to  $T_c g$ . Thus  $\tau \cap \zeta = v$ . But the point  $\tau \cap \zeta$  must be the point on  $\tau$  nearest the origin. Since  $\tau$  meets  $D_b$  the point  $\tau \cap \zeta = v$  must be in  $D_b$ , contrary to the fact that no point of  $T_c G_k$  is in  $D_b$ .

Since  $T^p$  is a radial expansion with center at the origin, statement (a<sub>3</sub>) is true for  $T_c^p G_k$ , since it is true for  $T_c G_k$ .

*Verification of (a<sub>4</sub>).* A simple, closed, regular, analytic arc  $H$  on which  $x^{(2)} > 0$ , admits infinitely many analytic polar coordinate representations (5.6). In (5.6)  $\Theta$  is uniquely determined up to a function

whose values are an integral multiple of  $2\pi$ . For no value of  $t \in [0, b]$  is  $\dot{\Theta}(t) = 0$ , since at a point  $x_0$  of  $H$  at which  $\dot{\Theta}(t_0) = 0$  the ray from the origin meeting  $x_0$  would then be tangent to  $H$  at  $x_0$ . This would be contrary to (a<sub>3</sub>).

*It remains to show that  $\dot{\Theta}(t) > 0$ .*

Since  $T^p$  is directly conformal and no arc  $T_c G_k$  meets the origin it is sufficient to prove that  $\dot{\Theta}(t) > 0$  in a representation (5.6) of  $T_c G_k$ . Since  $\dot{\Theta}(t) \neq 0$  it is sufficient to verify that  $\dot{\Theta}(a) > 0$ , making use of the relation

$$T_c G_k(t) = g(t) + (k-t)\dot{g}(t) + c \quad (\text{cf. (5.1)})$$

and the formula for  $g(t)$ . If  $T_c G_k$  has the representation  $x^{(1)} = \varphi(t)$ ,  $x^{(2)} = \phi(t)$  one finds that

$$\dot{\phi}(a)\varphi(a) - \dot{\varphi}(a)\phi(a) = (k-a)^2 > 0$$

from which it follows that  $\dot{\Theta}(a) > 0$ .

Thus the properties (a<sub>1</sub>) to (a<sub>4</sub>) of  $H$  hold as stated.

## 6. Proof of Theorem 2.1

The following lemma is a consequence of the properties of the mated evolutes and involutes

$$(6.1) \quad h = T_c^p g, \quad H = T_c^p G_k \quad (k > 3b)$$

as recorded in §5.

**Lemma 6.1.** *Let  $h = T_c^p g$  be prescribed in  $K$ , and positive integers  $n'$  and  $i$  given. Then for a sufficiently large  $k > 3b$  the involute  $H = T_c^p G_k$  of  $h$  is included in the interior of the annulus  $A(n', n'')$  for some  $n'' > n'$ , and  $H$  has a unique analytic polar coordinate representation*

$$(6.2) \quad \rho = \eta(\theta) > 0 \quad (\theta' \leq \theta \leq \theta'')$$

such that

$$(6.3) \quad [\theta', \theta''] \subset \overset{\circ}{I}(i).$$

*The choice of  $k$ .* Given  $h$  as in Lemma 6.1, whatever the choice of  $k > 3b$ , the corresponding involute  $T_c^p G_k$  of  $h$  satisfies (5.5). Let  $k > 3b$  be chosen so that  $p(k-2b) > n'$ . Then  $H = T_c^p G_k$  is included in the open interior of an annulus  $A(n', n'')$  for some choice of  $n'' > n'$ .

*The choice of  $\eta(\theta)$  in (6.2).* Let (5.6) be a representation of  $H$  so chosen (as is possible) that

$$(6.4) \quad (2i-2)\pi < \Theta(0) < \Theta(b) < (2i-1)\pi$$

and set  $\theta' = \Theta(0)$ ,  $\theta'' = \Theta(b)$ . Since  $\dot{\Theta}(t) > 0$  in a representation (5.6), a unique analytic representation of  $H$  exists of the form (6.2) with (6.3) holding.

This completes the proof of Lemma 6.1.

To prove Theorem 2.1 it is sufficient to prove Lemma 3.2.

*Proof of Lemma 3.2.* According to Lemma 4.1 there exists a sequence of circular arcs

$$(6.5) \quad h_i = T_{c_i}^{p_i} g \in K \quad (i = 1, 2, \dots)$$

whose point set union is everywhere dense in  $E_2$ .

According to Lemma 6.1, if  $k_1$  is sufficiently large the arc

$$H_1 = T_{c_1}^{p_1} G_{k_1}$$

is an involute mated to  $h_1$  as evolute, and admits an analytic representation (6.2) "belonging" to  $I(1)$  and to an annulus  $A(n_0, n_1)$  in which  $n_0 = 1$ .

Proceeding inductively, we assume that for some integer  $r > 0$  there exist integers

$$(6.6) \quad n_0 < n_1 < n_2 < \dots < n_r \quad (n_0 = 1)$$

and involutes

$$(6.7) \quad H_i = T_{c_i}^{p_i} G_{k_i} \quad (i = 1, \dots, r)$$

"mated" to the respective arcs  $h_i$  as evolutes, and admitting analytic representations of the form (6.2), "belonging" to the respective annuli  $A(n_{i-1}, n_i)$  over the corresponding intervals  $I(i)$ .

Corresponding to the given arc  $h_{r+1}$ , Lemma 6.1 implies the existence of a constant  $k_{r+1} > 3b$  so large that the involute

$$H_{r+1} = T_{c_{r+1}}^{p_{r+1}} G_{k_{r+1}}$$

"mated" to  $h_{r+1}$  is included in the annulus  $A(n_r, n_{r+1})$  for some choice of  $n_{r+1} > n_r$ . According to Lemma 6.1  $H_{r+1}$  has a unique analytic polar coordinate representation of form (6.2) "belonging" to  $A(n_r, n_{r+1})$  over  $I(r+1)$ . Thus there exist integers

$$(6.8) \quad n_0 < n_1 < n_2 < \dots \quad (n_0 = 1)$$

and involutes  $H_i = T_{c_i}^{p_i} G_{k_i}$  mated to the respective arcs  $h_i$  as evolutes, and "belonging" to the respective annuli  $A(n_{i-1}, n_i)$  over the corresponding intervals  $I(i)$ ,  $i > 0$ .

According to Lemma 3.1, for  $i > 0$ , the involute  $H_i$  of  $h_i$  admits an analytic extension  $\xi_i$  in polar coordinate form "spanning" the annulus  $A(n_{i-1}, n_i)$  over  $I(i)$ . The sequence of these extensions  $\xi_i$  defines a spiral  $A$  admissible in the sense of §2.

This completes the proof of Lemma 3.2.

*Proof of Theorem 2.1.* The set of focal points of the spiral  $\Lambda$  of Lemma 3.2 includes the set

$$X = h_1 \cup h_2 \cup h_3 \cup \dots$$

and so is everywhere dense in  $E_2$ .

This establishes Theorem 2.1.

## 7. Proof of Theorem 1.1 and comments

Let  $E_2$  be a 2-plane of coordinates  $x^1, x^2$  serving as a coordinate 2-plane in a euclidean  $n$ -space  $E_n, n > 2$ , of coordinates  $x^1, \dots, x^n$ . Let  $\pi$  be the projection of  $E_n$  onto  $E_2$  under which

$$\pi(x^1, \dots, x^n) = (x^1, x^2).$$

Let

$$(7.1) \quad x^1 = \varphi^1(t), \quad x^2 = \varphi^2(t) \quad (0 \leq t < \infty)$$

be a simple regular spiral  $\Lambda$  in  $E_2$  whose centers of curvature in  $E_2$  are everywhere dense. Such a curve exists by virtue of Theorem 2.1. We suppose that  $t$  is the arc length on the spiral  $\Lambda$ , measured from the initial point of  $\Lambda$ .

*The manifold  $M_1$ .* Let  $M_1$  be the regular manifold in  $E_2$  obtained by restricting the representation (7.1) to the open interval  $0 < t < \infty$ . Concerning  $M_1$  we shall prove the following lemma.

**Lemma 7.1.** *The antecedent  $\Gamma_{n-1}$  in  $E_n$  of  $M_1$  under the projection  $\pi$  is representable as a regular  $(n-1)$ -manifold  $M_{n-1}$  in  $E_n$  of class  $C^\infty$ . The focal points of  $M_{n-1}$  are everywhere dense in  $E_n$ .*

The set  $\Gamma_{n-1}$ , with a topology induced by that of  $E_n$ , is the image  $X$  of a single "regular presentation" ( $F: V, X$ ) in  $E_n$  in which  $V$  is the open subset of the euclidean space  $E_{n-1}$  of coordinates  $v^1, \dots, v^{n-1}$  on which  $v^1 > 0$ . We define the presentation  $v \rightarrow F(v): V \rightarrow X$  by setting

$$(7.2) \quad \begin{aligned} F^i(v) &= \varphi^i(v^1) & (i = 1, 2) \\ F^j(v) &= v^{j-1} & (j = 3, 4, \dots, n). \end{aligned}$$

The resulting presentation of  $X$  is regular and of class  $C^\infty$ .

A "focal mapping" based on  $M_{n-1}$  and representing *all* normals to  $M_{n-1}$  exists, with  $R \times V$  the domain of the parameters  $(s, v)$ , and with the form

$$(7.3) \quad \begin{aligned} (s, v) &\rightarrow F^1(v) + s\dot{\varphi}^2(v^1) = x^1 \\ (s, v) &\rightarrow F^2(v) - s\dot{\varphi}^1(v^1) = x^2 \\ (s, v) &\rightarrow F^j(v) = x^j, \end{aligned} \quad ((s, v) \in R \times V)$$

where  $j$  has the range  $3, \dots, n$ . Setting  $v^1 = t$ , the Jacobian

$$(7.4) \quad J(s, v) = \frac{D(x^1, \dots, x^n)}{D(s, v^1, \dots, v^{n-1})} = \begin{vmatrix} \dot{\varphi}^2(t), \dot{\varphi}^1(t) + s\ddot{\varphi}^2(t) \\ -\dot{\varphi}^1(t), \dot{\varphi}^2(t) - s\ddot{\varphi}^1(t) \end{vmatrix}$$

evaluated for arbitrary  $(s, v) \in R \times V$ , reduces to  $1 + sk(t)$  where

$$k(t) = \dot{\varphi}^1(t)\ddot{\varphi}^2(t) - \dot{\varphi}^2(t)\ddot{\varphi}^1(t).$$

The relation  $J(s, v) = 1 + sk(t)$  and the form of (7.3) show that the focal points of  $M_{n-1}$  are the antecedents under  $\pi$  of the centers of curvature of  $M_1$  in  $E_2$  and so are everywhere dense in  $E_n$ .

This establishes Lemma 7.1 as well as Theorem 1.1.

**Comments on the role of non-degenerate (ND) functions.**

Focal points of a regular connected  $C^\infty$ -manifold  $M_{n-1}$  in  $E_n$  are significant topologically largely because of their relation to ND functions on  $M_{n-1}$ . It follows from Theorem 1.3 that if  $c$  is not a focal point of  $M_{n-1}$ , nor on  $M_{n-1}$ , the restriction to points  $x \in M_{n-1}$  of the distance  $\|x - c\|$  between  $x \in E_n$  and  $c$  gives the values of a ND function  $f$  on  $M_{n-1}$ . Moreover the index of a critical point  $q$  of  $f$  is the number of centers of principal normal curvature (counted with their multiplicities) (cf. [2]) of  $M_{n-1}$ , based on  $q$ , and on the normal from  $q$  to  $c$  between  $q$  and  $c$ .

This theorem and the following extension are special types of "index theorems" for a "critical extremal" of an integral in the variational theory under admissible boundary conditions. In the book which the author is now writing the exposition of these results will be in detail and independent of the variational theory.

The preceding theorem has the following extension.

For  $0 < r < n$  let  $M_r$  be a regular  $C^\infty$ -manifold in  $E_n$  and let  $c = (c_1, \dots, c_n)$  be a point in  $E_n$ . Let  $q$  be a point in  $M_r - c$  such that the directed line  $\zeta = \bar{q}c$  is orthogonal to  $M_r$  at  $q$ . Let  $p \rightarrow f(p)$  be the  $C^\infty$ -function on  $M_n - c$  obtained by restricting the mapping  $x \rightarrow \|x - c\|$  to  $M_r - c$ . If  $q$  is a non-degenerate critical point of  $f$ , its index  $k$  can be evaluated as follows.

Let  $P_{r+1}$  be the  $(r+1)$ -plane determined by  $c$  and the  $r$ -plane tangent to  $M_r$  at  $q$ . Let  $\pi$  be the orthogonal projection of  $E_n$  onto  $P_{r+1}$ . The projection under  $\pi$  of a sufficiently small open neighborhood of  $q$ , relative to  $M_r$ , will be a regular  $C^\infty$ -manifold  $\hat{M}_r$  in  $P_{r+1}$ .

*The index  $k$  of the critical point  $q$  of  $f$  is the number of centers of principal normal curvature in  $P_{r+1}$  of  $\hat{M}_r$  on  $\zeta$  between  $q$  and  $c$  and based on  $q$ .*

Frankel and Andreotti [7] have made notable use of such properties in proving a fundamental theorem of Lefschetz on affine algebraic manifolds [8].

*A lacunary theorem.* One of the most useful theorems on ND functions on a compact differentiable manifold  $M_r$  is as follows. If  $f$  is a ND function on  $M_r$  which has  $m \geq 0$  critical points of index  $k$ , but no critical points of index  $k-1$  or  $k+1$ , then the  $k$ -th connectivity of  $M_r$  is  $m$ . This follows from the author's inequalities between the connectivities of  $M_r$  and the numbers  $m_k$  of critical points of index  $k$ . This result is well-illustrated in [7] and in Milnor's elegant derivation of the homology groups of a complex projective space [9].



**A homotopy theorem.** The existence of a non-degenerate function  $f$  of class  $C^\infty$  on a compact connected  $C^\infty$  manifold  $M_r$  sharply conditions homotopy relations on  $M_r$  in a way which we shall recall.

We can suppose that  $f$  has been so modified that it has just one critical point of index 0 and one of index  $r$  (cf. [12]) and that its critical values are all distinct. Let  $a < b$  be two critical values of  $f$  between which there are no critical values of  $f$ . Suppose that  $\alpha$  and  $\beta$  are the critical points of  $f$  with the respective critical values  $a$  and  $b$ . Let  $h$  and  $k$  be the indices of  $\alpha$  and  $\beta$  respectively. Set

$$(7.5) \quad \begin{aligned} A &= \{p \in M_r \mid f(p) \leq a\} \\ B &= \{p \in M_r \mid f(p) \leq b\}. \end{aligned}$$

There is then a *deformation*  $D$  of  $B - \beta$  on itself onto  $A$  leaving  $A$  pointwise fixed and with especially simple properties which we shall describe.

*A family  $F$  of  $f$ -arcs.* The deformation parameter under  $D$  is  $t$  and increases from 0 to 1. When  $t = 0$  each point is in its original position, and when  $t = 1$  in its final position.

By an  $f$ -arc  $\gamma$  on  $M_r$  we mean a simple arc on which the value of  $f$  at a point  $p \in \gamma$  is the parameter  $\tau$  of  $p$  on  $\gamma$ . Set

$$(7.6) \quad H = \{p \in M_r \mid a \leq f(p) \leq b\}.$$

There is a continuous family  $F$  of sensed  $f$ -arcs  $\gamma$ , with at least one arc meeting each point  $p \in H$ , and only one such arc, except for the points  $\alpha$  and  $\beta$  of  $H$ . An  $f$ -arc in  $F$  has an initial point at the  $f$ -level  $b$  and a terminal point at the  $f$ -level  $a$ . On each  $f$ -arc of  $F$ ,  $f$  decreases from  $b$  to  $a$ .

The *family  $F$*  can be so defined that it has the following characteristics.

Let  $c$  be a value such that  $a < c < b$ , and set

$$f^c = \{p \in M_r \mid f(p) = c\}.$$

We can make the arcs  $\gamma \in F$  correspond to the points  $q \in f^c$  in a 1-1-manner, with  $q$  corresponding to that unique arc  $\gamma \in F$  which meets  $q$ .  $F$  can then be so defined that there is a continuous map onto  $H$

$$(7.7) \quad (q, \tau) \rightarrow F^*(q, \tau) : f^c \times [a, b] \rightarrow H$$

with  $\gamma \in F$  defined by the “partial map” in which the parameter  $q$  of  $\gamma$  is fixed in (7.7). Moreover the family  $F$  can be so defined that the arcs of  $F$  which meet  $\beta$  and on which the parameter  $f = \tau$  increases from  $c$  to  $b$  are represented by rays of a topological  $k$ -disc, with all rays distinct except for their intersection at the topological center  $\beta$  when  $\tau = b$ . Similarly the arcs of  $F$  which meet the point  $\alpha$  and on which the parameter  $s = \tau$  decreases from  $c$  to  $a$  can be represented by rays of a topological  $(r - h)$ -disc, with all rays distinct except that they meet in  $\alpha$  as a topological center when  $\tau = a$ .

*Lemma 6.1 of [12].* The proof of the existence of the family  $F$  follows the methods of [12], in particular, one uses the fundamental Lemma

6.1 of [12]. A Riemannian metric is thereby assigned to  $M_r$  with the following important special property. Near each critical point  $z$  of  $f$  there is a special set of local coordinates  $u_1, \dots, u_r$  of  $M_r$  in terms of which the trajectories which are orthogonal in the *Riemannian* sense to the level manifolds of  $f$  near  $z$  have representations defined by the trajectories, orthogonal in the *euclidean* sense to quadric level manifolds of a form

$$-u_1^2 - \dots - u_s^2 + u_{s+1}^2 + \dots + u_r^2,$$

where the origin represents  $z$  and  $s$  is the index of  $z$ . It should be clear that the global definition in [12] of a Riemannian metric on  $M_r$  is a *global* extension of the author's earlier *local* euclidean "reduction theorem".

*Definition of the deformation D.* Let  $L$  be the non-singular linear transformation of the real axis onto itself such that  $L(b) = 0$  and  $L(a) = 1$ . Under  $D$  a point  $p \in B - \beta$  shall remain invariant for  $0 \leq t \leq L(f(p))$ . If  $f(p) \leq a$ ,  $p$  shall remain invariant for all  $t \in [0, 1]$ . If  $f(p) > a$  and  $p \in B - \beta$  let  $T_p$  be the unique arc of  $F$  meeting  $p$ . For each  $t$  on the interval  $1 \geq t \geq L(f(p))$   $p$  shall be replaced under  $D$  at the time  $t$  by the unique point  $p_t$  on  $T_p$  such that  $L(f(p_t)) = t$ .

**Homotopy Theorem 7.1.** *The resultant deformation  $D$  is then a continuous deformation of  $B - \beta$  on itself onto  $A$ , leaving  $A$  point-wise fixed, and having the special properties implied by the nature of the family  $F$ .*

In particular there is no essential difference between homotopy relations on  $A$  and on  $B - \beta$ . The *main problem* is how does the addition of the point  $\beta$  to  $B - \beta$  make  $B - \beta$  differ homotopically from  $B$ .

Thus Lemma 6.1 of [12] permits one to establish a fundamental basis for a study of homotopy relations on  $M_r$ . This is without use of the Whitehead theory of "homotopy equivalences".

In [5] the names of a few mathematicians are given who have contributed significantly to the critical point theory and its applications in recent years. The contribution [10] has just come to my attention. This paper bears on the Lie theory as does the work of R. Bott. The work of the Russian mathematician Vladimir Arnold [11] is of interest in this connection.

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