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Fock Representations and BRST Cohomology in $SL(2)$ Current Algebra

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Abstract. We investigate the structure of the Fock modules over $A_1^{(1)}$ introduced by Wakimoto. We show that irreducible highest weight modules arise as degree zero cohomology groups in a BRST-like complex of Fock modules. Chiral primary fields are constructed as BRST invariant operators acting on Fock modules. As a result, we obtain a free field representation of correlation functions of the $SU(2)$ WZW model on the plane and on the torus. We also consider representations of fractional level arising in Polyakov's $2D$ quantum gravity. Finally, we give a geometrical, Borel–Weil-like interpretation of the Wakimoto construction.

1. Introduction

Kac–Moody Lie algebras play a central role in two-dimensional conformal field theory [1]: it appears that most known examples of conformal field theory models can be understood in terms of WZW models [2, 3] (by a coset construction [4]) whose symmetry algebra is a Kac–Moody algebra. It is therefore important to understand the structure of their representations, and of chiral primary fields, which are tensor operators for these algebras. In this paper we focus on the algebra $A_1^{(1)}$, the central extension of the Lie algebra of loops in $sl(2, \mathbf{C})$.

A very powerful method for explicitly constructing representations, chiral primary fields and their correlation functions in conformal field theory is the Feigin–Fuchs construction [5] in terms of free fields, first considered in the case of the Virasoro algebra. It was recently shown [6] that this construction relies on a hidden BRST-like symmetry, which realizes the space of physical states as a subquotient of the free field Fock space. This observation led to an integral expression for correlation functions of minimal models on the torus.

Here we extend these results to the $A_1^{(1)}$ Lie algebra. The free field representation spaces (Fock modules) for this algebra were introduced by Wakimoto [7]. They are labeled by a spin J and a level K, which can be arbitrary complex numbers. The corresponding Feigin–Fuchs-like construction was proposed by Zamolodchikov [8]. The case of interest to us is the case where $K+2=p/p'$ is a positive rational number, and the spin is related to the level by the formula

$$2J+1=n-n'p/p', \quad 1 \leqq n \leqq p-1, \quad 0 \leqq n' \leqq p'-1. \tag{1.1}$$

Let us first discuss the Verma modules and the irreducible highest weight representations with spin and level given by (1.1). This class of representations has both mathematical and physical significance. Specifically, the Verma modules with highest weight J and level K contain infinitely many singular vectors, as can be seen from the Kac–Kazhdan determinant formula [9], and they present many striking similarities with the completely degenerate representations of the Virasoro algebra [10]. As shown by Kac and Wakimoto [11], the characters of the corresponding irreducible representations form, for each fixed level, a (finite dimensional) representation of the modular group. An important special case is $p'=1$. The irreducible representations (1.1) are then unitary and integrable. In physics they arise as the representations appearing in the Hilbert space of the $SU(2)$ WZW model [12]. The representations with $p'>1$ are needed for the coset construction of non-unitary minimal models [13]. They also arise in Polyakov's two-dimensional quantum gravity [14, 15] coupled to minimal models with central charge $1-6(p-p')^2/pp'$, as representations "dual" to the representations of $A_1^{(1)}$ (in the $sl(2,R)$ real form) appearing in the model.

In free field representations of conformal field theory, one must understand the relation between Fock modules, on which free fields act, and irreducible highest weight modules, which are the spaces of physical states of the models. In this paper we show that, for each pair J, K in (1.1), the corresponding Fock module is the degree zero element of a graded complex of Fock modules whose differential (the BRST operator) can be constructed out of free field operators. The cohomology of this complex vanishes except in degree zero, where it coincides with the irreducible highest weight module with the same highest weight. Chiral primary fields are represented as BRST invariant combinations of free field operators acting on Fock modules. These results give integral representations of conformal blocks on the plane and, by a Lefshetz formula, on the torus. In the case of the plane and for $p'=1$, these integral representations have been first given by Fateev and Zamolodchikov [12], who also solved for the structure constants.

As the free field representation of $A_1^{(1)}$ chiral primary fields are very closely related to the one of Virasoro chiral primary fields, many calculations done in the Virasoro case can be carried over to this case. In particular the calculation of the representation of the braid group describing the analytic continuation of conformal blocks [16] could be extended to this case, with almost identical details. One could also extend to the WZW case the result [17] that one-point functions of scaling fields of minimal models on the torus are modular covariant. This property is important if one wants to consider the WZW model on higher genus Riemann surfaces [18]. Indeed, it has been shown [19] that modular covariance of one-point functions on the torus and crossing symmetry on the sphere are sufficient conditions for a theory to have a consistent extension to arbitrary genus. As a last application of our results we mention the "quantum hamiltonian reduction" [20, 21]. It is shown in [21] that completely degenerate representations of the Virasoro algebra can be obtained from irreducible $A_1^{(1)}$ representations with weights (1.1) (and $n'>0$), by imposing the constraint $J^+(z)=1$ in a (conventional gauge theory) BRST manner. The proof of this relies on the fact that the BRST operator considered in this paper is BRST equivalent (with respect to the conventional BRST operator) to the Virasoro BRST operator of [6].

This paper is organized as follows: In Sect. 2 we introduce the Fock modules over $A_1^{(1)}$ following Wakimoto, and we describe the BRST complex and its structure. In Sect. 3 we introduce the free field representation of the chiral primary fields and of their correlation functions for genus zero and one. We also discuss the fusion rules. The structure of Fock modules and the computation of the BRST cohomology is done in Sect. 4. We conclude the paper with a Borel–Weil-like construction of the Wakimoto modules. This construction provides a natural setting for the extension of Wakimoto modules to arbitrary groups, and naturally leads to representations of highest weight with respect to an "infinite gradation." The meaning of these representations remains obscure.

In the notation of this paper, the algebra $A_1^{(1)}$ is given by generators $J_n^a, n\in\mathbf{Z}$, $a=+,0,-$, and K with non-vanishing commutators,

$$\begin{aligned}
[J_n^0, J_m^\pm] &= J_{n+m}^\pm,\\
[J_n^0, J_m^0] &= \frac{nK}{2}\delta_{n,-m},\\
[J_n^+, J_n^-] &= 2J_{n+m}^0 + nK\delta_{n,-m}.
\end{aligned} \tag{2.1}$$

It is customary to adjoin to the algebra a derivation d with $[d, J_n^a] = -nJ_n^a$ and $[d, K]=0$. Throughout this paper, $\mathbf{N}$ denotes the set of positive integers.

During the completion of this paper, we have learned of results obtained by other authors on related subjects. In particular Feigin and Frenkel [22] have obtained a generalization of the Wakimoto construction to more general Kac–Moody algebras. Their results on the structure of Fock modules overlap with ours. Gawędzki [23] has recently derived an integral representation for $G^{\mathbf{C}}/G$ correlation functions, which are "dual" to WZW correlation functions, by manipulating the functional integral. The precise relation with our results is not completely clear yet. Similar constructions have been proposed in [24]. The $SU(2)$ WZW model and related models have also been studied from a BRST point of view in [25, 26].

2. Free Field Representations

A free field representation of $A_1^{(1)}$ valid for any value of the central charge was first introduced by Wakimoto [7]. For any (untwisted) affine Kac–Moody algebra $g^{(1)}$ the field content of the free field representation can be extracted from the value of the central charge of the Sugawara operators: $c = K\dim g/(K+h^*)$. Here K is the level of the representation and h^* is the dual Coxeter number of g. Thanks to the Freudenthal-de Vries strange formula, $\dim g/24 = |\rho|^2/2h^*$ with ρ the Weyl vector of g, the Sugawara central charge admits the following decomposition:

$$c = \frac{K\dim g}{K+h^*} = (\#\text{ roots of } g) + \left(\operatorname{rank} g - \frac{12|\rho|^2}{K+h^*}\right). \tag{2.1}$$

To represent such a Virasoro algebra, we are naturally led to introduce a free field Φ taking values in a Cartan subalgebra of g and a pair of spin 0-spin 1 bosonic fields, $\omega_\alpha, \omega_\alpha^\dagger$, for each positive roots. The stress-tensor of the $\omega-\omega^\dagger$ system, $T(z)=\sum_{\alpha>0} T_\alpha(z)$, $T_\alpha=-\omega_\alpha^\dagger\partial\omega_\alpha$, has a central charge equal to the number of roots of g. The stress-tensor of the field Φ is given by a Chodos–Thorn construction, $T=\frac{1}{2}(i\partial\Phi)^2-(i/(\sqrt{K+h^*}))\rho\cdot\partial^2\Phi$. It gives the second bracket in the decomposition (2.1) of the central charge.

To be precise in the case of $A_1^{(1)}$ we introduce a free bosonic field $\Phi(z)$ and an $\omega(z)-\omega^\dagger(z)$ system,

$$i\partial\Phi(z)=\sum_n a_n z^{-n-1}, \quad \omega(z)=\sum_n \omega_n z^{-n}, \quad \omega^\dagger(z)=\sum_n \omega_n^\dagger z^{-n-1}. \tag{2.2}$$

They satisfy the following commutation relations:

$$[a_n, a_m]=n\delta_{n+m,0}, \quad [\omega_n, \omega_m^\dagger]=\delta_{n+m,0}. \tag{2.3}$$

Other commutators vanish.

The $\omega-\omega^\dagger$ system admits infinitely many non-equivalent Fock representations. We choose one of them by imposing the following condition on the vacuum:

$$\begin{aligned} a_n|J\rangle &= J\gamma^{-1}|J\rangle\delta_{n,0}, \\ \omega_n^\dagger|J\rangle &= 0, \quad n\geqq 0, \\ \omega_{n+1}|J\rangle &= 0 \end{aligned} \tag{2.4}$$

with $2\gamma^2=K+2$. We denote by F_J this Fock space.

In the Fock spaces F_J the affine algebra $A_1^{(1)}$ is represented by

$$\begin{aligned} J^+(z) &= \omega^\dagger(z), \\ J^0(z) &= :\omega(z)\omega^\dagger(z): + \gamma i\partial\Phi(z), \\ J^-(z) &= -:\omega^2(z)\omega^\dagger(z): - 2\gamma i\partial\Phi(z)\omega(z) - K\partial\omega(z). \end{aligned} \tag{2.5}$$

K is the level of the representation. By hypothesis, $K+2\neq 0$. Note that a_0 is in the center of this representation. Hence each Fock space F_J carries a representation of $A_1^{(1)}$. The vacuum $|J\rangle$ is a highest weight vector with weight $\Lambda_J=2J\Lambda_1+(K-2J)\Lambda_0$. Λ_0 and Λ_1 are the fundamental weights of $A_1^{(1)}$.

If no positive integers n and n' exist such that $2J+1=\pm[n-n'(K+2)]$ and $2J+1\notin\mathbf{N}$ the representation (2.5) is irreducible.

On the contrary if $2J+1=\pm[n-n'(K+2)]$ for some $n,n'\in\mathbf{N}$ or if $2J+1\in\mathbf{N}$ the representation (2.5) is reducible. For $n,n'\in\mathbf{Z}$ we set $2J_{n,n'}+1=n-n'(K+2)$ and we denote the corresponding weight by $\Lambda_{n,n'}$ and the Fock space $F_{J_{n,n'}}$ by $F(\Lambda_{n,n'})$.

As for the representation of the Virasoro algebra the analysis of the structure of the Fock modules is greatly simplified if one introduces a BRST-like operator. As we will soon explain, the irreducible representations arise as cohomology groups of this BRST operator. Let $V(z)$ be the operator defined by:

$$V(z)=\omega^\dagger(z):\exp\left(-\frac{i}{\gamma}\Phi(z)\right):. \tag{2.6}$$

$V(z)$ maps F_J into F_{J-1}. It is a screening operator; i.e. its OPE with the currents (2.5) are total derivatives. Acting on $F(\Lambda_{n,n'})$ the BRST charge is defined by

$$Q_n = \frac{1}{n}\frac{e^{i\pi n/\gamma^2}-1}{e^{i\pi/\gamma^2}-1}\oint_{|z|=R} dz \int_{\mathscr{C}} \prod_{i=2}^{n} dz_i V(z)V(z_2)\cdots V(z_n). \tag{2.7}$$

Q_n maps $F(\Lambda_{n,n'})$ to $F(\Lambda_{-n,n'})$. The integration contours $\mathscr{C}$ form a set of non-intersecting curves going counterclockwise from z to $ze^{i2\pi}$ and are localized in the neighborhood of the circle of radius $|z|$ centered at the origin (as in [6]). When acting on $F(\Lambda_{n,n'})$ the BRST current $S(z)=\int_{\mathscr{C}}\prod_{i=1}^{n} dz_i V(z)V(z_2)\cdots V(z_n)$ is single valued around the origin. Thus the z-contour can be closed and the BRST charge commutes with the currents (2.5):

$$[J^a_m, Q_n]=0 \quad \text{on } F(\Lambda_{n,n'}). \tag{2.8}$$

At this point two cases have to be distinguished according to the rationality of $K+2$.

a) $(\mathbf{K}+\mathbf{2})\notin\mathbf{Q}$

i) Let us first consider $F(\Lambda_{-n,-n'})$, with $n,n'\in\mathbf{N}$. We first describe the structure of the Fock space as an $A_1^{(1)}$ module and then solve the BRST cohomology. The Kac–Kazdhan determinant (see Sect. 4) tells us that $F(\Lambda_{-n,-n'})$ possesses one and only one vector $w_{-n,-n'}$ which is either singular or cosingular. (The definitions of singular and cosingular vectors are given in Sect. 4.) The weight of $w_{-n,-n'}$ is $\Lambda_{n,-n'} \bmod \mathbf{C}\delta$. We claim that $w_{-n,-n'}$ is singular. More precisely we prove that:

$$w_{-n,-n'} = Q_n|J_{n,-n'}\rangle. \tag{2.9}$$

To prove Eq. (2.9) we only need to show that $Q_n|J_{n,-n'}\rangle$ does not vanish, because $|J_{n,-n'}\rangle$ is singular in $F(\Lambda_{n,-n'})$ and so is $Q_n|J_{n,-n'}\rangle$ in $F(\Lambda_{-n,-n'})$ if it is not zero. This is proved in Appendix A. The structure of $F(\Lambda_{-n,-n'})$ can be pictured as follows:

$$\begin{array}{ccc} & & |J_{-n,-n'}\rangle \\ & & \downarrow \\ |J_{n,-n'}\rangle & \xrightarrow{Q_n} & w_{-n,-n'}. \end{array} \tag{2.10}$$

Let us now introduce the following complex,

$$0^{(-2)} \to F(\Lambda_{n,-n'})^{(-1)} \xrightarrow{Q_n} F(\Lambda_{-n,-n'})^{(0)} \to 0^{(+1)}. \tag{2.11}$$

Obviously it is a complex. By convention, the numbers in parentheses are the degrees. Because the BRST operator commutes with the currents (2.5), its cohomology groups are $A_1^{(1)}$ modules. Hence to solve the BRST cohomology we only have to look at the singular or cosingular vectors. From what has been deduced previously it follows that a non-trivial cohomology group exists only in degree zero. It is isomorphic to the irreducible representation of $A_1^{(1)}$ with weight $\Lambda_{-n,-n'}$:

$$H^i \equiv \operatorname{Ker} Q^{(i)}/\operatorname{Im} Q^{(i-1)} = \begin{cases} 0 & \text{if } i\neq 0 \\ L(\Lambda_{-n,-n'}) & \text{if } i=0 \end{cases}. \tag{2.12}$$

Here and henceforth, $L(\Lambda)$ denotes the irreducible highest weight representation of $A_1^{(1)}$ with highest weight Λ.

ii) Consider now $F(\Lambda_{n,n'})$, $n\in\mathbf{N}$, $n'\in\mathbf{N}\cup 0$. As above, $F(\Lambda_{n,n'})$ possesses a vector $w_{n,n'}$ of weight $\Lambda_{-n,n'} \bmod \mathbf{C}\delta$ which is either singular or cosingular. This follows from the Kac–Kazdhan determinant. We prove by contradiction that $w_{n,n'}$ is cosingular. Indeed if $w_{n,n'}$ is not cosingular there exists u in the universal enveloping algebra, such that $w_{n,n'}=u|J_{n,n'}\rangle$. In Appendix A we show that $Q_n w_{n,n'}\neq 0$. Thus $Q_n w_{n,n'}$ is proportional to $|J_{-n,n'}\rangle$. Therefore, if $w_{n,n'}$ is not cosingular, $|J_{-n,n'}\rangle = uQ_n|J_{n,n'}\rangle=0$. Hence $w_{n,n'}$ is cosingular. The structure of $F(\Lambda_{n,n'})$ is represented as follows:

$$\begin{array}{ccc} |J_{n,n'}\rangle & & \\ \uparrow & & \\ w_{n,n'} & \xrightarrow{Q_n} & |J_{-n,n'}\rangle . \end{array} \tag{2.13}$$

In this case the complex is the following:

$$0^{(-1)} \to F(\Lambda_{n,n'})^{(0)} \xrightarrow{Q_n} F(\Lambda_{-n,n'})^{(+1)} \to 0^{(+2)}. \tag{2.14}$$

As above non-trivial cohomology classes exist only in degree zero. We have:

$$H^i \equiv \operatorname{Ker} Q^{(i)}/\operatorname{Im} Q^{(i-1)} = \begin{cases} 0 & \text{if } i\neq 0 \\ L(\Lambda_{n,n'}) & \text{if } i=0 \end{cases}. \tag{2.15}$$

b) $(\mathbf{K}+\mathbf{2})\in\mathbf{Q}$

In the rational case, the structure of the Fock modules is more involved. In this section we only report the result, the proofs are given in Sect. 4.

Let $(K+2)=p/p'$ with p and p' coprime positive integers. Suppose that n and n' are two integers satisfying $0<n\leqq p-1$ and $0\leqq n'\leqq p'-1$. Then we have the following proposition.

Proposition 2.1. *The following infinite sequence*

$$\cdots \xrightarrow{Q_n} F(\Lambda_{-n+2p,n'}) \xrightarrow{Q_{p-n}} F(\Lambda_{n,n'}) \xrightarrow{Q_n} F(\Lambda_{-n,n'}) \xrightarrow{Q_{p-n}} F(\Lambda_{n-2p,n'}) \xrightarrow{Q_n} \cdots$$

is a complex; i.e. $Q_nQ_{p-n}=Q_{p-n}Q_n=0$.

The proof of this proposition follows from the Bernstein–Gelfand resolutions of the Fock spaces.

Let us assign a degree to the Fock spaces involved in the above complex by setting that $\operatorname{degree}(F(\Lambda_{n-2kp,n'}))=2k$ and $\operatorname{degree}(F(\Lambda_{-n-2kp,n'}))=2k+1$. The following theorem gives the cohomology groups of the complex introduced in Prop. 2.1:

Theorem 2.2.

$$H^i \equiv \operatorname{Ker} Q^{(i)}/\operatorname{Im} Q^{(i-1)} = \begin{cases} 0 & \text{if } i\neq 0 \\ L(\Lambda_{n,n'}) & \text{if } i=0 \end{cases}.$$

Theorem 2.2 allows us to derive the character formula for the irreducible highest weight representation $L(\Lambda_{n,n'})$. Let us denote this character by $\mathrm{Ch}\,L(\Lambda_{n,n'})(\tau,z)$. It is given by the trace of a homomorphism $\mathcal{O}$ of the BRST complex over the cohomology group H^0. The homomorphism $\mathcal{O}$ is $\mathcal{O}=\exp(i2\pi\tau d)\exp(i2\pi z J_0^0)$, where in physical notation the derivation d is $d=L_0-c/24$. Because H^0 is the unique nontrivial cohomology group in the complex, this trace is equal to the alternated sum of the traces over the Fock spaces:

$$\mathrm{Ch}\,L(\Lambda_{n,n'})(\tau,z)=\sum_{l\in Z}\mathrm{Ch}\,F(\Lambda_{n-2lp,n'})(\tau,z)-\sum_{l\in Z}\mathrm{Ch}\,F(\Lambda_{-n-2lp,n'})(\tau,z). \tag{2.16}$$

The character of the Fock spaces and of the Verma modules are the same

$$\mathrm{Ch}\,F(\Lambda_{m,m'})(\tau,z)=\frac{\exp(i2\pi\tau(mp'-m'p)^2/4pp')\exp(i\pi z(mp'-m'p)/p')}{\Pi(\tau,z)} \tag{2.17}$$

with, $q=\exp(i2\pi\tau)$, $\mathrm{Im}\,\tau>0$,

$$\Pi(\tau,z)=2i\sin(\pi z)q^{3/24}\prod_{n=1}^{\infty}(1-q^ne^{i2\pi z})(1-q^ne^{-i2\pi z})(1-q^n).$$

Gathering formulas (2.16) and (2.17) we obtain [11]:

$$\mathrm{Ch}\,L(\Lambda_{n,n'})(\tau,z)=\frac{\Theta^{pp'}_{np'-n'p}(\tau,z/p')-\Theta^{pp'}_{-np'-n'p}(\tau,z/p')}{\Pi(\tau,z)} \tag{2.18}$$

with

$$\Theta^a_b(\tau,z)=\sum_{l\in Z+b/2a}\exp(i2\pi a(\tau l^2+zl)).$$

In the irrational cases the characters can be easily deduced from Eqs. (2.12) and (2.15).

3. Conformal Blocks of the WZW Models on the Sphere and on the Torus

In the section we use the free field representation of the previous section to deduce explicit expression for the conformal blocks on the sphere and on the torus. Unless otherwise specified we suppose in what follows that $(K+2)\in\mathbf{Q}$, $2\gamma^2=K+2=p/p'$.

To write down expressions for the conformal blocks we have to introduce the chiral primary fields. A chiral primary field $\Phi^{k,k'}_{n,n'\;l,l'}(z)$ maps a highest weight representation with highest weight $\Lambda_{l,l'}$ into a highest weight representation with highest weight $\Lambda_{k,k'}$:

$$\Phi^{k,k'}_{n,n'\;l,l'}(z):L(\Lambda_{l,l'})\to L(\Lambda_{k,k'}). \tag{3.1}$$

It takes values in the highest weight $sl(2)$-representation $\rho_{J_{n,n'}}$ of spin $J_{n,n'}$ and it behaves like an affine primary field of spin $J_{n,n'}$; i.e. it satisfies the following commutation relations:

$$[J^a_m,\Phi^{k,k'}_{n,n'\;l,l'}(z)]=z^m\rho_{J_{n,n'}}(t^a)\Phi^{k,k'}_{n,n'\;l,l'}(z),$$

$$[d,\Phi^{k,k'}_{n,n'\;l,l'}(z)]=\left(z\frac{d}{dz}+\frac{J_{n,n'}(J_{n,n'}+1)}{K+2}\right)\Phi^{k,k'}_{n,n'\;l,l'}(z). \tag{3.2}$$

By $sl(2)$ invariance this field exists only if $J_{n,n'}+J_{l,l'}-J_{k,k'}\in\mathbf{N}\cup 0$.

The chiral primary operators can be explicitly constructed by using the screening charge (2.6) and vertex operators denoted by ${}_{\mu}V_J(z)$ for $(J-\mu)\in\mathbf{N}\cup 0$ and defined by:

$$ {}_{\mu}V_J(z) = :\omega^{J-\mu}(z)::\exp\left(i\frac{J}{\gamma}\Phi(z)\right):. \tag{3.3}$$

Then the μ-spin component of $\Phi^{k,k'}_{n,n'\,l,l'}(z)$, which we denote by ${}_{\mu}\Phi^{k,k'}_{n,n'\,l,l'}(z)$, is (up to a normalization constant):

$$ {}_{\mu}\Phi^{k,k'}_{n,n'\,l,l'}(z) = \int_{\mathscr{C}} \prod_{i=2}^{r} dz_i\, {}_{\mu}V_{J_{n,n'}}(z)V(z_2)\cdots V(z_r). \tag{3.4}$$

Here $r = J_{n,n'} + J_{l,l'} - J_{k,k'}$.

The vertex operator (3.4) is defined on the Fock module $F(\Lambda_{l,l'})$. To be a chiral primary field it must project to an operator acting on the irreducible highest weight module $L(\Lambda_{l,l'})$; i.e. on the non-trivial cohomology classes of the complex introduced in Sect. 2. In other words, it must be BRST invariant. Following the method used by one of the authors in the case of the Virasoro algebra [6], one can check that the chiral vertex operator (3.4) commutes (up to a phase) with the BRST charge. Namely, on $F(\Lambda_{l,l'})$ we have:

$$ Q_k\Phi^{k,k'}_{n,n'\,l,l'}(z) = \exp(-i\pi k J_{n,n'}\gamma^{-2})\Phi^{-k,k'}_{n,n'\ -l,l'}(z)Q_l. \tag{3.5}$$

This property can alternatively be illustrated by the following (up to a phase) commutative diagram.

$$\begin{array}{ccc} F(\Lambda_{l,l'}) & \xrightarrow{Q_k} & F(\Lambda_{-l,l'}) \\ \Phi^{k,k'}_{n,n'\,l,l'}(z)\downarrow & & \downarrow \Phi^{-k,k'}_{n,n'\ -l,l'}(z) \\ F(\Lambda_{k,k'}) & \xrightarrow{Q_k} & F(\Lambda_{-k,k'}). \end{array}$$

By definition, a non-vanishing chiral primary field exists only and only if the fusion rules are fulfilled. The chiral primary field does not vanish identically if and only if it can be normalized. By $sl(2)$ invariance all the fields ${}_{\mu}\Phi^{k,k'}_{n,n'\,l,l'}(z)$ are normalizable if the field ${}_{J_{n,n'}}\Phi^{k,k'}_{n,n'\,l,l'}(z)$ is. The normalization constant is:

$$ \mathcal{N}^{k,k'}_{n,n'\,l,l'}(m_k;m_l) = \langle J_{k,k'};m_k|\,{}_{J_{n,n'}}\Phi^{k,k'}_{n,n'\,l,l'}(1)|J_{l,l'};m_l\rangle, \tag{3.6}$$

where $|J;m\rangle$ are states with spin m in the $sl(2)$ representation of spin J. This constant can be explicitly calculated:

$$\begin{aligned} \mathcal{N}^{k,k'}_{n,n'\,l,l'}(m_k;m_l) &= \begin{bmatrix} J_{n,n'} & J_{l,l'} & J_{k,k'} \\ J_{n,n'} & m_l & m_k \end{bmatrix} \times \exp(i\pi\gamma^{-2}r(r-2J_{l,l'}-1)) \\ &\cdot \prod_{s=1}^{r} \frac{(1-\exp(-i\pi\gamma^{-2}(s-J_{n,n'}-1)))(1-e^{-i\pi\gamma^{-2}s})}{(1-e^{-i\pi\gamma^{-2}})} \prod_{s=1}^{r} \frac{\Gamma(s/2\gamma^2)}{\Gamma(1/2\gamma^2)} \\ &\cdot \prod_{s=0}^{r-1} \frac{\Gamma(1-(s-2J_{n,n'})/2\gamma^2)\Gamma((s-2J_{l,l'})/2\gamma^2)}{\Gamma(1+(s-1-J_{n,n'}-J_{l,l'}-J_{k,k'})/2\gamma^2)}. \end{aligned} \tag{3.7}$$

Here the first bracket denotes a Clebsh–Gordan coefficient and $r = J_{n,n'} + J_{l,l'} - J_{k,k'}$. From this explicit expression we deduce the fusion rules among the BRST invariant operators:

*) if $0 \leqq n' + l' \leqq p' - 1$:

$$\begin{cases} |n-l|+1 \leqq k \leqq \min(2p-n-l-1; n+l-1) \\ k' = n' + l' \end{cases}. \tag{3.8}$$

*) if $n' + l' \geqq p'$: There is no physical chiral vertex operator among the highest weight representations.

At this point it is better to distinguish between the unitary case ($K \in \mathbf{N}, p' = 1$) and the non-unitary one ($K \in \mathbf{Q} - \mathbf{N}, p' \neq 1$).

In the unitary case, because all the spin J are half-integers, all the chiral primary fields are described by the physical vertex operators (3.4) and (3.8).

On the plane the conformal block functions are expectation values of the chiral primary fields. For $|z_1| > |z_2| > \cdots > |z_N|$,

$$F^{[k_j]}_{[n_j]}(z_1, \ldots, z_N) = \text{const.} \left\langle \prod_{j=1}^{N} \Phi^{k_j}_{n_j\, k_{j+1}}(z_j) \right\rangle \tag{3.9}$$

with $k_1 = k_{n+1} = 1$. Because they all vanish in the unitary case, all the primed indices have been erased. The expectation values (3.9) can be explicitly evaluated by using the well known formula for the expectation values of vertex operators ${}_\mu V_J(z)$. They yield integral representations of the conformal block functions. The integration contours $\mathscr{C}$ can be deformed in a way that $F^{[k_j]}_{[n_j]}$ becomes a linear combination of integrals of the Dotsenko–Fateev type, with integration contours going from a point z_i to a different point z_j. In our approach, the relation between the integration contours and the intermediate states $[k_j]$ becomes transport. This completes the free field representation of the correlation functions on the sphere in the unitary case.

On the torus we have to worry about the non-physical states which should not propagate along the cycles. On the torus the conformal block functions are traces of product of primary fields:

$$F^{T[k_j]}_{n[n_j]}(z_1, \ldots, z_N | \tau, \nu) = \text{const.}\, \mathrm{Tr}_{L(\Lambda_n)} \left(q^{L_0 - c/24} e^{i2\pi\nu J^0_0} \prod_{j=1}^{N} \Phi^{k_j}_{n_j k_{j+1}}(z_j) \right) \tag{3.10}$$

with $k_1 = k_{N+1} = n$ and $|q| < |z_N| < \cdots < |z_1| < 1, q = \exp(i2\pi\tau)$.

The trick used to insure the non-propagation of the unphysical states consists in expressing the trace over the irreducible representation $L(\Lambda_n)$ as an alternated sum of traces over the Fock spaces $F(\Lambda_{\pm n - 2lp})$. To formulate it we have to extend the action of the product of the chiral primary fields involved in Eq. (3.10) to a homomorphism acting on the BRST complex introduced in Prop. 2.1. We introduce two family of operators, Ξ_{2l} and Ξ_{2l+1}, acting on the Fock spaces of even and odd degree, $F(\Lambda_{n-2lp})$ and $F(\Lambda_{-n-2lp})$ respectively:

$$\Xi_{2l} = q^{L_0 - c/24} e^{i2\pi\nu J^0_0} \prod_{j=1}^{N} \Phi^{k_j - 2lp}_{n_j}{}_{k_{j+1} - 2lp}(z_j), \tag{3.11a}$$

$$\Xi_{2l+1} = q^{L_0 - c/24} e^{i2\pi\nu J^0_0} e^{-i2\pi \sum_{j=1}^{N} J_{n_j} k_j / p} \prod_{j=1}^{N} \Phi^{-k_j - 2lp}_{n_j}{}_{-k_{j+1} - 2lp}(z_j). \tag{3.11b}$$

We introduced a phase in the definition (3.11) in order to get rid of the phase present in Eq. (3.5). With all these definitions we can write the following commutative diagram:

$$\begin{array}{ccccccc} \xrightarrow{Q_n} F(\Lambda_{-n+2p}) & \xrightarrow{Q_{p-n}} & F(\Lambda_n) & \xrightarrow{Q_n} & F(\Lambda_{-n}) & \xrightarrow{Q_{p-n}} & \\ \cdots \quad \Xi_{-1}\downarrow & & \Xi_0\downarrow & & \Xi_1\downarrow & \cdots & \\ \xrightarrow{Q_n} F(\Lambda_{-n+2p}) & \xrightarrow{Q_{p-n}} & F(\Lambda_n) & \xrightarrow{Q_n} & F(\Lambda_{-n}) & \xrightarrow{Q_{p-n}} & . \end{array} \tag{3.12}$$

The free field representation of the conformal block functions on the torus are then given by:

$$F^{T\,[k_j]}_{n\,[n_j]} = \sum_{l\in Z} \mathrm{Tr}_{F(\Lambda_{n-2lp})}(\Xi_{2l}) - \sum_{l\in Z} \mathrm{Tr}_{F(\Lambda_{-n-2lp})}(\Xi_{2l+1}). \tag{3.13}$$

Equation (3.13) gives the integral representation of the conformal blocks on the torus. The integrands are traces of vertex operators. We give the expressions of the traces that are needed to compute the conformal blocks (3.13):

$$\begin{aligned} &\mathrm{Tr}_{F(\Lambda_{m,m'})}\left(\prod_{i=1}^{N}\omega(z_i)\prod_{j=1}^{N}\omega^\dagger(w_j)\prod_{n=1}^{M}:\exp\left(i\frac{\alpha_n}{\gamma}\Phi(z_n)\right):q^{L_0-c/24}e^{i2\pi v J_0^0}\right)\\ &=\frac{q^{(mp'-m'p)^2/4pp'}e^{i\pi v(mp'-m'p)/p'}}{\Pi(\tau,v)}\prod_{n=1}^{M}e^{i2\pi\xi_n(\alpha_n^2/2+\alpha_n J_{m,m'})/\gamma^2}\\ &\cdot\prod_{n<m}^{M}\left(\frac{\Theta_1(\xi_n-\xi_m|\tau)}{\Theta_1'(0|\tau)}\right)^{\alpha_n\alpha_m/\gamma^2}\times\sum_{\sigma\ \mathrm{perm.}}\prod_{i=1}^{N}G(z_i,w_{\sigma(i)}|\tau,v), \end{aligned} \tag{3.14}$$

where $z=\exp(i2\pi\xi)$, $\Pi(\tau,v)$ is defined in Eq. (2.17) and

$$G(z,w|\tau,v)=\sum_{n\in Z}\left(\frac{1}{1-q^n e^{i2\pi v}}\right)\left(\frac{w^{n-1}}{z^n}\right). \tag{3.15}$$

In the rational case the situation is more subtle. Some of the spins $J_{n,n'}$ are no more half integers. Therefore not all the highest weight primary fields are self conjugated. The conjugated field of a primary field associated to a highest weight representation of $sl(2)$ is a lowest weight primary field. In other words tensor products of two highest weight $sl(2)$-representations of spins $J_{n,n'}$ with $n'\neq 0$ never reduce to the scalar representation of $sl(2)$. Thus as a consequence of the $sl(2)$ invariance it is not possible to write non-vanishing two-point functions involving only the fields $\Phi^{k,k'}_{n,n'\,l,l'}(z)$. Therefore constructing the non-unitary WZW models requires considering primary fields associated to both highest and lowest weight representations. This can also be seen in two different ways in the fusion algebra: i) as stated in Eq. (3.8), if $n'+l'\geq p'$ there is no physical vertex operator among the highest weight primary fields; or ii) the fusion rules deduced from the S-matrix of the characters (2.18) involve minus signs in them; these signs have actually to be reinterpreted as coming from the lowest weight representations.

Lowest weight primary fields can be constructed from a free field representation which is deduced from Eq. (2.5) by exchanging the positive and the negative roots of $sl(2)$. But then the relations between the free fields involved in the two representations are non-polynomial. The correlation functions mixing the highest and lowest weight primary fields are no longer computable in terms of free field expectation values. Nevertheless the operators defined in Eqs. (3.4) and (3.8) form a closed operator algebra and the correlation functions of these operators can be evaluated as in the unitary case.

4. Structure of Fock Modules and BRST Cohomology

In this section we derive the structure of submodules of Fock modules. The main tools are the Kac–Kazhdan formula [9] for the determinant of the Shapovalov form, and the Jantzen filtration [27] in the form described in [9]. The construction of the filtration of Fock and Verma modules closely parallels the construction done by Feigin and Fuchs [28] in the case of the Virasoro algebra, although the definition of the Jantzen filtration differs slightly from theirs. At the end of this section we use the results to compute the BRST cohomology.

Let $\mathscr{G} = sl(2, \mathbf{C}) \otimes \mathbf{C}[t, t^{-1}] \oplus \mathbf{C}K \oplus \mathbf{C}d$ be the Lie algebra $A_1^{(1)}$ [28]. Its root lattice is generated by the simple roots α_0, α_1, and the set of roots is $\Delta = \{n_0\alpha_0 + n_1\alpha_1, |n_0 - n_1| \leqq 1\}$. The Lie algebra $\mathscr{G}$ can be described by generators J_n^a, K, d with commutation relations (1.2). $\mathscr{G}$ is graded by the root lattice Γ:

$$\deg(J_n^a) = n\alpha_0 + (n+a)\alpha_1, \quad \deg(d) = \deg(K) = 0, \tag{4.1}$$

and it is convenient to introduce also a Z-grading (principal gradation), called depth:

$$\operatorname{depth}(J_n^a) = 2n + a, \quad \operatorname{depth}(d) = \operatorname{depth}(K) = 0. \tag{4.2}$$

The Cartan decomposition $\mathscr{G} = \mathscr{G}_- \oplus \mathscr{H} \oplus \mathscr{G}_-$ is then a decomposition into elements of positive, zero and negative depth. Let $U(\mathscr{G})$ be the (Γ and Z-graded) universal enveloping algebra of $\mathscr{G}$. The Verma module $V(\Lambda)$ of highest weight $\Lambda \in \mathscr{H}^*$ is generated by a highest weight vector $v_0(\Lambda)$ annihilated by $\mathscr{G}_+$, with the property $xv_0(\Lambda) = \Lambda(x)v_0(\Lambda)$, $x \in \mathscr{H}$. The Verma module is graded by the semigroup Γ_+ generated by the positive roots:

$$V(\Lambda) = \bigoplus_{\eta \in \Gamma_+} V(\Lambda)_\eta. \tag{4.3}$$

Vectors in $V(\Lambda)_\eta$ are called weight vectors (of degree η). Their weights differ from the highest weight by η. They are of the form $uv_0(\Lambda)$, $u \in U(\mathscr{G}_-)$, $\deg(u) = -\eta$. The Z-grading is defined by $\operatorname{depth}(uv_0) = -\operatorname{depth}(u)$.

The dimension of $V(\Lambda)_\eta$ is $P(\eta)$, the number of ways η can be written as a linear combination of positive roots with non-negative integer coefficients. The dual $V(\Lambda)^*$ is, as a graded vector space, the dual of $V(\Lambda)$:

$$V(\Lambda) = \bigoplus_\eta \operatorname{Hom}_{\mathbf{C}}(V(\Lambda)_\eta, \mathbf{C}), \tag{4.4}$$

and its structure of $U(\mathscr{G})$-module is given by

$$\langle \omega, u\xi \rangle = \langle \sigma(u)\omega, \xi \rangle, \tag{4.5}$$

for all $\omega \in V(\Lambda)^*$, $\xi \in V(\Lambda)$, and $u \in U(\mathscr{G})$, where σ is the involutive antiautomorphism

$$\sigma(J_n^\pm) = J_{-n}^\mp, \quad \sigma(J_n^0) = J_{-n}^0, \quad \sigma(K) = K. \tag{4.6}$$

The module $V(\Lambda)^*$ is an $U(\mathscr{G})$-module with highest weight vector $v_0(\Lambda)^*$, of weight Λ, normalized as $\langle v_0(\Lambda)^*, v_0(\Lambda)\rangle = 1$.

A *singular vector* in a graded $U(\mathscr{G})$-module M is a non-zero weight vector annihilated by $\mathscr{G}_+$. A *cosingular vector* $\xi \in M$ is a weight vector that cannot be written as $u\xi'$, $u \in U(\mathscr{G}_-)$, $\xi' \in M$. Two cosingular vectors are equivalent if they differ by an element of $U(\mathscr{G}_-)M$. Highest weight vectors are both singular and cosingular. There is a duality between singular vectors and equivalence classes of cosingular vectors: Let M^* be the dual of M, defined as above. Then

$$\operatorname{Ker} U(\mathscr{G}_+)^* \cong M^*/U(\mathscr{G}_-)M^*. \tag{4.7}$$

To prove this, it is sufficient to show that the duality pairing $M^* \times M \to \mathbf{C}$ projects to a non-degenerate pairing (in every degree)

$$M^*/U(\mathscr{G}_-)M^* \times \operatorname{Ker} U(\mathscr{G}_+) \to \mathbf{C}, \tag{4.8}$$

i.e. that $\operatorname{Ker} U(\mathscr{G}_+) = (U(\mathscr{G}_-)M^*)^\perp$. Let $\xi \in (U(\mathscr{G}_-)M^*)^\perp$. For all $\omega \in M^*$ and all $u \in U(\mathscr{G}_-)$,

$$0 = \langle u\omega, \xi\rangle = \langle \omega, \sigma(u)\xi\rangle. \tag{4.9}$$

Since σ maps $U(\mathscr{G}_-)$ onto $U(g_+)$ this condition is equivalent to $u'\xi = 0$ for all $u' \in U(\mathscr{G}_+)$, i.e. ξ is singular.

The Verma module has no cosingular vectors of positive degree. Thus $V(\Lambda)^*$ does not contain any singular vector of positive degree. The kernel of the canonical degree zero homomorphism of graded $U(\mathscr{G})$-modules

$$S(\Lambda): V(\Lambda) \to V(\Lambda)^*, \tag{4.10}$$

uniquely defined by $S(\Lambda)v_0(\Lambda) = v_0(\Lambda)^*$, is the submodule generated by the singular vectors of positive degree in $V(\Lambda)$. Upon choosing a basis of $U(\mathscr{G}_-)$, consisting of elements of well-defined degree, the map $S(\Lambda)_\eta: V(\Lambda)_\eta \to V(\Lambda)^*_\eta$ can be viewed as a $P(\eta) \times P(\eta)$ matrix, whose entries are polynomials in Λ. The determinant of $S(\Lambda)_\eta$ is given in general by the Kac–Kazhdan formula

$$\begin{gathered} \det S(\Lambda)_\eta = \text{const} \prod_{\alpha>0} \prod_{n=1}^{\infty} \Phi_{\alpha,n}(\Lambda)^{P(\eta - n\alpha)}, \\ \Phi_{\alpha,n}(\Lambda) = (\Lambda + \rho, \alpha) - \frac{n}{2}(\alpha, \alpha), \end{gathered} \tag{4.11}$$

where (,) is an invariant non-degenerate bilinear form on $\mathscr{H}^*$, ρ is the sum of the fundamental weights Λ_i, and it is understood that $P(0) = 1$ and $P(\eta) = 0$ if $\eta \notin \Gamma_+$. In the case of $A_1^{(1)}$, we parametrize weights Λ by the (iso-)spin J and the level K as

$$\Lambda = (K - 2J)\Lambda_0 + 2J\Lambda_1. \tag{4.12}$$

Weights are considered modulo $\mathbf{C}\delta$, with $\delta = \alpha_0 + \alpha_1$. We have

$$\Phi_{\alpha,n}(\Lambda) = \begin{cases} (K+2)n' - n - 2J - 1, & \text{if} \quad \alpha = n'\alpha_0 + (n'-1)\alpha_1, \quad n' \geqq 1, \\ (K+2)n', & \text{if} \quad \alpha = n'\delta, \quad n' \geqq 1, \\ (K+2)n' - n + 2J + 1, & \text{if} \quad \alpha = n'\alpha_0 + (n'+1)\alpha_1, \quad n' \geqq 0. \end{cases} \tag{4.13}$$

The Kac–Kazhdan formula implies that Verma modules are irreducible outside a set of lines $\Phi_{\alpha,n}=0$. On a generic point of such a line, the Verma module contains one singular vector of degree $n\alpha$, which generates an irreducible submodule. The structure of Fock modules was described in this case in Sect. 2. Here we are interested in the "completely degenerate" case, where Λ in the intersection of infinitely many lines $\Phi_{\alpha,n}=0$. This corresponds to the case of rational level. Specifically, we are interested in the case where $\Lambda=\Lambda_{n,n'}$ where in the parametrization (4.12),

$$\begin{aligned} &K+2=p/p', \quad p,p'\geqq 1, \quad \gcd(p,p')=1,\\ &2J+1=n-n'(K+2), \qquad n,n'\in Z, \quad 0\leqq n'\leqq p'-1, \quad n\neq 0 \bmod p. \end{aligned} \tag{4.14}$$

If $1\leqq n\leqq p-1$, the characters of the irreducible representations form a representation of the modular group, and are invariant under a finite index subgroup of $PSL(2,\mathbf{Z})$ [11]. The structure of the Verma module corresponding to the weight (4.12), (4.14) is described by the following diagram [29]:

$$\begin{array}{ccccccccccc} \cdots & \leftarrow & v_4 & \leftarrow & v_3 & \leftarrow & v_2 & \leftarrow & v_1 & \leftarrow & v_0 \\ & \nwarrow\!\!\!\!\swarrow & & \nwarrow\!\!\!\!\swarrow & & \nwarrow\!\!\!\!\swarrow & & \nwarrow\!\!\!\!\swarrow & & \swarrow & \\ \cdots & \leftarrow & v_{-4} & \leftarrow & v_{-3} & \leftarrow & v_{-2} & \leftarrow & v_{-1} & & \end{array} \tag{4.15}$$

and the v_i are the only singular vectors (up to proportionality) in the Verma module. Here and henceforth, an arrow, or a chain of arrows, goes from one vector to another if and only if the second vector is in the $U(\mathscr{G})$-submodule generated by the first one. The weights of the singular vectors v_i can be computed from the Kac–Kazhdan determinant formula. If $\Lambda=\Lambda_{nn'}$ with $1\leqq n\leqq p-1$, $0\leqq n'\leqq p'-1$, v_{2l} has weight $\Lambda_{n+2lp,n'}$ and v_{2l-1} has weight $\Lambda_{-n+2lp,n'}$. More generally, if $\Lambda=\Lambda_{\pm n-2lp,n'}$ and n,n' are as above, the Verma module is embedded in $V(\Lambda_{nn'})$ and we have the relations

$$\begin{aligned} \text{weight}\,(v_{\pm j}\in V(\Lambda_{n-2lp,n'}))&=\text{weight}\,(v_{\pm(j+|2l|)}\in V(\Lambda_{n,n'})),\\ \text{weight}\,(v_{\pm j}\in V(\Lambda_{-n-2lp,n'}))&=\text{weight}\,(v_{\mp(j+|2l+1|)}\in V(\Lambda_{n,n'})). \end{aligned} \tag{4.16}$$

The fact that the module generated by any of the vectors v_i contains infinitely many singular vectors is an immediate consequence of the Kac–Kazhdan determinant formula (4.11). What is non-trivial is that there is at most one singular vector (up to proportionality) in every degree. Comparing this diagram with [10] we note a striking similarity with the degenerate modules over the Virasoro algebra.

The structure of the dual $V(\Lambda)^*$ is given by "reversing the arrows." Let v_i^* be (cosingular) vectors in $V(\Lambda)^*$ such that $\deg(v_i^*)=\deg(v_i)$ and $\langle v_i^*,v_i\rangle=1$; v_i can be chosen so that $\langle v_i^*,U(\mathscr{G}_-)v_{-i}\rangle=0$, $|i|\geqq 1$.

Proposition 4.1. *Let $\Lambda=\Lambda_{nn'}$ be as in (4.12), (4.14).*
(i) *All vectors in $V(\Lambda)^*$ of depth $\leqq N$ are in the submodule generated by the vectors v_i^* with* depth$(v_i^*)\leqq N$.
(ii) *The submodule structure of $V(\Lambda)^*$ is given by the diagram*

$$\begin{array}{ccccccccccc} \cdots & \rightarrow & v_4^* & \rightarrow & v_3^* & \rightarrow & v_2^* & \rightarrow & v_1^* & \rightarrow & v_0^* \\ & \searrow\!\!\!\!\nearrow & & \searrow\!\!\!\!\nearrow & & \searrow\!\!\!\!\nearrow & & \searrow\!\!\!\!\nearrow & & \nearrow & \\ \cdots & \rightarrow & v_{-4}^* & \rightarrow & v_{-3}^* & \rightarrow & v_{-2}^* & \rightarrow & v_{-1}^* & & \end{array} \tag{4.17}$$

Proof. (i) For depth $=0$ the statement is trivially true. Suppose that (i) is proved up to depth N and let U_N be the submodule generated by $\{v_i^* \mid \mathrm{depth}(v_i^*) \leqq N\}$. Now, either $V(\Lambda)^*/U_N$ has no vectors of depth $N+1$, and the induction claim is proved, or there exists a weight vector ω in $V(\Lambda)^*$ of depth $N+1$ with non-zero projection in $V(\Lambda)^*/U_N$. Such a vector is necessarily cosingular (by induction hypothesis), and is thus proportional to v_i^* modulo $U(\mathscr{G})V(\Lambda)^*$.

(ii) Let $u \in U(\mathscr{G}_-)$ such that $uv_0 = v_i$. We have

$$1 = \langle v_i^*, v_i \rangle = \langle v_i^*, uv_0 \rangle = \langle \sigma(u)v_i^*, v_0 \rangle. \tag{4.18}$$

Thus $v_0^* = \sigma(u)v_i^*$. Similarly, let $v_i = uv_j, u \in U(\mathscr{G}_-)$. Then by the same argument, $v_j^* = \sigma(u)v_i^* + \omega$, where $\omega \in U(\mathscr{G}_-)V(\Lambda)^*$. By induction on the depth of v_j, we see, using (i), that ω is in the submodule generated by v_i^*, and the claim is proved.

Now other arrow can be inserted in the diagram, since $v_i^* = uv_j^*$ implies that $v_j = \sigma(u)v_i$. □

Next, we introduce the Jantzen filtration [27] following [9]. We first define it for the Verma module.

Let $z = z_0\Lambda_0 + z_1\Lambda_1$ with $z_1, z_2 \neq 0$, and consider the one-parameter family of weights $\Lambda(t) = \Lambda + tz, t \in \mathbf{C}$, where Λ is a weight corresponding to a degenerate representation. For $\varepsilon > |t| > 0, V(\Lambda(t))$ is irreducible. We study the behavior of

$$S(t) \equiv S(\Lambda(t)) : V(\Lambda(t)) \to V(\Lambda(t))^*, \tag{4.19}$$

in the vicinity of $t=0$. A choice of basis in $U(\mathscr{G}_-)$, consisting of elements of definite degree, induces a basis in $V(\Lambda(t))$ and $V(\Lambda(t))^*$. In this basis, the matrix elements of $S(t)$ and of any element of $\mathscr{G}$, are polynomials in t. Let $\hat{V}(\hat{V}^*)$ be the space of one-parameter families $(v(t))_{t\in\mathbf{C}}$ such that $v(t) \in V(\Lambda(t))$ $(V(\Lambda(t))^*)$ and whose components in the given basis depend polynomially on t. The spaces $\hat{V}, \hat{V}^*$ are graded $U(\mathscr{G})$-modules and $S(t)$ induces a homomorphism $\hat{S}: \hat{V} \to \hat{V}^*$. The map $\pi: \hat{V} \to V(\Lambda)$, $(v(t))_{t\in\mathbf{C}} \mapsto v(0)$, is a surjective homomorphism. Similarly we have a surjective homomorphism $\pi: \hat{V}^* \to V(\Lambda)^*$.

Setting

$$\hat{M}_k = \{v \in \hat{V} \mid S(t)v(t) \text{ is divisible by } t^k\}, \tag{4.20}$$

we define a filtration of $\hat{V}$ by $U(\mathscr{G})$-modules:

$$\hat{V} = \hat{M}_0 \supset \hat{M}_1 \supset \hat{M}_2 \supset \cdots. \tag{4.21}$$

The modules $M_k = \pi(\hat{M}_k)$ give a filtration of $V(\Lambda)$,

$$V(\Lambda) = M_0 \supset M_1 \supset M_2 \supset \cdots. \tag{4.22}$$

The submodule generated by the singular vectors in $V(\Lambda)$ is M_1. Thus V/M_1 is irreducible. The determinants of $S(t)$ give information on the dimensions of the modules M_i:

Proposition 4.2. *Let* $d_k = \dim(M_k \cap V(\Lambda)_\eta)$. *Then*

$$\det S(t)_\eta = ct^{\sum_1^\infty d_k}(1 + O(t)) = ct^{\sum_1^\infty k(d_k - d_{k-1})}(1 + O(t)), \tag{4.23}$$

where c is a non-zero constant. In particular, $M_k \cap V(\Lambda)_\eta = 0$ *for all sufficiently large k.*

Proof. We can choose a basis (ξ_j) of $V(\Lambda)_\eta$ such that $\xi_1, \ldots, \xi_{d_k}$ is a basis of $M_k \cap V(\Lambda)_\eta$. Let $\hat{\xi}_j = (\xi_j(t))_{t\in\mathbf{C}} \in \hat{V}_\eta$ such that $\xi_j(0) = \xi_j$, and $\hat{\xi}_1, \ldots, \hat{\xi}_{d_k} \in \hat{M}_k$. Then the vectors $\omega_j(t) = t^{-k} S(t)\xi_j(t)$ for $d_{k+1} < j \leqq d_k$ are well defined in $\hat{V}^*$ and are a basis of $\hat{V}(\Lambda)^*$ for small $|t|$. To prove this, it is sufficient to show that ω_j are linear independent for $t = 0$: suppose that $\sum_j \lambda_j \omega_j = 0$, $t \in \mathbf{C}$. Let i be the largest integer such that $\lambda_i \neq 0$, and suppose that $d_{k+1} < i \leqq d_k$. Multiplying by t^k the equation $\sum_j \lambda_j \omega_j(t) = O(t)$ and inserting the definition of $\omega_j(t)$, we get

$$S(t) \sum_{d_{k+1} < j \leqq d_k} \lambda_j \xi_j(t) = O(t^{k+1}). \tag{4.24}$$

Thus $\sum_{d_{k+1} < j \leqq d_k} \lambda_j \xi_j \in M_{k+1}$, a contradiction, unless all λ_j vanish. The order of the zero in the determinant does not depend on the (t-dependent) choice of bases in $V(\Lambda)_\eta$, $V(\Lambda)^*_\eta$. In the bases $(\xi_j(t)), (\omega_j(t)), S(t)_\eta$ is diagonal with j^{th} entry t^k, where k is defined by $d_{k-1} < j \leqq d_k$. □

Let $\hat{S}_k = t^{-k}\hat{S}$ defined on $\hat{M}_k$, and let $\hat{N}_k = \hat{V}^*/\hat{S}_{k-1}\hat{M}_{k-1}$. Since $\hat{S}_{k-1}\hat{M}_{k-1} \subset \hat{S}_k \hat{M}_k$, there is a sequence of surjective homomorphisms

$$\hat{V}^* = \hat{N}_0 \to \hat{N}_1 \to \hat{N}_2 \to \cdots, \tag{4.25}$$

and, applying π, a sequence of surjective homomorphisms

$$V(\Lambda)^* = N_0 \to N_1 \to N_2 \to \cdots, \tag{4.26}$$

where $N_k = V(\Lambda)^*/\pi(\hat{S}_{k-1}\hat{M}_{k-1})$. The point is that $\operatorname{Ker}(\pi : \hat{M}_k \to M_k) = t\hat{M}_{k-1}$ is mapped to $\hat{S}_{k-1}\hat{M}_{k-1}$ under $\hat{S}_k$. Thus $\hat{S}_k$ projects to a homomorphism

$$S_k : M_k \to N_k. \tag{4.27}$$

Lemma 4.3.

(i) $\operatorname{Ker} S_k = M_{k+1}, \operatorname{Coker} S_k = N_{k+1}$.
(ii) $\dim (M_k)_\eta = \dim (N_k)_\eta$.

Proof. (i) Let $x \in M_{k+1}$ and $\hat{x} \in \hat{M}_{k+1}$ such that $\pi\hat{x} = x$. Then $\hat{S}_k \hat{x} = O(t)$ and projects to zero. Thus $x \in \operatorname{Ker} S_k$. Conversely, let $x \in \operatorname{Ker} S_k$; there exists an $\hat{x} \in \hat{M}_k$, such that $\pi\hat{x} = x$ and $\hat{S}_k \hat{x} = t\hat{y} + \hat{S}_{k-1}\hat{z}$, for some $\hat{y} \in \hat{V}^*$ and $\hat{z} \in \hat{M}_{k-1}$. We have $\hat{S}(\hat{x} - t\hat{z}) = t^{k+1}\hat{y}$, and $x \in M_{k+1}$, since $\pi(\hat{x} - tz) = x$. The first claim in (i) is proved.

Let $\operatorname{Pr}_k : N_k \to N_{k+1}$ be the canonical projection. We have to show that $\operatorname{Ker}(\operatorname{Pr}_k) = \operatorname{Im} S_k$. Let x be a representative of a class in $\operatorname{Ker}(\operatorname{Pr}_k)$. Thus $x = \pi\hat{x}$, $\hat{x} = \hat{S}_k \hat{y}$ for some $\hat{y} \in \hat{M}_k$. Hence $x \in \operatorname{Im} S_k$. Conversely, let x be a representative of a class in $\operatorname{Im}(S_k : M_k \to N_k)$. Then $x = \pi\hat{x}$ with

$$\hat{x} = \hat{S}_k \hat{y} + \hat{S}_{k-1}\hat{z} = \hat{S}_k(\hat{y} + t\hat{z}), \tag{4.28}$$

and the class of x is in $\operatorname{Ker}(\operatorname{Pr}_k)$. Part (ii) follows recursively from (i), since $\dim (M_0)_\eta = \dim (N_0)_\eta$. □

The Jantzen filtration of the Verma module $V(\Lambda)$ in the case we are considering is given by the following result.

Proposition 4.4. *Let $\Lambda = \Lambda_{n,n'}$ be given by (4.12), (4.14). Let V_k be the submodule of $V(\Lambda)$ generated by the singular vector $v_k \in V(\Lambda)$, and U_k be the submodule of $V(\Lambda)^*$ generated by v_k^*. Then*

$$M_k = V_k + V_{-k}, \quad N_k = V(\Lambda)^*/(U_{k-1} + U_{-k+1}). \tag{4.29}$$

Moreover $S_k v_{\pm k}$ is proportional to the class of $v_{\pm k}^*$ in N_k.

Proof. Define k_i to be the largest integer such that $v_i \in M_{k_i}$. Thus $S_{k_i} v_i$ is a singular vector in the quotient N_{k_i}. The determinant of $S(t)_\eta$ can then be computed using Proposition 4.2 as

$$\det S(t)_\eta = c t^{N(\eta)}(1 + O(t)),$$
$$N(\eta) = \sum_j k_j \sum_{\substack{|i| \geqq |j| \\ i \neq -j}} P(\eta_i - \eta)(-1)^{|i|-|j|}. \tag{4.30}$$

The coefficient of k_j is the dimension of a subspace of V_j complementary to the proper submodules of V_j, and $\eta_i = \deg v_i$. From the Kac–Kazhdan determinant formula we get

$$\det S(t)_\eta = c t^{\sum_{i \text{ odd}} P(\eta_i - \eta)}(1 + O(t)). \tag{4.31}$$

Comparing the coefficients of $P(\eta_i - \eta)$ we obtain the equations

$$\sum_{\substack{|j| \leqq |i| \\ j \neq -i}} (-1)^{|i|-|j|} k_j = \begin{cases} 1, & i \text{ odd}, \\ 0, & i \text{ even}, \end{cases} \tag{4.32}$$

implying that $k_j = |j|$, and that v_j is in $M_{|j|}$ but not in $M_{|j|+1}$. Thus $S_j v_{\pm j}$ are non-vanishing singular vectors in N_j; they must thus coincide with $v_{\pm j}^*$, and $v_{\pm(|j|-1)}^*$ project to zero in N_j. □

Next, we discuss the structure of the Fock module $F(\Lambda)$. This module can also be given a Γ-grading by the formulae:

$$\deg(\omega_n) = -\alpha_1 + n\delta, \quad \deg(\omega_n^\dagger) = \alpha_1 + n\delta, \quad \deg(a_n) = \delta, \tag{4.33}$$

and for a monomial u acting on the highest weight vector w_0 of $F(\Lambda)$, $\deg(uw_0) = -\deg(u)$. Let $S'(\Lambda): V(\Lambda) \to F(\Lambda)$ be the canonical (degree zero) homomorphism of $U(\mathscr{G})$-modules mapping the highest weight vector to the highest weight vector. Similarly, we have a canonical homomorphism $S''(\Lambda): F(\Lambda) \to V(\Lambda)^*$, and the composite is

$$S''(\Lambda) S'(\Lambda) = S(\Lambda). \tag{4.34}$$

Introducing a basis of the oscillator algebra and of $U(\mathscr{G}_-)$, $S'(\Lambda)_\eta$, $S''(\Lambda)_\eta$ and $S(\Lambda)_\eta$ are all $P(\eta) \times P(\eta)$ matrices with polynomial dependence on Λ, and we can introduce, as above, $S'(t)_\eta$, $S''(t)_\eta$ and corresponding Jantzen filtrations M'_n, M''_n, N'_n and N''_n. Thus

$$\begin{aligned} S'_n: M'_n \to N'_n, \quad M'_{n+1} = \operatorname{Ker} S'_n, \quad N'_{n+1} = \operatorname{Coker} S'_n, \\ S''_n: M''_n \to N''_n, \quad M''_{n+1} = \operatorname{Ker} S''_n, \quad N''_{n+1} = \operatorname{Coker} S''_n. \end{aligned} \tag{4.35}$$

With the same notation as in Proposition 4.4, we have

Proposition 4.5. *If Λ is as in (4.12), (4.14),*

$$M'_k = V_{-2k+1}, \quad N''_k = V(\Lambda)^*/U_{-2k+1} = V^*_{2k-1}. \tag{4.36}$$

Therefore, we have the isomorphisms:

$$\begin{aligned}
&S'_0 : V(\Lambda)/V_{-1} \to \operatorname{Ker}(\operatorname{Pr}: F(\Lambda) \to N'_1),\\
&S'_k : V_{2k+1}/V_{-2k-1} \to \operatorname{Ker}(\operatorname{Pr}: N'_k \to N'_{k+1}),\\
&S''_0 : F(\Lambda)/M''_1 \to \operatorname{Ker}(\operatorname{Pr}: V(\Lambda)^* \to V^*_{-1}),\\
&S''_k : M''_k/M''_{k+1} \to \operatorname{Ker}(\operatorname{Pr}: V^*_{2k+1} \to V^*_{2k-1}).
\end{aligned} \tag{4.37}$$

Proof. We prove first a lower bound on the order of the zero of the determinants of $S'(t)$, $S''(t)$ and show that the bound is exact using the Kac–Kazhdan formula. Let k'_i be the largest integer with the property that $v_i \in M'_{k'_i}$, and k''_i be the smallest integer such that there exists a representative of the class of v^*_i in $\pi(\operatorname{Im}\hat{S}''_{k''_i})$. The determinant of $S'(t)$ is then, as in Proposition 4.4,

$$\begin{aligned}
&\det S'(t)_\eta = c' t^{N'(\eta)}(1 + O(t)),\\
&N'(\eta) = \sum_j k'_j \sum_{\substack{|i| \geqq |j| \\ i \neq -j}} P(\eta_i - \eta)(-1)^{|i|-|j|}.
\end{aligned} \tag{4.38}$$

The order $N''(\eta)$ of the zero of the determinant of $S''(t)_\eta$ is given by the same formula, but with k'_j replaced by k''_j. The weight Λ lies in the intersection of infinitely many lines $\Phi_{\alpha,m}(\Lambda') = 0$ on which the Kac–Kazhdan determinant vanishes. For infinitely many Λ' with $\Phi_{\alpha,m}(\Lambda') = 0$ and $\alpha = -\alpha_1 + m'\delta$, the Fock space $F(\Lambda')$ contains a singular vector of the form $Q_m w_0$, where w_0 is the highest weight vector of the Fock space $F(\Lambda' - m\alpha)$. Similarly, for infinitely many points on $\Phi_{\alpha,m}(\Lambda') = 0$ with $\alpha = \alpha_1 + m'\delta, F(\Lambda')$ contains a cosingular vector ρ with $Q_m\rho = w_0$. Since the condition of having a singular (or a cosingular) vector of a given degree in a Fock module is a polynomial condition on the highest weight Λ', it follows that for *all* Λ' on the line $\Phi_{m,\alpha}(\Lambda') = 0, F(\Lambda')$ contains either a singular or a cosingular vector. In particular, $F(\Lambda)$ contains infinitely many singular vectors χ_j of some degree as $v_{2j-1}, j = 1, 2, \ldots$ and infinitely many cosingular vectors ρ_j of degree $\deg(v_{-2j+1})$, $j = 1, 2, \ldots$.

Let k be the smallest integer such that $\rho_j \in \pi(\operatorname{Im}\hat{S}'_k)$. Thus $\rho_j = S'_k\xi$, for some (non-zero) cosingular vector ξ in $M'_k \subset V(\Lambda)$. Since M'_k is a submodule of $V(\Lambda)$, ξ must be a singular vector. As $\deg(\rho_j) = \deg(v_{-2j+1}), \xi$ is proportional to v_{-2j+1}, $k = k'_{-2j+1}$, and $v_{\pm(2j-2)} \notin M'_k$. Therefore

$$k'_{-2j+1} \geqq k'_{\pm(2j-2)} + 1, \quad j \geqq 1. \tag{4.39}$$

A dual estimate of the integers k''_j holds: Let k be the largest integer such that $\chi_i \in M''_k$. Thus $S''_k\chi_i$ is a singular vector in the quotient N'_k. Therefore $S''_k\chi_i$ is proportional to the class of v^*_{2j-1}, and $V^*_{\pm(2j-2)}$ project to zero in N'_{k-1}. Hence

$$k''_{2j-1} \geqq k''_{\pm(2j-2)} + 1, \quad j \geqq 1. \tag{4.40}$$

From the relation $\det S'(t)_\eta \det S''(t)_\eta = \det S(t)_\eta$ and the Kac–Kazhdan determinant formula, we get by comparing coefficients of $P(\eta_i - \eta)$

$$\sum_{\substack{|j| \leqq |i| \\ i \neq -j}} (-1)^{|i|-|j|}(k'_j + k''_j) = \begin{cases} 1, & i \text{ odd}, \\ 0, & i \text{ even}. \end{cases} \tag{4.41}$$

It is easy to see inductively that the only sequences (k'_j), (k''_j) such that $k'_j \leqq k'_i$, $k''_j \leqq k''_i$ whenever $|j| < |i|$ and with (4.39), (4.40) and (4.41) are

$$k'_{-2j} = k'_{2j+1} = |j|, \quad k''_{-2j} = k''_{-2j-1} = |j|, \tag{4.42}$$

which is equivalent to the claim. □

By Proposition 4.4 the classes in N'_k of the vectors $v^*_{2k-1}, v^*_{\pm 2k}, v^*_{2k+1}$ are in $\operatorname{Im} S'_k$. Let $w_{2k-1}, w_{\pm 2k}, w_{2k+1}$ be such that $S'_k w_i = v^*_i$. The following proposition gives a more complete description of the structure of $F(\Lambda)$. A module is called completely reducible if it is the direct sum of irreducible highest weight modules.

Proposition 4.6. (i) *(Bernstein–Gel'fand–Gel'fand resolution of $F(\Lambda)$). The space of singular vectors of positive degree in $F(\Lambda)$ is $\bigoplus_1^\infty \mathbf{C}w_{2k-1}$. They generate a completely reducible submodule $SF(\Lambda)$. The singular vectors in $F[1] = F(\Lambda)/SF(\Lambda)$ are (the projection of) $\bigoplus_{-\infty}^\infty \mathbf{C}w_{2k} \equiv SF[1]$. They also generate a completely reducible submodule. Finally, $F[2] = F[1]/SF[1]$ is generated by the singular vectors $\bigoplus_1^\infty \mathbf{C}w_{-2k+1}$ and is completely reducible.*
(ii) *Every submodule of $F(\Lambda)$ is generated by a family of vectors belonging to the set $\{w_i\}_{i \in \mathbf{Z}}$.*
(iii) *The structure of $F(\Lambda)$ is described by the diagram*

$$\begin{array}{ccccccccccc} \cdots & \leftarrow & w_4 & \rightarrow & w_3 & \leftarrow & w_2 & \rightarrow & w_1 & \leftarrow & w_0 \\ & \nwarrow\!\!\nearrow & & \nwarrow\!\!\nearrow & & \nwarrow\!\!\nearrow & & \nwarrow\!\!\nearrow & & \nearrow & \\ \cdots & \rightarrow & w_{-4} & \leftarrow & w_{-3} & \rightarrow & w_{-2} & \leftarrow & w_{-1} & & \end{array} . \tag{4.43}$$

Proof. (i) Let χ be a singular vector of positive degree in $F(\Lambda)$, and let k be the largest integer such that $\chi \in M''_k$. Then, by Proposition 4.5, $S''_k\chi$ is a singular vector in $N''_k = V(\Lambda)^*/U_{-2k+1}$, and must therefore be proportional to $cl(v^*_{2k-1})$, the class of v^*_{2k-1}. Hence χ is proportional to w_{2k-1}. The submodule generated by w_{2k-1} is irreducible since there are no singular vectors of degree $\deg(v_{\pm 2k})$ in $F(\Lambda)$. Let now χ be a representative of a singular vector in $F(\Lambda)/SF(\Lambda)$ and let k be the largest integer with $\chi \in M''_k$. Then the image of χ by $S''_k : M''_k/M''_k \cap SF(\Lambda) \to V(\Lambda)^*/(U_{2k-1} + U_{-2k+1})$ is singular. Thus $S''_k\chi$ is proportional to v^*_{2k} or to v^*_{-2k} and χ is proportional to w_{2k} or w_{-2k} (modulo $SF(\Lambda)$). The same argument can be repeated for a singular vector in $F[1]/SF[1]$, with the conclusion that χ is proportional to (the class of) some w_{-2k+1}.
(ii) Let H be a submodule of $F(\Lambda)$, and let $H_i = H \cap \operatorname{Ker}(\Pr : F(\Lambda) \to F[i])$. The module H_i/H_{i+1} is a submodule of the completely reducible module $SF[i]$. Thus H_i/H_{i+1} is generated by a subset of w_i's. Therefore every vector in H_i can be written as $\sum u_j w_j$ plus a vector in H_{i+1}, $u_j \in U(\mathscr{G})$. Iteration of this argument concludes the proof.

(iii) By construction of the vectors w_i we have the arrows of the diagram

$$\begin{array}{ccccccccccc} \cdots & & w_4 & \rightarrow & w_3 & & w_2 & \rightarrow & w_1 & & w_0 \\ & \nearrow & & \nearrow & & \nearrow & & \nearrow & & \nearrow & \\ \cdots & \rightarrow & w_{-4} & & w_{-3} & \rightarrow & w_{-2} & & w_{-1} & & \end{array} . \tag{4.44}$$

We can assume by induction that N'_k is the quotient of $F(\Lambda)$ by the submodule generated by $w_{-2k+2}, w_{-2k+3}, \ldots, w_{2k-2}, w_{2k-1}$. Then the image of v_{2k+1} is proportional to the class of w_{-2k+1} (this being the only non-zero vector of that degree in N'_k). The singular vectors $S'_k v_{\pm 2k}$ generate submodules of N'_k that do not contain the class of $w_{-2k+1} = S'_k v_{-2k+1}$. They must be proportional to $cl(w_{2k+1})$. Thus $\operatorname{Im} S'_k$ is generated by the classes of $w_{2k-1}, w_{\pm 2k}, w_{2k+1}$ in N'_k, and the induction assumption on N'_k is extended to N'_{k+1}. We have proved the validity of the arrows going left, but only in the quotient N'_k. We have, for some $u \in U(\mathscr{G})$,

$$w_{2k} = u w_{-2k+1} \text{ modulo } \operatorname{Ker}(\operatorname{Pr}: F(\Lambda) \to N'_k). \tag{4.45}$$

But since $w_{2k} - u w_{-2k+1} \in M''_{k-1}$ and $M'_{k-1} \cap \operatorname{Ker}(\operatorname{Pr} F(\Lambda) \to N'_k)$ is generated by $w_{2k-3}, w_{\pm 2(k-1)}$ and w_{2k-1} which are all in $U(\mathscr{G}) w_{-2k+1}$, we conclude that $w_{2k} = u' w_{-2k+1}$ for some $u' \in U(\mathscr{G})$. Similarly, $w_{-2k} \in U(\mathscr{G}) w_{-2k+1}$. The same argument works for $w_{2k+1}: w_{2k+1} - u_{\pm} w_{\pm 2k} \in \operatorname{Ker}(\operatorname{Pr}: F(\Lambda) \to N'_k) \cap M''_k$, a submodule generated by $w_{2k-1} \in U(\mathscr{G}) w_{2k}$. Thus $w_{2k+1} \in U(\mathscr{G}) w_{\pm 2k}$. $\square$

Finally, we compute the BRST cohomology of the complex of Fock modules introduced in Sect. 1.

Proof of Proposition 2.1 and Theorem 2.2. Let us fix p, p', n, n', and denote by $F^{(l)}$ the Fock space of degree l. Thus $F^{(0)}$ is $F(\Lambda_{n,n'})$. Let $W_i^{(l)}$ be the submodule generated by the vector $w_i = w_i^{(l)} \in F^{(l)}$ introduced in Proposition 4.6. As shown there, all submodules of $F^{(l)}$ are direct sums of $W^{(l)}$'s. We claim that

$$\begin{aligned} \operatorname{Ker} Q^{(l)} &= \begin{cases} \bigoplus_{i=1}^{\infty} W_{2i}^{(l)} & \text{if } l < 0, \\ \bigoplus_{i=0}^{\infty} W_{2i}^{(l)} & \text{if } l \geqq 0, \end{cases} \\ \operatorname{Im} Q^{(l)} &= \begin{cases} \bigoplus_{i=1}^{\infty} W_{2i}^{(l)} & \text{if } l \leqq 0, \\ \bigoplus_{i=0}^{\infty} W_{2i}^{(l)} & \text{if } l > 0, \end{cases} \end{aligned} \tag{4.46}$$

from which both Proposition 2.1 and Theorem 2.2 follow. We prove (4.46) for $l = 0$. The proof in the general case is identical except for the range of i. The singular vectors of positive degree in $F^{(0)}$ are $w_{2i+1}^{(0)}$, $i = 0, 1, 2, \ldots$ Since there are no singular vectors with the same weight as $w_{2i+1}^{(1)}$ in $F^{(1)}, Q^{(0)} w_{2i+1}^{(0)}$ vanishes. Thus $\operatorname{Ker} Q^{(0)} \supset \bigoplus_0^{\infty} W_{2i+1}^{(0)}$, and $Q^{(0)}$ projects to a homomorphism $F^{(0)} \Big/ \bigoplus_0^{\infty} W_{2i+1}^{(0)} \to F^{(1)}$. By the same argument, the vectors $w_{2i}^{(0)}$, $i = 0, 1, 2, \ldots$, which have singular projection in $F^{(0)}\Big/\bigoplus_0^\infty W^{(0)}_{2i+1}$, are in the kernel of $Q^{(0)}$. This proves that $\operatorname{Ker} Q^{(0)} \supset \bigoplus_0^\infty W^{(0)}_{2i}$ (recall that $W^{(0)}_{2i+1} \subset W^{(0)}_{2i}, i = 0, 1, 2, \ldots$).

Now view Q_n as a map from $F^{(0)}\Big/\bigoplus_0^\infty W^{(0)}_{2i}$ to $F^{(1)}$, and let π be the canonical projection $F^{(0)} \to F^{(0)}\Big/\bigoplus_0^\infty W^{(0)}_{2i}$. To complete the proof of (4.46) it is sufficient to show that

$$Q_n \pi W^{(0)}_{i+1} = W^{(1)}_i, \tag{4.47}$$

$i = 0, 1, 2, \ldots$. The computation in Appendix A shows that $\pi w^{(0)}_1$ is mapped to $w^{(1)}_0$, which proves (4.47) for $i = 0$. Next, suppose inductively that (4.47) is proved for all $i \leqq 2j$; $W^{(1)}_{2j+1}$ is a proper submodule of $W^{(1)}_{2j} = Q^{(0)} W^{(0)}_{2j-1}$. Thus there must exist a submodule of $\pi W^{(0)}_{2j-1}$ which is mapped to $W^{(1)}_{2j+1}$ under $Q^{(0)}$. The only submodule of $W^{(0)}_{2j-1}$ that can have this property is $W^{(0)}_{2j}$. This shows that $W^{(1)}_{2j+1} = Q^{(0)} \pi W^{(0)}_{2j}$. Moreover, since $\pi W^{(0)}_{2j} \subset \pi W^{(0)}_{2j+1}$, there is a submodule of $F^{(1)}$ containing $W^{(1)}_{2j+1}$ which is equal to the image of $\pi W^{(0)}_{2j+1}$ by $Q^{(0)}$. This shows that $W^{(1)}_{2j+2} = Q^{(0)} W^{(0)}_{2j+1}$. The proof is complete. □

5. A Geometrical Interpretation

In this section we illustrate how a free field representation of $A^{(1)}_1$ analogue to the one given in Eq. (2.5) arises naturally in a Borel–Weil like construction. Usually the Borel–Weil construction on a compact group G begins by considering the flag variety $G^{\mathbf{C}}/B_-$, where B_- is a Borel subgroup of $G^{\mathbf{C}}$. In the case we are considering we have to twist a little bit the construction by changing the definition of the gaussian decomposition. Usually the gaussian decomposition consists in splitting the elements g of G into $g = g_+ g_0 g_-$, where $g_\pm$ belong to the Borel subgroups and g_0 to the Cartan subgroup. In the case of the loop group $\widehat{SL(2)}$ there are infinitely many possible choices of Borel subgroups. To each choice is associated a gradation of the algebra $A^{(1)}_1$ or equivalently each choice corresponds to a choice of simple roots of $A^{(1)}_1$. Because the affine Weyl group is infinite there are infinitely many possible choices of simple roots: in the notation of Sect. 4, for any positive integer k the set of roots $\alpha^{(k)}_0, \alpha^{(k)}_1$, with $\alpha^{(k)}_0 = (1+k)\alpha_0 + k\alpha_1, \alpha^{(k)}_1 = (1-k)\alpha_1 - k\alpha_0$, form a set of complete roots of $A^{(1)}_1$. Our gaussian decomposition is defined by choosing the "gradation" which is the inductive limit of these gradations. To be precise, the Borel subalgebra that we are considering is spanned by the elements $J^+_m, m \in \mathbf{Z}$ and $J^0_n, n \in \mathbf{N}$. Elements g_+ of the borel subgroup $\hat{B}_+$ are parametrized by two infinite set of coordinates, $x_m; m \in \mathbf{Z}$ and $y_n; n \in \mathbf{N}$:

$$g_+(x_m; y_n) = \exp\left(\sum_{m\in\mathbf{Z}} x_m J^+_m\right) \exp\left(\sum_{n\in\mathbf{N}} y_n J^0_n\right). \tag{5.1}$$

Note the normal order prescription in Eq. (5.1). This definition of the Borel subgroup would arise if we did a gaussian decomposition on the element $g(z)$ of the group $G^{\mathbf{C}}$ for fixed value of z.

In an analogous way one defines the Borel subgroup $\hat{B}_-$. The Cartan subgroup

H is unchanged; i.e. it is generated by J_0^0 and the central element K. The gaussian decomposition $g = g_+ g_0 g_-$ with $g_\pm \in \hat{B}_\pm$ and $g_0 \in H$ holds in the formal group $\widehat{SL(2)}$.

A character $\chi_{K,J}$ of H is specified by two numbers: the central charge K and the spin J. Each character of H defines a line bundle $\mathscr{L}_{K;J}$ over $\widehat{SL(2)}/\hat{B}_-$. Sections of $\mathscr{L}_{K;J}$ can be thought of as functions $f(g)$ defined on $\widehat{SL(2)}$ with the transformation properties:

$$f(gg_-) = f(g), \quad f(gg_0) = f(g)\chi_{K;J}(g_0) \tag{5.2}$$

for $g_- \in \hat{B}_-$ and $g_0 \in H$.

The action $D_{K;J}$ of the formal group $\widehat{SL(2)}$ on the sections of $\mathscr{L}_{K;J}$ is defined by

$$(D_{K;J}(g_0)f)(g) = f(g_0^{-1}g) \tag{5.3}$$

for any $g_0 \in \widehat{SL(2)}$.

The action of the Lie algebra $A_1^{(1)}$ is the infinitesimal version of Eq. (5.3):

$$D_{K;J}(J_n^a) = \frac{d}{dt} D_{K;J}(e^{tJ_n^a})|_{t=0}. \tag{5.4}$$

Sections of $\mathscr{L}_{K;J}$ are uniquely specified by their values on elements of $\hat{B}_+$. Therefore they can be understood as functions of the variables x_m; $m \in \mathbf{Z}$ and y_n; $n \in \mathbf{N}$, see Eq. (5.1). On these functions the operators $D_{K;J}(J_n^a)$ act as differential operators. Namely,

$$D_{K;J}(J_n^+) = -\frac{\partial}{\partial x_n},$$

$$D_{K;J}(J_n^0) = -\sum_m x_{m-n}\frac{\partial}{\partial x_m} - \theta(n)\frac{\partial}{\partial y_n} + \theta(-n)n\frac{K}{2}y_{-n} + \delta_{n,0}J,$$

$$D_{K;J}(J_n^-) = \sum_{l,m} x_l x_{m-l-n}\frac{\partial}{\partial x_m} + 2\sum_{m>0} x_{m-n}\frac{\partial}{\partial y_m} + K\sum_{m>0} m y_m x_{-m-n} + Knx_{-n} - 2Jx_{-n}. \tag{5.5}$$

The representation (5.5) is a free field representation analogous to the representation (2.5). Indeed if we set $a_0 = J/\gamma$ and

$$\omega_n^\dagger = -\frac{\partial}{\partial x_n}, \qquad \omega_n = x_{-n},$$

$$a_n = -\sqrt{\frac{2}{K}}\frac{\partial}{\partial y_n}; \quad n > 0, \tag{5.6}$$

$$a_n = n\sqrt{\frac{K}{2}}y_{-n}; \quad n < 0,$$

then the representation (5.5) can be written exactly as Eq. (2.5) but with $2\gamma^2 = K$. The discrepancy between the two values of γ comes from the fact that the vacua of the $\omega - \omega^\dagger$ system and hence the normal order prescriptions are not the same in the two representations. In the representation (5.5) the vacuum is the constant function **1** and it satisfies

$$\omega_n^\dagger \mathbf{1} = 0 \quad \text{for all} \quad n \in \mathbf{Z}. \tag{5.7}$$

This choice of the vacuum has been induced by our choice of the Borel subgroup $\hat{B}_+$. This construction can be generalized to the other affine algebras.

Appendix

A key property of the BRST operator Q_n is that it is a homomorphism of $U(\mathscr{G})$-modules. This property allows one to construct singular vectors as images under Q_n of highest weight vectors, and cosingular vectors as vectors that are mapped by Q_n to highest weight vectors. For this procedure to work one must show that the vectors constructed this way do not vanish.

Lemma. *Let* $K \neq -2$ *and* $\Lambda = \Lambda_{nn'} = (K - 2J_{nn'})\Lambda_0 + 2J_{nn'}\Lambda_1, n, n' \in \mathbf{Z}$. *Denote by* $|J_{nn'}\rangle$ *the highest weight vector of the Fock space* $F(\Lambda_{nn'})$. *Then*

(i) *If* $p - 1 \geqq n \geqq 1$ *and* $n' > 0$, *then* $Q_n|J_{n,-n'}\rangle \neq 0$.
(ii) *If* $p - 1 \geqq n \geqq 1$ *and* $n' \geqq 0$, *then there exists a vector* $|w\rangle \in F(\Lambda_{nn'})$ *such that* $Q_n|w\rangle = |J_{-n,n'}\rangle$.

Here $K + 2 = p/p'$ *if* $K \in Q$ *and* $p = \infty$ *if* $K \notin Q$.

Proof. (i) First rewrite $Q_n|J_{n,-n'}\rangle$ by changing integration variables $z_i = zu_i, i \geqq 2$:

$$Q_n|J_{n,-n'}\rangle = \frac{1}{n}\frac{e^{\pi i n\gamma^{-2}} - 1}{e^{\pi i\gamma^{-2}} - 1} \oint \frac{dz}{z} z^{n(1-n')} \int \prod_{i=2}^{n} du_i u_i^{-n'-(1-n)/2\gamma^2}(1 - u_i)^{\gamma^{-2}}$$
$$\prod_{2 \leqq i \leqq j \leqq n} (u_i - u_j)^{\gamma^{-2}} \omega^\dagger(z) \prod_{i=2}^{n} \omega^\dagger(zu_i) \exp\left[-\alpha \sum_{j=1}^{\infty} \frac{a_{-j}}{j} z^j \left(1 + \sum_{i=2}^{n} u_i^j\right)\right] |J_{n,-n'}\rangle. \tag{A.1}$$

The u_i-integrations start and end at the singularity $u_i = 1$. In order for the z-integration to give a non-vanishing result, n' has to be strictly positive. We evaluate (A.1) against the covector $\langle w| = \langle J_{-n,-n'}|(\omega_{n'})^n$:

$$\langle w|Q_n|J_{n,-n'}\rangle = 2\pi i(n-1)! \frac{e^{\pi i n\gamma^{-2}} - 1}{e^{\pi i\gamma^{-2}} - 1}$$
$$\cdot \int \prod_{i=2}^{n} du_i u_i^{-1+(1-n)/2\gamma^2}(1 - u_i)^{\gamma^{-2}} \prod_{2 \leqq i < j \leqq n} (u_i - u_j)^{\gamma^{-2}}$$
$$= (2\pi i)^n \frac{\Gamma(1 + n/2\gamma^2)}{\Gamma(1 + 1/2\gamma^2)} \prod_{j=1}^{n} \frac{\sin(\pi j/2\gamma^2)}{\sin(\pi/2\gamma^2)}, \tag{A.2}$$

where we have used the Dotsenko–Fateev formulas (Appendix A in the second paper in [5]) to evaluate the multiple integral. We see that if γ^2 is not rational, (A.2) does never vanish, and if $K + 2 = p/p'$ we have the non-vanishing condition $1 \leqq n \leqq p - 1$.

(ii) Let $|w\rangle = (\omega_{-n'})^n|J_{nn'}\rangle$. This is a non-vanishing vector if n' is non-negative. The same calculation as above gives that $\langle J_{-n,n'}|Q_n|w\rangle$ is equal to the right-hand side of (A.2) up to a factor $(-1)^n$. For the values of n we are considering, this expression does not vanish, so that $|w\rangle$ can be rescaled to give the desired vector. □

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Note added in proof. After having submitted the paper we learned that the representations of $A_1^{(1)}$ described in Sect. 5 have been also considered by H. P. Jacobsen and V. G. Kac in ref. [31] in special case of vanishing central charge, $K = 0$.

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