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Fock Representations of the Affine Lie Algebra $A_1^{(1)}$

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Abstract. The aim of this note is to show that the affine Lie algebra $A_1^{(1)}$ has a natural family $\pi_{\mu,\nu}$ of Fock representations on the space $\mathbf{C}[x_i, y_j;\ i \in \mathbb{Z}$ and $j \in \mathbb{N}]$, parametrized by $(\mu, \nu) \in \mathbf{C}^2$. By corresponding the highest weight $\Lambda_{\mu,\nu}$ of $\pi_{\mu,\nu}$ to each (μ, ν), the parameter space $\mathbf{C}^2$ forms a double cover of the weight space $\mathbf{C}\Lambda_0 \oplus \mathbf{C}\Lambda_1$ with singularities at linear forms of level -2; this number is (-1)-times the dual Coxeter number. Our results contain explicit realizations of irreducible non-integrable highest weight $A_1^{(1)}$-modules for generic (μ, ν).

1. Introduction

It is well known that the basic representations of affine Lie algebras have concrete realizations in terms of vertex operators, and that they have close connections to some areas in mathematical physics such as soliton theory and dual resonance models. But there seems to be only a few results on constructions of highest weight modules other than basic representations even in the case of the affine Lie algebra $A_1^{(1)}$.

In this note our interests are concentrated especially on *non-integrable* representations of $A_1^{(1)}$. We shall show that almost all irreducible non-integrable highest weight $A_1^{(1)}$-modules are realized in a unified way on the Fock space $\mathbf{C}[x_i, y_j;\ i \in \mathbb{Z}$ and $j \in \mathbb{N}]$. Among them, the most interesting case is the case when the representation is of level -2; we shall construct the representation of level -2 on the Fock space $\mathbf{C}[x_i;\ i \in \mathbb{Z}]$, and derive a character formula which is related to the one conjectured by Kac and Kazhdan [5].

2. On the space $\mathbf{C}[x] = \mathbf{C}[x_j; j \in \mathbf{Z}]$ of polynomial functions in x_j's, we introduce the following differential operators:

$$a_j = Y_+(j)x_j - Y_-(j)\frac{\partial}{\partial x_j},$$

$$a_j^* = Y_+(j)\frac{\partial}{\partial x_j} + Y_-(j)x_j,$$

where $Y_{\pm}$ are functions on $\mathbf{Z}$ defined by

$$Y_+(j)=\begin{cases}1 & \text{if } j\geqq 0\\ 0 & \text{if } j<0\end{cases},$$

and

$$Y_-(j)=1-Y_+(j).$$

Define the normal products : : by

$$:a_i a_j^*:=a_i a_j^* + Y_-(i)\delta_{i,j},$$

$$:a_i a_j^* a_k^*:=a_i a_j^* a_k^* + Y_-(i)\ (\delta_{i,j}a_k^* + \delta_{i,k}a_j^*),$$

and we put

$$E(m)=\sum_{j\in\mathbf{Z}} :a_{j+m}a_j^*:$$

for every $m\in\mathbf{Z}$.

Lemma 1.

1) $[a_i^*, a_j]=\delta_{i,j}$,
2) $[:a_i a_j^*:, a_k]=\delta_{j,k}a_i$,
   $[:a_i a_j^*:, a_k^*]=-\delta_{i,k}a_j^*$,
3) $[:a_i a_j^*:, :a_k a_l^*:]=\delta_{j,k}:a_i a_l^*:-\delta_{i,l}:a_k a_j^*:+\delta_{i,l}\delta_{j,k}(Y_-(j)-Y_-(i))$.

Lemma 2.

1) $[E(m), a_j]=a_{j+m}$,
   $[E(m), a_j^*]=-a_{j-m}^*$,
2) $[E(m), :a_i a_j^*:]=:a_{i+m}a_j^*:-:a_i a_{j-m}^*:+\delta_{i+m,j}(Y_-(i)-Y_-(j))$,
3) $[E(m), E(n)]=m\delta_{m,-n}$.

We set

$$:E(m)a_j^*:=E(m)a_j^*+Y_-(m)a_{j-m}^*,$$

$$:E(m)a_j^*a_k^*:=E(m)a_j^*a_k^*+Y_-(m)\ (a_{j-m}^*a_k^*+a_j^*a_{k-m}^*).$$

Note that the above definition of normal products : : differs from usual ones. Namely $E(m)$ is brought to the left of a_j^* and $a_j^*a_k^*$ for $m\geqq 0$ and to the right for $m<0$ in our definition.

Then it is also easy to check the following:

Lemma 3.

1) $[:E(m)a_j^*:, a_k]=:a_{k+m}a_j^*:+\delta_{j,k}E(m)+\delta_{j,k+m}(Y_-(m)-Y_-(j))$,
2) $[:E(m)a_j^*:, a_k^*]=-a_j^*a_{k-m}^*$,

3) $[:E(m)a_j^*:, :a_k a_l^*:] = :a_{k+m}a_l^* a_j^*: - :a_k a_{l-m}^* a_j^*:$
$+\delta_{j,k}\{:E(m)a_l^*: + (Y_-(j) - Y_-(m))a_{l-m}^*\}$
$+\delta_{j,k+m}(Y_-(m) - Y_-(j))a_l^* + \delta_{l,k+m}(Y_-(k) - Y_-(l))a_j^*$,

4) $[:E(m)a_j^*:, E(n)] = :E(m)a_{j-n}^*: + m\delta_{m,-n}a_j^*$,

5) $[:E(m)a_j^*:, :E(n)a_k^*:] = :E(m)a_{j-n}^* a_k^*: - :E(n)a_j^* a_{k-m}^*:$
$+(Y_-(n) - Y_-(m))a_{j-n}^* a_{k-m}^* + m\delta_{m,-n}a_j^* a_k^*$.

Next we take the space $\mathbf{C}[y] = \mathbf{C}[y_j;\ j \in \mathbf{N}]$, and we set $b_0 = 0$, $b_j = jy_j$, $b_{-j} = -\dfrac{\partial}{\partial y_j}$ for every $j \in \mathbf{N}$.

Now recall the affine Lie algebra $\mathfrak{g} = \mathfrak{g}(A_1^{(1)})$;

$$\mathfrak{g} = \mathbf{C}[t, t^{-1}] \otimes \mathfrak{sl}(2, \mathbf{C}) \oplus \mathbf{C}c \oplus \mathbf{C}d,$$

with commutation relations

$$[A(m), B(n)] = [A, B]\ (m+n) + m\mathrm{Tr}(AB)\delta_{m,-n}c,$$
$$[d, A(m)] = mA(m),$$

and

$$[c, \mathfrak{g}] = \{0\},$$

where $A(m) = t^m \otimes A$. Its Chevalley generators are chosen such that

$$e_0 = X(1), \qquad e_1 = Y(0),$$
$$f_0 = Y(-1), \qquad f_1 = X(0),$$
$$\alpha_0^\vee = H(0) + c, \qquad \alpha_1^\vee = -H(0),$$

where

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Fix any complex numbers μ and ν, and we set

$$V(\mu, \nu) = \begin{cases} \mathbf{C}[x] \otimes \mathbf{C}[y] & \text{if} \quad \nu \neq 0 \\ \mathbf{C}[x] & \text{if} \quad \nu = 0 \end{cases},$$

and define the operators on $V(\mu, \nu)$ as follows:

i) in the case when $\nu = 0$,

$$\pi_{\mu,0}(X(n)) = a_{-n},$$
$$\pi_{\mu,0}(Y(n)) = (\mu - 1 - n)a_n^* - \sum_{j \in \mathbf{Z}} :E(j)a_{j+n}^*:,$$
$$\pi_{\mu,0}(H(n)) = 2E(-n) + (1-\mu)\delta_{n,0},$$
$$\pi_{\mu,0}(c) = -2,$$
$$\pi_{\mu,0}(d) = -\sum_{j \in \mathbf{Z}} j :a_j a_j^*:,$$

ii) in the case when $\nu \neq 0$,

$$\pi_{\mu,\nu}(X(n)) = a_{-n},$$

$$\pi_{\mu,\nu}(Y(n)) = \left[\mu - 1 - \left(\frac{\nu^2}{2} + 1\right) n\right] a_n^* : E(j) a_{j+n}^* : + \nu \sum_{j \in \mathbf{Z}} b_j a_{j+n}^*,$$

$$\pi_{\mu,\nu}(H(n)) = 2E(-n) - \nu b_{-n} + (1-\mu)\delta_{n,0},$$

$$\pi_{\mu,\nu}(c) = -\left(2 + \frac{\nu^2}{2}\right),$$

$$\pi_{\mu,\nu}(d) = -\sum_{j \in \mathbf{Z}} j : a_j a_j^* : + \sum_{j=1}^{\infty} b_j b_{-j}.$$

Theorem 1. *$(\pi_{\mu,\nu}, V(\mu,\nu))$ is a representation of $\mathfrak{g}(A_1^{(1)})$.*

The proof of the above theorem is straightforward; one needs only to note that the following sums

$$\sum_{m,n \in \mathbf{Z}} : a_{i+m+n} a_{j+m}^* a_{k+n}^* :$$

and

$$\sum_{m,n \in \mathbf{Z}} : E(m+n) a_{j+m}^* a_{k+n}^* :$$

are locally finite on the space $\mathbf{C}[x]$.

The constant function $\psi_{\mu,\nu} = 1$ in $V(\mu,\nu)$ is a highest weight vector satisfying

$$\pi_{\mu,\nu}(\alpha_0^\vee)\psi_{\mu,\nu} = -\left(\mu + 1 + \frac{\nu^2}{2}\right)\psi_{\mu,\nu},$$

$$\pi_{\mu,\nu}(\alpha_1^\vee)\psi_{\mu,\nu} = (\mu - 1)\psi_{\mu,\nu},$$

and

$$\pi_{\mu,\nu}(d)\psi_{\mu,\nu} = 0.$$

So the irreducible component containing $\psi_{\mu,\nu}$ is isomorphic to $L(\Lambda_{\mu,\nu})$, where $\Lambda_{\mu,\nu}$ is the linear form on the Cartan subalgebra $\mathfrak{h}$ defined by

$$\Lambda_{\mu,\nu} = -\left(1 + \mu + \frac{\nu^2}{2}\right)\Lambda_0 - (1-\mu)\Lambda_1.$$

The character of the space $V(\mu,\nu)$ is easily computed:

$$\operatorname{ch} V(\mu,\nu) = \begin{cases} e^{\Lambda_{\mu,\nu}} \prod_{\alpha \in \Delta^+} \dfrac{1}{(1-e^{-\alpha})} & \text{if} \quad \nu \neq 0, \\[2ex] e^{\Lambda_{\mu,0}} \prod_{\alpha \in \Delta_R^+} \dfrac{1}{(1-e^{-\alpha})} & \text{if} \quad \nu = 0, \end{cases}$$

where Δ^+ (respectively Δ_R^+) is the set of all positive (respectively positive real) roots, since the weight of a monomial

$$x_{i_1} \cdots x_{i_l} x_{-j_1} \cdots x_{-j_m} y_{k_1} \cdots y_{k_n}$$

is equal to

$$\Lambda_{\mu,\nu}-(i_1+\ldots+i_l+j_1+\ldots+j_m+k_1+\ldots+k_n)\delta+(m-l)\alpha_1$$

for $i_1,\ldots,i_l \geqq 0$ and $j_1,\ldots,j_m$, $k_1,\ldots,k_n \geqq 1$.

The most interesting case would be the case when $\nu=0$. In this case, it may be expected that $(\pi_{\mu,0}, V(\mu,0))$ is irreducible for every $\mu \in (\mathbf{C}-\mathbf{Z})\cup\{0\}$. At present we can only dominate its character by the majorant series in the following way. The ring $\mathbf{C}[[e^{-\alpha_0}, e^{-\alpha_1}]]$ of formal power series in $e^{-\alpha_0}$ and $e^{-\alpha_1}$ is made a partially ordered set by putting

$$f=\sum f_\lambda e^\lambda \leqq g=\sum g_\lambda e^\lambda$$

if and only if $g_\lambda - f_\lambda \geqq 0$ for every λ. Then

Corollary. *The character of any irreducible highest weight $A_1^{(1)}$-module $L(\Lambda)$ of level -2 satisfies the inequality*

$$e^{-\Lambda}\operatorname{ch}L(\Lambda) \leqq \prod_{\alpha\in\Delta_R^+} \frac{1}{(1-e^{-\alpha})}.$$

Finally, in view of Proposition 3.1 in [5] (or Proposition 9.10 in [4]), we obtain

Theorem 2. *Let μ and ν be complex numbers satisfying the following two conditions:*

i) $\nu \neq 0$,

ii) *for any integer $n \geqq 0$, neither $\mu - n\left(2+\frac{\nu^2}{2}\right)$ nor $-\left(\mu+\frac{\nu^2}{2}\right)-n\left(2+\frac{\nu^2}{2}\right)$ is a positive integer.*

Then the $A_1^{(1)}$-module $V(\mu,\nu)$ is irreducible and isomorphic to $L(\Lambda_{\mu,\nu})$, where

$$\Lambda_{\mu,\nu} = -\left(1+\mu+\frac{\nu^2}{2}\right)\Lambda_0-(1-\mu)\Lambda_1 .$$

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