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Focusing and twinkling: critical exponents from catastrophes in non-Gaussian random short waves

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Abstract. In the limit of geometrical optics, when the wavenumber k becomes infinite, the nth moment $\overline{I^n}$ of the intensity $I = |\psi|^2$ of a random wave ψ diverges when $n \geqslant 2$ as

$$\overline{I^n} \sim k^{\nu_n}$$

(apart from possible logarithmic factors). We call ν_n the nth critical exponent. The divergence indicates strong non-Gaussian fluctuations (twinkling) in ψ, and arises from *caustics* (focusing) of the associated family of rays. We compute ν_n from the Thom–Arnol'd classification of caustics as catastrophes (generic singularities of gradient mappings). A crucial transformation is to study the caustics not in space–time but on the torus whose coordinates are the N random phases $\theta_1 \ldots \theta_N$ for members of the ensemble of functions describing the phase screen or inhomogeneous medium responsible for the disorder of the wave.

The results indicate the following 'universality': when $N \to \infty$ (Gaussian random medium) the exponents ν_n depend only on whether the waves propagate in two or three space dimensions. For two space dimensions all ν_n are calculated; for three space dimensions ν_n are calculated for $n \leqslant 13$, and some uncertainties arise from the incompleteness of the classification of catastrophes of corank 2. When N is finite the higher exponents ν_n differ from the universal values. The simplest case is $N = 1$, corresponding to a (non-random) sinusoidal phase screen, and we show that the analysis of $\overline{I^2}$ based on caustics is very accurate when compared with an exact treatment.

Experimental tests of the theory are feasible, and measurements of critical exponents could provide means of probing deeply into the structure of random media.

1. Introduction

This paper is about intensity fluctuations in a monochromatic plane wave with strong phase variations introduced by encounter with a random medium. The refracting irregularities of the medium may be confined to its boundary (as with the surface of the sea) or distributed throughout its volume (as with turbulence in the atmosphere).

For a complex wave ψ the intensity I is defined by $|\psi|^2$ and those statistics not involving correlations are described by the probability distribution $P(I)$ as embodied in the *moments*

$$\overline{I^n} \equiv \overline{|\psi|^{2n}} \tag{1}$$

where the bar denotes an average over whatever ensemble defines the randomness of the medium. In one case commonly considered (Beckmann and Spizzichino 1963), Re ψ and Im ψ are distributed with *Gaussian* (Ga) statistics, and

$$\overline{I^n_{(\mathrm{Ga})}} = n!(\bar{I})^n. \tag{2}$$

This arises in weakly irregular media, or at great distances beyond finite slabs of media with irregularities of arbitrary strength, whatever their statistics (Mercier 1962); it is a kind of 'universality'. By contrast, as the phase variations get stronger so do the intensity fluctuations, and the moments ($n > 1$) exceed the values given by the equa(2). In numerical calculations of the second moment $\overline{I^2}$ this *non-Gaussian* behaviour was discovered by Mercier (1962) (see also Bramley and Young 1967); it was observed experimentally in twinkling starlight by Jakeman *et al* (1976) and in turbulent liquid crystals by Pusey and Jakeman (1975). The purpose of this paper is to draw attention to a different kind of universality that emerges in the limit of infinitely strong phase variations.

For a given medium that is not highly dispersive, one way to produce strong phase variations is by increasing the wavenumber k ($\equiv 2\pi/$wavelength, λ) of the incident radiation. Therefore the limit being discussed is the *geometrical optics limit* $k \to \infty$ (or for quantum waves the classical limit). In this limit the wave is dominated by the *caustics* of the associated family of rays. Caustics are focal surfaces, that is, envelopes of the rays; more recently it has been appreciated that they can be classified in terms of *catastrophes*, that is singularities of gradient mappings (Thom 1969, 1975, Duistermaat 1974, Arnol'd 1975, Berry 1976). On caustics I is infinite according to geometrical optics, and it was realised by Salpeter (1967) that this implies that all moments except the first are infinite. His argument ran as follows: let x be a coordinate running through the caustic from its dark side ($x < 0$) to its bright side ($x > 0$). Then, for the simplest caustic, the intensity according to geometrical optics varies as

$$I(x) \propto \operatorname{Re} x^{-1/2} \tag{3}$$

so that the contribution of this caustic to the nth moment is

$$\overline{I^n} \propto \int I^n(x)\,\mathrm{d}x \propto \int_0^a \frac{\mathrm{d}x}{x^{n/2}} \tag{4}$$

where a is some estimate of the spacing between caustics. Obviously the integrals diverge for $n > 1$, giving $\overline{I^n} \to \infty$.

Salpeter (1967) also argued that for finite k diffraction would soften the singularities at caustics, the effect being to replace the lower limit in (4) by a k-dependent cut-off. For the second moment this leads to the estimate

$$\overline{I^2} \propto \ln k. \tag{5}$$

This result has been confirmed, and the proportionality constant derived, in intricate analyses by Shishov (1971), Buckley (1971), Taylor and Infosino (1975) and Jakeman and McWhirter (1977).

These authors did not make use of the concept of caustics, but instead worked with the diffraction integral for ψ given by the *random phase screen model.* On this model the random medium is confined to the plane $z = 0$, and impresses a phase $\phi(\xi, \eta, t)$ on a plane wave of unit intensity travelling in the positive z direction (ξ, η are coordinates in the plane $z = 0$ and t is time). It is convenient here to write

$$\phi(\xi, \eta, t) = kf(\xi, \eta, t) - \omega t, \tag{6}$$

where $\omega(\equiv ck)$ is the angular frequency of the incident wave, so that $-f$ is approximately the deviation from the plane $z = 0$ that the screen produces in the wavefront at ξ, η, t. In the simple case where the slopes $|\partial f/\partial \xi|$, $|\partial f/\partial \eta|$ are small

(paraxiality) and $\partial f/\partial t \ll \omega/k$ (quasi-monochromaticity) it is shown in appendix 1 that at points (x, y, z, t) beyond the screen the wave is

$$\psi(x, y, z, t) = \frac{-\mathrm{i}k\,\mathrm{e}^{\mathrm{i}(kz-\omega t)}}{2\pi z} \int \mathrm{d}\xi \int \mathrm{d}\eta \exp\left[\mathrm{i}k\left(f\left(\xi+x, \eta+y, t-\frac{z}{c}-\frac{\xi^2}{2cz}-\frac{\eta^2}{2cz}\right)\right.\right.$$
$$\left.\left.+\frac{\xi^2+\eta^2}{2z}\right)\right] \qquad (z>0). \tag{7}$$

The authors quoted above proceeded by taking f as a smooth Gaussian random function and calculating the ensemble average of $|\psi|^4$. (On this model the first moment $\bar{I}$ is trivially unity, a result that is also a consequence of the conservation of energy.) This direct method is difficult enough for $\overline{I^2}$ but appears quite hopeless for higher moments because of the high-order multiple integrals involved.

Here, however, we do want to study these higher moments, to establish just how they diverge as $k \to \infty$. The divergences will be described by *critical exponents* ν_n, defined as

$$\nu_n \equiv \lim_{k\to\infty} \mathrm{d}(\ln \overline{I^n})/\mathrm{d}\ln k. \tag{8}$$

This definition is insensitive to factors $\ln k$ in $\overline{I^n}$ and gives for $\overline{I^2}$ on the basis of equation (5) a critical exponent of zero, completely failing to describe the large-k behaviour for this case. But the higher ν_n are not zero and there is some evidence suggesting that logarithmic factors occur only occasionally, if at all, when $n > 2$, so that (8) can be written as

$$\overline{I^n} \xrightarrow{k\to\infty} C_n k^{\nu_n}. \tag{9}$$

The critical exponents will be derived not from global diffraction integrals like equation (7) but directly from the catastrophe structure of the geometrical rays as softened by diffraction, on principles explained in § 3. First, however, it is necessary to set up the problem in terms of the parameters of the ensemble defining the random medium; this formulation, which illuminates the relation between ensemble and other averages, constitutes § 2. For media whose disorder depends on infinitely many parameters (e.g. a Gaussian random phase screen) the moments possess universality in that the ν_n depend only on whether the system is spatially two-dimensional (e.g. a 'corrugated' phase screen) or three-dimensional. The two-dimensional case (§ 4) is relatively straightforward and calculated ν_n are shown in table 1. However, the three-dimensional case (§ 5) depends on the extensive but still incomplete classifications of catastrophes developed by Arnol'd (1973, 1974, 1975) and the calculations of ν_n (table 3) only reach $n = 13$. The constants C_n in equation (9) are not universal but depend on the details of the random medium. When the random medium is specified by a finite number of parameters the higher critical exponents differ from the universal values. The simplest case corresponds to the (non-random) sinusoidal phase screen specified by a single parameter; $\overline{I^2}$ for this system is calculated in § 6 on a 'catastrophe' basis and found to agree very well with an exact calculation. In § 7 the experimental implications of the theory are discussed.

The theory presented here is strongly evocative of the 'renormalisation group' technique developed by Wilson (1975) for problems in statistical mechanics. Wilson's theory also deals with critical exponents that are universal for each dimension and certain 'relevant', 'irrelevant' and 'marginal' variables appear for which we shall find

analogues in § 3. It appears that Gaussian random wave fields correspond to 'mean-field' many-particle systems, while the short-wave limit considered here corresponds to many-particle systems approaching their critical points; some support for these analogies can be found in a review by Jona-Lasinio (1975) about connections between the renormalisation group and the breakdown of the central limit theorem of probability theory.

2. The torus of random phases

For definiteness the discussion of the random media will be in terms of the function $f(\xi, \eta, t)$ appearing in equations (6) and (7) of the phase screen model. Let f be a stationary random function with Fourier expansion

$$f(\xi, \eta, t) = \sum_{i=1}^{N} f_i \cos(u_i\xi - v_i\eta - \Omega_i t + \theta_i). \tag{10}$$

Let the ensemble for f have fixed amplitudes f_i, wavenumbers u_i and v_i and frequencies Ω_i and let members of the ensemble correspond to different choices of the *independent random phases* $\theta_1 \ldots \theta_N$ which can range from $-\pi$ to π with equal probability. The phases thus inhabit an *N-dimensional torus* T_N each point of which defines a function f in the ensemble. If N tends to infinity and u_i, v_i and Ω_i become densely distributed while f_i tends to a smooth function of u, v and Ω then $f(\xi, \eta, t)$ becomes a *Gaussian random function* (Rice 1944, 1945, Longuet-Higgins 1956). The simplest random function, however, is not Gaussian but has $N = 2$ and at least one of the ratios u_1/u_2, v_1/v_2, Ω_1/Ω_2 irrational.

For each member of the ensemble the diffraction integral (7) defines a wave $\psi(x, y, z, t; \theta_1 \ldots \theta_N)$ and ensemble averaging consists of integrating over T_n whatever function is to be averaged. Thus the moments (1) are

$$\overline{I^n} = \frac{1}{(2\pi)^N} \int \mathrm{d}\theta_1 \ldots \int \mathrm{d}\theta_N |\psi(x, y, z, t; \theta_1 \ldots \theta_N|^{2n}, \tag{11}$$

where x, y, z and t are held fixed. Thus the wave intensity is considered as a function on T_n rather than in space–time. This can be pictured as follows: the wave at x, y, z, t is painted on T_N with a density equal to $|\psi|^{2n}/(2\pi)^N$, and then $\overline{I^n}$ equals the mass of this decorated torus.

Because the randomness of f is stationary in ξ, η and t, the wave ψ will be a stationary random function of t for fixed x, y, z and a stationary random function of x, y for fixed t and z, so that alternative ways of writing the ensemble average (11) are as time or space averages over any single member of the ensemble:

$$\overline{I^n} = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{+T/2} \mathrm{d}t |\psi|^{2n} = \lim_{L\to\infty} \frac{1}{L^2} \int_{-L/2}^{+L/2} \mathrm{d}x \int_{-L/2}^{+L/2} \mathrm{d}y |\psi|^{2n}, \tag{12}$$

where now $\theta_1 \ldots \theta_N$ are fixed. The relation between (11) and (12) can be understood by inspecting (7) and (10). Time averaging corresponds to averaging along the following path on T_N:

$$\theta_i = -\Omega_i t. \tag{13}$$

This path winds round the torus and as $t \to \infty$ covers it densely and uniformly if the

frequencies Ω_i are incommensurable as they generically will be (Arnol'd and Avez 1968, appendix 1, Born 1960, appendix 1). Space averaging corresponds to averaging over the following surface on T_N:

$$\theta_i = u_i x + v_i y. \tag{14}$$

This surface will fill T_N densely and uniformly if the u_i and the v_i are incommensurable (this is obvious for T_3 considered as a cell of a lattice in $\theta_1\theta_2\theta_3$ with spacing 2π, in which case (14) is an irrationally oriented plane). It should not be forgotten, however, that the time and space averages (12) are valid only when f is a stationary random function, whereas the average (11) over the parameters of the ensemble is more fundamental and holds when (12) breaks down (e.g. when the random medium is illuminated with a spherical wave, or is rigidly translating or rotating or changing in some other determinate fashion).

The advantage of the formulation in terms of T_N, which will be crucial in extracting the short-wave behaviour, is that it transforms a *random* function ψ of the infinite-range variables x, y, t into a *deterministic* function on the compact manifold T_N. The randomness arises during repeated irrational windings described by equations (13) and (14).

3. Catastrophes on the torus

As $k \to \infty$ the principal contributions to ψ come from the rays of geometrical optics. On the random phase screen model the rays for given $x, y, z, t; \theta_1 \ldots \theta_N$ come from those points ξ, η on the screen for which the phase in the integral of equation (7) is extremal. Denoting by v that part of the phase not involving k, i.e.

$$v(\xi, \eta; x, y, z, t, \theta_1 \ldots \theta_N) \equiv f\left(\xi + x, \eta + y, t - \frac{z}{c} - \frac{(\xi^2+\eta^2)}{2cz}, \theta_1 \ldots \theta_N\right) + \frac{\xi^2+\eta^2}{2z}, \tag{15}$$

the rays are defined by

$$\frac{\partial v}{\partial \xi} = 0, \qquad \frac{\partial v}{\partial \eta} = 0. \tag{16}$$

In the language of catastrophe theory (Poston and Stewart 1976) v is a 'potential function', and (16) defines a 'gradient mapping' between the 'state variables' ξ, η and the 'control parameters' $x, y, z, t, \theta_1 \ldots \theta_N$. In the ensemble averaging over T_N oly the phases θ_i, vary, and x, y, z, t can be omitted from the list of control variables (in fact x, y and t can be set equal to zero in view of (13) and (14)). *Caustics* on T_N occur for θ_i-values for which the mapping is singular so that in addition to (16) the equation

$$\frac{\partial^2 v}{\partial \xi^2}\frac{\partial^2 v}{\partial \eta^2} - \left(\frac{\partial^2 v}{\partial \xi\, \partial \eta}\right)^2 = 0 \tag{17}$$

holds. According to Jänich (1974) the Hamiltonian nature of ray propagation ensures that caustics will be determined by a similar formalism in spatially inhomogeneous media where the phase screen model does not apply.

The caustics dominate the integral over T_N in the ensemble average (11). To work out how they contribute as $k \to \infty$ it is necessary to understand their structure, and here we quote results from catastrophe theory (Thom 1975, Arnol'd 1975). In the

generic case (i.e. for 'almost all' potential functions v) the caustics are singularities on T_N in the form of 'surfaces' of dimension $d = N-1$. These 'surfaces' are themselves singular on 'lines' with $d = N-2$. The 'lines' have singularities with $d = N-3$ and the hierarchy extends to the highest generic singularities which are points in T_N with $d = 0$. The *codimension* K of each singularity is defined by

$$K = N - d \tag{18}$$

and measures the dimension of submanifolds on T_N that may typically intersect the singularities in isolated points. Thus T_N can have singularities with codimensions 1 to N.

Near any singularity two new state variables X and Y, and K new control variables $\Theta_1 \ldots \Theta_K$ related to the original variables by a smooth transformation (a diffeomorphism), can be found, taking v into one of a list of *normal forms*, or *catastrophes*, denoted by V_j and expanded thus:

$$V_j(X, Y; \Theta_1 \ldots \Theta_K) = G_j(X, Y) + \sum_{l=1}^{K} \Theta_l U_{lj}(X, Y). \tag{19}$$

The new controls Θ_l can be considered as K canonical local coordinates on T_N normal to the singularity of codimension K. On the singularity itself all Θ_l vanish and the potential function is the 'germ' G_j which for the simpler catastrophes is the sum of two monomials in X and Y, i.e.

$$G_j(X, Y) = X^{p_{1j}} Y^{q_{1j}} + X^{p_{2j}} Y^{q_{2j}} \tag{20}$$

(more complicated cases, involving more terms, arise from 'modality' (Arnol'd 1975) and will be discussed in § 5). The linear terms in Θ_l describe how the catastrophe 'unfolds' away from the singularity on T_N, and the U_{lj} are monomials, i.e.

$$U_{lj}(X, Y) = X^{a_{lj}} Y^{b_{lj}}. \tag{21}$$

The exponents p, q, a and b are all non-negative integers.

The jth catastrophe gives a separate contribution $\overline{I_j^n}$ to the nth moment $\overline{I^n}$ as $k \to \infty$ provided the catastrophe is 'local' in a sense to be described below. To evaluate this contribution from the diffraction integral (7) and the ensemble average (11), set up coordinates θ_i on T_N such that $\theta_1 \ldots \theta_K$ are locally 'normal' to the singularity j and $\theta_{K+1} \ldots \theta_N$ are locally 'parallel' to it. Then transform $\theta_1 \ldots \theta_K$ into the canonical $\Theta_1 \ldots \Theta_K$ appearing in the normal form (19) and ξ, η into the canonical X, Y. Then $\overline{I_j^n}$ can be written as

$$\overline{I_j^n} = B_j \int \mathrm{d}\Theta_1 \ldots \int \mathrm{d}\Theta_K \left| k \int \mathrm{d}X \int \mathrm{d}Y \exp(\mathrm{i}kV_j(X, Y; \Theta_1 \ldots \Theta_K)) \right|^{2n}, \tag{22}$$

where B_j corresponds to the 'measure' of the jth catastrophe on T_N and results from transforming variables and integrating over $\theta_{K+1} \ldots \theta_N$ along the singularity. (It is assumed that B_j is finite.)

It is now only a matter of algebra to extract the k dependence of (22). The substitutions

$$X = k^{-\mu_j} X', \qquad Y = k^{-\lambda_j} Y' \tag{23}$$

where

$$\mu_j = \frac{q_{2j} - q_{1j}}{q_{2j}p_{1j} - q_{1j}p_{2j}}, \qquad \lambda_j = \frac{p_{2j} - p_{1j}}{q_{1j}p_{2j} - q_{2j}p_{1j}} \tag{24}$$

effect the transformation

$$kG_j(X, Y) = G_j(X', Y'). \tag{25}$$

This follows from equation (20) and has the effect of eliminating k from the germ of the integrand of ψ. This leaves within the modulus signs in (22) the factor k^{β_j}, where

$$\beta_j = 1 - \mu_j - \lambda_j = 1 + \frac{q_{1j} - q_{2j} - p_{1j} + p_{2j}}{q_{2j}p_{1j} - q_{1j}p_{2j}}. \tag{26}$$

Arnol'd (1973) called the number β_j the 'singularity index' of the catastrophe j and evaluated it for many cases; the calculations have been simplified by Varchenko (1976). It is a measure of the 'strength' of diffraction at a caustic locally equivalent to the jth catastrophe, and is defined in general by

$$\beta_j = \lim_{k\to\infty} \frac{\mathrm{d}\ln\psi}{\mathrm{d}\ln k} \tag{27}$$

where ψ is evaluated at a point on the catastrophe.

The moments, however, depend not only on the 'strength' of the catastrophe but also on its 'width', because of the Θ integrations in the ensemble average (22). The transformation (23) sends this into the following expression, which is a consequence of (19) and (21):

$$\overline{I_j^n} = B_j k^{2n\beta_j} \int \mathrm{d}\Theta_1 \ldots \int \mathrm{d}\Theta_K \left| \int \mathrm{d}X' \int \mathrm{d}Y' \exp(\mathrm{i}V_j(X', Y'; k^{\sigma_{1j}}\Theta_1 \ldots k^{\sigma_{Kj}}\Theta_K)) \right|^{2n}, \tag{28}$$

where

$$\sigma_{lj} \equiv 1 - \mu_j a_{lj} - \lambda_j b_{lj}. \tag{29}$$

Usually σ_{lj} will be positive and we shall say that Θ_l is a 'relevant' variable for a reason soon to become apparent. Then (28) shows that as $k \to \infty$ ψ oscillates increasingly rapidly as the phases Θ_l vary, and the diffraction pattern 'condenses' onto the caustic. The k dependence can be extracted by the obvious transformation

$$\Theta_l = k^{-\sigma_{lj}}\Theta_l', \tag{30}$$

and (28) becomes, finally

$$\overline{I_j^n} = B_j J_{jn} k^{2n\beta_j - \gamma_j}, \tag{31}$$

where

$$\gamma_j \equiv \sum_{l=1}^{K} \sigma_{lj} = \sum_{l=1}^{K} (1 - \mu_j a_{lj} - \lambda_j b_{lj}) \tag{32}$$

and

$$J_{jn} \equiv \int \mathrm{d}\Theta_1 \ldots \int \mathrm{d}\Theta_K \left| \int \mathrm{d}X \int \mathrm{d}Y \exp(\mathrm{i}V_j(X, Y; \theta_1 \ldots \Theta_K)) \right|^{2n}. \tag{33}$$

For each catastrophe the numbers J_{jn}, β_j and γ_j are invariant under the equivalence of unfoldings, but the numbers B_j depend on how the catastrophe is embedded in T_N, i.e. B_j are not invariants under equivalence of unfoldings.

The exponent in equation (31), namely

$$\nu_{nj} \equiv 2n\beta_j - \gamma_j, \tag{34}$$

gives the contribution of the jth catastrophe to $\overline{I^n}$. The dominant contribution to the nth moment comes from the catastrophe for which ν_{nj} is largest. Therefore the quantities towards the evaluation of which the whole analysis has been directed, the critical ν_n defined by (8) and (9), are given by

$$\nu_n = \max(\nu_{nj}), \tag{35}$$

and we have arrived at the central result of this paper. Given a list of all catastrophes j with codimension K less than or equal to the number N of random phases defining the medium on which the wave is incident, the indices p_{1j}, p_{2j}, q_{1j}, q_{2j} characterising the germ (equation (20)) and the indices a_{lj}, b_{lj} characterising the unfolding (equation (21)) can be determined by inspection. Then μ_j and λ_j (equation (24)) and the singularity index β_j (equation (26)) can be computed as well as γ_j (equation (32)) and finally ν_{nj} from equation (34).

Before carrying out this programme in the next two sections some remarks must be made. First, for the general random phase screen varying in ξ, η and t the associated catastrophe theory involves just two state variables ξ and η. Therefore catastrophes whose 'corank'—i.e. number of 'essential' state variables (see Arnol'd 1975)—exceeds two can be ignored. This result also holds in three-dimensional spatially inhomogeneous time-varying media, since a ray passing through x, y, z, t can be labelled with the two coordinates ξ, η of the point where it intersected any fixed reference plane prior to striking the medium. If, however, the medium is translationally invariant in one direction, say η (e.g. a random phase screen corrugated along ξ) then the wave propagates essentially in two space dimensions x and z; there is just one essential state variable and the only catastrophes that need be considered are those with corank unity, for which a complete list exists so that all critical exponents ν_n can be computed (§ 4).

Second, for some catastrophes of corank two (to be studied in § 5) one or more of the indices σ_{lj} (equation (29)) is negative. In these cases it follows from equation (28) that the corresponding control parameters Θ_l do not contribute to the variation of the wave across T_N as $k \to \infty$. We shall call such Θ_l 'irrelevant' variables since they do not contribute to the 'weight' of the catastrophe and should be transferred to the set of variables $\Theta_{K+1} \ldots \Theta_N$ whose integration 'along' the caustic contributes to the constant B_j in equation (22). The negative σ_{lj} from these irrelevant variables must be omitted from the summation (32) for γ_j. Occasionally an index σ_{lj} is zero. We shall call the corresponding Θ_l a 'marginal' variable; its effect on the variation of the wave across the catastrophe does not depend on k, and so does not contribute to the critical exponents ν_n.

Third, if our picture of the short-wave limit is to be valid the catastrophe j that satisfies (35) and thus dominates $\overline{I^n}$ must indeed condense onto the appropriate singularity in T_N as $k \to \infty$. To ensure this it is not sufficient that the indices σ_{lj} are positive: the catastrophe must also be 'local', in that the 'weight' integral J_{jn} (equation (33)) must converge, that is it must not be dominated by large values of the variables Θ_l. For corank two catastrophes such convergence seems very difficult to analyse, but in § 4 we show that for catastrophes of corank unity the integrals do in fact converge for the j satisfying (35).

Fourth, the critical exponents depend on the number N of random phases. For a Gaussian random medium $N = \infty$ and all catastrophes contribute to the hierarchy of exponents. We prove in the next section that for catastrophes of corank unity the codimension K of the dominant catastrophe increases with the order n of the

moment considered. The results of § 5 suggest that catastrophes of corank two behave similarly. This means that when N is finite the lower-order moments have the same critical exponents as in the 'universal' Gaussian random medium case. However there is a certain order $n = n_c$ for which the dominant catastrophe has codimension $K = N$. Higher moments $n > n_c$ are dominated by the same catastrophe and so their critical exponents will be smaller than in the Gaussian random medium case. In these circumstances (at least for the random phase screen problem) the dependence of $\overline{I^n}$ on the extra control parameter z, which we have not emphasised until now, becomes important, for there will be one value of z equal to the smallest radii of curvature of the initial wavefront, at which catastrophes of codimension $N+1$ occur, altering the critical exponents. Section 6 will contain an example of this. (The dependence on z noted by previous workers in the Gaussian medium case corresponds to variations in the geometric factor B_j in equation (22) rather than changes in critical exponents and reflects the different densities of caustics in xy planes with different z.)

4. Two space dimensions

Here only catastrophes of corank unity can occur; these are the *cuspoids*. The first four members of this family are the fold, cusp, swallowtail and butterfly in the list of Thom (1975). In the more extensive classification of Arnol'd (1974, 1975) the cuspoid catastrophes are denoted by A_j and their potential function (equation (19)) is

$$V_j(X, Y; \Theta_1 \ldots \Theta_{j-1}) = X^{j+1} + Y^2 + \sum_{l=1}^{j-1} \Theta_l X^l. \tag{36}$$

The suffix j denotes the 'multiplicity' of the catastrophe; this is the number of rays touching at the most singular place $\Theta_l = 0$ on the caustic, i.e. the degree of degeneracy of the germ G_j. The codimension of A_j, is $K = j-1$. It is only as a matter of convenience that the 'inessential' term Y^2 has been included in equation (36); this cannot affect the unfolding of the catastrophes (since Y^2 has only a single non-degenerate extremum) but it enables the formalism of § 3, in terms of two state variables, to be employed here without modification.

By inspection of (36) the indices defined in equations (20) and (21) have the values

$$p_{1j} = j+1, \qquad q_{1j} = 0, \qquad p_{2j} = 0, \qquad q_{2j} = 2, \qquad a_{lj} = l, \qquad b_{lj} = 0, \tag{37}$$

so that μ_j and λ_j (equation (24)) are

$$\mu_j = 1/(j+1), \qquad \lambda_j = \tfrac{1}{2}. \tag{38}$$

From equation (26) the singularity index is

$$\beta_j = \frac{j-1}{2(j+1)} = \frac{K}{2(K+2)}, \tag{39}$$

while equation (32) leads to

$$\gamma_j = \frac{(j+2)(j-1)}{2(j+1)} = \frac{K^2+3K}{2(K+2)} \tag{40}$$

(all σ_{lj} (equation (29)) are positive so that all Θ_l are relevant variables). Therefore the

exponent governing the contribution of A_j to $\overline{I^n}$ is, from equation (34),

$$\nu_{nj} = \frac{(j-1)(2n-j-2)}{2(j+1)} = \frac{K(2n-K-3)}{2(K+2)}. \tag{41}$$

From the graph of ν_{nj} as a function of K it is obvious that as $k \to \infty$ $\overline{I^n}$ will contain contributions from catastrophes with codimensions 1 to $2n-3$. The maximum on the graph occurs at

$$K = K_{\max}(n) \equiv 2(\sqrt{n-\tfrac{1}{2}}-1). \tag{42}$$

Therefore the dominant catastrophe has codimension $[K_{\max}(n)]$ or $1+[K_{\max}(n)]$ (using square brackets to denote 'the integral part of') and the critical exponent is, from equation (35),

$$\nu_n = \max\left(\frac{[K_{\max}(n)](2n-[K_{\max}(n)]-3)}{2([K_{\max}(n)]+2)}, \frac{(1+[K_{\max}(n)])(2n-[K_{\max}(n)]-4)}{2([K_{\max}(n)]+3)}\right). \tag{43}$$

Although ν_n cannot be expressed in analytic form its determination for any n involves only two elementary arithmetic computations. Table 1 shows ν_n for $n \leqslant 13$ together with the codimension K of the dominant catastrophe A_{K+1}; the values of ν_n against n are plotted on figure 1. An analytic approximation to ν_n, obtained by substituting $K_{\max}(n)$ into equation (41), is

$$\nu_n(\text{approx}) = n - 2\sqrt{n-\tfrac{1}{2}} + \tfrac{1}{2}. \tag{44}$$

Table 1. Catastrophes of corank unity. For the moment $\overline{I^n}$, K is the codimension of the dominant catastrophe(s), ν_n the critical exponent and ν_n(approx) the value given by equation (44).

n	2	3	4	5	6	7	8	9	10	11	12	13
K	1	1	2	2	3	3	3 and 4	4	4	4 and 5	5	5
ν_n	0	$\frac{1}{3}$	$\frac{3}{4}$	$\frac{5}{4}$	$\frac{9}{5}$	$\frac{12}{5}$	3	$\frac{11}{3}$	$\frac{13}{3}$	5	$\frac{40}{7}$	$\frac{45}{7}$
ν_n(approx)	0·051	0·388	0·758	1·257	1·810	2·401	3·023	3·669	4·336	5·019	3·718	6·429

Values of ν_n(approx) are also shown on table 1 and as the continuous curve on figure 1; the approximation is clearly very accurate. It appears from table 1 that higher catastrophes dominate more moments $\overline{I^n}$ than lower ones, and indeed it follows from equation (42) that as $K \to \infty$ the number of moments dominated by the catastrophe of codimension K is $K/2$. Occasionally two catastrophes give equal exponents; for example $\overline{I^8}$ is dominated by the swallowtail ($K=3$) and the butterfly ($K=4$).

These calculations of ν_n are valid only if the contributing catstrophes are 'local', that is if the 'weight' integral J_{jn} (equation (33)) converges. To establish convergence it is sufficient to study the behaviour of the integral along the manifolds of slowest decay away from the origin in the space of control parameters $\Theta_1 \ldots \Theta_K$. These manifolds are lines of codimension $K-1$, defined in terms of the potential function (36) by

$$\frac{\partial Y_j}{\partial Y} = 0; \qquad \frac{\partial^s V_j}{\partial X^s} = 0 \qquad (1 \leqslant s \leqslant K). \tag{45}$$

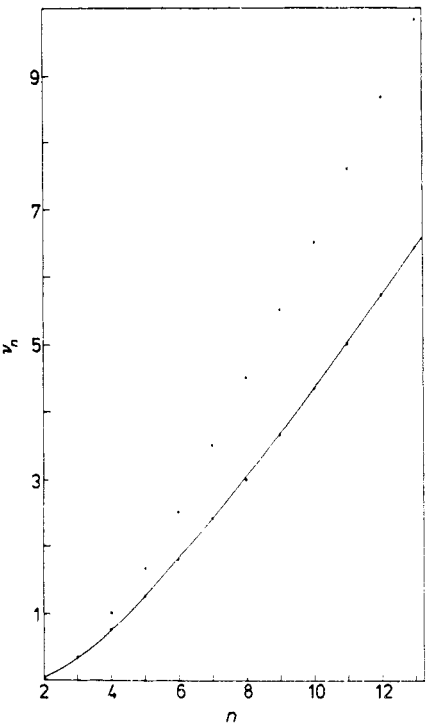

Figure 1. Critical exponents for two space dimensions (lower points) and three space dimensions (upper points). The full curve is the approximation (44).

The last of these equations ($s = K$) gives

$$X = X_c \equiv \pm\left(\frac{-2\Theta_K}{(K+2)(K+1)}\right)^{1/2}, \tag{46}$$

and then the penultimate ($s = K-1$) and earlier ($s = K-2, K-3\ldots$) equations lead successively to

$$\begin{aligned} &\Theta_{K-1} = \pm(-2\Theta_K)^{3/2}K/[(K+1)(K+2)]^{1/2}, \qquad \Theta_{K-2} = O(\Theta_K^2), \\ &\Theta_{K-l} < O(\Theta_K^2) \qquad (l > 2). \end{aligned} \tag{47}$$

Therefore the manifolds of slowest decay of the catastrophes A_{K+1} are *cusped lines* approaching $\Theta_l = 0$ in the plane Θ_K, Θ_{K-1}; an example is the cusp-shaped line of cusp edges ($K = 2$) in the swallowtail catastrophe ($K = 3$).

For Θ_l on these cusped lines, V_j in (33) has a stationary point of order $K-1$ (i.e. K coalescing zeros) at X_c (equation (46)) so that the integral over X, Y can be evaluated using the expansion

$$\begin{aligned} V_j &\approx \text{constant} + Y^2 + \frac{(X-X_c)^{K+1}}{K!}\frac{\partial^{K+1}V_j(X_c)}{\partial X^{K+1}} \\ &= \text{constant} + Y^2 + (X-X_c)^{K+1}(K+2)(K+1)X_c. \end{aligned} \tag{48}$$

This gives

$$\int \mathrm{d}X \int \mathrm{d}Y \exp(\mathrm{i}\, V_j(X, Y; \Theta_1 \ldots \Theta_K)) = \mathrm{O}(X_c^{-1/(K+1)}) = \mathrm{O}(\Theta_K^{-1/2(K+1)}) \tag{49}$$

where the Θ_l are related by (47). Therefore the contribution to J_{jn} (equation (33)) arising from integration along the cusped lines is

$$J_{jn} \propto \int \frac{\mathrm{d}\Theta_K}{\Theta_K^{n/K+1}} = \begin{cases} \text{finite} & \text{if } K < n-1 \\ \text{infinite} & \text{if } K > n-1. \end{cases} \tag{50}$$

Inspection of table 1 or reasoning based on equation (42) shows that the catastrophe dominating $\overline{I^n}$ always has $K < n-1$ so that J_{jn} is always finite and the catastrophe does indeed have the necessary 'local' property, except for the anomalous case $n = 2$ (cf § 1, especially (5)).

5. Three space dimensions

Here catastrophes of corank two occur in addition to those of corank unity. The classification of normal forms V_j is still incomplete and greatly complicated by the phenomenon of 'modality'. This is the generic occurrence of germs G_j (equation (19)) depending smoothly on one or more 'moduli' a in such a way that the unfoldings for different values of a are not equivalent under a diffeomorphism, although they do have a weaker topological equivalence.

We illustrate this by an imaginary example (not structurally stable) in which a two-dimensional caustic surface in T_3 has a cusped edge, along which a is the coordinate and normal to which θ_2 and θ_3 are coordinates, and where the cusped sections of the caustic have equations

$$\theta_3 = \begin{cases} \pm C(a)\theta_2^{\alpha(a)} & (\theta_2 > 0) \\ 0 & (\theta_2 < 0), \end{cases} \tag{51}$$

$C(a)$ and $\alpha(a)$ being smooth functions and $\alpha > 1$. If $\alpha(a)$ were constant (e.g. $\frac{3}{2}$ as for the standard cusp) then the sections for different a would be equivalent under diffeomorphism. But this will not be the case if $\alpha(a)$ is not constant, even though each section still has the topology of a cusped curve (i.e. it reverses direction at a point where it crosses its tangent). (For a discussion based on a realistic example see pp 69–73 of Poston and Stewart (1976).)

In table 2 we list all the 43 catastrophes with codimension $K \leqslant 11$, extracted from the lists of Arnol'd (1973, 1974, 1975). Of these A_j, D_j, E_6, E_7 and E_8 have no modulus a and the rest are unimodal. For $K \geqslant 12$ some catastrophes are bimodal, trimodal, etc and we shall not consider them. Sometimes Arnol'd employs different symbols to denote the same catastrophe; in table 2 we use what seems the simplest notation in these cases. The unfolding monomials U_{lj} (equations (19) and (21)) in table 2 are taken from Arnol'd (1974) with the exception of those for J_{j+4} which were worked out using techniques explained by Poston and Stewart (1976). No distinction is made between catastrophes equivalent under complex transformations of variables even though their caustics may be topologically different (e.g. the elliptic and hyperbolic umbilics D_4); in the present application this will not lead to error.

Table 2. Catastrophes of corank unity and two and codimension $K \leqslant 11$. G_j is the germ, β_j the singularity index, U_{lj} the unfolding monomials and γ_j the index defined by equation (32). All these catastrophes are either non-modal or else have the single modulus a.

K	Symbol	$G_j(X, Y)$	β_j	$U_{lj}(X, Y)$	γ_j
$j-1$	$A_j (j \geqslant 2)$	$X^{j+1}+Y^2$	$\frac{j-1}{2(j+1)}$	$X \ldots X^{j-1}$	$\frac{(j+2)(j-1)}{2(j+1)}$
$j-1$	$D_j (j \geqslant 4)$	X^2Y+Y^{j-1}	$\frac{j-2}{2(j-1)}$	$X, Y \ldots Y^{j-2}$	$\frac{j^2-2j+2}{2(j-1)}$
5	E_6	X^3+Y^4	$\frac{5}{12}$	X, XY, XY^2, Y, Y^2	$\frac{5}{2}$
6	E_7	X^3+XY^3	$\frac{4}{9}$	$X, XY, Y \ldots Y^4$	$\frac{26}{9}$
7	E_8	X^3+Y^5	$\frac{7}{15}$	$X, XY \ldots XY^3, Y \ldots Y^3$	$\frac{49}{15}$
7	X_9	$X^4+X^2Y^2+aY^4$	$\frac{1}{2}$	$X, X^2, XY, XY^2, X^2Y, Y, Y^2$	$\frac{7}{2}$
$j+3$	$X_{j+5} (j \geqslant 5)$	$X^4+X^2Y^2+aY^j$	$\frac{1}{2}$	$X \ldots X^3, XY, Y \ldots Y^{j-1}$	$\frac{7}{2}$
8	J_{10}	$X^3+X^2Y^2+aY^6$	$\frac{1}{2}$	$X, XY \ldots XY^3, Y \ldots Y^4$	4
$j+2$	$J_{j+4} (j \geqslant 11)$	$X^3+X^2Y^2+aY^j$	$\frac{1}{2}$	$X, XY, XY^2, Y \ldots Y^{j-1}$	4
$p+q-1$	$Y_{p,q} (5 \leqslant p \leqslant q)$	$X^p+X^2Y^2+aY^q$	$\frac{1}{2}$	$X \ldots X^{p-1}, XY, Y \ldots Y^{q-1}$	$\frac{p^2+12}{2p}$
9	Z_{11}	$X^3Y+Y^5+aXY^4$	$\frac{8}{15}$	$X, X^2, XY \ldots XY^3, Y \ldots Y^4$	$\frac{21}{5}$
10	Z_{12}	$X^3Y+XY^4+aX^2Y^3$	$\frac{6}{11}$	$X, X^2, XY \ldots XY^3,$ $X^2Y, X^2Y^2, Y \ldots Y^3$	$\frac{50}{11}$
11	Z_{13}	$X^3Y+Y^6+aXY^5$	$\frac{5}{9}$	$X, X^2, XY \ldots XY^4, Y \ldots Y^5$	$\frac{44}{9}$
10	W_{12}	$X^4+Y^5+aX^2Y^3$	$\frac{11}{20}$	$X, X^2, XY \ldots XY^3, X^2Y,$ $X^2Y^2, Y \ldots Y^3$	$\frac{9}{2}$
11	W_{13}	$X^4+XY^4+aY^6$	$\frac{9}{16}$	$X, X^2, XY, XY^2, X^2Y, X^2Y^2,$ $Y \ldots Y^5$	$\frac{77}{16}$
10	E_{12}	$X^3+Y^7+aXY^5$	$\frac{11}{21}$	$X, XY \ldots XY^4, Y \ldots Y^5$	$\frac{100}{21}$
11	E_{13}	$X^3+XY^5+aY^8$	$\frac{8}{15}$	$X, XY \ldots XY^3, Y \ldots Y^7$	$\frac{77}{15}$

The contribution ν_{nj} of the catastrophe j to $\overline{I^n}$ depends on the indices β_j and γ_j defined by equations (26) and (32), and these quantities are shown on table 2. For the zero-modal catastrophes the calculation of β_j and γ_j is straightforward as described in § 3, and there are no irrelevant or marginal variables. The unimodal germs each have three terms and it is not obvious which pair to choose in order to define the indices p and q in equation (20). Moreover the modulus a, although not a control parameter in the sense of catastrophe theory, is nevertheless a variable on T_N and must be integrated over in the ensemble averaging for $\overline{I^n}$. In fact the correct procedure is simply to ignore the terms involving a, thus making the calculation of β_j and γ_j straightforward as before. Of course this requires justification which will now be given.

Consider first the catastrophe Z_{11}. The indices in equation (24) are $\mu = \frac{4}{15}$ and $\nu = \frac{1}{5}$, so that the transformation (23) changes the modal term as follows:

$$kaXY^4 = k^{-1/15} aX'Y'^4. \tag{52}$$

Therefore the modal term becomes insignificant as $k \to \infty$ and a is an irrelevant variable in the sense explained in § 3 and so does not contribute to ν_{nj}. All control parameters Θ_l are relevant variables. If instead of X^3Y+Y^5 either of the pairs of monomials in G_j involving a had been chosen, then after the transformation (23) the remaining monomial would have been multiplied by a positive power of k and could not be neglected as $k \to \infty$. These results for Z_{11} apply also to Z_{12}, Z_{13}, W_{12}, W_{13}, E_{12}

and E_{13}, thus justifying the corresponding entries in table 2. (The apparent alternative choices in the cases of W_{13} and E_{13} of $XY^4 + aY^6$ and $XY^5 + aY^8$ respectively, for which the terms X^4 and X^3 respectively would appear irrelevant, are in fact forbidden since the resulting integrals over X, Y are divergent when $\Theta_l = 0$.)

The remaining catastrophes are those of 'hyperbolic' type, and make up the families X, J and Y on table 2. The simplest two are X_9 and J_{10}, for which the transformation (23) neatly eliminates k from the modal terms aY^4 and aY^6. Therefore a is a marginal variable in the sense explained in § 3; all control parameters Θ_l are relevant. (For J_{10} Arnol'd (1974) gives a ninth unfolding monomial XY^4; the corresponding control parameter would be marginal.) In the remaining hyperbolic catastrophes application of (23) shows that a is an irrelevant variable. However if the modal term is simply ignored the integral over X, Y (with $\Theta_l = 0$) diverges, and this state of affairs cannot be avoided by choosing one of the two pairs of monomials in G_j involving a instead of the pair not involving a. A careful analysis shows that the term a does not affect the singularity index but contributes a factor $\ln k$ to the asymptotic behaviour of ψ when $\Theta_l = 0$. For X_{j+5}, J_{j+4}, $Y_{p,q}$ the control parameters multiplying respectively Y^4, Y^6 and $Y^{2p/p-2}$ are marginal, while those multiplying higher powers of Y are irrelevant.

This is as far as we can go in justifying the entries in table 2. A complete justification would involve an analysis of the convergence of the integrals J_{jn} (equation (33)) similar to that carried out for the catastrophes of corank unity at the end of § 4. It seems very difficult to do this, especially for the hyperbolic catastrophes, so that the remainder of this section is conjecture.

Each critical exponent ν_n is the result of competition between all the catastrophes j in table 2 according to (35) and (34). Some catastrophes do not contribute to any moment $\overline{I^n}$ since for any two catastrophes j and j' if $\beta_j > \beta_{j'}$ and $\gamma_j < \gamma_{j'}$ then $\nu_{nj} > \nu_{nj'}$ for all n. For example, this criterion eliminates E_{12} in favour of W_{12}, E_{13} in favour of W_{13}, and the families J and Y in favour of the family X. The surviving catastrophes determine ν_n by the following geometric construction. Plot ν_{nj} (equation (34)) as a function of n for each j; then ν_n is the highest envelope of the resulting family of straight lines. In practice this is difficult to carry out with sufficient accuracy, and it is easier to proceed by inspection. The result is table 3 and the upper line of points on figure 1.

Table 3. Catastrophes of corank two or less dominating the moment $\overline{I^n}$. The symbol identifies the catastrophe on the list in table 2, K is the codimension and ν_n the critical exponent.

n	2	3	4	5	6	7	8	9	10	11	12	13
Symbol	A	A_2 and D_4	D_4	D_4 and E_6	E_6 and X	X	X	X	X and W_{12}	W_{12}	W_{12}	W_{13}
K	1	1 and 3	3	3 and 5	5 and $\geqslant 7$	$\geqslant 7$	$\geqslant 7$	$\geqslant 7$	$\geqslant 7$ and 10	10	10	11
ν_n	0	$\frac{1}{3}$	1	$\frac{5}{3}$	$\frac{5}{2}$	$\frac{7}{2}$	$\frac{9}{2}$	$\frac{11}{2}$	$\frac{13}{2}$	$\frac{38}{5}$	$\frac{87}{10}$	$\frac{157}{16}$

By comparison with table 1 it is clear that ν_n increases faster and less regularly than for waves in two space dimensions. There are several sources of uncertainty in table 3. First, in view of the difficulties already mentioned connected with the hyperbolic catastrophes it is not clear whether all or some members of the family X dominate the moments $\overline{I^6}$ to $\overline{I^{10}}$. Second, in view of the incompleteness of existing classifications of catastrophes of corank two it is possible that some catastrophe with

$K > 11$ might have a β_j so large and a γ_j so small as to overwhelm some or all of those in table 3 and dominate some or all moments of order $n \leqslant 13$; the trend of table 3 makes this seem unlikely. And third, the form of ν_n as $n \to \infty$ (cf equation (44)) is unknown, although since β_j increases with K and from (26) can never exceed unity we conjecture

$$\lim_{n\to\infty} \nu_n/n = 2 \tag{53}$$

(actually in the computations of Varchenko 1976, $\beta_j \leqslant \frac{3}{4}$ for catastrophes of corank two, so the highest critical exponents presumably come from the unclassified catastrophes N at the end of the list of Arnol'd 1975).

6. Example: the sinusoidal wavefront

The simplest phase screen produces a stationary corrugated wavefront with a single Fourier component:

$$f(\xi) = F\cos(u\xi + \theta). \tag{54}$$

Ensemble averaging consists in integrating over the 'torus' T_1, which in this non-random case is a circle with coordinate θ. The second moment $\overline{I^2}$ can be evaluated exactly using the diffraction integral (7) with the result

$$\overline{I^2}(z) = \sum_{m=-\infty}^{\infty} J_m^2\left(2kF\sin\frac{zu^2m}{2k}\right), \tag{55}$$

where J_m denotes the ordinary Bessel function. Details of the calculation are given in appendix 2.

The behaviour of $\overline{I^2}$ as $k \to \infty$ can be found for different regions z by direct asymptotic evaluation of (55); it is more instructive, though, to employ a physical argument leading to the same result. With the single control parameter θ we expect at worst fold catastrophe A_2 for typical z. However as z varies through the 'focal' value

$$z = z_f = \frac{1}{\max(-\partial^2 f/\partial\xi^2)} = \frac{1}{Fu^2} \tag{56}$$

a cusp catastrophe A_3 occurs at $\theta = 0$ with three extrema of $f(\xi)$ coalescing at $\xi = 0$. In space there are cusps at points $x = 2\pi l/u$ (l integer), $z = z_f$ from which caustic lines recede to $z = +\infty$. Here we restrict ourselves to $z = z_f$, returning to more general z in the last paragraph of this section.

On the assumption of § 3 that the catastrophe is local on T_1 as $k \to \infty$ we expand the phase $v(\xi, \theta)$ (equation (15)) for small ξ and θ, setting $x = 0$ without loss of generality. Then performing the trivial integration over η gives the following result (which is an instance of equation (22)):

$$\overline{I^2}(z_f) = \left(\frac{1}{2\pi z_f}\right)^2 \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\theta \left| k^{1/2}\int_{-\infty}^{\infty} \mathrm{d}\xi \exp\left[\mathrm{i}kF\left(\frac{u^4\xi^4}{24} - \theta u\xi\right)\right]\right|^4. \tag{57}$$

In real space this corresponds to integrating through the 'cusp' diffraction function of

Pearcey (1946). Scaling and evaluating the θ integral gives

$$\overline{I^2}(z_t)=\frac{(2kF)^{1/4}3^{3/4}}{(2\sqrt{2})\pi^2}\mathscr{I}, \tag{58}$$

where $\mathscr{I}$ is a number defined by

$$\mathscr{I}\equiv\int \mathrm{d}X_1\int \mathrm{d}X_2\int \mathrm{d}X_3\int \mathrm{d}X_4\{\exp[\tfrac{1}{4}\mathrm{i}(X_1^4-X_2^4+X_3^4-X_4^4)]\}\,\delta(X_1-X_2+X_3-X_4), \tag{59}$$

δ being the Dirac delta function.

The expression (58) shows that the critical exponent ν_2 is $\frac{1}{4}$ and it is not hard to calculate that on this model the nth critical exponent is

$$\nu_n=\frac{n}{2}-\frac{3}{4}\qquad (z=z_t). \tag{60}$$

It is interesting to compare this with the exponents in table 1 for the Gaussian random corrugated phase screen. For $n\leqslant 9$ the non-generic occurrence of infinitely many cusp points in the plane of observation $z=z_t$ causes the moments for the sinusoidal screen to exceed those for the Gaussian screen, but for $n\geqslant 10$ this effect is outweighed by the higher catastrophes possible in the Gaussian case.

To evaluate $\mathscr{I}$ (equation (59)) in terms of special functions, the following transformation is made:

$$\begin{aligned} S&\equiv X_1-X_2, & t&\equiv X_3-X_4\\ U&\equiv\frac{X_1+X_2}{2}, & V&\equiv\frac{X_3+X_4}{2}. \end{aligned} \tag{61}$$

Then use of the delta function gives

$$\begin{aligned}\mathscr{I}&=\int_{-\infty}^{\infty}\mathrm{d}S\left|\int_{-\infty}^{\infty}\mathrm{d}U\,\exp[\mathrm{i}(U^3S+\tfrac{1}{4}US^3)]\right|^2\\ &=\frac{8\pi^2}{3^{2/3}}\int_0^{\infty}\frac{\mathrm{d}S}{S^{2/3}}\mathrm{Ai}^2\left(\frac{S^{8/3}}{4\times 3^{1/3}}\right),\end{aligned} \tag{62}$$

where Ai is the Airy function (Abramowitz and Stegun 1964). Combining this with (58), integrating by parts and then performing an elementary numerical integration gives, finally

$$\overline{I^2}(z_t)=(2kF)^{1/4}2^{11/4}3^{9/8}\int_0^{\infty}\mathrm{d}z\,z^{1/8}\mathrm{Ai}(z)\left|\frac{\mathrm{dAi}(z)}{\mathrm{d}z}\right|=1{\cdot}2834(2kF)^{1/4}. \tag{63}$$

In figure 2 this short-wave asymptotic form is compared with the exact second moment (55) computed for $z=z_t$ by Dr M Tabor. The exact values lie close to a straight line whose intercept and slope fit formula (63) if the multiplier 1·2834 is replaced by 1·303 and the critical exponent $\frac{1}{4}$ by 0·247. This means that (63) as well as being the exact asymptotic form of $\overline{I^2}$ as $2kF\to\infty$ is actually an extremely accurate approximation even when $2kF$ is of order unity.

For z-values close to the plane of focus, that is

$$z=z_t(1+h)\qquad (|h|\ll 1), \tag{64}$$

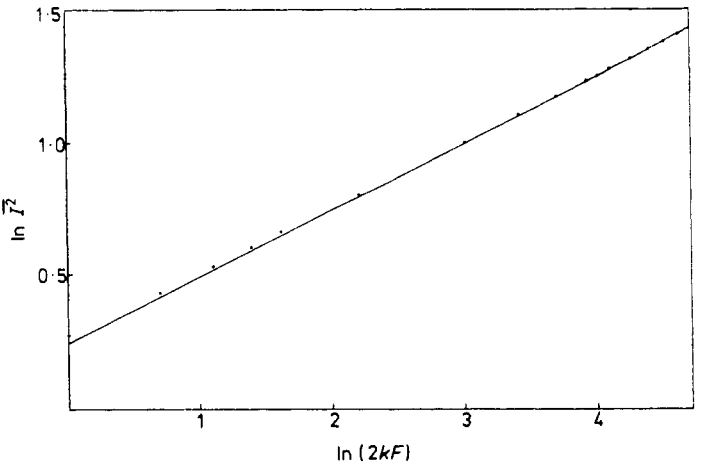

Figure 2. $\overline{I^2}$ in the focusing plane of a sinusoidal phase screen, as a function of the maximum phase shift $2kF$. The points are exact values computed from equations (55) by Dr M Tabor, and the line is the short-wave asymptotic form (63).

a similar analysis gives the short-wave limiting form of $\overline{I^2}$ as

$$\overline{I^2}(z_f(1+h)) = (3\sqrt{2})(2kF)^{1/4} \int_0^\infty \frac{\mathrm{d}y}{y^{1/2}} \mathrm{Ai}^2\left(\frac{y^4}{12} - hy\sqrt{(2kF)}\right). \tag{65}$$

Of course when $h = 0$ this simply reproduces the result (63). But when $|h\sqrt{(2kF)}| \gg 1$, that is at distances from the focal plane that are large on the scale of variation of the 'cusp' diffraction pattern, the form of $\overline{I^2}$ depend on the sign of h, and asymptotic analysis of equation (65) yields

$$\overline{I^2} \to \begin{cases} \dfrac{1}{\sqrt{(2|h|)}} & (h\sqrt{(2kF)} \ll -1) \\ \dfrac{1}{\pi}\left(\dfrac{2}{h}\right)^{1/2} \ln(2kF\sqrt{h}) & (h\sqrt{(2kF)} \gg 1). \end{cases} \tag{66}$$

For negative h, $\overline{I^2}$ does not depend on k and so remains finite in the geometrical optics limit; this is because there are no caustics for $z < z_f$. For positive h the presence of caustic lines (fold catastrophes) emanating from the cusps causes $\overline{I^2}$ to diverge logarithmically as discussed in § 1; this corresponds to a critical exponent $\nu_2 = 0$ and since $N = 1$ the other critical exponents are given from equation (41) as

$$\nu_n = \tfrac{1}{3}(n-2) \qquad (z > z_f) \tag{67}$$

(cf equation (60)).

7. Discussion

The analysis presented here reveals a quite extraordinary complexity in the short-wave asymptotic structure of the intensity moments of a random field. In addition to the dominant term which for most cases has the form of equation (9) there will be terms similar in form but with smaller exponents from the catastrophes that fail to

dominate. Moreover each of these terms will be the multiplier of an asymptotic series in falling powers of k. If the multiplier C_n in the dominant term is small k must be extremely large before the dominance becomes apparent; for k smaller (but still large enough for the asymptotic analysis to hold) $\overline{I^n}$ will seem to be dominated by the catastrophe with largest ν_{nj} in the set with large multipliers.

An experimental test of the 'universal' exponents would require a source of disorder with a continuous spectrum (and hence an infinite number of random phases) whose smallest scale is much larger than the largest wavelength employed. An obvious source of this type of disorder is turbulence in a refracting gas or liquid, and the illuminating wave could be light from a tunable laser or starlight viewed through a range of coloured filters on a night of 'bad seeing' (i.e. strong presumably non-Gaussian intensity fluctuations). Such an experiment, in which $\overline{I^n}$ was measured as a function of k, would provide a test of the predicted critical exponents for waves in three-dimensional space as listed in table 3.

The assumption that the incident wave must be plane is necessary in practice rather than in principle. For an incoherent bundle of plane waves, such as light from the full $\frac{1}{2}^\circ$ width of the Sun's disc, the caustics produced by refraction in a random medium move so rapidly that practical detectors are unable to respond to all the variations in intensity and therefore record a smoothed intensity whose moments will remain finite as $k \to \infty$, unlike the moments $\overline{I^n}$ of the true intensity. It is this 'incoherence' effect, rather than the 'diffraction' discussed throughout this paper, that is responsible for the blurring of fine detail in the random caustic patterns on the bottom of swimming pools (Hannay 1976†, Berry and Nye 1977). The intensity moments from scintillating extended sources of radio waves were discussed by Salpeter (1967). If the random medium or phase screen is dispersive, any non-monochromaticity in the incident wave will cause blurring of the caustics in the direction of the incident wave (chromatic aberration) and will also reduce the moments observed with practical detectors; this effect is responsible for red stars appearing to twinkle more strongly than white ones (Minnaert 1954).

The dependence of ν_n on the dimensionality of T_N could form the basis of an experimental technique for distinguishing media whose disorder has a discrete spectrum of wavenumbers u, v (equation (10)) from media whose disorder is Gaussian, with a continuous spectrum ($N = \infty$). However a delicate question of limits arises for media with disorder whose spectrum is continuous but sharply concentrated near a finite number (N) of u, v values. In such cases the critical exponents would have the universal values of table 2 or table 3, but these would appear only in the limit of large k; for k smaller (but still large) the moments would appear to have the critical exponents characteristic of media with N random phases.

It should be obvious that the work described here is the beginning of a substantial programme rather than a finished theory. There are four directions in particular in which further study should be concentrated. First, the disposition of caustics on the tori T_N should be studied in detail, starting with T_2 which is certainly tractable and corresponds to the simplest random phase screen (with two random phases and incommensurable wavenumbers). Second, techniques should be developed for calculating the measures of the different types of catastrophe on T_N (i.e. B_j in equation (22)). Third, the convergence of the 'weight' integrals J_{jn} (equation (33)) should be

† Hannay J H 1976 *Paraxial Optics and Statistical Problems of Wave Propagation*, University of Cambridge, Hamilton Prize Essay, unpublished.

examined and those that converge should be computed. And fourth, the subtle question of which hyperbolic catastrophes X contribute to critical exponents in three space dimensions (§ 5) should be resolved.

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Appendix 1

This is the derivation of the formula (7) for the wave ψ beyond the random phase screen given by (6). All plane wave components of ψ either travel or decay towards $z=+\infty$ for $z>0$, so ψ can be expanded in the form

$$\psi(\boldsymbol{R}, z, t)=\iint \mathrm{d}\boldsymbol{Q} \int \mathrm{d}\Omega\, a(\boldsymbol{Q}, \Omega) \exp[\mathrm{i}(\boldsymbol{Q}\cdot\boldsymbol{R}-\Omega t+q_z z)], \tag{A.1}$$

where $\boldsymbol{R}$ denotes (x, y), $\boldsymbol{Q}$ denotes (Q_x, Q_y) and

$$q_z \equiv \begin{cases} +\left(\dfrac{\Omega^2}{c^2}-Q^2\right)^{1/2} & (\Omega>cQ) \\ +\mathrm{i}\left(Q^2-\dfrac{\Omega^2}{c^2}\right)^{1/2} & (\Omega<cQ). \end{cases} \tag{A.2}$$

By Fourier inversion of (A.1) at $z=0$ and use of the boundary condition

$$\psi(\boldsymbol{R}, 0, t)=\exp[\mathrm{i}(kf(x, y, t)-\omega t)], \tag{A.3}$$

$a(\boldsymbol{Q}, \Omega)$ is found to be

$$a(\boldsymbol{Q}, \Omega)=\frac{1}{(2\pi)^3}\int \mathrm{d}\xi \int \mathrm{d}\eta \int \mathrm{d}\tau\, \exp\{\mathrm{i}[kf(\xi, \eta, \tau)-Q_x\xi-Q_y\eta+(\Omega-\omega)\tau]\}, \tag{A.4}$$

which when substituted into (A.1) gives ψ. The paraxial approximation $|\partial f/\partial\xi|\ll 1$ and $|\partial f/\partial\eta|\ll 1$ means that all significant plane wave components travel in directions making small angles with the z direction, so that g_z can be expanded as

$$q_z \approx \frac{\Omega}{c}-\frac{Q^2 c}{2\Omega}. \tag{A.5}$$

The $\boldsymbol{Q}$ integration in (A.1) is now elementary and gives

$$\psi=\frac{-\mathrm{i}}{(2\pi)^2 cz}\int \mathrm{d}\xi \int \mathrm{d}\eta \int \mathrm{d}\tau\, \exp[\mathrm{i}(kf(\xi, \eta, \tau)-\omega\tau)] \int \mathrm{d}\Omega\, \Omega \exp\left[\mathrm{i}\Omega\left(\tau-t+\frac{z}{c}+\frac{(x-\xi)^2+(y-\eta)^2}{2cz}\right)\right]. \tag{A.6}$$

The integration over Ω gives the derivative of a delta function and leads to

$$\psi = \frac{-\mathrm{i}}{2\pi cz} \int \mathrm{d}\xi \int \mathrm{d}\eta \left(\mathrm{i}\frac{\partial}{\partial \tau} \exp[\mathrm{i}(kf(\xi, \eta, \tau) - \omega\tau)] \right)_{\tau = t - z/c - \{[(x-\xi)^2 + (y-\eta)^2]/2cz\}} \tag{A.7}$$

Now it is assumed that the phase screen shivers much more slowly than the wave travels, i.e. $\partial f/\partial \tau \ll \omega/k$ (quasi-monochromaticity) and then differentiating with respect to τ immediately gives (7).

Appendix 2

This is the derivation of the expression (55) for the second moment beyond the sinusoidal phase screen whose form is given by equation (54). The diffraction integral (7) and ensemble average (11) give

$$\overline{I^2} = \frac{k^2}{4\pi^2 z^2} \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{d}\theta \left| \int_{-\infty}^{\infty} \mathrm{d}\xi \exp\left[\mathrm{i}k\left(F \cos(u\xi + \theta) + \frac{\xi^2}{2z} \right) \right] \right|^4. \tag{A.8}$$

On using the standard relation

$$\exp(\mathrm{i}\alpha \cos \beta) = \sum_{n=-\infty}^{\infty} \mathrm{i}^n [\exp(\mathrm{i}n\beta)] J_n(\alpha) \tag{A.9}$$

the ξ integration becomes elementary and leads to

$$\overline{I^2} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{d}\theta \sum_{n_1} \sum_{n_2} \sum_{n_3} \sum_{n_4} \exp\left[\mathrm{i}\left(\theta(n_1 - n_2 + n_3 - n_4) - \frac{u^2 z}{2k}(n_1^2 - n_2^2 + n_3^2 - n_4^2) \right) \right]$$
$$\times J_{n_1}(kF) J_{n_2}(kF) J_{n_3}(kF) J_{n_4}(kF). \tag{A.10}$$

The θ integration gives a Kronecker delta over the n and rearranging the summations and setting $n_1 - n_2 = m$ gives

$$\overline{I^2} = \sum_{m=-\infty}^{\infty} \left[\sum_{n_2=-\infty}^{\infty} J_{n_2}(kF) J_{n_2+m}(kF) \exp\left(\frac{\mathrm{i}u^2 z}{k} n_2 m \right) \right]$$
$$\times \left[\sum_{n_3=-\infty}^{\infty} J_{n_3}(kF) J_{n_3+m}(kF) \exp\left(\frac{-\mathrm{i}u^2 z}{k} n_3 m \right) \right]. \tag{A.11}$$

The expression (55) follows immediately on using the standard relation

$$\sum_{n=-\infty}^{\infty} J_n(q) J_{n+m}(q) \exp(\mathrm{i}pn) = J_m\left(2q \sin \frac{p}{2} \right) \exp\left(\mathrm{i}\frac{p}{2} m \right) \tag{A.12}$$

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